Facility Location on Planar Graphs with Unreliable Links

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Facility Networks

Expected Coverage

Problem Formulation

Facility Networks

Graph representation of facility network with vertices $v_1$, $v_2$, $v_3$, and $v_4$. Weights $w(v_1) = 10$, $w(v_2) = 4$, $w(v_3) = 6$, and $w(v_4) = 5$. Budget $B_p: E \rightarrow [0, 1]$. Budget constraints: $0.6$, $0.3$, $0.7$, $0.5$, $0.4$. The graph shows the connections and weights between the vertices.
Facility Networks

\[ w(v_1) = 10 \]
\[ w(v_2) = 4 \]
\[ w(v_3) = 6 \]
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\[ w : V \rightarrow \mathbb{R}^+ \]
Facility Networks

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Budget \( \mathcal{B} \)
Facility Networks

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\[ \mathcal{C} \]

Budget \[ \mathcal{B} \]
Sub-graph Realizations

\[ Q_1, P(Q_1) \]

\[ Q_2, P(Q_2) \]
Sub-graph Realizations

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Sub-graph Realizations

$Q_1, P(Q_1)$

$Q_2, P(Q_2)$
The **Max-Exp-Cover** Problem

**Input:**

A graph $G = (V, E)$

**Demand $w$:** $V \to \mathbb{R}^+$

**Survival probability $p$:** $E \to [0, 1]$

**Budget $B$:**

Assume: $Q \subseteq 2^E$ with $P: Q \to [0, 1]$

Compute:

$$\max_{F \subseteq V, |F| \leq B} \sum_{Q \in Q} P(Q) \sum_{v \in V} w(v) \cdot I(Q, F, v)$$

$I(Q, F, v) = \begin{cases} 
1 & \text{if } v \in N_Q[F] \\
0 & \text{otherwise}
\end{cases}$
The \textbf{Max-Exp-Cover} Problem

Input: A graph $G = (V, E)$
Demand $w : V \rightarrow \mathbb{R}^+$
Survival probability $p : E \rightarrow [0, 1]$
Budget $\mathcal{B}$
The **Max-Exp-Cover** Problem

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The **Max-Exp-Cover** Problem

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- A graph \( G = (V, E) \)
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- Survival probability \( p : E \rightarrow [0, 1] \)
- Budget \( \mathcal{B} \)

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\[
\max_{F \subseteq V, |F| \leq \mathcal{B}} \sum_{Q \in \mathcal{Q}} P(Q) \sum_{v \in V} w(v) \cdot I(Q, F, v)
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Coverage Function

\[ C(v, F) = \sum_{Q \in Q} \sum_{v \in V} w(v) \cdot I(Q, F, v) \]

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\[ = \sum_{v \in V} w(v) \sum_{Q \in Q} P(Q) \cdot I(Q, F, v) |_{v \in N(Q)[F]} \]

\[ = C(V, F) \]
The coverage function $C$

Given a set $F \subseteq V$ and a vertex $v \in V$, the function $C(v, F)$ is the expected coverage of $v$ by $F$.

$$C(v, F) = w(v) \cdot \sum_{Q \in \mathcal{Q}} P(Q) \cdot I(Q, F, v)$$
The coverage function $C$

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$$\sum_{Q \in Q} P(Q) \sum_{v \in V} w(v) \cdot I(Q, F, v) = \sum_{v \in V} w(v) \sum_{Q \in Q} P(Q) \cdot I(Q, F, v)$$
Coverage Function

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$$= \sum_{v \in V} w(v) \sum_{Q \in Q : v \in N_Q[F]} P(Q)$$

$$= \sum_{v \in V} C(v, F) = C(V, F)$$
LRO Model

Failure Model

LRO Model

Vulnerability-Based Dependency [Hassin et al., 2009]

Given $e_i$ and $e_j$ such that $p(e_i) > p(e_j)$

$$\Pr[e_j \text{ fails} \mid e_i \text{ fails}] = 1$$

If an edge $e_i$ fails then the weaker edges than $e_i$ surely fails.

