Quadratically Tight Relations for Randomized Query Complexity

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Outline

Query Complexity

- We want to compute some Boolean function $f : \{0,1\}^n \to \{0,1\}$.
- The input is $x = (x_1, \ldots, x_n)$.
- With a single query we can ask the value of any x_i.
- The cost of the computation is the number of queries made.

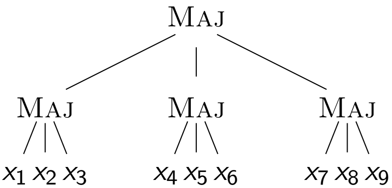
Query Complexity

- Determistic query complexity $\mathrm{D}(f)$ (minimum worst-case number of queries).
- Randomized query complexity $\mathrm{R}(f)$ (correct with probability $\geq 2/3$).
- Exact randomized query complexity $\mathrm{R}_0(f)$ (minimum worst-case expected number of queries).

$$\mathrm{R}(f) \leq \mathrm{R}_0(f) \leq \mathrm{D}(f).$$

Example: Recursive Majority

- $3\text{-}\textsc{Maj}(x_1, x_2, x_3) = 1 \iff x_1 + x_2 + x_3 \geq 2.$
- $\mathsf{D}(3\text{-}\textsc{Maj}_h) = 3^h.$
- $\mathsf{R}_0(3\text{-}\textsc{Maj}_h) \leq (8/3)^h.$



Query Complexity

- In this work, we study which measures $\mathrm{M}(f)$ can characterize $\mathrm{R}_0(f)$ or $\mathrm{R}(f)$ quadratically: $\mathrm{M}(f) \leq \mathrm{R}(f) \leq \mathrm{M}(f)^2$?
- We show two results:
  1. The *expectational certificate* complexity bounds $\mathrm{R}_0(f)$ quadratically:

$$\mathrm{EC}(f) \leq \mathrm{R}_0(f) \leq O(\mathrm{EC}(f)^2).$$

  2. The *partition bound* bounds $\mathrm{R}(f)$ quadratically for product distributions μ:

$$\mathrm{D}^{\mu}_{1/3}(f) \leq O(\mathrm{prt}_{1/3}(f)^2).$$

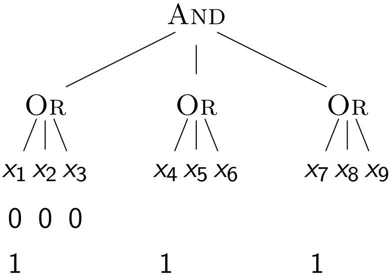
Outline

Certificate Complexity

- A *certificate* for an input x is a set of positions of x that have to be revealed to know the value of $f(x)$ with certainty.
- The *length* of a certificate is the number of positions revealed.
- A *minimal* certificate of x is a certificate of smallest length $\mathrm{C}(f,x)$.
- The *certificate complexity* of f is $\mathrm{C}(f) = \max_x \mathrm{C}(f,x)$.
- It is known that $\mathrm{C}(f) \leq \mathrm{R}_0(f) \leq \mathrm{C}(f)^2$.

Example: AND-OR

- $\mathrm{C}(\textsc{And-Or}_n) = \sqrt{n}$.



Fractional Certificate Complexity

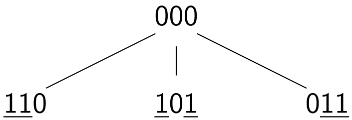
- Fractional certificate complexity $\mathrm{FC}(f)$ is given by the optimal value of the following LP: [Tal / Gilmer, Saks, Srinivasan]

$$\begin{array}{ll} \text{minimize} & \max\limits_{x} \sum\limits_{i \in [n]} w_x(i) \\ \text{subject to} & \forall x, y \text{ s.t. } f(x) \neq f(y) : \sum\limits_{i : x_i \neq y_i} w_x(i) \geq 1 \\ & \forall x, i : 0 \leq w_x(i) \leq 1. \end{array}$$

- $\mathrm{FC}(f) \leq \mathrm{C}(f)$.
- It is known that $\mathrm{FC}(f) \leq \mathrm{R}(f) \leq \mathrm{R}_0(f) \leq \mathrm{FC}(f)^3$.

