SLOPES OF 3D SUBSHIFTS OF FINITE TYPE

CSR, June 9, 2018

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UNDECIDABILITY AND ITS FRIENDS
"Informal" question:

P: "Does $A(x) = B$?"

Set of objects that satisfies a property:

$$P = \{x \mid A(x) = B\}$$
A problem is:

→ **Recursively Enumerable** (RE) if a Turing machine accepts its elements

→ **Decidable** if it is RE and its complement is RE

→ **Undecidable** if not decidable
Δ₀ = Σ₀ = Π₀: recursive problems.

→ Σₙ: recursively enumerable with oracle Πₙ₋₁

→ Πₙ: complement of Σₙ or recursively enumerable with oracle Σₙ₋₁

→ Δₙ = Σₙ ∩ Πₙ

\[ \Delta_0 = \Sigma_0 = \Pi_0: \text{recursive problems.} \]

\[ \rightarrow \Sigma_n: \text{recursively enumerable with oracle } \Pi_{n-1} \]

\[ \rightarrow \Pi_n: \text{complement of } \Sigma_n \text{ or recursively enumerable with oracle } \Sigma_{n-1} \]

\[ \rightarrow \Delta_n = \Sigma_n \cap \Pi_n \]
SUBSHIFTS AND PERIODICITY
\[ \mathcal{A} = \{\text{■ □ ▢ ▤}\} \text{ a finite alphabet} \]

Configuration \( c \in \mathcal{A}^{\mathbb{Z}^2} \)
SUBSHIFTS OF FINITE TYPE (SFT)

\[ \mathcal{A} = \{\text{■■■■} \} \text{ a finite alphabet} \]

\[ F = \{\text{■■■■} \} \subset \mathcal{A}^\mathbb{Z}^d \text{ a finite set of finite patterns} \]
SUBSHIFTS OF FINITE TYPE (SFT)

→ $\mathcal{A} = \{\text{■ □ □ □} \}$ a finite alphabet

→ $F = \{\text{■ ■ ■ ■} \} \subset \mathcal{A}^{\mathbb{Z}^d}$ a finite set of finite patterns

$$X_F = \{ c \in \mathcal{A}^{\mathbb{Z}^d} \mid \forall p \in F, c \text{ does not contain } p \}$$

$c \in X_F$
Undecidability and its Friends

Subshifts and Periodicity

Slopes of Periodicity

SUBSHIFTS AND COMPUTABILITY
SUBSHIFTS AND COMPUTABILITY
UNDECIDABLE PROBLEMS

Emptiness Problem:

"Is $X_F$ empty?"
Undecidability and its Friends

Subshifts and Periodicity

Slopes of Periodicity

UNDECIDABLE PROBLEMS

Emptiness Problem:

"Is $X_F$ empty?"

→ **Undecidable** ($\in \Sigma_1$) for $d \geq 2$ [Berger, 1964]
UNDECIDABLE PROBLEMS

Emptiness Problem:

"Is $X_F$ empty?"

→ **Undecidable** $(\in \Sigma_1)$ for $d \geq 2$ [Berger, 1964]
→ **Decidable** for $d = 1$
A configuration $c$ is $\nu$-periodic if

$$\forall x \in \mathbb{Z}^2, c(x) = c(x + \nu)$$

A configuration without periodicity vector is aperiodic.
SLOPES OF PERIODICITY
\( v = (p, q) \).

The **slope** of \( v \) is \( \theta = \frac{p}{q} \).
2D CASE: DEFINITIONS

\( v = (p, q). \)

The **slope** of \( v \) is \( \theta = \frac{p}{q}. \)

\( c \) 1-periodic with slope \( \theta \):

\( c \) has **slope of periodicity** \( \theta \)

\( S_X = \{ \theta \text{ slope of periodicity of } c \mid c \in X \} \) is the **set of slopes** of the SFT \( X \)
2D CASE: DEFINITIONS

\[ A = \{\text{red square, blue square}\} \quad F = \{\text{red square, blue square, red square}\} \]

\[ X_F = \{\text{pattern 1}, \text{pattern 2}, \text{pattern 3}\} \]

\[ S_{X_F} = \{0\} \]
2D CASE: DEFINITIONS

\[ \mathbf{v} = (p, q) . \]

The slope of \( \mathbf{v} \) is \( \theta = \frac{p}{q} \).

\( c \) \( \mathbf{1} \)-periodic with slope \( \theta \):

\( c \) has slope of periodicity \( \theta \)

\( S_X = \{ \theta \text{ slope of periodicity of } c \mid c \in X \} \) is the set of slopes of the SFT \( X \)

\( S = \{ S_X \mid X \text{ a 2D SFT} \} \) is the set of all set of slopes
2D CASE: MAIN THEOREM

Theorem [Jeandel, Vanier 2010]

In dimension 2, the problem

"Does SFT $X$ have slope $\theta$?"

is $\Sigma_1$-complete.
2D CASE: MAIN THEOREM

Theorem [Jeandel, Vanier 2010]

\[ S = \Sigma_1 \cap P(\mathbb{Q} \cup \infty). \]
3D CASE: CONJECTURE

Conjecture [Jeandel, Vanier 2010]

In dimension 3, The problem

"Does SFT $X$ have slope $\theta$?"

is $\Sigma_2$-complete.
3D CASE: DEFINITIONS

\[ v = (p, q, r). \]

The **slope** of \( v \) is \((\theta_1, \theta_2) = (\frac{p}{q}, \frac{p}{r}).\)
3D CASE: DEFINITIONS

\[ v = (p, q, r). \]

The **slope** of \( v \) is \( (\theta_1, \theta_2) = \left( \frac{p}{q}, \frac{p}{r} \right) \).

