Maximum colorful cycles in vertex-colored graphs

Yannis Manoussakis, Giuseppe F. Italiano, Nguyen Kim Thang, HongPhong PHAM

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Outline

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1. Introduction
2. Hardness results
3. Algorithm for bipartite chain graphs
4. Algorithm for threshold graphs
5. Future works

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- Some past works: tropical dominating sets[JA Angles,2015], tropical connected components[JA Angles, 2016], maximum colorful matchings[J Cohen,2017] and maximum colorful paths[J Cohen, 2017].
- This work: Maximum Colorful Cycle Problem (MCCP): look for a cycle with the maximum number of colors possible in a vertex-colored graph.

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- The longest path problem is easy for threshold graphs [Mahadev,1994] and bipartite chain graphs [Uehara, 2007]

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  - Reduce from the MAX-SAT problem
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    - x_i appears in clauses $b_{i1}, b_{i2}, \ldots, b_{i\alpha_i}$, $\overline{x_i}$ appears in clauses $b'_{i1}, b'_{i2}, \ldots, b'_{i\beta_i}$

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  - Construct an intersection model and its interval graph G^c, each clause by a distinct color.
  - Assign *true/false* values for $x_i's$ to obtain the maximum number of satisfied clauses $\equiv$ a cycle with the maximum number of colors in G^c.

Hardness results for MCCP

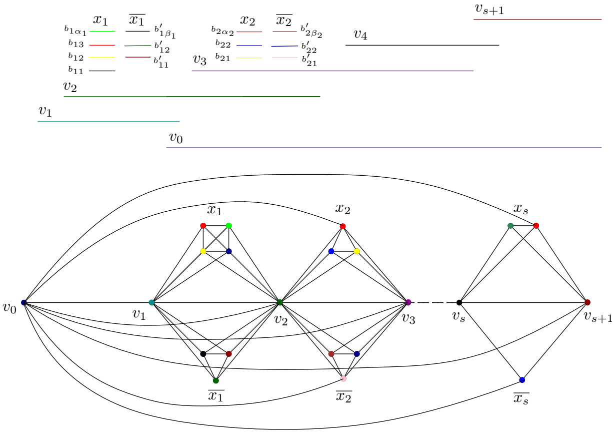

Figure: Reduction of the MAX SAT problem to MCCP for interval graphs.

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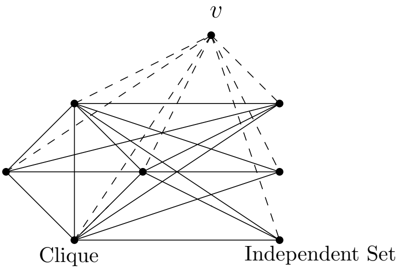
  - Use the fact: the longest path problem is NP-hard for split graphs.



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Algorithm for bipartite chain graphs

- A bipartite chain graph $G^c = (X, Y, E)$ if: $N(x_1) \supseteq N(x_2) \supseteq \ldots \supseteq N(x_{|X|})$. Similarly, we have $N(y_1) \supseteq N(y_2) \supseteq \ldots \supseteq N(y_{|Y|})$.

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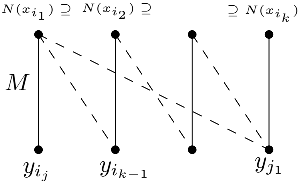
- C_c and C_m : the maximum number colors of cycles and matchings of G^c, respectively.
- *Lemma 2*: $C_m - 2 \leq C_c \leq C_m$



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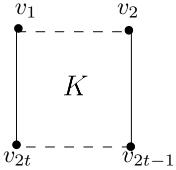

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- Define $X_{m,n}^{\ell} = \{x_{\min[c_{11}]}, x_{\min[c_{12}]}, \ldots, x_{\min[c_{1\ell}]}\}$: ℓ first vertices with distinct colors in $X_{m,n}$.

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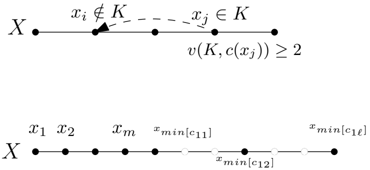
- Key lemma: Let K be a maximum colorful cycle of G^c, then there exists another maximum colorful cycle K' where $V(K') = \{x_1, x_2, \ldots, x_m\} \cup X^{\ell}_{m,n} \cup \{y_1, y_2, \ldots, y_n\} \cup Y^{\ell'}_{m,n}$ with $1 \leq m \leq |X|$, $1 \leq n \leq |Y|$ and $0 \leq \ell = |X_K| - m, 0 \leq \ell' = |Y_K| - n$.

