Can We Create Large k-Cores by Adding Few Edges?

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Joint work with Nimrod Talmon (Ben Gurion University)

CSR, Moscow

10 June 2018

THE UNIVERSITY OF WARWICK

Outline of Talk

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- What is a k-core?

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- Finding k-cores

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- Hence, we can talk about **the** k-core

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What is a *k*-core?

- Applications of *k*-cores [via Wikipedia]

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- Anchored k-cores
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- Why do we even want large *k*-cores?

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- The subgraph that remains is exactly the k-core.

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Game-theoretic interpretation for k-cores

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  - Chwe ['99]
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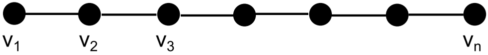

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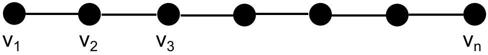

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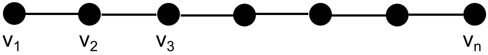

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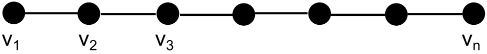

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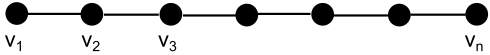

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- How can we prevent this unraveling?

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- To prevent the unraveling described in the previous slide, we use the concept of anchors.

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- We motivate some of the users in the social network with "external incentives", so that they remain engaged even if they have less than k friends.

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- To prevent the unraveling described in the previous slide, we use the concept of anchors.
- We motivate some of the users in the social network with "external incentives", so that they remain engaged even if they have less than k friends.
- The role of the anchors is to augment the degrees of the other vertices.

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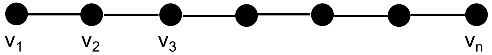

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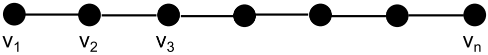

- If we set v_1 and v_n to be the anchors, then the entire social network survives.

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The ANCHORED k-CORE problem: Bhawalkar et al. [ICALP '12]

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- W[1]-hard parameterized by p
- FPT on planar graphs parameterized by b

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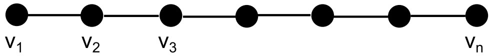

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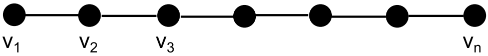

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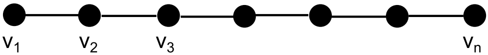

- Instead of anchoring v_1 and v_n, we could add an edge between v_1 and v_n
- Then we get a cycle of length n, and the whole network survives

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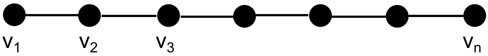
Obtaining k-cores via edge-additions?



- Instead of anchoring v_1 and v_n, we could add an edge between v_1 and v_n
- Then we get a cycle of length n, and the whole network survives
- Does not really make sense in case of social networks!
  - Can't add edges between random people

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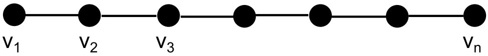

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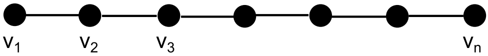

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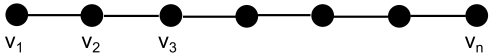

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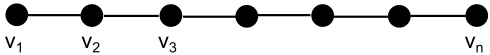

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- Next slide: Formal definition of EDGE k-CORE problem

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Input: An undirected graph $G = (V, E)$ and integers b, k, p

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- First we see that there is no relation between EDGE k-CORE and ANCHORED k-CORE.
- **Edge Additions** $>$ **Anchored Vertices**:
  - Let G be a disjoint union of two components G_1 and G_2, where $G_1 = K_{z_1}$ and G_2 is a z_2-regular graph on n_2 vertices
  - Choose $z_1 \ll z_2 \ll n_2$. If $k = z_2$ and $p = z_1 + n_2$,
  - Then number of anchors is $b_v = z_1$, while number of edge deletions is $b_e \geq \frac{z_1 \cdot (z_2 - z_1 + 1)}{2} \gg z_1 = b_v$