Linear Reliable Ordering [Hassin et al., 2017]

Every pair of edges are following VB-dependency.

$m + 1$ realizations are possible.

Let $G_0, G_1, \ldots, G_m$ be all the possible realizations.
LRO Model

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- Let $G_0, G_1, \ldots, G_m$ be all the possible realizations.
LRO Model

- Order the edges \( e_1, e_2, \ldots, e_m \) in descending order of survival probability.
- \( G_0 \) - Empty graph. When \( e_1 \) fails.
- \( G_i \) occurs when \( e_i \) is the weakest link that survives.

![Diagram of LRO Model]
Failure Model

LRO Instance

\[ p_1 \rightarrow \cdots \rightarrow p_7 \]

\[ a \rightarrow b \rightarrow c \rightarrow d \rightarrow e \rightarrow f \]
LRO Instance

$G, G_7, P_7$
LRO Instance

\[ G, G_7, p_7 \quad \text{and} \quad G_6, p_6 - p_7 \]
LRO Instance

\( G, G_7, p_7 \)
\( G_6, p_6 - p_7 \)
\( G_5, p_5 - p_6 \)
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\( G_3, p_3 - p_4 \)
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Results
The \textbf{Max-Exp-Cover} problem with LRO Model - \textbf{Existing}

- When $R = 1$, NP-Hard [Hassin et al., 2009]
- When $R = \infty$, $O(m+n)$ [Hassin et al., 2009]
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The **Max-Exp-Cover** problem with LRO Model - **In this work**

- When $R = 1$, FPT on bounded treewidth graph,
## Results

### The **Max-Exp-Cover** problem with LRO Model - **Existing**

- When $R = 1$, NP-Hard [Hassin et al., 2009]
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### The **Max-Exp-Cover** problem with LRO Model - **In this work**

- When $R = 1$, FPT on bounded treewidth graph, and PTAS on planar graph
Results

The \textbf{Max-Exp-Cover} problem with LRO Model - \textit{Existing}

- When $R = 1$, NP-Hard [Hassin et al., 2009]
- When $R = \infty$, $O(m+n)$ [Hassin et al., 2009]

The \textbf{Max-Exp-Cover} problem with LRO Model - \textit{In this work}

- When $R = 1$, FPT on bounded treewidth graph, and PTAS on planar graph
- Observed that, the problem has greedy approximation algorithm $(1 - \frac{1}{e})$. 
Tree Decomposition

$H$ - Tree, $X = \{X_i \subseteq V \mid i \in H\}$ - Bag
A pair $(H, X)$ satisfy the following conditions.

1. $\forall v \in V, \exists i \in H \mid v \in X_i$.
2. $\forall uv \in E, \exists i \in H \mid u, v \in X_i$.
3. $\forall v \in V$, let $T_v = \{i \in H \mid v \in X_i\}$, then $H[T_v]$ is connected.
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- $width = \max_{i \in H} |X_i| - 1$. 
Nice Tree Decomposition
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- **Leaf:** $i$ has no child and $X_i = \{\}$. 

\[ i \]
Nice Tree Decomposition

- **Leaf**: $i$ has no child and $X_i = \{\}$.  
- **Introduce**: $i$ has a child $j$ : $X_i = X_j \cup \{v\}$ for some $v \notin X_j$.  

![Diagram showing a tree decomposition with nodes labeled $i$, $j$, $u$, $v$, and $w$.]
### Nice Tree Decomposition

- **Leaf:** $i$ has no child and $X_i = \{\}$. 
- **Introduce:** $i$ has a child $j : X_i = X_j \cup \{v\}$ for some $v \notin X_j$. 
- **Forget:** $i$ has a child $j : X_i = X_j \setminus \{v\}$ for some $v \in X_j$. 

