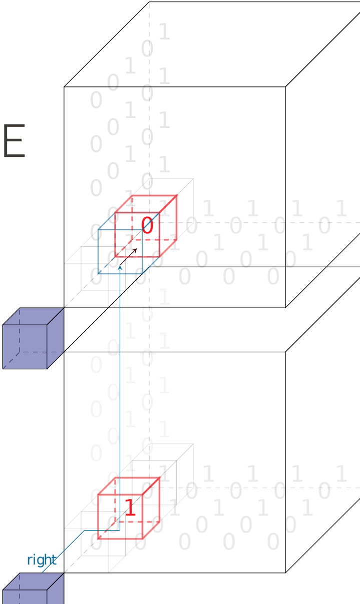
SLOPES OF 3D
SUBSHIFTS OF FINITE TYPE

CSR, June 9, 2018



Etienne Moutot, Pascal Vanier

UNDECIDABILITY AND ITS FRIENDS

DECISION PROBLEMS

"Informal" question:

P: "Does $A(x) = B$?"

Set of objects that satisfies a property:

$$P = \{x \mid A(x) = B\}$$

UNDECIDABLE PROBLEMS

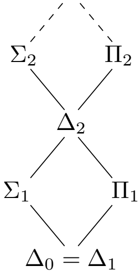
A problem is:

- → **Recursively Enumerable** (RE) if a Turing machine accepts its elements
- → **Decidable** if it is RE and its complement is RE
- → **Undecidable** if not decidable

ARITHMETICAL HIERARCHY

$\Delta_0 = \Sigma_0 = \Pi_0$: **recursive** problems.

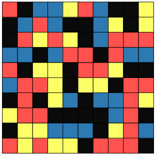
- Σ_n: **recursively enumerable with oracle** Π_{n-1}
- Π_n: **complement** of Σ_n or recursively enumerable with oracle Σ_{n-1}
- $\Delta_n = \Sigma_n \cap \Pi_n$



SUBSHIFTS AND PERIODICITY

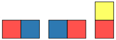
SUBSHIFTS OF FINITE TYPE (SFT)

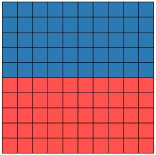
→ $\mathcal{A} = \{$■ ■ ■ ■$\}$ a **finite alphabet**



Configuration $c \in \mathcal{A}^{\mathbb{Z}^2}$

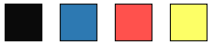
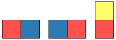
SUBSHIFTS OF FINITE TYPE (SFT)

- $\mathcal{A} = \{$  $\}$ a **finite alphabet**
- $F = \{$  $\} \subset \mathcal{A}^{\mathbb{Z}^d}$ a **finite set of finite patterns**

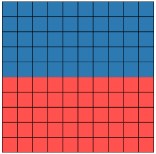


Valid configuration

SUBSHIFTS OF FINITE TYPE (SFT)

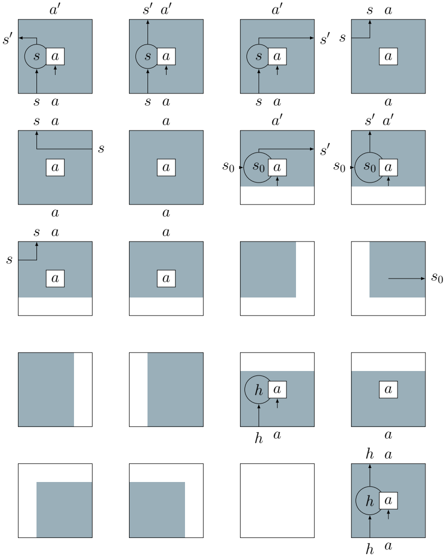
- → $\mathcal{A} = \{$$\}$ a **finite alphabet**
- → $F = \{$$\} \subset \mathcal{A}^{\mathbb{Z}^d}$ a **finite set of finite patterns**

$$X_F = \{c \in \mathcal{A}^{\mathbb{Z}^d} \mid \forall p \in F, c \text{ does not contain } p\}$$

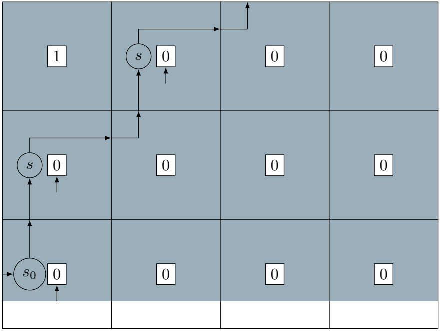


$c \in X_F$

SUBSHIFTS AND COMPUTABILITY



SUBSHIFTS AND COMPUTABILITY



UNDECIDABLE PROBLEMS

Emptiness Problem :

"Is X_F empty ?"

