A Tight Lower Bound for Steiner Orientation

Rajesh Chitnis

Joint work with Andreas Emil Feldmann

CSR, Moscow

7 June 2018

THE UNIVERSITY OF WARWICK

Outline of Talk

Outline of Talk

- Steiner Orientation

Outline of Talk

- Steiner Orientation
- Upper Bound

Outline of Talk

- Steiner Orientation
- Upper Bound
- Non-Tight Lower Bound

Outline of Talk

- Steiner Orientation
- Upper Bound
- Non-Tight Lower Bound
- Tight Lower Bound

Outline of Talk

- Steiner Orientation
- Upper Bound
- Non-Tight Lower Bound
- Tight Lower Bound
- Some new results...

Outline of Talk

- Steiner Orientation
- Upper Bound
- Non-Tight Lower Bound
- Tight Lower Bound
- Some new results...

Steiner Orientation
Input: A mixed graph G, and a set $\mathcal{T}$ of k terminal pairs

Steiner Orientation
Input: A mixed graph G, and a set $\mathcal{T}$ of k terminal pairs
Question: Is there an orientation of the undirected egdes of G such that the resulting graph has an $s \rightsquigarrow t$ path for each $(s, t) \in \mathcal{T}$

A Tight Lower Bound for Steiner Orientation

The Steiner Orientation problem

Steiner Orientation
Input: A mixed graph G, and a set $\mathcal{T}$ of k terminal pairs
Question: Is there an orientation of the undirected egdes of G such that the resulting graph has an $s \rightsquigarrow t$ path for each $(s, t) \in \mathcal{T}$
Parameter: k

Steiner Orientation
Input: A mixed graph G, and a set $\mathcal{T}$ of k terminal pairs
Question: Is there an orientation of the undirected egdes of G such that the resulting graph has an $s \leadsto t$ path for each $(s, t) \in \mathcal{T}$
Parameter: k

- Solvable in polynomial time if G is undirected

A Tight Lower Bound for Steiner Orientation

The Steiner Orientation problem

Steiner Orientation
Input: A mixed graph G, and a set $\mathcal{T}$ of k terminal pairs
Question: Is there an orientation of the undirected egdes of G such that the resulting graph has an $s \rightsquigarrow t$ path for each $(s, t) \in \mathcal{T}$
Parameter: k

- Solvable in polynomial time if G is undirected
  - Hassin and Meggido ['89]

A Tight Lower Bound for Steiner Orientation

The Steiner Orientation problem

Steiner Orientation
Input: A mixed graph G, and a set $\mathcal{T}$ of k terminal pairs
Question: Is there an orientation of the undirected egdes of G such that the resulting graph has an $s \rightsquigarrow t$ path for each $(s, t) \in \mathcal{T}$
Parameter: k

- Solvable in polynomial time if G is undirected
  - Hassin and Meggido ['89]
- NP-hard if G is actually mixed

A Tight Lower Bound for Steiner Orientation

The Steiner Orientation problem

Steiner Orientation
Input: A mixed graph G, and a set $\mathcal{T}$ of k terminal pairs
Question: Is there an orientation of the undirected egdes of G such that the resulting graph has an $s \rightsquigarrow t$ path for each $(s,t) \in \mathcal{T}$
Parameter: k

- Solvable in polynomial time if G is undirected
  - Hassin and Meggido ['89]
- NP-hard if G is actually mixed
  - Arkin and Hassin ['02]

A Tight Lower Bound for Steiner Orientation

The Steiner Orientation problem

Steiner Orientation
Input: A mixed graph G, and a set $\mathcal{T}$ of k terminal pairs
Question: Is there an orientation of the undirected egdes of G such that the resulting graph has an $s \rightsquigarrow t$ path for each $(s, t) \in \mathcal{T}$
Parameter: k

- Solvable in polynomial time if G is undirected
  - Hassin and Meggido ['89]
- NP-hard if G is actually mixed
  - Arkin and Hassin ['02]
- How fast can we solve STEINER ORIENTATION?

A Tight Lower Bound for Steiner Orientation

The Steiner Orientation problem

Steiner Orientation
Input: A mixed graph G, and a set $\mathcal{T}$ of k terminal pairs
Question: Is there an orientation of the undirected egdes of G such that the resulting graph has an $s \rightsquigarrow t$ path for each $(s, t) \in \mathcal{T}$
Parameter: k

- Solvable in polynomial time if G is undirected
  - Hassin and Meggido ['89]
- NP-hard if G is actually mixed
  - Arkin and Hassin ['02]
- How fast can we solve STEINER ORIENTATION?
  - $n^{O(k)}$

A Tight Lower Bound for Steiner Orientation

The Steiner Orientation problem

Steiner Orientation
Input: A mixed graph G, and a set $\mathcal{T}$ of k terminal pairs
Question: Is there an orientation of the undirected egdes of G such that the resulting graph has an $s \rightsquigarrow t$ path for each $(s, t) \in \mathcal{T}$
Parameter: k

- Solvable in polynomial time if G is undirected
  - Hassin and Meggido ['89]
- NP-hard if G is actually mixed
  - Arkin and Hassin ['02]
- How fast can we solve STEINER ORIENTATION?
  - $n^{O(k)}$
  - $f(k) \cdot n^{O(1)}$

Outline of Talk

- Steiner Orientation
- Upper Bound
- Non-Tight Lower Bound
- Tight Lower Bound
- Some new results...

