# Facility Location on Planar Graphs with Unreliable Links

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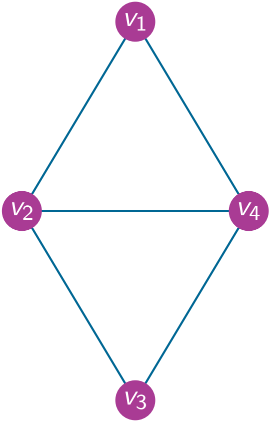
# Facility Networks



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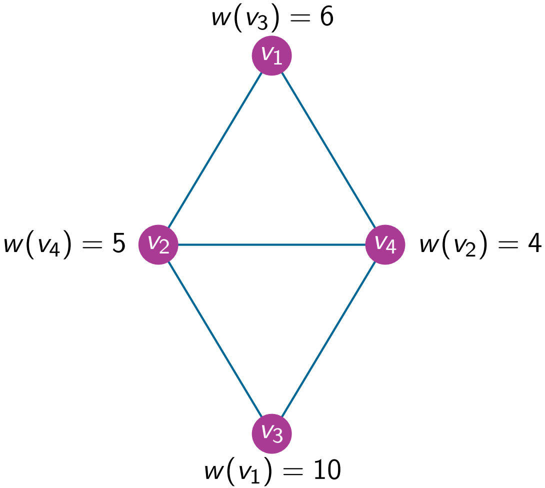


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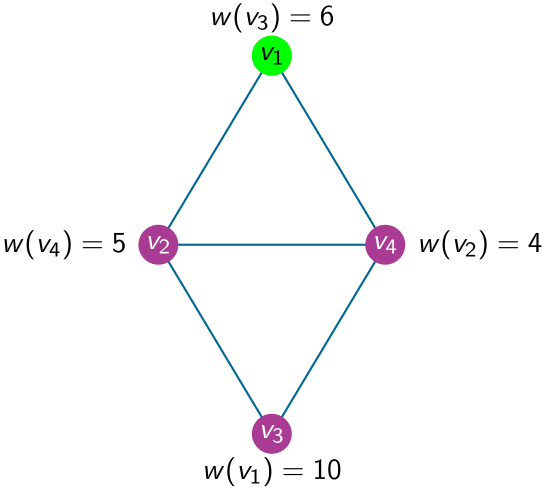


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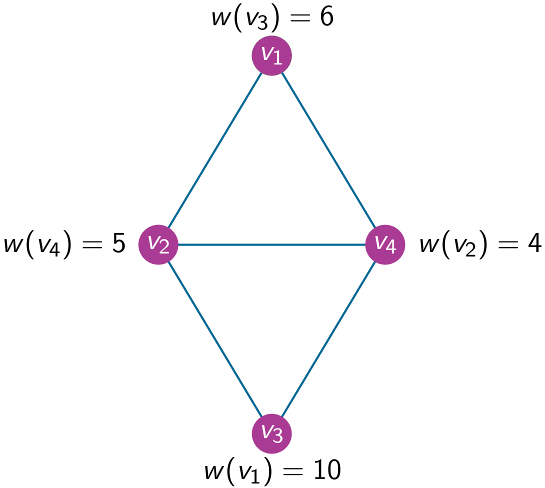


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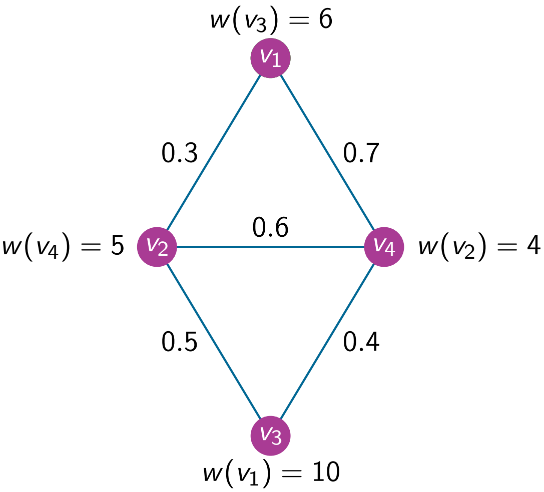


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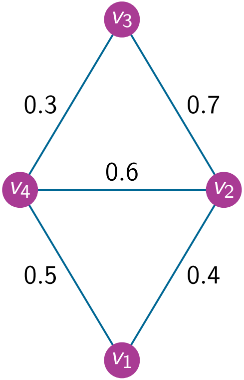
$p : E \rightarrow [0, 1]$