![Diagram](image)
Nice Tree Decomposition

- **Leaf:** $i$ has no child and $X_i = \{\}$. 
- **Introduce:** $i$ has a child $j : X_i = X_j \cup \{v\}$ for some $v \notin X_j$. 
- **Forget:** $i$ has a child $j : X_i = X_j \setminus \{v\}$ for some $v \in X_j$. 
- **Join:** $i$ has two children $j$ and $k : X_i = X_j = X_k$. 

![Diagram](attachment:image.png)
Best Neighbour

Given a vertex \( u \in V \) and a set \( S \subseteq V \),

\[
bn(u, S) = \begin{cases} 
    u & \text{if } u \in S \\
    v = \max_{v' \in N(u) \cap S} \quad \left( \text{if } u \not\in S \land N(u) \cap S \neq \emptyset \right) \\
    p(u, v) & \text{otherwise}
\end{cases}
\]
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bn(u, S) = \begin{cases} 
  u & \text{if } u \in S \\
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  \text{undefined} & \text{otherwise}
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Best Neighbour - In LRO

$G, G_7, p_7$

$G_6, p_6 - p_7$

$G_5, p_5 - p_6$

$G_4, p_4 - p_5$

$G_3, p_3 - p_4$

$G_3, p_2 - p_3$

$G_1, p_1 - p_2$

$G_0, 1 - p_1$
Best Neighbour - In LRO

\[ G, G_7, p_7 \]
\[ G_6, p_6 - p_7 \]
\[ G_5, p_5 - p_6 \]
\[ G_4, p_4 - p_5 \]

\[ G_3, p_3 - p_4 \]
\[ G_3, p_2 - p_3 \]
\[ G_1, p_1 - p_2 \]
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Coverage using Best Neighbour

**Lemma**

*Let* \( u \in V \) *be a vertex and* \( S \subseteq V \) *be a set. If the coverage* \( C(u, S) > 0 \), *then there is a vertex* \( v \in S \) *such that* \( C(u, S) = C(u, v) \).*
Coverage using Best Neighbour

Lemma

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Proof.

- Since \( C(u, S) > 0 \), \( N[v] \cap S \neq \emptyset \). Then \( S' = N[v] \cap S \).
Coverage using Best Neighbour

Lemma

Let $u \in V$ be a vertex and $S \subseteq V$ be a set. If the coverage $C(u, S) > 0$, then there is a vertex $v \in S$ such that $C(u, S) = C(u, v)$.

Proof.

- Since $C(u, S) > 0$, $N[v] \cap S \neq \emptyset$. Then $S' = N[v] \cap S$.
- When $u \in S$, then $C(u, S) = C(u, u) = w(u)$. 
Coverage using Best Neighbour

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- Suppose $u \notin S$, then $S' = \{v_1, v_2, \ldots, v_\ell\}$ for some $0 < \ell \leq d_u$. 
Coverage using Best Neighbour

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**Proof.**

- Since $C(u, S) > 0$, $N[v] \cap S \neq \emptyset$. Then $S' = N[v] \cap S$.
- When $u \in S$, then $C(u, S) = C(u, u) = w(u)$.
- Suppose $u \notin S$, then $S' = \{v_1, v_2, \ldots, v_\ell\}$ for some $0 < \ell \leq d_u$.
- Assume $p(uv_1) > p(uv_2) > \cdots > p(uv_\ell)$. 
Proof. (Cont)

\[ C(u, S) = C(u, S') \]

\[ S' = N(v) \cap S \]

Hence the proof. \( \square \)
Proof. (Cont)

\[ C(u, S) = C(u, S') \]
\[ = w(u) \cdot \sum_{G_i | u \in N_{G_i} [S']} p(G_i) \]

Hence the proof. □
Proof. (Cont)

\[ C(u, S) = C(u, S') \]
\[ = w(u) \cdot \sum_{G_i | u \in N_{G_i}[S']} p(G_i) \]
\[ = w(u) \cdot \sum_{i=j}^{m} p(G_i) \]

\[ G_j \mid (u, v_1) \text{ survives} \]
Proof. (Cont)

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\[ = C(u, v_1) \]
Proof. (Cont)

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\[ G_j \mid (u, v_1) \text{ survives} \]
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\[ = C(u, v_1) \]
\[ = C(u, bn(u, S)) \]

Hence the proof. \(\Box\)
Structure of a Solution

Let $i$ be a node in $H$ with bag $X_i$ and vertex set $V_i$. Let $S = A \cup Z$ be a solution such that $A = S \cap X_i$ and $Z = S \setminus A$. 