A Tight Lower Bound for Steiner Orientation

Sketch of $n^{O(k)}$ algorithm of Cygan, Kortsarz and Nutov ['13]

- Lemma 1: Let C be a subgraph which admits a strongly-connected orientation. Then we can obtain an equivalent instance by contracting C to a single node.

A Tight Lower Bound for Steiner Orientation

Sketch of $n^{O(k)}$ algorithm of Cygan, Kortsarz and Nutov ['13]

- Lemma 1: Let C be a subgraph which admits a strongly-connected orientation. Then we can obtain an equivalent instance by contracting C to a single node.
- Lemma 2: Let G' be graph obtained from G by contracting each undirected component into a single vertex. If G' has a directed cycle C' then we can find it in polytime and use it to find an oriented cycle in G

A Tight Lower Bound for Steiner Orientation

Sketch of $n^{O(k)}$ algorithm of Cygan, Kortsarz and Nutov ['13]

- Lemma 1: Let C be a subgraph which admits a strongly-connected orientation. Then we can obtain an equivalent instance by contracting C to a single node.
- Lemma 2: Let G' be graph obtained from G by contracting each undirected component into a single vertex. If G' has a directed cycle C' then we can find it in polytime and use it to find an oriented cycle in G
- From Lemma 1 and Lemma 2, can assume G is a DAG

A Tight Lower Bound for Steiner Orientation

Sketch of $n^{O(k)}$ algorithm of Cygan, Kortsarz and Nutov ['13]

- Lemma 1: Let C be a subgraph which admits a strongly-connected orientation. Then we can obtain an equivalent instance by contracting C to a single node.
- Lemma 2: Let G' be graph obtained from G by contracting each undirected component into a single vertex. If G' has a directed cycle C' then we can find it in polytime and use it to find an oriented cycle in G
- From Lemma 1 and Lemma 2, can assume G is a DAG
- Guess second and second-last vertices of satisfying path for each terminal pair

A Tight Lower Bound for Steiner Orientation

Sketch of $n^{O(k)}$ algorithm of Cygan, Kortsarz and Nutov ['13]

- Lemma 1: Let C be a subgraph which admits a strongly-connected orientation. Then we can obtain an equivalent instance by contracting C to a single node.
- Lemma 2: Let G' be graph obtained from G by contracting each undirected component into a single vertex. If G' has a directed cycle C' then we can find it in polytime and use it to find an oriented cycle in G
- From Lemma 1 and Lemma 2, can assume G is a DAG
- Guess second and second-last vertices of satisfying path for each terminal pair
  - This gives $n^{O(k)}$ possibilities

A Tight Lower Bound for Steiner Orientation

Sketch of $n^{O(k)}$ algorithm of Cygan, Kortsarz and Nutov ['13]

- Lemma 1: Let C be a subgraph which admits a strongly-connected orientation. Then we can obtain an equivalent instance by contracting C to a single node.
- Lemma 2: Let G' be graph obtained from G by contracting each undirected component into a single vertex. If G' has a directed cycle C' then we can find it in polytime and use it to find an oriented cycle in G
- From Lemma 1 and Lemma 2, can assume G is a DAG
- Guess second and second-last vertices of satisfying path for each terminal pair
  - This gives $n^{O(k)}$ possibilities
- Use topological order of G (since it is a DAG) and some clever dynamic programming

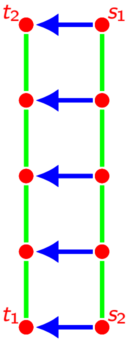
Outline of Talk

- Steiner Orientation
- Upper Bound
- Non-Tight Lower Bound
- Tight Lower Bound
- Some new results...

A Tight Lower Bound for Steiner Orientation

Non-tight $f(k) \cdot n^{o(\sqrt{k})}$ Lower Bound of Wahlstrom and Pilipczuk ['16]

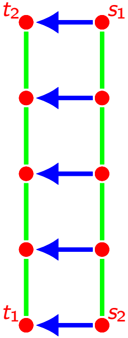
Basic gadget: Attempt I



A Tight Lower Bound for Steiner Orientation

Non-tight $f(k) \cdot n^{o(\sqrt{k})}$ Lower Bound of Wahlstrom and Pilipczuk ['16]

Basic gadget: Attempt I

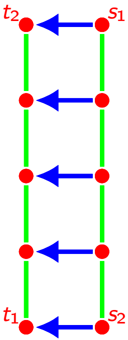


How can we satisfy both the pairs (s_1, t_1) and (s_2, t_2)?

A Tight Lower Bound for Steiner Orientation

Non-tight $f(k) \cdot n^{o(\sqrt{k})}$ Lower Bound of Wahlstrom and Pilipczuk ['16]

Basic gadget: Attempt I

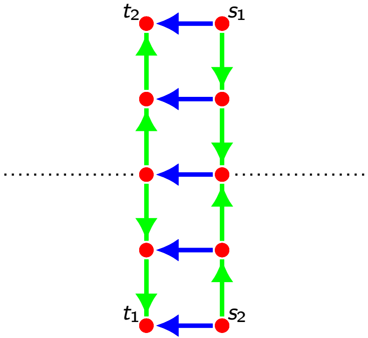


How can we satisfy both the pairs (s_1, t_1) and (s_2, t_2)?
Note that the only edges to orient are the green paths!