# Sub-graph Realizations



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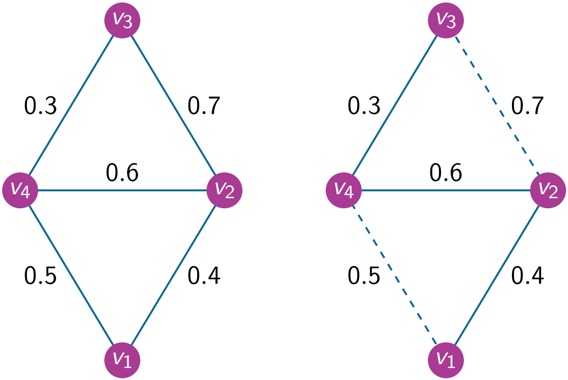


$Q_1$, $P(Q_1)$

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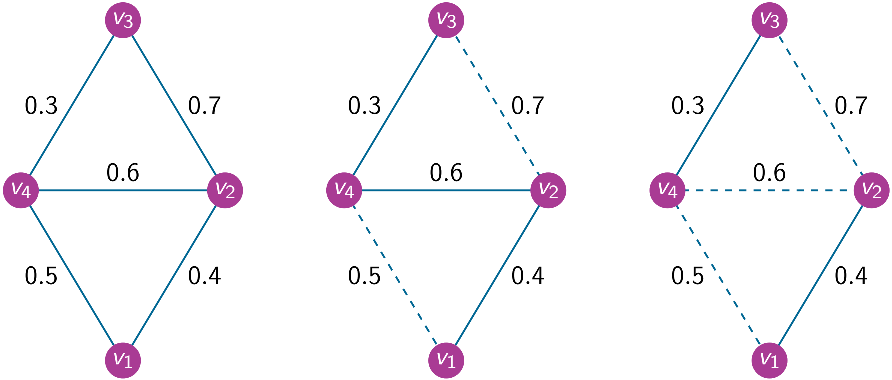


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Input: A graph $G = (V, E)$

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$$I(Q, F, v) = \begin{cases} 1 & \text{if } v \in N_Q[F] \\ 0 & \text{otherwise} \end{cases}$$

# Coverage Function

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## The coverage function $\mathcal{C}$

Given a set $F \subseteq V$ and a vertex $v \in V$, the function $\mathcal{C}(v, F)$ is the expected coverage of $v$ by $F$.

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&= \sum_{v \in V} \mathcal{C}(v, F) = \mathcal{C}(V, F)
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**Vulnerability-Based Dependency [Hassin et al., 2009]**

- Given $e_i$ and $e_j$ such that $p(e_i) > p(e_j)$
- $\Pr[e_j \text{ fails} \mid e_i \text{ fails}] = 1$

If an edge $e_i$ fails then the weaker edges than $e_i$ surely fails.

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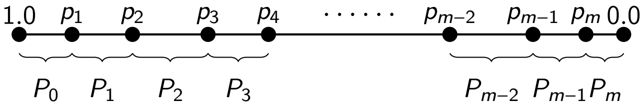
## Linear Reliable Ordering [Hassin et al., 2017]

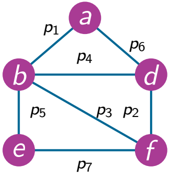
- Every pair of edges are following VB-dependency.
- $m + 1$ realizations are possible.
- Let $G_0, G_1, \ldots, G_m$ be all the possible realizations.

# LRO Model

- Order the edges $e_1, e_2, \ldots, e_m$ in descending order of survival probability.
- $G_0$ - Empty graph. When $e_1$ fails.
- $G_i$ occurs when $e_i$ is the weakest link that survives.



# LRO Instance



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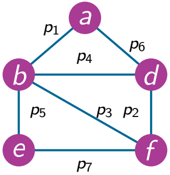


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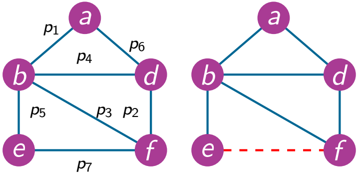


$G$, $G_7$, $p_7$ $\qquad$ $G_6$, $p_6 - p_7$

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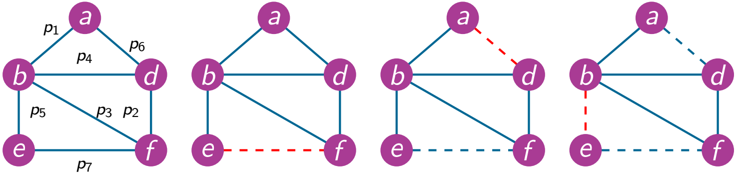
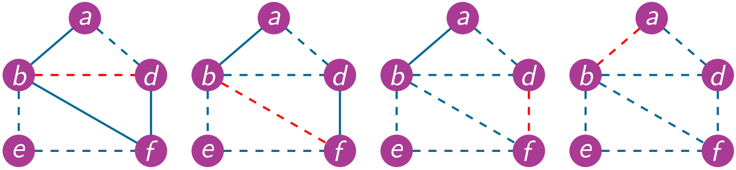



$G, G_7, p_7$ | $G_6, p_6 - p_7$ | $G_5, p_5 - p_6$ | $G_4, p_4 - p_5$




$G_3, p_3 - p_4$ | $G_3, p_2 - p_3$ | $G_1, p_1 - p_2$ | $G_0, 1 - p_1$

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**The MAX-EXP-COVER problem with LRO Model - Existing**

- When $R = 1$, NP-Hard [Hassin et al., 2009]
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**The MAX-EXP-COVER problem with LRO Model - In this work**

- When $R = 1$, FPT on bounded treewidth graph, and PTAS on planar graph
- Observed that, the problem has greedy approximation algorithm $(1 - \frac{1}{e})$.

