Recognizing read-once functions from depth-3 formulas

Alexander Kozachinskiy

National Research University Higher School of Economics

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  NO: check all formulas.

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Special case: read-once (read-1) formulas.

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Complexity of ROR

ROR $\in \Sigma_2^P$ (observation).

ROR $\in$ coNP (follows from [Gur77]).

More detalied complexity of ROR:

- depth-2 ROR $\in$ P, $O(nl)$-time algorithm [GMR08].
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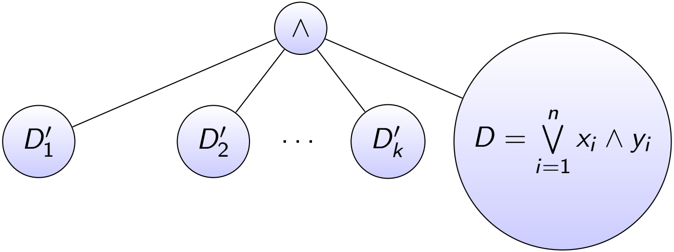

Here $D'_1, \ldots, D'_k$ are read-once disjoint DNF's over $x_1, y_1, \ldots, x_n, y_n$.

Corollary

ROR *is coNP-complete even for inputs of the form* $A \wedge D$*, where* A *is a monotone read-once* $\bigwedge - \bigvee - \bigwedge$ *formula and* $D = \bigvee_{i=1}^{n} x_i \wedge y_i$.

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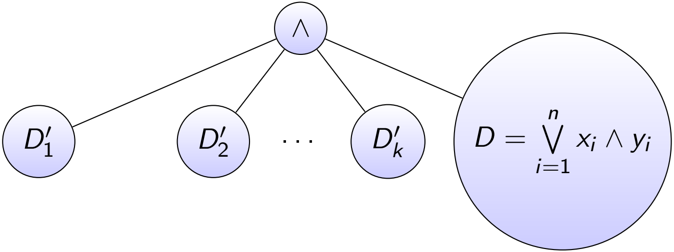

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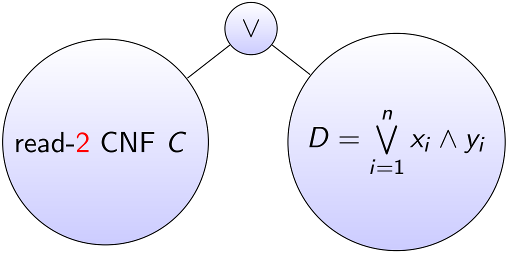
Gurvich's hardness result in more detail.



Theorem (Gur10)

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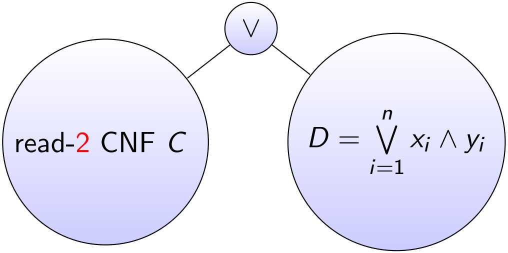

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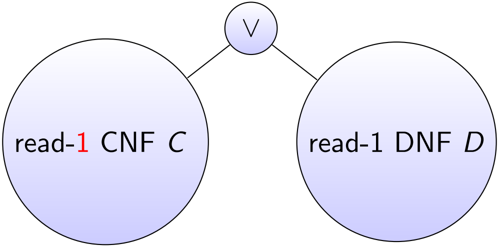
Tractable subclass of depth-3 formulas



Theorem (this work)

ROR *is solvable in polynomial time for inputs of the form* $C \vee D$*, where* C *is a monotone read-1 CNF,* D *is a monotone read-1 DNF and every variable of* C *occurs also in* D*.*

Reduction

Reduction from
CO-CLIQUE$= \{(G, k)$: there is no clique of size k in $G\}$.

$$(G = (V, E), k) \impliedby \text{(Alice} \wedge \text{Bob) vs Merlin game}$$
$$\impliedby \text{depth-3 read-2 } \Phi.$$

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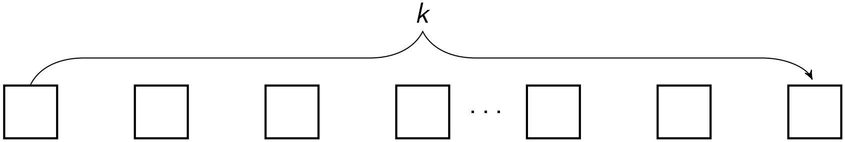