A Tight Lower Bound for Steiner Orientation

Non-tight $f(k) \cdot n^{o(\sqrt{k})}$ Lower Bound of Wahlstrom and Pilipczuk ['16]

Basic gadget: Attempt I

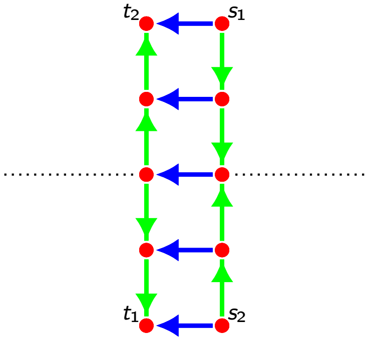


This is the only way to satisfy both the pairs (s_1, t_1) and (s_2, t_2)

A Tight Lower Bound for Steiner Orientation

Non-tight $f(k) \cdot n^{o(\sqrt{k})}$ Lower Bound of Wahlstrom and Pilipczuk ['16]

Basic gadget: Attempt I



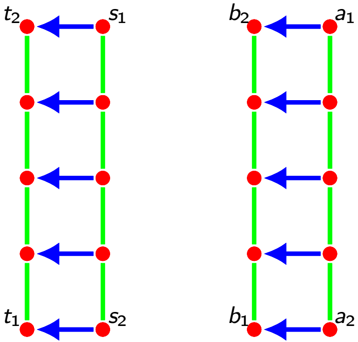
This is the only way to satisfy both the pairs (s_1, t_1) and (s_2, t_2)

- Except the choice of which unique blue edge is used by both paths

A Tight Lower Bound for Steiner Orientation

Non-tight $f(k) \cdot n^{o(\sqrt{k})}$ Lower Bound of Wahlstrom and Pilipczuk ['16]

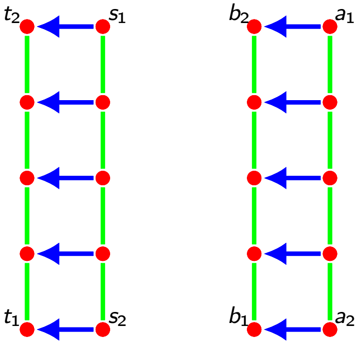
Basic gadget: Attempt II



A Tight Lower Bound for Steiner Orientation

Non-tight $f(k) \cdot n^{o(\sqrt{k})}$ Lower Bound of Wahlstrom and Pilipczuk ['16]

Basic gadget: Attempt II

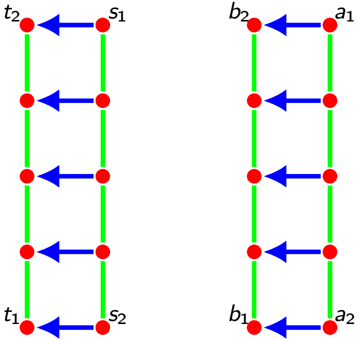


The pairs are $(s_1, t_1), (s_2, t_2), (a_1, b_1)$ and (a_2, b_2)

A Tight Lower Bound for Steiner Orientation

Non-tight $f(k) \cdot n^{o(\sqrt{k})}$ Lower Bound of Wahlstrom and Pilipczuk ['16]

Basic gadget: Attempt II

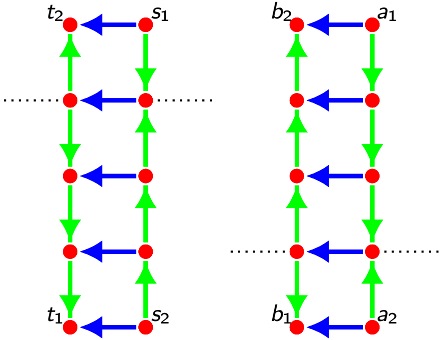


The pairs are $(s_1, t_1), (s_2, t_2), (a_1, b_1)$ and (a_2, b_2)
Hence, by last slide we have the orientations of green paths:

A Tight Lower Bound for Steiner Orientation

Non-tight $f(k) \cdot n^{o(\sqrt{k})}$ Lower Bound of Wahlstrom and Pilipczuk ['16]

Basic gadget: Attempt II

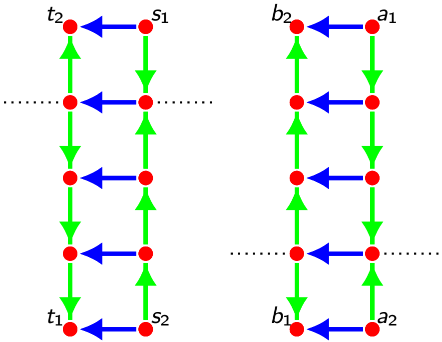


The pairs are $(s_1, t_1), (s_2, t_2), (a_1, b_1)$ and (a_2, b_2)
Hence, by last slide the orientations of green paths are as shown!

A Tight Lower Bound for Steiner Orientation

Non-tight $f(k) \cdot n^{o(\sqrt{k})}$ Lower Bound of Wahlstrom and Pilipczuk ['16]

Basic gadget: Attempt II

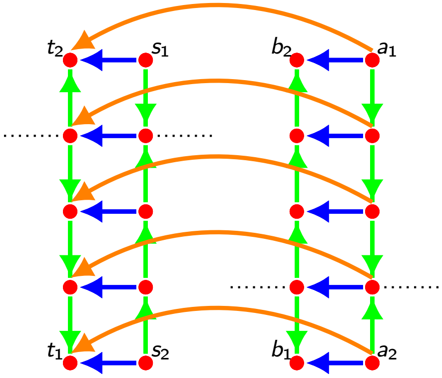


The pairs are $(s_1, t_1), (s_2, t_2), (a_1, b_1)$ and (a_2, b_2)
Hence, by last slide the orientations of green paths are as shown!
Can we make the two gadgets use blue edges on same level?