# Tree Decomposition

$H$ - Tree, $\mathcal{X} = \{X_i \subseteq V \mid i \in H\}$ - Bag
A pair $(H, \mathcal{X})$ satisfy the following conditions.

1. $\forall v \in V, \exists i \in H \mid v \in X_i$.
2. $\forall uv \in E, \exists i \in H \mid u, v \in X_i$.
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- $width = \max_{i \in H} |X_i| - 1$.

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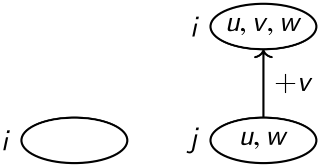

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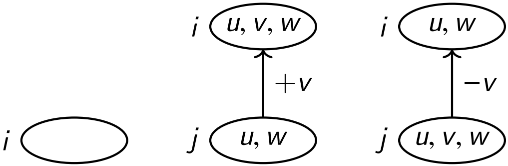

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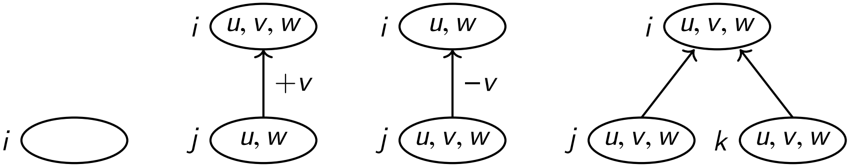
- **Join:** $i$ has two children $j$ and $k : X_i = X_j = X_k$.



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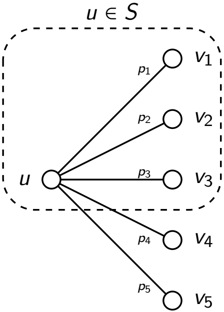

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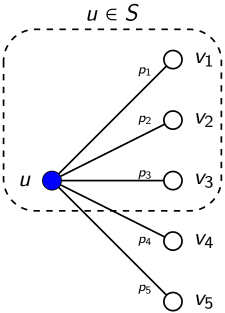

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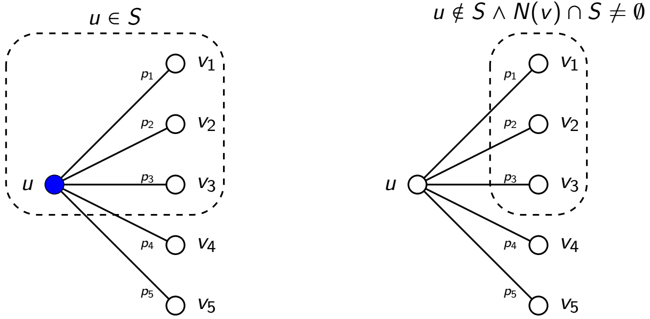

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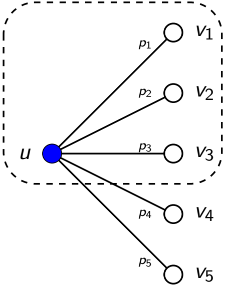
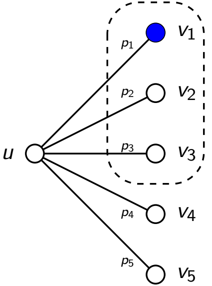
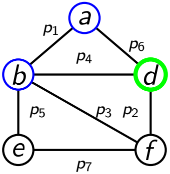



$u \notin S \wedge N(v) \cap S \neq \emptyset$



# Best Neighbour - In LRO

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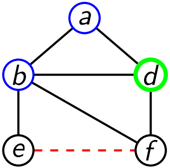


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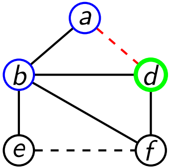


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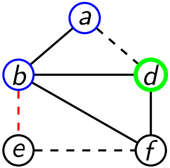


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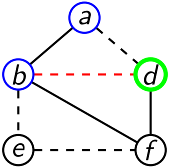
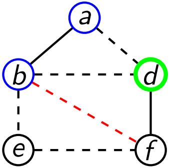