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- **Edge Additions $<$ Anchored Vertices**
  - Let $G_1 = K_{2n}$, and G_2 be K_{2n} with a perfect matching removed. Add a matching of size $2n$ between G_1 and G_2.
  - Set $k = 2n$ and $p = 4n$
  - Then $b_v = 2n$, and $b_e = n < 2n = b_v$

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- FPT parameterized by $k + b + \mathbf{tw}$, even for $k = 3$

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Obtaining k-cores via edge-additions: Polytime algorithms

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- $k = 0$: Answer YES if and only if $p \leq n$

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  - Say input graph is three disjoint edges and $b = 2 = k$

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  - Our algorithm: Preprocessing + (surprisingly complicated) greedy algorithm!

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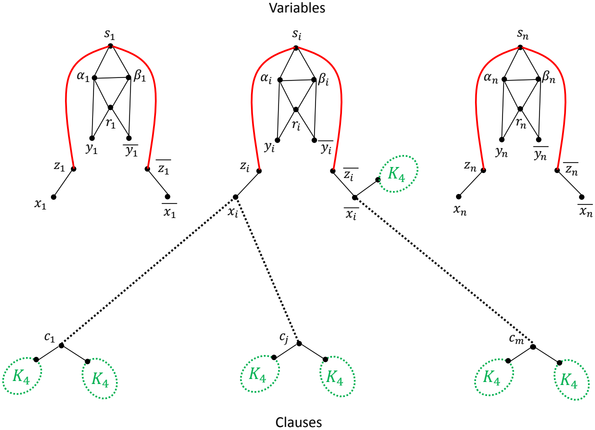
  - Let $\alpha_{\mathbf{0}}, \alpha_{\geq \mathbf{1}}$ be the number of vertices of degree $0, \geq 1$ respectively
  - If $b \leq \lfloor \frac{\alpha_{\mathbf{0}}}{\mathbf{2}} \rfloor$, then the maximum size of a 1-core is $\alpha_{\geq \mathbf{1}} + 2b$.
  - Otherwise, if $b > \lfloor \frac{\alpha_{\mathbf{0}}}{\mathbf{2}} \rfloor$, then the entire vertex set can become a 1-core by adding a matching between the vertices of degree 0 (and an extra edge if $\alpha_{\mathbf{0}}$ is odd).
- **k=2**
  - Simple greedy doesn't work
  - Say input graph is three disjoint edges and $b = 2 = k$
  - Then greedy might just make it into a path on 6 vertices
  - Optimal solution is to make a 4-cycle and get $p = 4$
  - So, only degree of vertex is not only thing to consider! Degrees of their neighbors are important too
  - Our algorithm: Preprocessing + (surprisingly complicated) greedy algorithm!
- Next slide: NP-hardness for $k \geq 3$

Can we create large k-cores by adding few edges?

Obtaining k-cores via edge-additions: NP-hardness for $k = 3$ via 3-SAT



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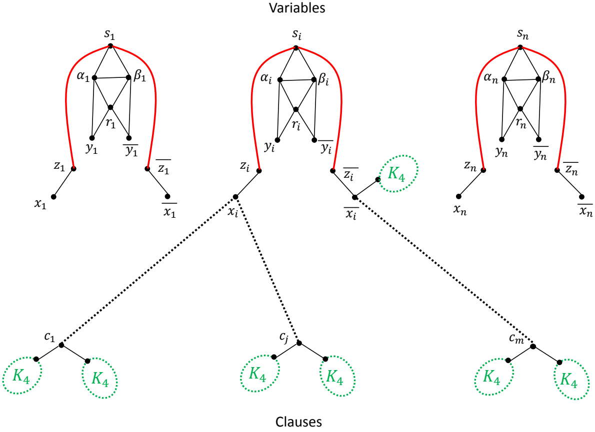