Example: Majority

- $3\text{-}\mathrm{MAJ}(x_1, x_2, x_3) = 1 \iff x_1 + x_2 + x_3 \geq 2$.
- $\mathrm{C}(f, 000) = 2$.
- $\mathrm{FC}(f, 000) = 3/2$.
  Weights $w_1 = w_2 = w_3 = 1/2$.



Fractional Certificate Complexity

- Hypothesis: $\mathrm{R}_0(f) \leq \mathrm{FC}(f)^2$.
- If that is true, then $\mathrm{R}_0(f) \leq \mathrm{Q}(f)^4$. (Quantum query complexity.) Currently the best upper bound is $\mathrm{R}_0(f) \leq \mathrm{Q}(f)^6$.
- A quadratic separation is known, $\mathrm{R}(\textsc{And-Or}_n) = \Omega(n)$, $\mathrm{FC}(\textsc{And-Or}_n) = \sqrt{n}$.

Expectational Certificate Complexity

- Expectational certificate complexity $\mathrm{EC}(f)$ is given by the optimal value of the following program:

$$\begin{aligned} \text{minimize} \quad & \max_x \sum_{i \in [n]} w_x(i) \\ \text{subject to} \quad & \forall y \text{ s.t. } f(x) \neq f(y) : \sum_{i : x_i \neq y_i} w_x(i) w_y(i) \geq 1, \\ & \forall x, i : 0 \leq w_x(i) \leq 1. \end{aligned}$$

- Not a linear program anymore!

Expectational Certificate Complexity

$\mathrm{R}(f) = O(\mathrm{EC}(f)^2)$ algorithm:

- Repeat $O(\mathrm{EC}(f))$ times:
  - Pick any consistent (with previous queries) input z s.t. $f(z) = 1$;
  - If there is no such z, **return 0**.
  - Independently query each x_i with probability $w_z(i)$;
- **Return 1**.

Each round takes $\sum_{i=1}^{n} w_z(i) \leq \mathrm{EC}(f)$ queries on expectation; hence query complexity is $O(\mathrm{EC}(f)^2)$.

Expected amount of weight removed from w_x each round is $\sum_{i:x_i \neq z_i} w_x(i) w_z(i) \geq 1$; hence, $O(\mathrm{EC}(f))$ many rounds is enough.

Expectational Certificate Complexity

Properties:

- $\mathrm{FC}(f) \leq \mathrm{EC}(f) \leq \mathrm{C}(f)$.
- $\mathrm{EC}(f) \leq \mathrm{C}(f)^2$, tight!
- $\mathrm{EC}(f) \leq \mathrm{R}_0(f) \leq O(\mathrm{EC}(f)^2)$.
- $\mathrm{EC}(f) \leq O(\mathrm{FC}(f)^{3/2})$.
- $\mathrm{EC}(f)^{2/3} \leq \mathrm{R}(f) \leq O(\mathrm{EC}(f)^2)$.

Outline

1 Query Complexity

2 Expectational Certificate Complexity

3 Partition Bound

Partition Bound

- The ϵ-partition bound of f (denoted by $\mathrm{prt}_\epsilon(f)$), is given by the $\log_2$ of the optimal value of the following LP: [Jain, Klauck]

$$\text{minimize} \sum_{z,A} w_{z,A} \cdot 2^{|A|} \quad \text{subject to} \quad \begin{aligned} &\forall x : \sum_{A \ni x} w_{f(x),A} \geq 1 - \epsilon, \\ &\forall x : \sum_{z, A \ni x} w_{z,A} = 1, \\ &\forall z, A : w_{z,A} \geq 0. \end{aligned}$$

- Lower bound, $\frac{1}{2}\,\mathrm{prt}_\epsilon(f) \leq \mathrm{R}_\epsilon(f)$.