$G_3$, $p_3 - p_4$

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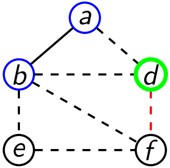
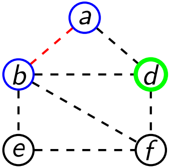



$G_1$, $p_1 - p_2$

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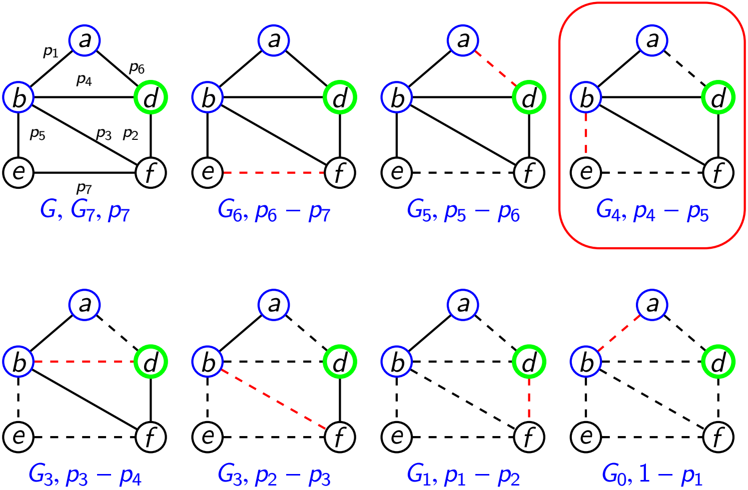
# Best Neighbour - In LRO



# Coverage using Best Neighbour

**Lemma**

*Let $u \in V$ be a vertex and $S \subseteq V$ be a set. If the coverage $\mathcal{C}(u, S) > 0$, then there is a vertex $v \in S$ such that $\mathcal{C}(u, S) = \mathcal{C}(u, v)$.*

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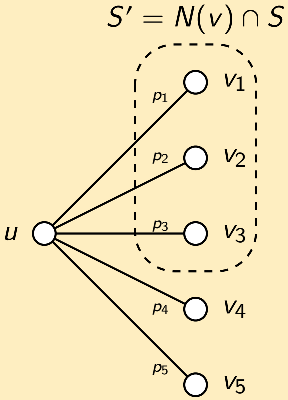
- Assume $p(uv_1) > p(uv_2) > \cdots > p(uv_\ell)$.

## Proof. (Cont)

$$\mathcal{C}(u, S) \;=\; \mathcal{C}(u, S')$$



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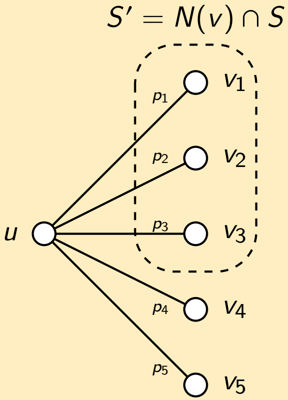
$$\begin{aligned} \mathcal{C}(u, S) &= \mathcal{C}(u, S') \\ &= w(u) \cdot \sum_{G_i \mid u \in N_{G_i}[S']} p(G_i) \end{aligned}$$



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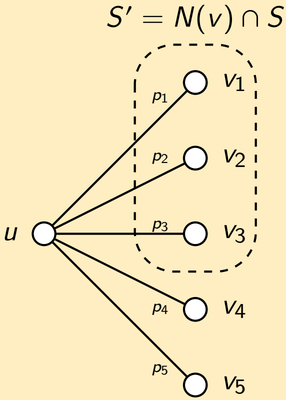
$G_j \mid (u, v_1)$ survives



## Proof. (Cont)

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\begin{aligned}
\mathcal{C}(u,S) &= \mathcal{C}(u,S') \\
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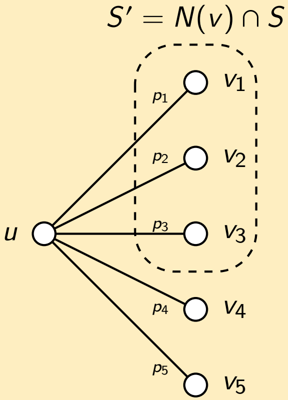
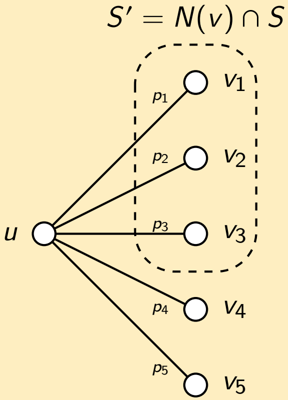
&= w(u)\cdot \sum_{i=j}^{m} p(G_i) \\
&\quad G_j \mid (u,v_1) \text{ survives} \\
&= w(u)\cdot p(uv_1)
\end{aligned}
$$