\( c \) 1-periodic with slope \( \theta \):

\[ c \text{ has slope of periodicity } \theta \]

\[ S_X = \{ \theta \text{ slope of periodicity of } c \mid c \in X \} \text{ is the set of slopes of the SFT } X \]

\[ S = \{ S_X \mid X \text{ a 3D SFT} \} \text{ is the set of all set of slopes} \]
## 3D CASE

### Theorem

In dimension 3, $S \supseteq \Sigma_2 \cap P(\mathbb{Q} \cup \infty)$.

### Theorem [Grandjean, Hellouin, Vanier 2018]

In dimension 3, $S \subseteq \Sigma_2 \cap P(\mathbb{Q} \cup \infty)$. 
COMPLEXITY GAP: INTUITION

→ $X$ a 2D SFT

→ $c \in X$

→ $\theta \in \mathbb{Q} \cup \infty$

$\Rightarrow \exists Y_{c,\theta}$ of dimension 1 such that:

"Is $c$ periodic along $\theta$ ?"

$\iff$

"Is $Y_{c,\theta}$ empty ?"

Decidable
COMPLEXITY GAP: INTUITION

→ \( X \) a 2D SFT
→ \( c \in X \)
→ \( \theta \in \mathbb{Q} \cup \infty \)

\[ \Rightarrow \exists Y_{c,\theta} \text{ of dimension 1 such that:} \]

"Is \( \theta \) slope of \( X \)?"
\[ \Leftrightarrow \]
"\( \exists c \in X \) such that \( Y_{c,\theta} \) empty?"

\[ \in \Sigma_1 \]
COMPLEXITY GAP: INTUITION

→ $X$ a **3D** SFT
→ $c \in X$
→ $\theta \in (\mathbb{Q} \cup \infty)^2$

$\Rightarrow \exists Y_{c,\theta}$ of dimension 2 such that:

"Is $c$ periodic along $\theta$ ?"
$\Leftrightarrow$
"Is $Y_{c,\theta}$ empty ?"

$\in \Sigma_1$
COMPLEXITY GAP: INTUITION

\[ \Rightarrow X \text{ a 3D SFT} \]
\[ \Rightarrow c \in X \]
\[ \Rightarrow \theta \in (\mathbb{Q} \cup \infty)^2 \]
\[ \Rightarrow \exists Y_{c,\theta} \text{ of dimension } 2 \text{ such that:} \]

"Is \( \theta \) slope of \( X \)?"
\[ \Leftrightarrow \]
"\( \exists c \in X \) such that \( Y_{c,\theta} \) empty?"

\[ \in \Sigma_2 \]
PROOF IDEAS

Theorem

Let \( R \in \Sigma_2 \cap P(\mathbb{Q} \cup \infty) \). Then there exists an SFT \( X \) such that \( R = S_X \).

Let \( M \) be a Turing machine \( \Sigma_2 \) such that \( R = \{ \theta \mid M \text{ accepts } \theta \} \).

**Goal:** Construct \( X \) such that \( S_X = R \).

\[ \iff \]

Any 1-periodic configuration of \( X \) has slope \( \theta = \left( \frac{p}{q}, \frac{p}{r} \right) \in R \).
Slopes of Periodicity

The diagram illustrates a cube with arrows indicating the directions of vectors $p$, $q$, and $r$. The cube is aligned with the axes $x$, $y$, and $z$. The vectors $p$, $q$, and $r$ are depicted as projections along the respective axes.

- $p$ is the vector along the $x$-axis.
- $q$ is the vector along the $y$-axis.
- $r$ is the vector along the $z$-axis.
\[ X = B \times B' \times B'' \times C \times W \times P \times S \times T_O \times T_M \times A \]

\rightarrow B, B' and B'' create cuboids with pieces of aperiodic background in them

\rightarrow C forces cubes to appear

\rightarrow W creates a periodicity vector, and writes the input in the cubes

\rightarrow P reduces the size of the output

\rightarrow S synchronizes aperiodic background between cubes

\rightarrow T_O encodes the oracle \( \Pi_1 \)

\rightarrow T_M encodes the Turing machine \( \Sigma_2 \)

\rightarrow A ensures the existence of configurations with unique periodicity
WHAT NEXT?

→ $\Sigma_2$-hardness seems to work for higher dimensions
→ ... But not the proof of $\in \Sigma_2$. 
Thank you!