A Tight Lower Bound for Steiner Orientation

Non-tight $f(k) \cdot n^{o(\sqrt{k})}$ Lower Bound of Wahlstrom and Pilipczuk ['16]

Basic gadget: Attempt II

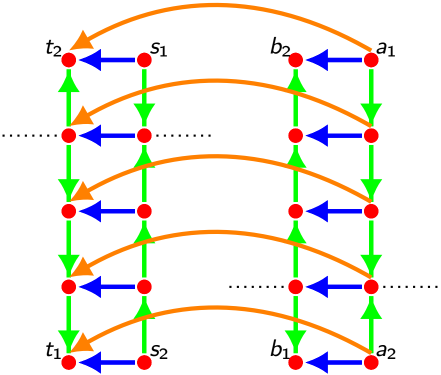


The pairs are $(s_1, t_1), (s_2, t_2), (a_1, b_1)$ and (a_2, b_2)
Hence, by last slide the orientations of green paths are as shown!
Can we make the two gadgets use blue edges on same level?

A Tight Lower Bound for Steiner Orientation

Non-tight $f(k) \cdot n^{o(\sqrt{k})}$ Lower Bound of Wahlstrom and Pilipczuk ['16]

Basic gadget: Attempt II

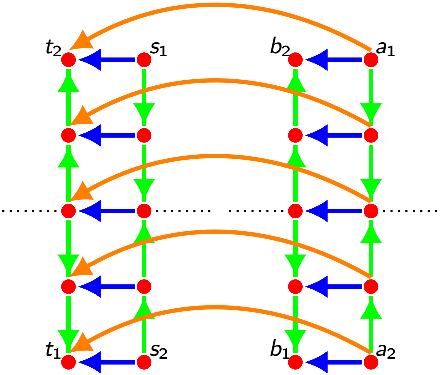


The pairs are $(s_1, t_1), (s_2, t_2), (a_1, b_1)$ and (a_2, b_2)
Hence, by last slide the orientations of green paths are as shown!
Can we make the two gadgets use blue edges on same level?
Add orange edges, and the pairs (a_1, t_1) and (a_2, t_2)

A Tight Lower Bound for Steiner Orientation

Non-tight $f(k) \cdot n^{o(\sqrt{k})}$ Lower Bound of Wahlstrom and Pilipczuk ['16]

Basic gadget: Attempt II



The pairs are $(s_1, t_1), (s_2, t_2), (a_1, b_1)$ and (a_2, b_2)
Hence, by last slide the orientations of green paths are as shown!
Can we make the two gadgets use blue edges on same level?
Add orange edges, and the pairs (a_1, t_1) and (a_2, t_2)

A Tight Lower Bound for Steiner Orientation

Non-tight $f(k) \cdot n^{o(\sqrt{k})}$ Lower Bound of Wahlstrom and Pilipczuk ['16]

Multicolored k-Clique
Input: An undirected graph $G = (V_1 \cup V_2 \cup \ldots V_k, E)$
Question: Does G have a clique of size k which contains exactly one vertex from each V_i
Parameter: k

A Tight Lower Bound for Steiner Orientation

Non-tight $f(k) \cdot n^{o(\sqrt{k})}$ Lower Bound of Wahlstrom and Pilipczuk ['16]

Multicolored k-Clique
Input: An undirected graph $G = (V_1 \cup V_2 \cup \dots V_k, E)$
Question: Does G have a clique of size k which contains exactly one vertex from each V_i
Parameter: k

- Solvable in $\binom{n}{k} = n^{O(k)}$ time

A Tight Lower Bound for Steiner Orientation

Non-tight $f(k) \cdot n^{o(\sqrt{k})}$ Lower Bound of Wahlstrom and Pilipczuk ['16]

Multicolored k-Clique
Input: An undirected graph $G = (V_1 \cup V_2 \cup \ldots V_k, E)$
Question: Does G have a clique of size k which contains exactly one vertex from each V_i
Parameter: k

- Solvable in $\binom{n}{k} = n^{O(k)}$ time
- Under ETH, there is a $f(k) \cdot n^{o(k)}$ lower bound by Chen et al. ['06] where $n = |V(G)|$

A Tight Lower Bound for Steiner Orientation

Non-tight $f(k) \cdot n^{o(\sqrt{k})}$ Lower Bound of Wahlstrom and Pilipczuk ['16]

Multicolored k-Clique
Input: An undirected graph $G = (V_1 \cup V_2 \cup \ldots V_k, E)$
Question: Does G have a clique of size k which contains exactly one vertex from each V_i
Parameter: k

- Solvable in $\binom{n}{k} = n^{O(k)}$ time
- Under ETH, there is a $f(k) \cdot n^{o(k)}$ lower bound by Chen et al. ['06] where $n = |V(G)|$
  - ETH: 3-SAT cannot be solved in time $2^{o(N)}$, where N is number of variables

A Tight Lower Bound for Steiner Orientation

Non-tight $f(k) \cdot n^{o(\sqrt{k})}$ Lower Bound of Wahlstrom and Pilipczuk ['16]

Multicolored k-Clique
Input: An undirected graph $G = (V_1 \cup V_2 \cup \ldots V_k, E)$
Question: Does G have a clique of size k which contains exactly one vertex from each V_i
Parameter: k