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→ **Undecidable** ($\in \Sigma_1$) for $d \geq 2$ [Berger, 1964]

UNDECIDABLE PROBLEMS

Emptiness Problem :

"Is X_F empty ?"

- → **Undecidable** ($\in \Sigma_1$) for $d \geq 2$ [Berger, 1964]
- → **Decidable** for $d = 1$

(A)PERIODICITY

A configuration c is v**-periodic** if

$$\forall x \in \mathbb{Z}^2, c(x) = c(x+v)$$

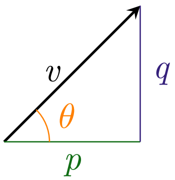
A configuration without periodicity vector is **aperiodic**.

SLOPES OF PERIODICITY

2D CASE: DEFINITIONS

$v = (p, q)$.

The **slope** of v is $\theta = \frac{p}{q}$.



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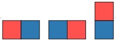
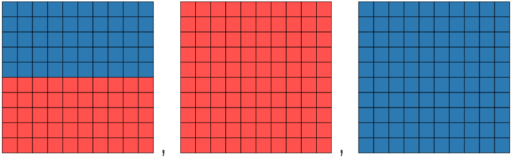
c 1-periodic with slope θ:

c has **slope of periodicity** θ

$S_X = \{\theta$ slope of periodicity of $c \mid c \in X\}$ is the **set of slopes** of the SFT X

2D CASE: DEFINITIONS

$\mathcal{A} = \{$  $\}$ $F = \{$  $\}$

$X_F = \{$  $\}$

$$S_{X_F} = \{0\}$$

2D CASE: DEFINITIONS

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$S = \{S_X \mid X$ a 2D SFT$\}$ is the set of all set of slopes

2D CASE: MAIN THEOREM

Theorem [Jeandel, Vanier 2010]

In dimension 2, The problem

"Does SFT X have slope θ ?"

is Σ_1-complete.

2D CASE: MAIN THEOREM

Theorem [Jeandel, Vanier 2010]

In dimension 2, $S = \Sigma_1 \cap P(\mathbb{Q} \cup \infty)$.

3D CASE: CONJECTURE

Conjecture [Jeandel, Vanier 2010]

In dimension **3**, The problem

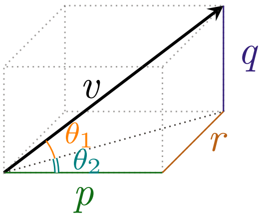
"Does SFT X have slope θ ?"

is Σ_2-complete.

3D CASE: DEFINITIONS

$v = (p, q, r)$.

The **slope** of v is $(\theta_1, \theta_2) = (\frac{p}{q}, \frac{p}{r})$.



3D CASE: DEFINITIONS

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3D CASE

Theorem

In dimension 3, $S \supseteq \Sigma_2 \cap P(\mathbb{Q} \cup \infty)$.

Theorem [Grandjean, Hellouin, Vanier 2018]

In dimension 3, $S \subseteq \Sigma_2 \cap P(\mathbb{Q} \cup \infty)$.

COMPLEXITY GAP: INTUITION

- → X a 2D SFT
- → $c \in X$
- → $\theta \in \mathbb{Q} \cup \infty$

$\Rightarrow \exists Y_{c,\theta}$ of dimension 1 such that:

"Is c periodic along θ ?"

$\Leftrightarrow$

"Is $Y_{c,\theta}$ empty ?"

Decidable

COMPLEXITY GAP: INTUITION

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"Is θ slope of X ?"

$\Leftrightarrow$

"$\exists c \in X$ such that $Y_{c,\theta}$ empty ?"