Partition Bound

- Example: $\mathrm{prt}(\textsc{And-Or}_n) = \Omega(n)$.
- Known that $\mathrm{R}(f) = O(\mathrm{prt}(f))^3$.
- Best separation is quadratic, $\mathrm{R}(f) = \Omega(\mathrm{prt}(f)^2)$. [Ambainis, Kokainis, Kothari]
- Is $\mathrm{prt}(f)$ quadratically tight for $\mathrm{R}(f)$?

Distributional Query Complexity

- Let μ be a probability distribution over inputs $\{0,1\}^n$.
- Distributional query complexity $\mathrm{D}_\epsilon^\mu(f)$ is the minimum worst-case cost of a deterministic algorithm $\mathcal{A}$ such that

$$\Pr_{x\sim\mu}[\mathcal{A}(x) = f(x)] \geq 1 - \epsilon.$$

- Yao's theorem:

$$\mathrm{R}_\epsilon(f) = \max_\mu \mathrm{D}_\epsilon^\mu(f).$$

Block Sensitivity

- An input x is *sensitive* on a subset of positions $B \subseteq [n]$, if $f(x) \neq f(x^B)$.
- The *block sensitivity* of x, denoted by $\mathrm{bs}(f, x)$, is the maximum number of disjoint sensitive blocks.
- The block sensitivity of f is $\max_x \mathrm{bs}(f, x)$.

Corruption Bound

- Let μ be a probability distribution over the inputs.
- Let A be an ϵ-error b-certificate under μ, if

$$\Pr_{x \sim \mu}[f(x) \neq b \mid x \in A] \leq \epsilon.$$

- Query corruption bound:

$$\mathrm{corr}_{\epsilon}^{b,\mu}(f) = \min\{|A| \mid A \text{ is an } \epsilon\text{-error } b\text{-certificate under } \mu\}.$$

- Query corruption bound:

$$\mathrm{corr}_{\epsilon}(f) = \max_{\mu} \max_{b} \mathrm{corr}_{\epsilon}^{b,\mu}(f).$$

Corruption Bound

- Minimum query corruption bound over product distributions:

$$\mathrm{corr}^{\times}_{\min,\epsilon}(f) = \max_{\mu} \min_{b} \mathrm{corr}^{b,\mu}_{\epsilon}(f),$$

  where μ is a product distribution.

- μ is a bit-wise product distribution if for all x,

$$\mu(x) = \prod_{i=1}^{n} \mu_i(x_i).$$

Corruption Bound

- We adapt the proof of $\mathrm{D}(f) \leq \mathrm{C}(f)\,\mathrm{bs}(f)$ to prove that

$$\mathrm{D}^{\mu}_{4\epsilon}(f) = O(\mathrm{corr}^{\times}_{\min,\epsilon}(f) \cdot \mathrm{bs}(f))$$

  for product distributions.

- Since $\mathrm{corr}^{\times}_{\min,\epsilon}(f) \leq \mathrm{corr}_{\epsilon}(f)$ and $\mathrm{bs}(f) \leq \mathrm{corr}_{\epsilon}(f)$, we get

$$\mathrm{D}^{\mu}_{4\epsilon}(f) = O(\mathrm{corr}_{\epsilon}(f)^2).$$

Partition Bound

- Since $\mathrm{bs}(f) = O(\frac{1}{\epsilon}\,\mathrm{prt}_\epsilon(f))$ and $\mathrm{corr}^{\times}_{\min,2\epsilon}(f) \le \mathrm{prt}_\epsilon(f)$, we get

$$\mathrm{D}^{\mu}_{8\epsilon}(f) = O\left(\frac{1}{\epsilon}\mathrm{prt}_\epsilon(f)^2\right).$$

- A polylogarithmic improvement over previous best upper bound; constant error instead of inverse polynomial error. [Harsha, Jain, Radhakrishnan]

Lower Bounds

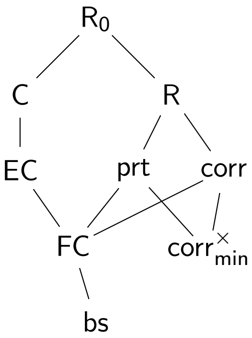


Figure: Lower bounds on $R_0(f)$ and $R(f)$.

Thank you!

Questions?