$C_A = \{ u \in X_i | bn(u, S) \in A \}$ and $C_Z = \{ u \in X_i | bn(u, S) \in Z \}$.

$U = X_i \setminus (A \cup C_A \cup C_Z)$. 
Structure of a Solution

- Let $i$ be a node in $H$ with bag $X_i$ and vertex set $V_i$. 
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- $U = X_i \setminus (A \cup C_A \cup C_Z)$.
Solution Structure

\[ C(V_i, S) = C((V_i \setminus X_i) \cup X_i, S) \]
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\[ = C(V_i \setminus X_i, S) + C(C_A, S) + C(C_Z, S) + C(A, S) \]
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Partition

For every feasible solution \( S \subseteq V_i \), there is a four-way partition \( P = (A, C_A, C_Z, U) \) of \( X_i \) such that

\[ C(V_i, S) = C(V_i \setminus X_i, S) + C(A \cup C_A, A) + C(C_Z, Z) \]
For each feasible solution, there exists a unique partition $P$ such that the coverage of $S$ can be expressed using $P$. 

Maintains a table of size $(B+1)^4$. For each $0 \leq b \leq B$ and each four-way partition $P = (A, C_A, C_Z, U)$ of $X_i$, $T_i[b, P]$ is a tuple $(\text{solution}, \text{value})$ such that $T_i[b, P].\text{solution} = S \subseteq V_i \mid |S| = b$ and $S \cap X_i = A$. 

$C(A \cup C_A, A) + C(C_Z, Z) + C(V_i \setminus X_i, S)$ is maximized over all possible such $S \subseteq V_i$. 

Dynamic Programming
Dynamic Programming

- For each feasible solution, there exists an unique partition $P$ such that the coverage of $S$ can be expressed using $P$.
- We explore all four way partitions, and find the optimal solution $w.r.t$ the partition.
Dynamic Programming

- For each feasible solution, there exists an unique partition $P$ such that the coverage of $S$ can be expressed using $P$.
- We explore all four way partitions, and find the optimal solution \( w.r.t \) the partition.
- Maintains a table of size \( (B + 1)4^{|X_i|} \).
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- For each $0 \leq b \leq B$ and each four-way partition $P = (A, C_A, C_Z, U)$ of $X_i$, $T_i[b, P]$ is a tuple $(\text{solution}, \text{value})$ such that
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  - $|S| = b$ and $S \cap X_i = A$. 

$C(A \cup C_A, A) + C(C_Z, Z) + C(V_i \setminus X_i, S)$ is maximized over all possible such $S \subseteq V_i$. 
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Introduce Node

- Let $i$ be an introduce node with child $j$ such that $X_i = X_j \cup \{v\}$ for some $v \notin X_j$. 

![Diagram]

Let $0 \leq b \leq B$ be a budget and $P = (A, C_A, C_Z, U)$ be a four way partitioning of $X_i$. We consider two cases that

(i) $-v \in A$

(ii) $-v \in A$
Introduce Node

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Let $0 \leq b \leq B$ be a budget and $P = (A, C_A, C_Z, U)$ be a four way partitioning of $X_i$. 