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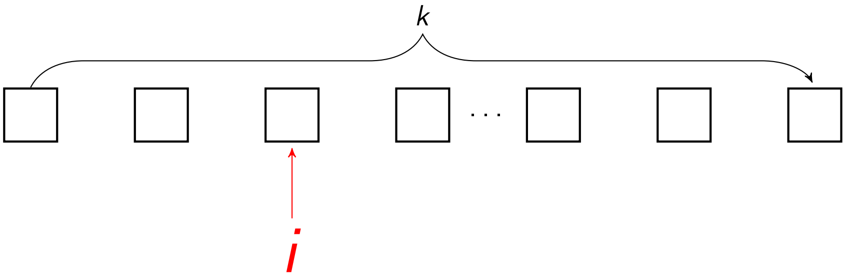

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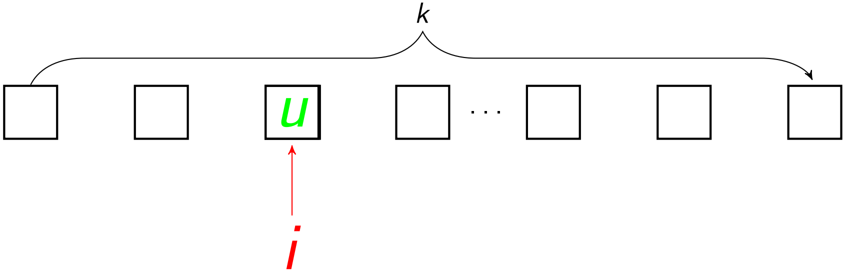

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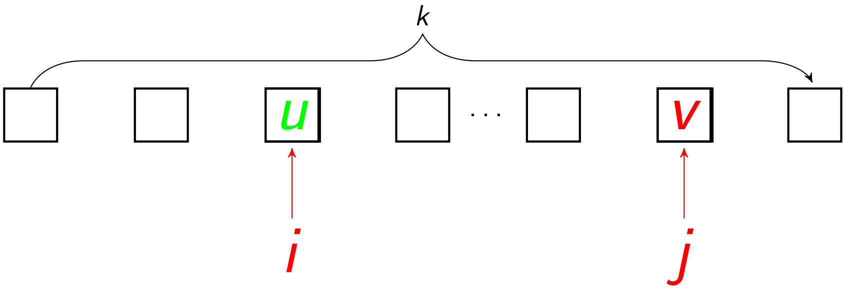

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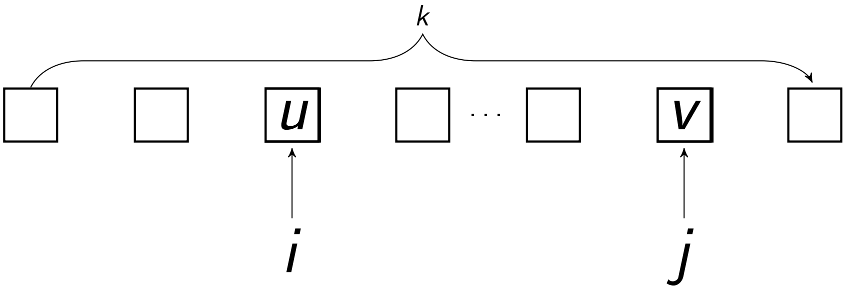

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4. Bob comes in and *tries to guess a cell which was touched first.*

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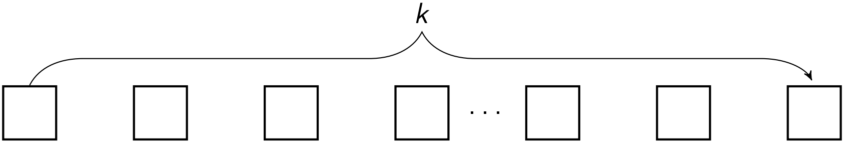
The first direction

There is a k-clique in $G \implies$ Alice and Bob have a winning strategy



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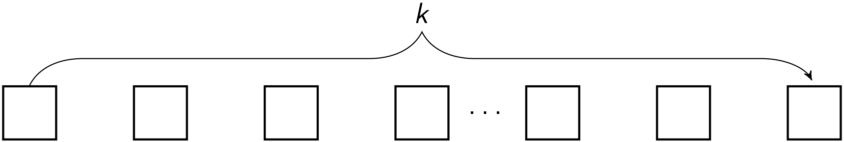