## Proof. (Cont)

$$
\begin{aligned}
C(u,S) &= C(u,S') \\
&= w(u) \cdot \sum_{G_i \mid u \in N_{G_i}[S']} p(G_i) \\
&= w(u) \cdot \sum_{i=j}^{m} p(G_i) \\
& \quad G_j \mid (u, v_1) \text{ survives} \\
&= w(u) \cdot p(uv_1) \\
&= C(u, v_1)
\end{aligned}
$$



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\mathcal{C}(u,S) &= \mathcal{C}(u,S') \\
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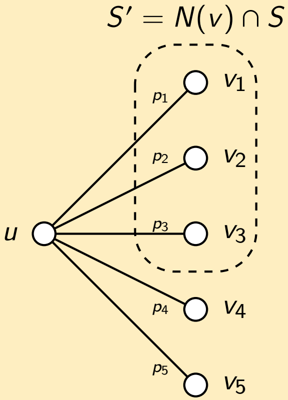
&= w(u)\cdot \sum_{i=j}^{m} p(G_i) \\
& \quad G_j \mid (u,v_1) \text{ survives} \\
&= w(u)\cdot p(uv_1) \\
&= \mathcal{C}(u,v_1) \\
&= \mathcal{C}(u, bn(u,S))
\end{aligned}
$$




Hence the proof. □

# Structure of a Solution

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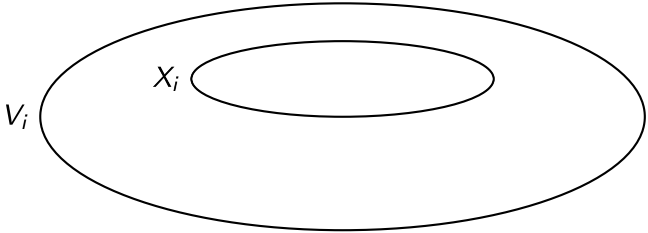

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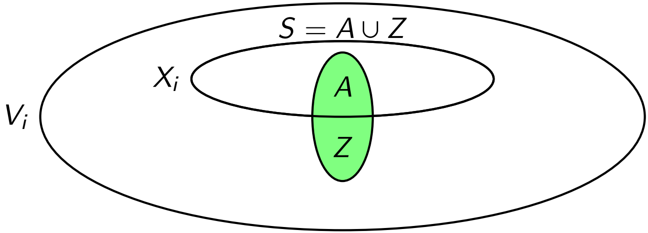

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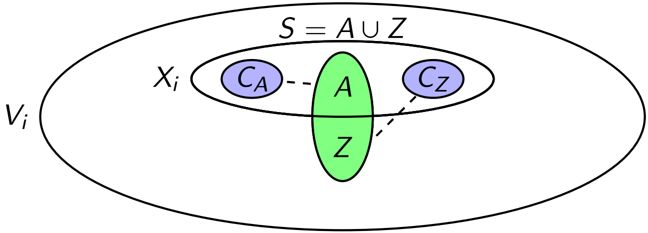
- $U = X_i \setminus (A \cup C_A \cup C_Z)$.



# Solution Structure

$$\mathcal{C}(V_i, S) \quad = \quad \mathcal{C}((V_i \setminus X_i) \cup X_i, S)$$

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&= \mathcal{C}(V_i \setminus X_i, S) + \mathcal{C}(C_A, {\color{red}A}) + \mathcal{C}(C_Z, {\color{red}Z}) + \mathcal{C}(A, {\color{red}A})
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\end{aligned}
$$

## Partition

For every feasible solution $S \subseteq V_i$, there is a four-way partition $\mathtt{P} = (A, C_A, C_Z, U)$ of $X_i$ such that

$$\mathcal{C}(V_i, S) = \mathcal{C}(V_i \setminus X_i, S) + \mathcal{C}(A \cup C_A, A) + \mathcal{C}(C_Z, Z)$$

# Dynamic Programming

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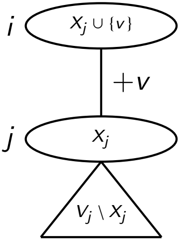
  - $T_i[b, \mathtt{P}].\textbf{solution} = S \subseteq V_i$.
  - $|S| = b$ and $S \cap X_i = A$.
  - $\mathcal{C}(A \cup C_A, A) + \mathcal{C}(C_Z, Z) + \mathcal{C}(V_i \setminus X_i, S)$ is maximized over all possible such $S \subseteq V_i$.

# Introduce Node

- Let $i$ be an introduce node with child $j$ such that $X_i = X_j \cup \{v\}$ for some $v \notin X_j$.