Algorithm for bipartite chain graphs

- Main ideas for key lemma: Replace x_j by x_i if: $i < j$ and $x_i \notin K$ and $x_j \in K$ and $v(K, c(x_j)) \geq 2$ => obtain $x_1, x_2, \ldots, x_m$.



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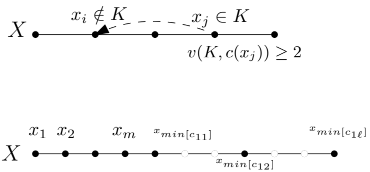
  - Replace $\ell = |X_K| - m$ remaining vertices of X_K by vertices $X^{\ell}_{m,n} = \{x_{\min[c_{11}]}, x_{\min[c_{12}]}, \ldots, x_{\min[c_{1\ell}]}\}$.



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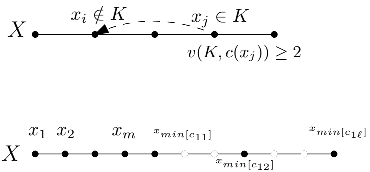

- Similarly for Y ...

Algorithm for bipartite chain graphs

Algorithm 2 Maximum colorful cycle in vertex-colored bipartite chain graphs.

$C_m \leftarrow$ the number of colors of a maximum colorful matching (using algorithm [5])
for $C_m \geq C_c \geq C_m - 2$ **do**
  for $1 \leq m \leq |X|, 1 \leq n \leq |Y|$ **do**
    $X_{m,n} \leftarrow \{x_i \in X | c(x_i) \notin \mathcal{C}(\{x_1, x_2, \ldots, x_m\}) \cup \mathcal{C}(\{y_1, y_2, \ldots, y_n\})\}$
    $Y_{m,n} \leftarrow \{y_j \in Y | c(y_j) \notin \mathcal{C}(\{x_1, x_2, \ldots, x_m\}) \cup \mathcal{C}(\{y_1, y_2, \ldots, y_n\})\}$
    Denote $\mathcal{C}(X_{m,n}) := \{c_{11}, c_{12}, \ldots, c_{1k_1}\}$ and $\mathcal{C}(Y_{m,n}) := \{c_{21}, c_{22}, \ldots, c_{2k_2}\}$.
    for $0 \leq \ell \leq C_c$ **do**
      $\ell' \leftarrow \max\{C_c - \ell - |\mathcal{C}(x_1, x_2, ..., x_m)| - |\mathcal{C}(y_1, y_2, ..., y_n)|, 0\}$
      $X^{\ell}_{m,n} \leftarrow \{x_{\min[c_{11}]}, x_{\min[c_{12}]}, \ldots, x_{\min[c_{1\ell}]}\}$ the set of ℓ first vertices (in the ordering of x-vertices) with distinct colors in $X_{m,n}$.
      $Y^{\ell'}_{m,n} \leftarrow \{y_{\min[c_{21}]}, y_{\min[c_{22}]}, \ldots, y_{\min[c_{2\ell'}]}\}$ the set of ℓ' first vertices (in the ordering of y-vertices) with distinct colors in $Y_{m,n}$
      if $\exists$ a Hamiltonian cycle K of $\{x_1, x_2, \ldots, x_m\} \cup X^{\ell}_{m,n} \cup \{y_1, y_2, \ldots, y_n\} \cup Y^{\ell'}_{m,n}$ **then**
        return K as the maximum colorful cycle
      end if
    end for
  end for
end for

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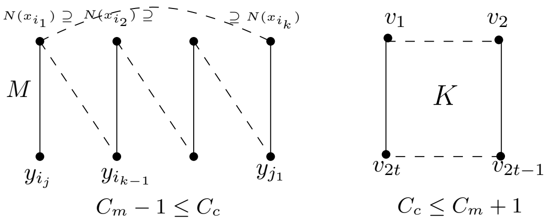
  - Lemma 4: $C_m - 1 \leq C_c \leq C_m + 1$



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- $\mathcal{C}_1 := C(Y) \setminus C(X) = \{c_{11}, c_{12}, \ldots, c_{1k_1}\}$
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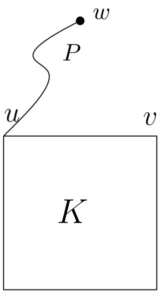

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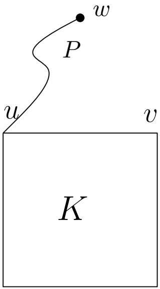

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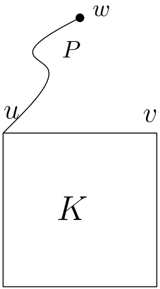
  - Consider $P' = P \cup K \setminus (u, w)$, then P' has $|P'| \geq C_m + 2$ colors and $M \in P'$.