- Solvable in $\binom{n}{k} = n^{O(k)}$ time
- Under ETH, there is a $f(k) \cdot n^{o(k)}$ lower bound by Chen et al. ['06] where $n = |V(G)|$
  - ETH: 3-SAT cannot be solved in time $2^{o(N)}$, where N is number of variables
- Wahlstrom and Pilipczuk gave a reduction from Multicolored k-Clique to Steiner Orientation with $O(k^2)$ pairs

A Tight Lower Bound for Steiner Orientation

Non-tight $f(k) \cdot n^{o(\sqrt{k})}$ Lower Bound of Wahlstrom and Pilipczuk ['16]

Multicolored k-Clique
Input: An undirected graph $G = (V_1 \cup V_2 \cup \ldots V_k, E)$
Question: Does G have a clique of size k which contains exactly one vertex from each V_i
Parameter: k

- Solvable in $\binom{n}{k} = n^{O(k)}$ time
- Under ETH, there is a $f(k) \cdot n^{o(k)}$ lower bound by Chen et al. ['06] where $n = |V(G)|$
    - ETH: 3-SAT cannot be solved in time $2^{o(N)}$, where N is number of variables
- Wahlstrom and Pilipczuk gave a reduction from Multicolored k-Clique to Steiner Orientation with $O(k^2)$ pairs
    - This gives a $f(k) \cdot n^{o(\sqrt{k})}$ lower bound for Steiner Orientation under ETH

A Tight Lower Bound for Steiner Orientation

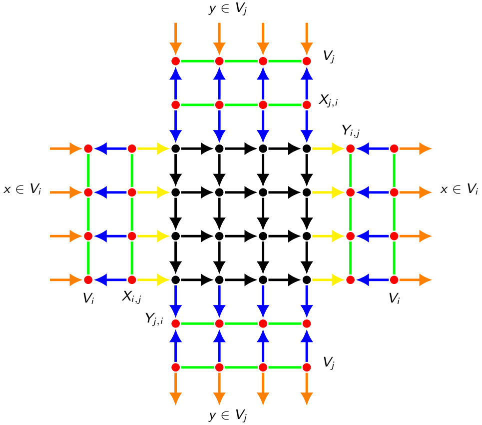
Non-tight $f(k) \cdot n^{o(\sqrt{k})}$ Lower Bound of Wahlstrom and Pilipczuk ['16]

Multicolored k-Clique
Input: An undirected graph $G = (V_1 \cup V_2 \cup \ldots V_k, E)$
Question: Does G have a clique of size k which contains exactly one vertex from each V_i
Parameter: k

- Solvable in $\binom{n}{k} = n^{O(k)}$ time
- Under ETH, there is a $f(k) \cdot n^{o(k)}$ lower bound by Chen et al. ['06] where $n = |V(G)|$
    - ETH: 3-SAT cannot be solved in time $2^{o(N)}$, where N is number of variables
- Wahlstrom and Pilipczuk gave a reduction from Multicolored k-Clique to Steiner Orientation with $O(k^2)$ pairs
    - This gives a $f(k) \cdot n^{o(\sqrt{k})}$ lower bound for Steiner Orientation under ETH
    - Next slide...

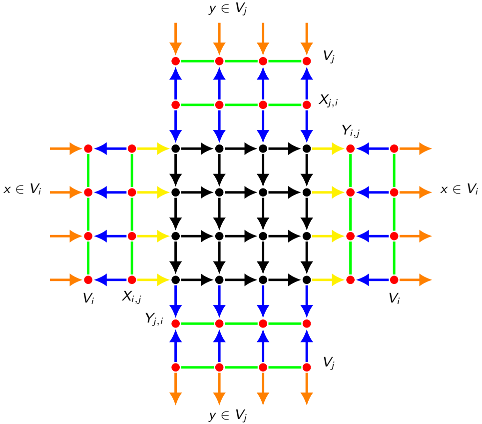
A Tight Lower Bound for Steiner Orientation

Non-tight $f(k) \cdot n^{o(\sqrt{k})}$ Lower Bound of Wahlstrom and Pilipczuk ['16]



A Tight Lower Bound for Steiner Orientation

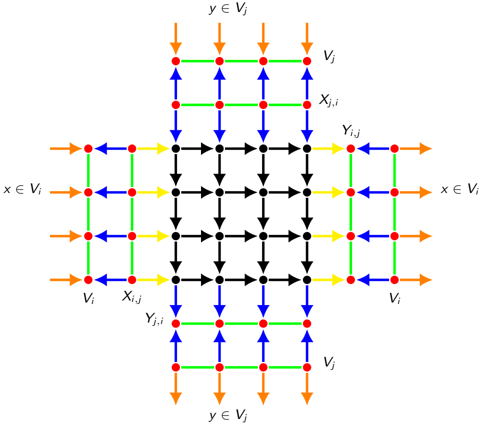
Non-tight $f(k) \cdot n^{o(\sqrt{k})}$ Lower Bound of Wahlstrom and Pilipczuk ['16]



In addition to usual terminal pairs, also add the pairs $(X_{i,j}, Y_{i,j})$ and $(X_{j,i}, Y_{j,})$

A Tight Lower Bound for Steiner Orientation

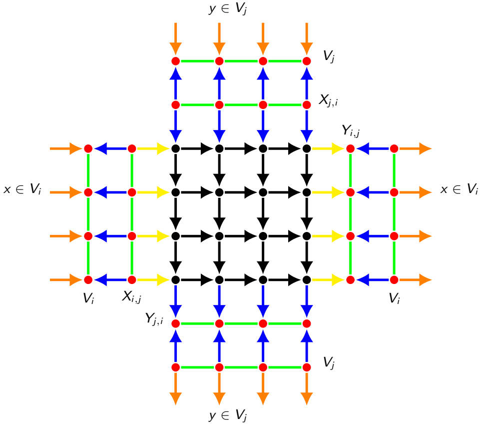
Non-tight $f(k) \cdot n^{o(\sqrt{k})}$ Lower Bound of Wahlstrom and Pilipczuk ['16]



In addition to usual terminal pairs, also add the pairs $(X_{i,j}, Y_{i,j})$ and $(X_{j,i}, Y_{j,})$
So we have to take one horizontal black row and one vertical black column!