$\in \Sigma_1$

COMPLEXITY GAP: INTUITION

- X a **3D** SFT
- $c \in X$
- $\theta \in (\mathbb{Q} \cup \infty)^2$

$\Rightarrow \exists Y_{c,\theta}$ of dimension **2** such that:

"Is c periodic along θ ?"

$\Leftrightarrow$

"Is $Y_{c,\theta}$ empty ?"

$\in \Sigma_1$

COMPLEXITY GAP: INTUITION

- X a **3D** SFT
- $c \in X$
- $\theta \in (\mathbb{Q} \cup \infty)^2$

$\Rightarrow \exists Y_{c,\theta}$ of dimension **2** such that:

"Is θ slope of X ?"

$\Leftrightarrow$

"$\exists c \in X$ such that $Y_{c,\theta}$ empty ?"

$\in \Sigma_2$

PROOF IDEAS

Theorem

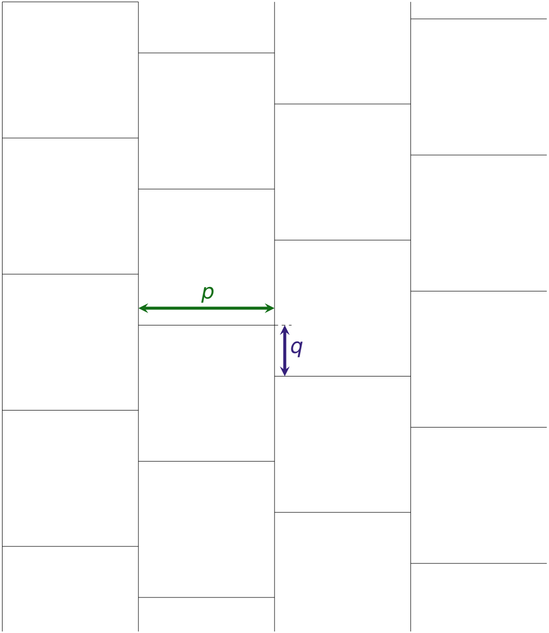
Let $R \in \Sigma_2 \cap P(\mathbb{Q} \cup \infty)$. Then there exists an SFT X such that $R = S_X$.

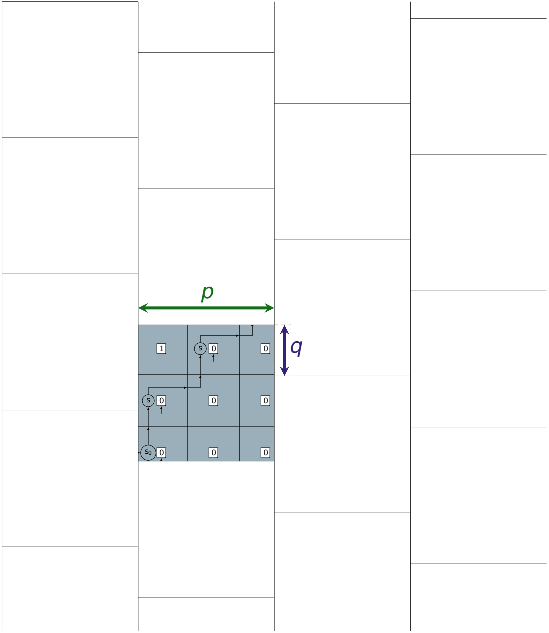
Let M be a Turing machine Σ_2 such that $R = \{\theta \mid M \text{ accepts } \theta\}$.

Goal: Construct X such that $S_X = R$.

$\Leftrightarrow$

Any 1-periodic configuration of X has slope $\theta = (\frac{p}{q}, \frac{p}{r}) \in R$.