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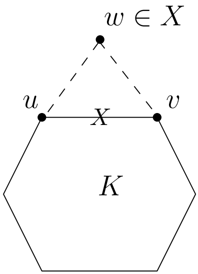

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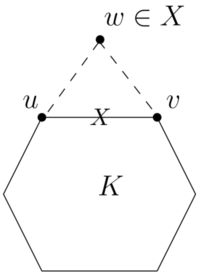

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    Moreover, the set of dominating vertices $X_K = X^+(v^*)$ and the number of isolated vertices $|Y_K| = |X^+(v^*)|$.

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  - Denote $\{c'_{11}, c'_{12}, \ldots, c'_{1k'}\} = C(Y) \setminus C(X^+(v^*))$.

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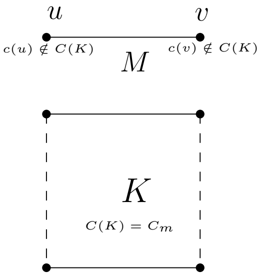
    $$|X^+(v^*)| + |C(X^+(v^*))| = C_m + 1$$

    Moreover, the set of dominating vertices $X_K = X^+(v^*)$ and the number of isolated vertices $|Y_K| = |X^+(v^*)|$.

  - Denote $\{c'_{11}, c'_{12}, \ldots, c'_{1k'}\} = C(Y) \setminus C(X^+(v^*))$.
  - *Lemma 9: There exists another maximum colorful cycle K' where $V(K') = X^+(v^*) \cup \{v_{\min[c'_{11}]}, v_{\min[c'_{12}]}, \ldots, v_{\min[c'_{1|X^+(v^*)|}]}\}$.*

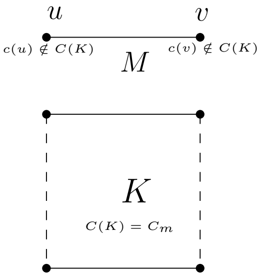
Case 2: $C_c = C_m$

- *Lemma 10: For any maximum colorful cycle K, there exists at most one dominating vertex v such that $v \notin K$ and $c(v) \notin (K)$.*



Case 2: $C_c = C_m$

- *Lemma 10: For any maximum colorful cycle K, there exists at most one* <u>*dominating*</u> *vertex v such that $v \notin K$ and $c(v) \notin (K)$.*
  - By contradiction, then there exists a matching M with $\geq C_m + 1$ colors.



Case 2: $C_c = C_m$

- *Case 2.1: There exists a maximum colorful cycle K such that there exists exactly one dominating vertex v^{**} such that $v^{**} \notin K$ and $c(v^{**}) \notin C(K)$.*

Case 2: $C_c = C_m$

- *Case 2.1: There exists a maximum colorful cycle K such that there exists exactly one dominating vertex v^{**} such that $v^{**} \notin K$ and $c(v^{**}) \notin C(K)$.*
    - *Lemma 11: There exists another colorful cycle K' such that $V(K') = V(X) \setminus \{v^{**}\} \cup \{v_{\min[c'_{\mathbf{11}}]}, v_{\min[c'_{\mathbf{12}}]}, \ldots, v_{\min[c'_{\mathbf{1}\ell}]}\}$ where $\ell = C_m - |X| + 1$.*

Case 2: $C_c = C_m$

- *Case 2.1: There exists a maximum colorful cycle K such that there exists exactly one dominating vertex v^{**} such that $v^{**} \notin K$ and $c(v^{**}) \notin C(K)$.*
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- *Case 2.2: For any maximum colorful cycle K, there does not exist any dominating vertex v such that $v \notin K$ and $c(v) \notin C(K)$.*

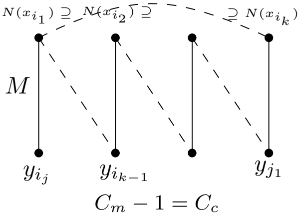
Case 2: $C_c = C_m$

- *Case 2.1: There exists a maximum colorful cycle K such that there exists exactly one dominating vertex v^{**} such that $v^{**} \notin K$ and $c(v^{**}) \notin C(K)$.*
  - *Lemma 11: There exists another colorful cycle K' such that $V(K') = V(X) \setminus \{v^{**}\} \cup \{v_{\min[c'_{11}]}, v_{\min[c'_{12}]}, \ldots, v_{\min[c'_{1\ell}]}\}$ where $\ell = C_m - |X| + 1$.*