A Tight Lower Bound for Steiner Orientation

Non-tight $f(k) \cdot n^{o(\sqrt{k})}$ Lower Bound of Wahlstrom and Pilipczuk ['16]

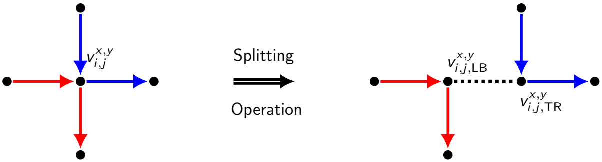


In addition to usual terminal pairs, also add the pairs $(X_{i,j}, Y_{i,j})$ and $(X_{j,i}, Y_{j,})$
So we have to take one horizontal black row and one vertical black column!

Still need to encode edge-relations in vertices of black grid (next slide)

A Tight Lower Bound for Steiner Orientation

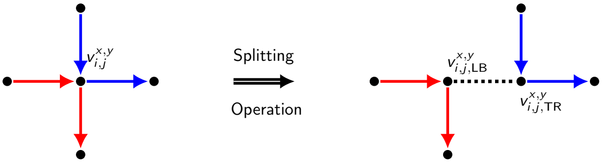
Non-tight $f(k) \cdot n^{o(\sqrt{k})}$ Lower Bound of Wahlstrom and Pilipczuk ['16]



- The splitting operation for vertex $v_{i,j}^{x,y}$ when $(x, y) \notin E(G)$ where $x \in V_i$ and $y \in V_j$.

A Tight Lower Bound for Steiner Orientation

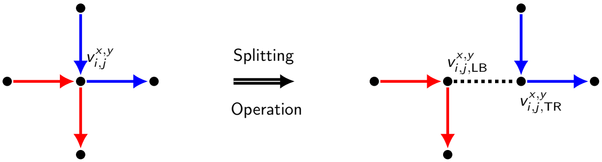
Non-tight $f(k) \cdot n^{o(\sqrt{k})}$ Lower Bound of Wahlstrom and Pilipczuk ['16]



- The splitting operation for vertex $v_{i,j}^{x,y}$ when $(x, y) \notin E(G)$ where $x \in V_i$ and $y \in V_j$.
- The idea behind this splitting is that no matter which way we orient the undirected dotted edge we **cannot** go **both** from left to right and from top to bottom.

A Tight Lower Bound for Steiner Orientation

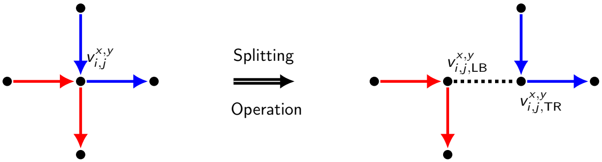
Non-tight $f(k) \cdot n^{o(\sqrt{k})}$ Lower Bound of Wahlstrom and Pilipczuk ['16]



- The splitting operation for vertex $v_{i,j}^{x,y}$ when $(x, y) \notin E(G)$ where $x \in V_i$ and $y \in V_j$.
- The idea behind this splitting is that no matter which way we orient the undirected dotted edge we **cannot** go **both** from left to right and from top to bottom.
- However, if we **just** want to go from left to right (top to bottom) then it **is possible** by orienting the dotted edge to the right (left), respectively.

A Tight Lower Bound for Steiner Orientation

Non-tight $f(k) \cdot n^{o(\sqrt{k})}$ Lower Bound of Wahlstrom and Pilipczuk ['16]



- The splitting operation for vertex $v_{i,j}^{x,y}$ when $(x,y) \notin E(G)$ where $x \in V_i$ and $y \in V_j$.
- The idea behind this splitting is that no matter which way we orient the undirected dotted edge we **cannot** go **both** from left to right and from top to bottom.
- However, if we **just** want to go from left to right (top to bottom) then it **is possible** by orienting the dotted edge to the right (left), respectively.
- So, if we use a horizontal black row and vertical black column then the unique black vertex they meet in cannot be split, i.e., the two corresponding vertices form an edge!

□

A Tight Lower Bound for Steiner Orientation

Improved $f(k) \cdot n^{o(k/\log k)}$ Lower Bound of Wahlstrom and Pilipczuk ['16]

- The $f(k) \cdot n^{o(\sqrt{k})}$ lower bound had $O(1)$ pairs per edge gadget

A Tight Lower Bound for Steiner Orientation

Improved $f(k) \cdot n^{o(k/\log k)}$ Lower Bound of Wahlstrom and Pilipczuk ['16]

- The $f(k) \cdot n^{o(\sqrt{k})}$ lower bound had $O(1)$ pairs per edge gadget
- Standard way to improve this lower bound to $f(k) \cdot n^{o(k/\log k)}$ is to reduce from a slightly different problem

A Tight Lower Bound for Steiner Orientation

Improved $f(k) \cdot n^{o(k/\log k)}$ Lower Bound of Wahlstrom and Pilipczuk ['16]