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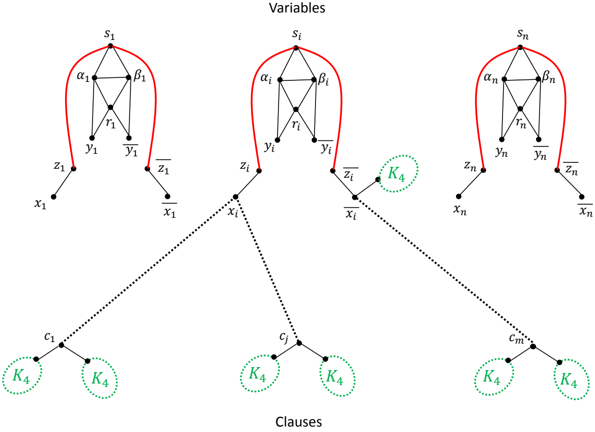

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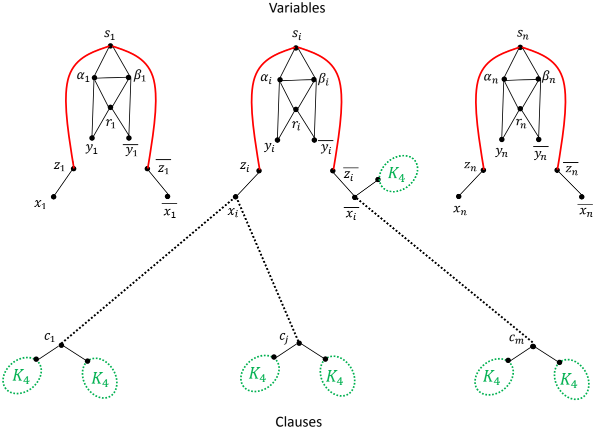

- Each variable is used at most 3 times
- Each variable is used at least once in positive, and at least once in negative
- SAT $\Leftrightarrow$ (EKC answers YES with $k = 3, b = n$ and $p = |G| - 3n$)

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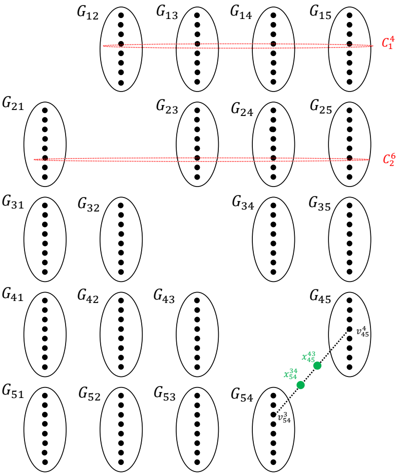
Obtaining k-cores via edge-additions: W[1]-hardness for $k = 3$ w.r.t $(b + p)$ via Clique



The graph G' when $n = 8$ and $\ell = 5$.

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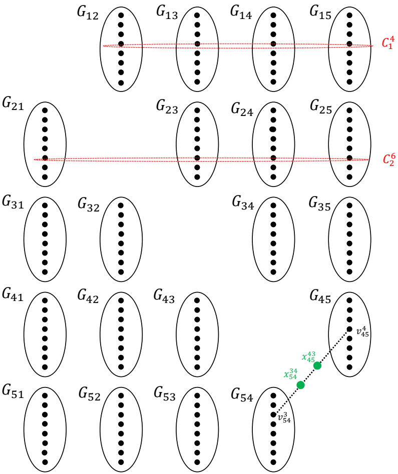

The graph G' when $n = 8$ and $\ell = 5$.

- G has a ℓ-clique $\Leftrightarrow$ EKC answers YES on G' with $k = 3$, $b = \binom{\ell}{2}$ and $p = 4b$

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- **Main technical result of our paper**: Explicit DP algorithm which runs in time $(k + \mathbf{tw})^{O(\mathbf{tw}+b)} \cdot \mathrm{poly}(n)$.

Thank You

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Questions?