- *Case 2.2: For any maximum colorful cycle K, there does not exist any dominating vertex v such that $v \notin K$ and $c(v) \notin C(K)$.*
  - *Lemma 12: There exists another colorful cycle K' where $V(K') = X_t(G^c) \cup \{v_{\max[c_{21}]}, v_{\max[c_{22}]}, \ldots, v_{\max[c_{2k_2}]}\}$ $\cup\{v_{\min[c_{11}]}, v_{\min[c_{12}]}, \ldots, v_{\min[c_{1\ell}]}\}$ where $\ell = C_m - |X|$ and $t = |V(K)| - k_2 - C_m + |X|$.*

Case 2: $C_c = C_m - 1$

- Easily construct a cycle with $C_m - 1$ colors from any maximum colorful matching based on the order of dominating vertices



Algorithm for threshold graphs

Algorithm 1 Maximum colorful cycle in vertex-colored threshold graphs.

$C_m \leftarrow$ the number of colors of a maximum colorful matching (using algorithm [5])
$\mathcal{C}_1 := \mathcal{C}(Y) \setminus \mathcal{C}(X) = \{c_{11}, c_{12}, \ldots, c_{1k_1}\}$ and $\mathcal{C}_2 := \mathcal{C}(X) = \{c_{21}, c_{22}, \ldots, c_{2k_2}\}$
if $\exists$ a Hamiltonian cycle K of $V(X) \cup \{v_{\min[c_{11}]}, \ldots, v_{\min[c_{1k_1}]}\}$ **then** # `Case 1.1`
  return K as the maximum colorful cycle # `Lemma 4`
else # `Case 1.2`
  $v^* \leftarrow$ the unique vertex satisfying $|X^+(v^*)| + |C(X^+(v^*))| = C_m + 1$
  $X^+(v^*) \leftarrow$ set of dominating vertices added to the graph after v^*
  $\{c'_{11}, c'_{12}, \ldots, c'_{1k'}\} \leftarrow \mathcal{C}(Y) \setminus \mathcal{C}(X^+(v^*))$
  if $\exists$ a Hamiltonian cycle K of $X^+(v^*) \cup \{v_{\min[c_{11}]}, v_{\min[c_{12}]}, \ldots, v_{\min[c_{1k_1}]}\}$ **then**
    return K as the maximum colorful cycle # `Lemma 6`
  end if
end if
for $v^{**} \in V(G^c)$ **do** # `Case 2.1`
  $\{c'_{11}, c'_{12}, \ldots, c'_{1k'}\} \leftarrow \mathcal{C}(Y) \setminus \mathcal{C}(X) \cup \{c(v^{**})\}$ and $\ell \leftarrow C_m - |\mathcal{C}(X)| + 1$
  if $\exists$ a Hamiltonian cycle K of $V(X) \setminus \{v^{**}\} \cup \{v_{\min[c'_{11}]}, v_{\min[c'_{12}]}, \ldots, v_{\min[c'_{1\ell}]}\}$ **then**
    return K as the maximum colorful cycle # `Lemma 8`
  end if
end for
for $0 \leq t \leq |V(X) \setminus \{v_{\max[c_{21}]}, v_{\max[c_{22}]}, \ldots, v_{\max[c_{2k_2}]}\}|$ **do** # `Case 2.2`
  $X_t(G^c) \leftarrow t$ last vertices in $V(X) \setminus \{v_{\max[c_{21}]}, v_{\max[c_{22}]}, \ldots, v_{\max[c_{2k_2}]}\}$
  if $\exists$ a Hamiltonian cycle K of $X_t(G^c) \cup \{v_{\max[c_{21}]}, v_{\max[c_{22}]}, \ldots, v_{\max[c_{2k_2}]}\} \cup \{v_{\min[c_{11}]}, v_{\min[c_{12}]}, \ldots, v_{\min[c_{1\ell}]}\}$ where $\ell = C_m - |\mathcal{C}(X)|$ **then**
    return K as the maximum colorful cycle # `Lemma 9`
  end if
end for
return K as a maximum colorful cycle constructed from any maximum colorful matching based on Lemma 1 # `Case 3`

Outline

1. Introduction
2. Hardness results
3. Algorithm for bipartite chain graphs
4. Algorithm for threshold graphs
5. Future works

Future works

- Study on sufficient and necessary conditions for the existence of tropical structures.
  - Parameters: the number of colors, degrees of vertices, the number of vertices and edges, etc.

Future works

- Study on sufficient and necessary conditions for the existence of tropical structures.
  - Parameters: the number of colors, degrees of vertices, the number of vertices and edges, etc.

- Work on others unsolved maximum colorful problems such as maximum colorful independent set/cliques, or for edge-colored graphs.

Thank you for your attention!