- The $f(k) \cdot n^{o(\sqrt{k})}$ lower bound had $O(1)$ pairs per edge gadget
- Standard way to improve this lower bound to $f(k) \cdot n^{o(k/\log k)}$ is to reduce from a slightly different problem

Colored Subgraph Isomorphism (CSI)
Input: An undirected graph $G = (V_1 \cup V_2 \cup \ldots V_\ell, E(G))$ and an undirected graph $H = ([\ell], E(H))$
Question: Is there an injective function $\phi : [\ell] = V(H) \to V(G)$ such that $\phi(i) \in V_i$ for each $i \in [\ell]$ and for each $1 \leq i \neq j \leq \ell$ we have $i - j \in E(H)$ implies $\phi(i) - \phi(j) \in E(G)$
Parameter: $r = |E(H)|$

A Tight Lower Bound for Steiner Orientation

Improved $f(k) \cdot n^{o(k/\log k)}$ Lower Bound of Wahlstrom and Pilipczuk ['16]

- The $f(k) \cdot n^{o(\sqrt{k})}$ lower bound had $O(1)$ pairs per edge gadget
- Standard way to improve this lower bound to $f(k) \cdot n^{o(k/\log k)}$ is to reduce from a slightly different problem

Colored Subgraph Isomorphism (CSI)
Input: An undirected graph $G = (V_1 \cup V_2 \cup \ldots V_\ell, E(G))$ and an undirected graph $H = ([\ell], E(H))$
Question: Is there an injective function $\phi : [\ell] = V(H) \to V(G)$ such that $\phi(i) \in V_i$ for each $i \in [\ell]$ and for each $1 \leq i \neq j \leq \ell$ we have $i - j \in E(H)$ implies $\phi(i) - \phi(j) \in E(G)$
Parameter: $r = |E(H)|$

- Marx ['07] showed a lower bound of $f(r) \cdot n^{o(r/\log r)}$ for CSI under ETH

A Tight Lower Bound for Steiner Orientation

Improved $f(k) \cdot n^{o(k/\log k)}$ Lower Bound of Wahlstrom and Pilipczuk ['16]

- The $f(k) \cdot n^{o(\sqrt{k})}$ lower bound had $O(1)$ pairs per edge gadget
- Standard way to improve this lower bound to $f(k) \cdot n^{o(k/\log k)}$ is to reduce from a slightly different problem

Colored Subgraph Isomorphism (CSI)
Input: An undirected graph $G = (V_1 \cup V_2 \cup \ldots V_\ell, E(G))$ and an undirected graph $H = ([\ell], E(H))$
Question: Is there an injective function $\phi : [\ell] = V(H) \to V(G)$ such that $\phi(i) \in V_i$ for each $i \in [\ell]$ and for each $1 \leq i \neq j \leq \ell$ we have $i - j \in E(H)$ implies $\phi(i) - \phi(j) \in E(G)$
Parameter: $r = |E(H)|$

- Marx ['07] showed a lower bound of $f(r) \cdot n^{o(r/\log r)}$ for CSI under ETH
- The previous reduction can be easily modified to start from CSI instead of Multicolored k-Clique

A Tight Lower Bound for Steiner Orientation

Improved $f(k) \cdot n^{o(k/\log k)}$ Lower Bound of Wahlstrom and Pilipczuk ['16]

- The $f(k) \cdot n^{o(\sqrt{k})}$ lower bound had $O(1)$ pairs per edge gadget
- Standard way to improve this lower bound to $f(k) \cdot n^{o(k/\log k)}$ is to reduce from a slightly different problem

Colored Subgraph Isomorphism (CSI)
Input: An undirected graph $G = (V_1 \cup V_2 \cup \dots V_\ell, E(G))$ and an undirected graph $H = ([\ell], E(H))$
Question: Is there an injective function $\phi : [\ell] = V(H) \to V(G)$ such that $\phi(i) \in V_i$ for each $i \in [\ell]$ and for each $1 \leq i \neq j \leq \ell$ we have $i - j \in E(H)$ implies $\phi(i) - \phi(j) \in E(G)$
Parameter: $r = |E(H)|$

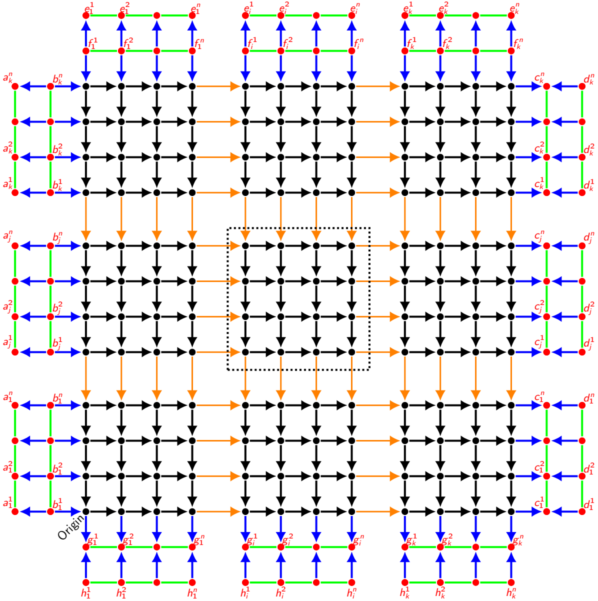
- Marx ['07] showed a lower bound of $f(r) \cdot n^{o(r/\log r)}$ for CSI under ETH
- The previous reduction can be easily modified to start from CSI instead of Multicolored k-Clique
- The lower bound follows since the number of terminal pairs is $O(|V(H)| + |E(H)|) = O(|E(H)|)$

Outline of Talk

- Steiner Orientation
- Upper Bound
- Non-Tight Lower Bound
- Tight Lower Bound
- Some new results...