![Diagram](attachment:image.png)
Introduce Node

- Let $i$ be an introduce node with child $j$ such that $X_i = X_j \cup \{v\}$ for some $v \notin X_j$.
- Let $0 \leq b \leq B$ be a budget and $P = (A, C_A, C_Z, U)$ be a four way partitioning of $X_i$.
- We consider two cases that (i) $v \notin A$ and (ii) $v \in A$. 

\[ i \quad \frac{X_j \cup \{v\}}{\downarrow v} \quad \frac{X_j}{j} \quad \frac{V_j \setminus X_j}{\downarrow v} \]

\[ i \quad \frac{X_i}{\quad \begin{array}{cccc} A & C_A & C_Z & U \\ v & \vrule & \vrule & \vrule \\ \vrule & \vrule & \vrule & \vrule \\ \vrule & \vrule & \vrule & v \end{array} \quad \frac{\vrule}{\vrule} \quad \frac{\vrule}{\vrule} \quad \frac{\vrule}{\vrule} \]

\[ j \quad \frac{X_j}{\downarrow v} \quad \frac{V_j \setminus X_j}{\Downarrow v} \]
Introduce Node (Cont...)

\[ P_i = (A_i, C_A, C_Z, U) \]

\[ P_j = (A_i, C_{A\{v\}}, C_{Z\{v\}}, U\{v\}) \]

\[ T_i[b, P] = \text{solution} = T_j[b, P_j] \]

\[ T_j[b, P_j].\text{value} = \begin{cases} T_j[b, P_j].\text{value} & \text{if } v / \in C_A \\ T_j[b, P_j].\text{value} + C(v, A) & \text{if } v \in C_A \end{cases} \]
Introduce Node (Cont...)

Case: \( v \notin A \)
Case: \( v \notin A \)

\[
P = (A, C_A, C_Z, U) \rightarrow P_j = (A, C_A \setminus \{v\}, C_Z \setminus \{v\}, U \setminus \{v\})
\]
Introduce Node (Cont...)

Case: $v \notin A$

$$P = (A, C_A, C_Z, U) \rightarrow P_j = (A, C_A \setminus \{v\}, C_Z \setminus \{v\}, U \setminus \{v\})$$

$$T_i[b, P].solution = T_j[b, P_j].solution$$

$$T_i[b, P].value = \begin{cases} 
T_j[b, P_j].value & \text{if } v \notin C_A \\
T_j[b, P_j].value + C(v, A) & \text{if } v \in C_A
\end{cases}$$
Introduce Node (Cont...)

Case: $v \in A$

- Let $C_{Av} = \{ u \in C_A \mid bn(u, A) = v \}$. 
Case: \( v \in A \)

- Let \( C_{Av} = \{ u \in C_A | bn(u, A) = v \} \).
- Let \( P_j = (A \setminus \{ v \}, C_A \setminus C_{Av}, C_Z, U \cup C_{Av}) \).
Case: $v \in A$

- Let $C_{AV} = \{u \in C_A \mid bn(u, A) = v\}$.
- Let $P_j = (A \setminus \{v\}, C_A \setminus C_{AV}, C_Z, U \cup C_{AV})$.

\[
T_i[b, P].solution = T_j[b - 1, P_j].solution \cup \{v\}
\]
\[
T_i[b, P].value = T_j[b - 1, P_j].value + C(\{v\} \cup C_{AV}, v)
\]
Correctness – case $v \notin A$

- Assume $v \in C_A$.
- Let $T_i[b, P].solution = T_j[b, P'].solution = A \cup Z$ where $P' = (A, C_A \setminus \{v\}, C_Z, U)$.
- By contradiction, assume $S' = A \cup Z'$ optimal than $S$. That is $T_i[b, P].value < C(V_i \setminus X_i, S') + C(A \cup C_A, A) + C(C_Z, Z')$.

\[
T_j[b, P'].value = T_i[b, P].value - C(v, A) < C(V_i \setminus X_i, S') + C(A \cup C_A, A) + C(C_Z, Z') - C(v, A) < C(V_j \setminus X_j, S') + C(A \cup C_A \setminus \{v\}, A) + C(C_Z, Z')
\]