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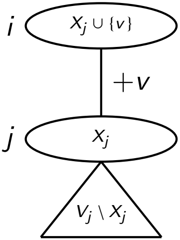
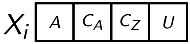
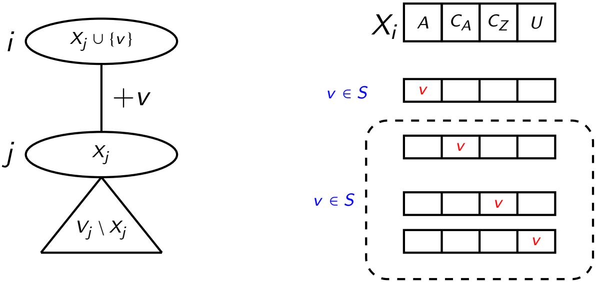


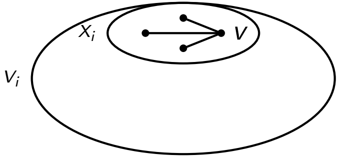



# Introduce Node

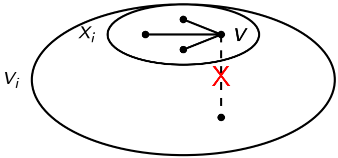
- Let $i$ be an introduce node with child $j$ such that $X_i = X_j \cup \{v\}$ for some $v \notin X_j$.
- Let $0 \le b \le \mathcal{B}$ be a budget and $\mathtt{P} = (A, C_A, C_Z, U)$ be a four way partitioning of $X_i$.
- We consider two cases that $(i) - v \notin A$ and $(ii) - v \in A$.



# Introduce Node (Cont...)

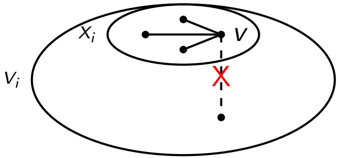


# Introduce Node (Cont...)




## Case: $v \notin A$

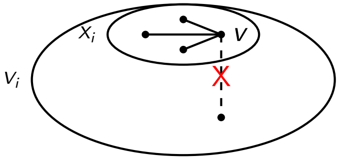
# Introduce Node (Cont...)




## Case: $v \notin A$

$$\mathtt{P} = (A, C_A, C_Z, U) \rightarrow \mathtt{P}_j = (A, C_A \setminus \{v\}, C_Z \setminus \{v\}, U \setminus \{v\})$$

# Introduce Node (Cont...)




**Case: $v \notin A$**

$\mathrm{P} = (A, C_A, C_Z, U) \rightarrow \mathrm{P}_j = (A, C_A \setminus \{v\}, C_Z \setminus \{v\}, U \setminus \{v\})$

$$T_i[b, \mathrm{P}].\textbf{solution} = T_j[b, \mathrm{P}_j].\textbf{solution}$$

$$T_i[b, \mathrm{P}].\textbf{value} = \begin{cases} T_j[b, \mathrm{P}_j].\textbf{value} & \text{if } v \notin C_A \\ T_j[b, \mathrm{P}_j].\textbf{value} + \mathcal{C}(v, A) & \text{if } v \in C_A \end{cases}$$

# Introduce Node (Cont...)

**Case: $v \in A$**

- Let $C_{A_v} = \{u \in C_A \mid bn(u, A) = v\}$.

# Introduce Node (Cont...)

**Case: $v \in A$**

- Let $C_{Av} = \{u \in C_A \mid bn(u, A) = v\}$.
- Let $\mathtt{P}_j = (A \setminus \{v\}, C_A \setminus C_{Av}, C_Z, U \cup C_{Av})$.

# Introduce Node (Cont...)

**Case: $v \in A$**

- Let $C_{Av} = \{u \in C_A \mid bn(u, A) = v\}$.
- Let $\mathtt{P}_j = (A \setminus \{v\}, C_A \setminus C_{Av}, C_Z, U \cup C_{Av})$.

$$\begin{aligned} T_i[b, \mathtt{P}].\textbf{solution} &= T_j[b-1, \mathtt{P}_j].\textbf{solution} \cup \{v\} \\ T_i[b, \mathtt{P}].\textbf{value} &= T_j[b-1, \mathtt{P}_j].\textbf{value} + \mathcal{C}(\{v\} \cup C_{Av}, v) \end{aligned}$$

# Correctness – case $v \notin A$

- Assume $v \in C_A$.
- Let $T_i[b, \mathtt{P}]$.**solution** $= T_j[b, \mathtt{P}']$.**solution** $= A \cup Z$ where $\mathtt{P}' = (A, C_A \setminus \{v\}, C_Z, U)$
- By contradiction, assume $S' = A \cup Z'$ optimal than $S$. That is $T_i[b, \mathtt{P}].\textbf{value} < \mathcal{C}(V_i \setminus X_i, S') + \mathcal{C}(A \cup C_A, A) + \mathcal{C}(C_Z, Z')$.

$$\begin{aligned} T_j[b, \mathtt{P}'].\textbf{value} &= T_i[b, \mathtt{P}].\textbf{value} - \mathcal{C}(v, A) \\ &< \mathcal{C}(V_i \setminus X_i, S') + \mathcal{C}(A \cup C_A, A) + \mathcal{C}(C_Z, Z') - \mathcal{C}(v, A) \\ &< \mathcal{C}(V_j \setminus X_j, S') + \mathcal{C}(A \cup C_A \setminus \{v\}, A) + \mathcal{C}(C_Z, Z') \end{aligned}$$

- Contradicts optimality of $T_j[b, \mathtt{P}']$ by $S'$.