p
q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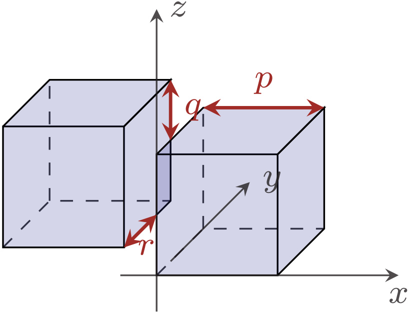
z
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y
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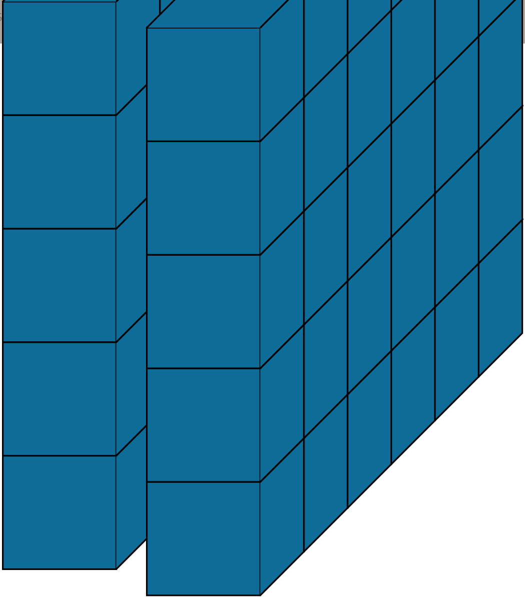

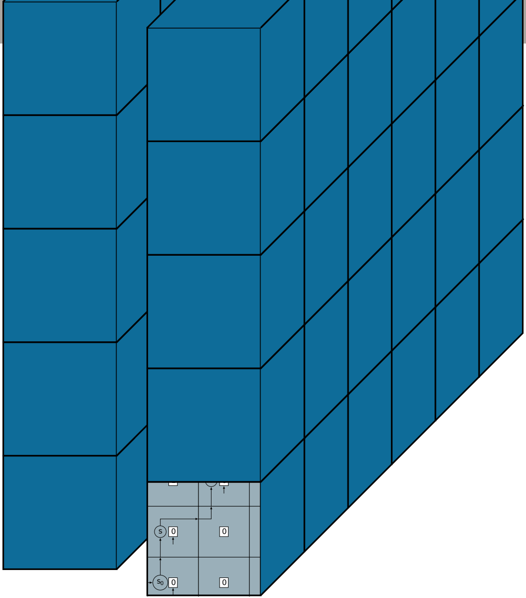
s
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s_0
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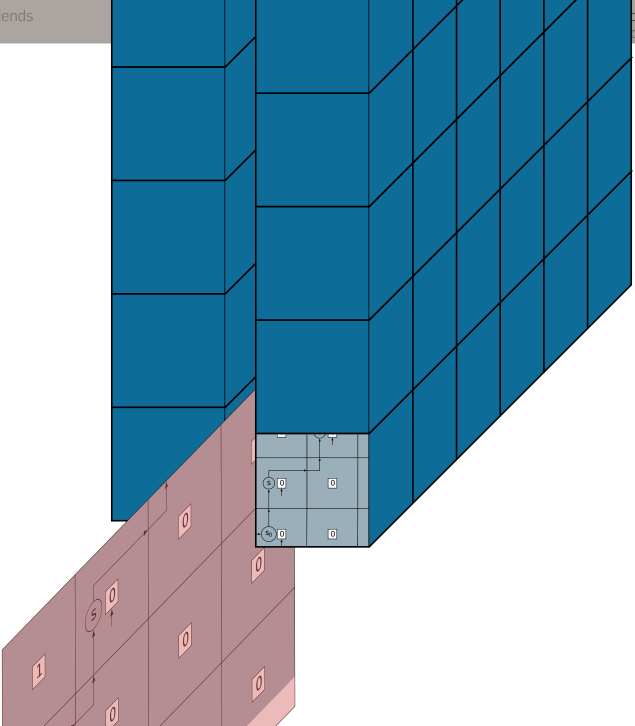
S
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0
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0
0
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1
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0
0

$$X = B \times B' \times B'' \times C \times W \times P \times S \times T_O \times T_M \times A$$

- → B, B' and B'' create cuboids with pieces of aperiodic background in them
- → C forces cubes to appear
- → W creates a periodicity vector, and writes the input in the cubes
- → P reduces the size of the output
- → S synchronizes aperiodic background between cubes
- → T_O encodes the oracle Π_1
- → T_M encodes the Turing machine Σ_2
- → A ensures the existence of configurations with unique periodicity

WHAT NEXT ?

- → Σ_2-hardness seems to work for higher dimensions
- → … But not the proof of $\in \Sigma_2$.

Thank you !