A Tight Lower Bound for Steiner Orientation

Our $f(k) \cdot n^{O(k)}$ Lower Bound

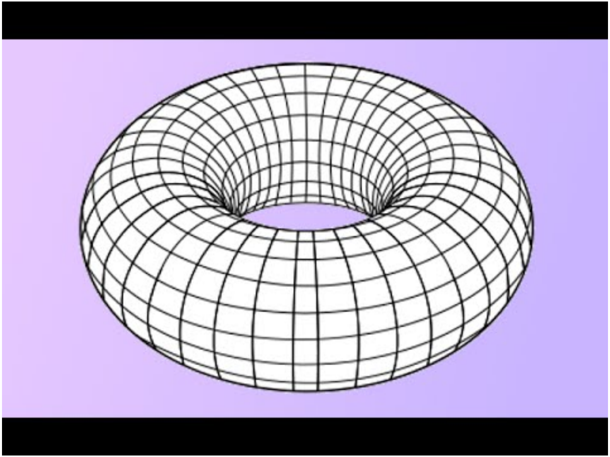


Our graph has genus 1, i.e., can be drawn on a torus.

A Tight Lower Bound for Steiner Orientation

Our $f(k) \cdot n^{O(k)}$ Lower Bound

Our graph has genus 1, i.e., can be drawn on a torus.



Outline of Talk

- Steiner Orientation
- Upper Bound
- Non-Tight Lower Bound
- Tight Lower Bound
- Some new results...

A Tight Lower Bound for Steiner Orientation

New results since conference submission

A Tight Lower Bound for Steiner Orientation

New results since conference submission

- Since our graph has genus 1, the main open question was what happens for genus 0 graphs, i.e., planar graphs.

A Tight Lower Bound for Steiner Orientation

New results since conference submission

- Since our graph has genus 1, the main open question was what happens for genus 0 graphs, i.e., planar graphs.
- Recently managed to modify the construction to make the graph planar

A Tight Lower Bound for Steiner Orientation

New results since conference submission

- Since our graph has genus 1, the main open question was what happens for genus 0 graphs, i.e., planar graphs.
- Recently managed to modify the construction to make the graph planar
  - We only had to "remove"either vertical or horizontal crossing edges.

A Tight Lower Bound for Steiner Orientation

New results since conference submission

- Since our graph has genus 1, the main open question was what happens for genus 0 graphs, i.e., planar graphs.
- Recently managed to modify the construction to make the graph planar
  - We only had to "remove"either vertical or horizontal crossing edges.

A Tight Lower Bound for Steiner Orientation

New results since conference submission

- Since our graph has genus 1, the main open question was what happens for genus 0 graphs, i.e., planar graphs.
- Recently managed to modify the construction to make the graph planar
  - We only had to "remove"either vertical or horizontal crossing edges.
- FPT inapproximability result: Under Gap-ETH, there is a constant $\epsilon > 0$ such that STEINER ORIENTATION has no $(1 + c)$-approximation in FPT time (even on planar graphs):

A Tight Lower Bound for Steiner Orientation

New results since conference submission

- Since our graph has genus 1, the main open question was what happens for genus 0 graphs, i.e., planar graphs.
- Recently managed to modify the construction to make the graph planar
  - We only had to "remove"either vertical or horizontal crossing edges.
- FPT inapproximability result: Under Gap-ETH, there is a constant $\epsilon > 0$ such that STEINER ORIENTATION has no $(1 + c)$-approximation in FPT time (even on planar graphs):
  - k-Clique $\Rightarrow$ All $16k$ pairs are satisfied

A Tight Lower Bound for Steiner Orientation

New results since conference submission

- Since our graph has genus 1, the main open question was what happens for genus 0 graphs, i.e., planar graphs.
- Recently managed to modify the construction to make the graph planar
  - We only had to "remove"either vertical or horizontal crossing edges.
- FPT inapproximability result: Under Gap-ETH, there is a constant $\epsilon > 0$ such that STEINER ORIENTATION has no $(1 + c)$-approximation in FPT time (even on planar graphs):
  - k-Clique $\Rightarrow$ All $16k$ pairs are satisfied
  - Every k-vertex graph has at most $\frac{1}{1+\epsilon} \cdot \binom{k}{2}$ edges $\Rightarrow$ at most $\frac{16k}{1+\epsilon}$ pairs are satisfied

A Tight Lower Bound for Steiner Orientation

New results since conference submission

- Since our graph has genus 1, the main open question was what happens for genus 0 graphs, i.e., planar graphs.
- Recently managed to modify the construction to make the graph planar
  - We only had to "remove"either vertical or horizontal crossing edges.
- FPT inapproximability result: Under Gap-ETH, there is a constant $\epsilon > 0$ such that $\textsc{Steiner Orientation}$ has no $(1 + c)$-approximation in FPT time (even on planar graphs):
  - k-Clique $\Rightarrow$ All $16k$ pairs are satisfied
  - Every k-vertex graph has at most $\frac{1}{1+\epsilon} \cdot \binom{k}{2}$ edges $\Rightarrow$ at most $\frac{16k}{1+\epsilon}$ pairs are satisfied
- Open question: Is there $O(1)$-approximation in FPT time? At least on planar graphs?

Thank You
Спасибо

Thank You
Спасибо

Questions?