# Correctness – case $v \in A$

- Let $C_{Av} = \{u \in C_A \mid bn(u, A) = v\}$.
- Let $T_i[b, \mathtt{P}]$.**solution** $= T_j[b-1, \mathtt{P}']$.**solution** $\cup \{v\} = A \cup Z$ where $\mathtt{P}' = (A \setminus \{v\}, C_A \setminus C_{Av}, C_Z, U \cup C_{Av})$
- By contradiction, assume $S' = A \cup Z'$ optimal than $S$. That is $T_i[b, \mathtt{P}]$.**value** $< \mathcal{C}(V_i \setminus X_i, S') + \mathcal{C}(A \cup C_A, A) + \mathcal{C}(C_Z, Z')$.

$$
\begin{aligned}
T_j[b, \mathtt{P}'].\textbf{value} &= T_i[b, \mathtt{P}].\textbf{value} - \mathcal{C}(\{v\} \cup C_{Av}, v) \\
&< \mathcal{C}(V_i \setminus X_i, S') + \mathcal{C}(A \cup C_A, A) + \mathcal{C}(C_Z, Z') \\
&\quad - \mathcal{C}(\{v\} \cup C_{Av}, v) \\
&< \mathcal{C}(V_j \setminus X_j, S') + \mathcal{C}((A \cup C_A) \setminus (\{v\}), A \setminus \{v\}) \\
&\quad + \mathcal{C}(C_Z, Z')
\end{aligned}
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- Contradicts optimality of $T_j[b, \mathtt{P}']$ by $S'$.

# Forget Node

- Let $0 \leq b \leq \mathcal{B}$ be a budget and $\texttt{P} = (A, C_A, C_Z, U)$ be a four way partitioning of $X_i$.

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  $\mathtt{P}_3 = (A, C_A, C_Z \cup \{v\}, U)$ $\quad$ $\mathtt{P}_4 = (A, C_A, C_Z, U \cup \{v\})$

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- Let $\mathtt{P}_j = \max_{\mathtt{P}' \in \{\mathtt{P}_1, \mathtt{P}_2, \mathtt{P}_3, \mathtt{P}_4\}} T_j[b, \mathtt{P}'].\textbf{value}$

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- Let $\mathtt{P}_j = \max_{\mathtt{P}' \in \{\mathtt{P}_1, \mathtt{P}_2, \mathtt{P}_3, \mathtt{P}_4\}} T_j[b, \mathtt{P}'].\textbf{value}$

$$T_i[b, \mathtt{P}].\textbf{solution} = T_j[b, \mathtt{P}_j].\textbf{solution}$$
$$T_i[b, \mathtt{P}].\textbf{value} = T_j[b, \mathtt{P}_j].\textbf{value}$$

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- Let $C_{Z\,k} = \{u \in C_Z \mid bn(u, S) \in Z_k\}$
- Let $b_j = |Z_j|$ and $b_k = |Z_k|$.

$$(b', C_{Z\,j}, C_{Z\,k}) = \max_{\substack{0 \le b_1 \le b - |A|, \\ C_{Z\,1} \cup C_{Z\,2} = C_Z}} T_j[b_1 + |A|, \mathtt{P}'_j].\textbf{value} + T_k[b - b_1, \mathtt{P}'_k].\textbf{value}$$

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where $\mathtt{P}'_j = (A, C_A, C_{Z1}, U \cup C_{Z2})$ and $\mathtt{P}'_j = (A, C_A, C_{Z2}, U \cup C_{Z1})$

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$$(b', C_{Z\,j}, C_{Z\,k}) = \max_{\substack{0 \le b_1 \le b - |A|, \\ C_{Z\,1} \cup C_{Z\,2} = C_Z}} T_j[b_1 + |A|, \mathtt{P}'_j].\textbf{value} + T_k[b - b_1, \mathtt{P}'_k].\textbf{value}$$

where $\mathtt{P}'_j = (A, C_A, C_{Z\,1}, U \cup C_{Z\,2})$ and $\mathtt{P}'_j = (A, C_A, C_{Z\,2}, U \cup C_{Z\,1})$

$$\begin{aligned} T_i[b, \mathtt{P}].\textbf{solution} &= T_j[b' + |A|, \mathtt{P}_j].\textbf{solution} \cup T_k[b - b', \mathtt{P}_k].\textbf{solution} \\ T_i[b, \mathtt{P}].\textbf{value} &= T_j[b' + |A|, \mathtt{P}_j].\textbf{value} + T_k[b - b', \mathtt{P}_k].\textbf{value} \\ & \quad - \mathcal{C}(A \cup C_A, A) \end{aligned}$$

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- For a $k$-outerplanar graph $G$, a tree decomposition with width at most $3k + 1$ can be computed in linear time using [Shmoys and Williamson, 2011].