- Contradicts optimality of $T_j[b, P']$ by $S'$. 
Correctness – case $\nu \in A$

- Let $C_{Av} = \{u \in C_A \mid bn(u, A) = \nu\}$.
- Let $T_i[b, P].solution = T_j[b - 1, P'].solution \cup \{\nu\} = A \cup Z$ where $P' = (A \setminus \{\nu\}, C_A \setminus C_{Av}, C_Z, U \cup C_{Av})$.
- By contradiction, assume $S' = A \cup Z'$ optimal than $S$. That is $T_i[b, P].value < C(V_i \setminus X_i, S') + C(A \cup C_A, A) + C(C_Z, Z')$.

$$T_j[b, P'].value = T_i[b, P].value - C(\{\nu\} \cup C_{Av}, \nu)$$
$$< C(V_i \setminus X_i, S') + C(A \cup C_A, A) + C(C_Z, Z') - C(\{\nu\} \cup C_{Av}, \nu)$$
$$< C(V_j \setminus X_j, S') + C((A \cup C_A) \setminus (\{\nu\}), A \setminus \{\nu\}) + C(C_Z, Z')$$

- Contradicts optimality of $T_j[b, P']$ by $S'$. 
Let $0 \leq b \leq B$ be a budget and $P = (A, C_A, C_Z, U)$ be a four way partitioning of $X_i$. 
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- $P_1 = (A \cup \{v\}, C_A, C_Z, U)$
- $P_2 = (A, C_A \cup \{v\}, C_Z, U)$
- $P_3 = (A, C_A, C_Z \cup \{v\}, U)$
- $P_4 = (A, C_A, C_Z, U \cup \{v\})$
Let \( 0 \leq b \leq B \) be a budget and \( P = (A, C_A, C_Z, U) \) be a four way partitioning of \( X_i \).

\[
\begin{align*}
P_1 &= (A \cup \{v\}, C_A, C_Z, U) \\
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\end{align*}
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Let \( P_j = \max_{P' \in \{P_1, P_2, P_3, P_4\}} T_j[b, P'].value \)
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$$T_i[b, P].solution = T_j[b, P_j].solution$$
$$T_i[b, P].value = T_j[b, P_j].value$$
Join Node

- Let $C_{Zj} = \{ u \in C_Z | bn(u, S) \in Z_j \}$
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- Let $C_{Zk} = \{ u \in C_Z \mid bn(u, S) \in Z_k \}$
- Let $b_j = |Z_j|$ and $b_k = |Z_k|$.

$\left(b', C_{Zj}, C_{Zk}\right) = \max_{0 \leq b_1 \leq b - |A|, \ C_{Z1} \cup C_{Z2} = C_Z} T_j[b_1 + |A|, P'_j].value + T_k[b - b_1, P'_k].value$
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where $P'_j = (A, C_A, C_{Z1}, U \cup C_{Z2})$ and $P'_j = (A, C_A, C_{Z2}, U \cup C_{Z1})$
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$T_i[b, P].\text{solution} = T_j[b' + |A|, P_j].\text{solution} \cup T_k[b - b', P_k].\text{solution}$

$T_i[b, P].\text{value} = T_j[b' + |A|, P_j].\text{value} + T_k[b - b', P_k].\text{value} - C(A \cup C_A, A)$
PTAS in planar graphs

- We are given with an instance of \textsc{Max-Exp-Cover-1} problem and an $\epsilon > 0$. 

For a $k$-outerplanar graph $G$, a tree decomposition with width at most $3k + 1$ can be computed in linear time using [Shmoys and Williamson, 2011].
We are given with an instance of \textsc{Max-Exp-Cover-1} problem and an $\epsilon > 0$.