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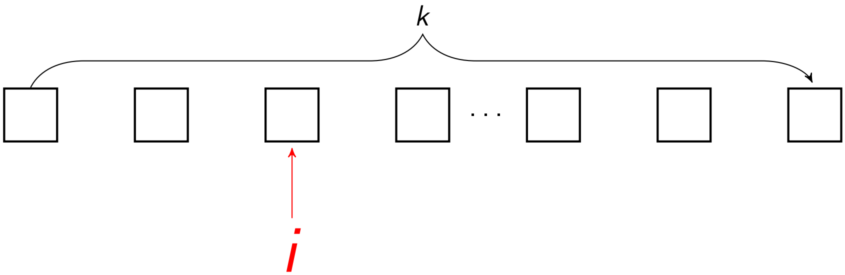

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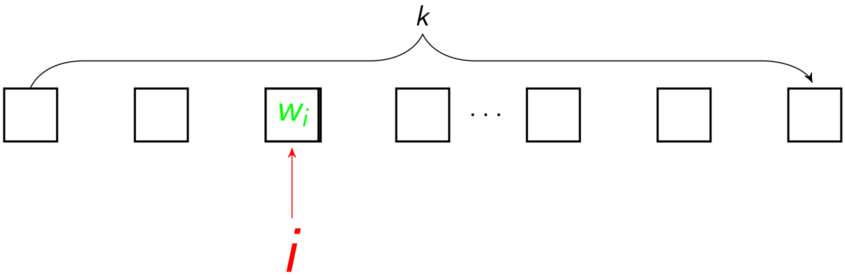

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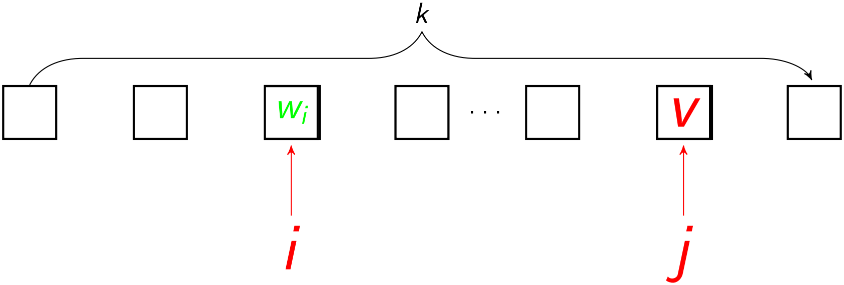

Bob outputs a cell with a corresponding vertex of a clique in it.

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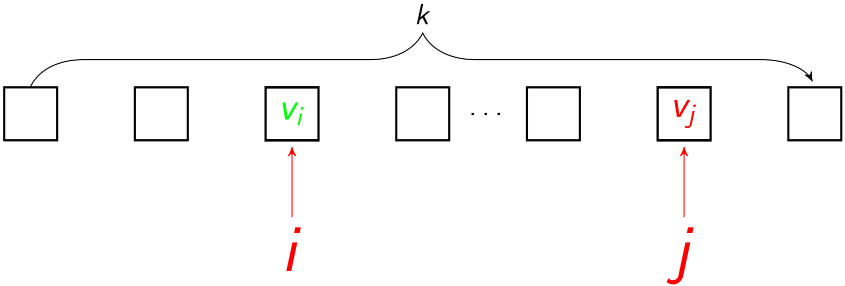

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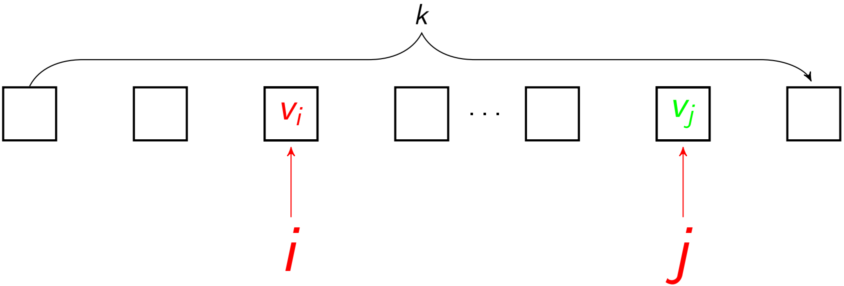

From game to a formula

Constructing a formula:

- Variables correspond to possible outcomes of Alice-Merlin interaction:

$$x_{j,v}^{i,u} \qquad \text{over all } i \neq j,\ \{u,v\} \notin E.$$

- Assignments of variables encode Bob's possible strategies.

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Final remarks

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G \text{ has } k\text{-clique} &\iff \text{Alice and Bob have a winning strategy} \\
&\iff A \to B \text{ is not a tautology} \\
&\iff A \wedge (w_1 w_3 \vee w_2 w_4) \wedge (B \vee w_1 w_2 \vee w_3 w_4) \notin \mathrm{ROR}
\end{aligned}
$$

Thank you! Any questions?