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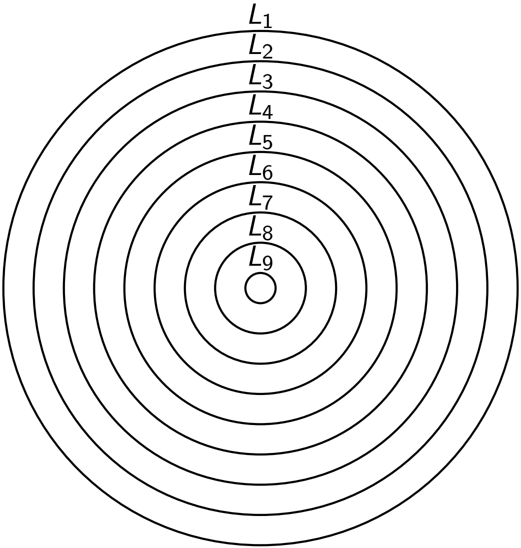

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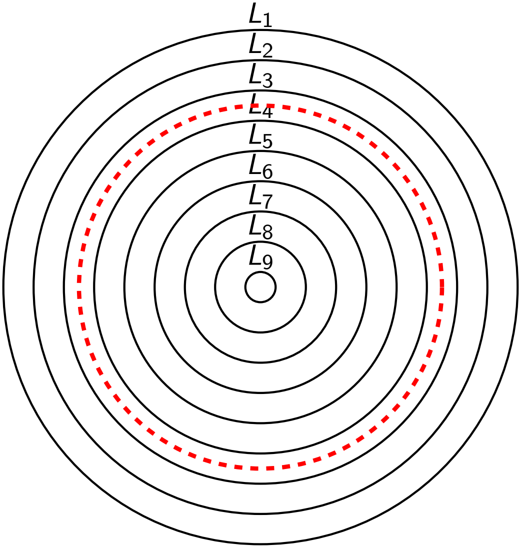

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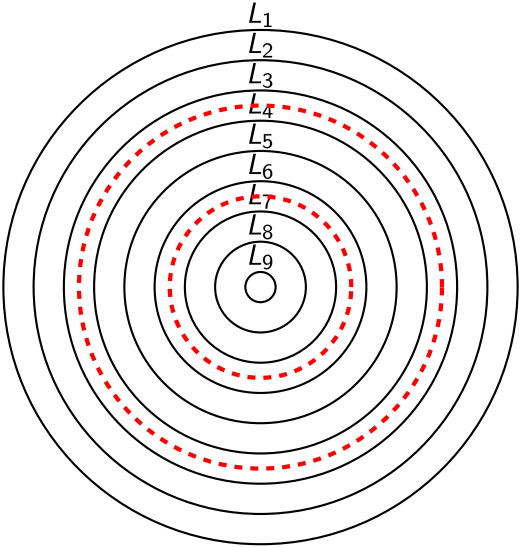

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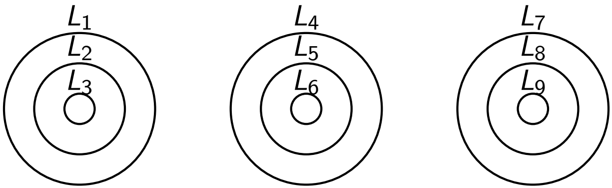

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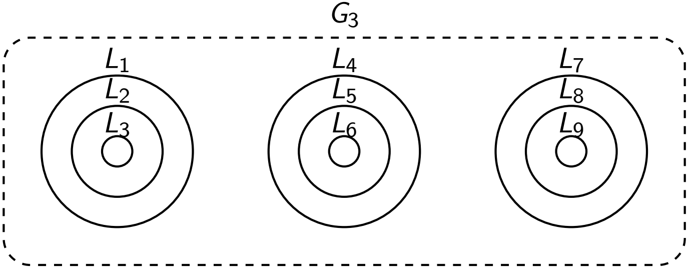

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$$S = \max_{S' \in \{S_1, S_2, \ldots, S_k\}} \mathcal{C}(V, S')$$

- Output $S$ such that $\mathcal{C}(V, S)$ is at least $1 - \frac{1}{\epsilon}$ times of optimum expected coverage.

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**Theorem**

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□


THANK
YOU!

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