Compute an $l$-outerplanar embedding of $G$, for a minimum $l$. Level ordering such that $V = \bigcup_{i=1}^{l} L_i$. 

\[ k = \frac{1}{\epsilon} \] 

For $i = 1$ to $k$:

Let $G_i = (V, E_i)$ where $E_i = E \setminus \{ (u, v) \in E | \text{level}(u) \equiv i \mod k, \text{level}(v) \equiv (i+1) \mod k \}$

$G_i$ is a collection of $k$-outerplanar graphs.

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Subgraph construction
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Subgraph construction

$L_1$, $L_2$, $L_3$

$L_4$, $L_5$, $L_6$

$L_7$, $L_8$, $L_9$
Subgraph construction

$G_3$

$L_1$
$L_2$
$L_3$

$L_4$
$L_5$
$L_6$

$L_7$
$L_8$
$L_9$
Solution Construction

- We have \( k \) many \( k \)-outerplanar graphs \((G_1, G_2, \ldots, G_k)\).
We have $k$ many $k$-outerplanar graphs ($G_1, G_2, \ldots, G_k$).

Compute the optimal solution $S_i$ in each graph $G_i$. 

Let $S$ be the set achieving maximum expected coverage.

$S = \max \{ S_1, S_2, \ldots, S_k \}$

Output $S$ such that $C(V, S)$ is at least $1 - \frac{1}{\epsilon}$ times of optimum expected coverage.
Solution Construction

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\[
S = \max_{S' \in \{S_1, S_2, \ldots, S_k\}} C(V, S')
\]

- Output \( S \) such that \( C(V, S) \) is at least \( 1 - \frac{1}{\epsilon} \) times of optimum expected coverage.
Approximation Analysis

Theorem

The set $S$ is $(1 - \frac{1}{k})$-approximate solution for the Max-Exp-Cover-1 problem.
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Proof.

- Let $OPT = \{v_1, v_2, \ldots, v_B\}$. 
Approximation Analysis

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Proof.

- Let $OPT = \{v_1, v_2, \ldots, v_B\}$.
- Let $L_i = \{v \in V \mid \text{level}(v) \equiv i \mod k\}$. 
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- Let $OPT = \{v_1, v_2, \ldots, v_B\}$.
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- Also, $L_i$ is the collection of level-$k$ vertices of $G_i$. 
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The set $S$ is $(1 - \frac{1}{k})$-approximate solution for the Max-Exp-Cover-1 problem.

Proof.

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- Also, $L_i$ is the collection of level-$k$ vertices of $G_i$.
- Since $L_1, L_2, \ldots L_k$ is a partition of $V$, $C(V, OPT) = \sum_{i=1}^{k} C(L_i, OPT)$.
Proof.

- $\exists j$ such that $C(L_j, OPT) \leq \frac{1}{k} \cdot C(V, OPT)$.
Approximation Analysis

Proof.

- \( \exists j \) such that \( C(L_j, OPT) \leq \frac{1}{k} \cdot C(V, OPT) \).
- Then, \( C(V \setminus L_j, OPT) \geq (1 - \frac{1}{k}) \cdot C(V, OPT) \).
Proof.

- ∃ j such that $C(L_j, \text{OPT}) \leq \frac{1}{k} \cdot C(V, \text{OPT})$.
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- Let $G_j$ be the graph corresponding to $j$. 
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- Let \( G_j \) be the graph corresponding to \( j \).
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\[
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\[
C(V, S) \geq C(V, S_j) \geq C(G_j, V, S_j) \geq C(G_j, V, \text{OPT}) \\
\geq C(G_j, V \setminus L_j, \text{OPT}) = C(G, V \setminus L_j, \text{OPT})
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Approximation Analysis

Proof.

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- Since \( C(V, S) \geq C(V, S_j) \).

\[
C(V, S) \geq C(V, S_j) \geq C(G_j, V, S_j) \geq C(G_j, V, \text{OPT}) \\
\geq C(G_j, V \setminus L_j, \text{OPT}) = C(G, V \setminus L_j, \text{OPT}) \\
\leq (1 - \frac{1}{k}) \cdot C(V, \text{OPT})
\]

\( \square \)
