Lower Bounds for Unrestricted Boolean Circuits: Open Problems

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Computing Boolean Functions

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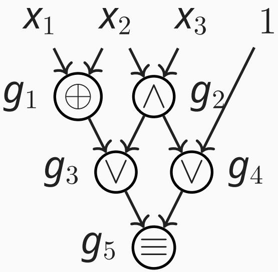

Fundamental Question

Given a Boolean function $f\colon \{0,1\}^n \to \{0,1\}$, what is the minimum number of gates needed to compute f?

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Does there exist an infinite sequence of functions $f_1, f_2, \ldots$ such that f_n has n inputs, $\bigcup_{n=1}^{\infty} f_n^{-1}(1) \in \mathrm{NP}$, and f_n requires superpoly(n) gates? (This would mean that $\mathrm{P} \neq \mathrm{NP}$)

Exponential Bounds

Lower Bound

Counting shows that almost all functions of n variables have circuit size $\Omega(2^n/n)$ [S49]

Upper Bound

Any function can be computed by circuits of size $(1 + o(1))2^n/n$ [L58]

Explicit Lower Bounds

The lower bound $\Omega(2^n/n)$ is non-constructive: it does not give an explicit function (i.e., a function from NP) with superpolynomial circuit size.

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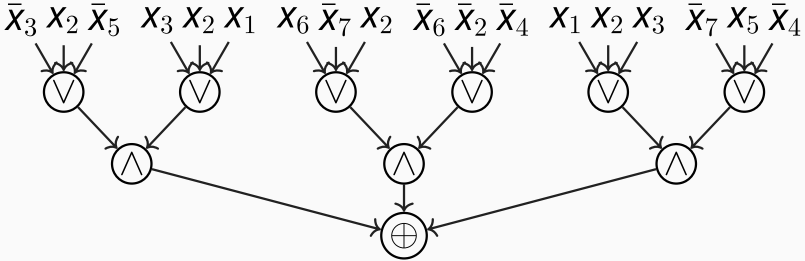
What can we prove for explicit functions? What about restricted circuit classes?

Remainder of the Talk

- (Very brief) Overview of known lower bounds for restricted circuits
- (Brief) Overview of various approaches that could potentially lead to improved lower bounds for unrestricted circuits

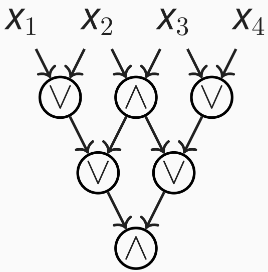
Restricted classes: constant depth circuits



- depth: constant, fan-in: unbounded
- exponential lower bounds: switching lemma [A83, FSS84, Y85, H86, R95], approximating polynomials [RS87]

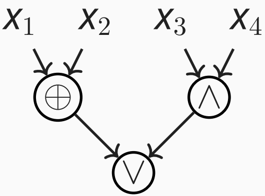
Restricted classes: monotone circuits

- fanin: 2
  fanout: unbounded
  operations: $\{\wedge, \vee\}$
- exponential lower bounds: method of approximations [R85, A85, AB87]



Restricted classes: formulas

- fanin: 2, fanout: 1
- n^2, n^3 lower bounds: random restrictions, universal functions, formal complexity measures [S61, N66, K71, A85, IN93, PZ93, H98]



$(x_1 \oplus x_2) \vee (x_3 \wedge x_4)$

Restricted vs Unrestricted

Restricted circuits

lower bounds:
n^3, $2^{n^{1/8}}$, $2^{n-o(n)}$

many beautiful techniques are known

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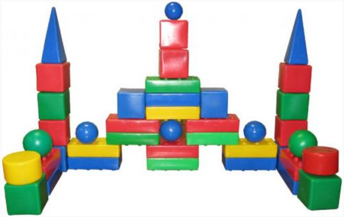
Unrestricted circuits

lower bounds:
$2n$, $2.5n$, $3n$

just one simple technique is known



Quote

“This may seem quite depressing. It is.”

Saxena, Seshadhri, 2010. From Sylvester–Gallai Configurations to Rank Bounds: Improved Blackbox Identity Test for Depth-3 Circuits

Outline

1. Gate Elimination
2. Multi-Output Functions
3. Non-Gate-Elimination Lower Bounds
4. Symmetric Functions
5. Satisfiability Algorithms
6. Mass Production
7. Logarithmic Depth Circuits

Outline

1. Gate Elimination

   How to prove, say, a $3n$ lower bound for a Boolean function f?

2. Multi-Output Functions
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Gate Elimination Method

- Show that f is resistant to about n substitutions

- Show that one can always find a substitution eliminating at least 3 gates

Lower Bounds

- The currently best known lower bound $\left(3+\frac{1}{86}\right)n$ is proved by gate elimination [FGHK16]
- The corresponding function f is affine disperser for sublinear dimension: f is non-constant on any affine subspace of $\{0,1\}^n$ of large enough dimension
- Explicit constructions of such functions were found relatively recently [BK12]

Linear Size Circuits for Affine Dispersers

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Open problem: Do there exist affine dispersers for sublinear dimension of linear circuit size?

Quadratic Dispersers

Open problem: Construct an explicit "quadratic" disperser f (even in NP, even with $o(n)$ outputs) that is not constant on any set $S \subseteq \{0,1\}^n$ of size at least $2^{n/100}$ that can be defined as

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This will give an improved lower bound (about $3.1n$) [GK16]

Limitations of Gate Elimination

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- Formally, there exist circuits such that any substitution of the form $x \leftarrow g$, where g is an arbitrary function, removes no more than five gates from the circuit [GHkk16]. Therefore, one definitely needs new ideas to get something stronger than $5n$

Outline

1. Gate Elimination
2. Multi-Output Functions

   Can one prove stronger lower bounds for functions with multiple outputs?

3. Non-Gate-Elimination Lower Bounds
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Open problem: How to prove a $5n$ lower bound for an n-to-n function?

Outline

1. Gate Elimination
2. Multi-Output Functions
3. Non-Gate-Elimination Lower Bounds

   Are there approaches other than gate elimination for proving lower bounds for unrestricted circuits?

4. Symmetric Functions
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Open problem: Can any of these non-gate-elimination methods be extended to get stronger than $2n$ lower bounds?

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1. Gate Elimination
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4. Symmetric Functions

   Can one prove a superlinear lower bound for a symmetric function?

5. Satisfiability Algorithms
6. Mass Production
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Symmetric Functions

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Open problem: What is $C(\mathrm{SUM}_n)$?

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   Given a circuit, how hard is it to find an assignment making this circuit to output 1?

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Open problem: Do non-trivial satisfiability algorithms for circuits of size cn imply cn circuit lower bounds?

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   Can one take a function of 20 bits of circuit size 100 and cook out of it a family of functions of circuit size $5n$*?*

7. Logarithmic Depth Circuits

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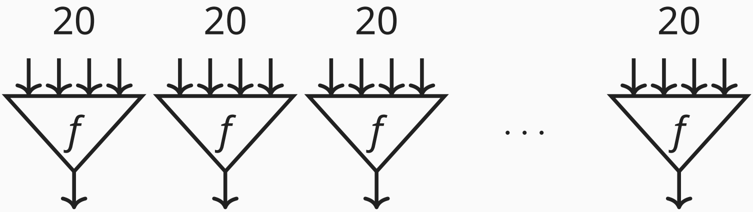

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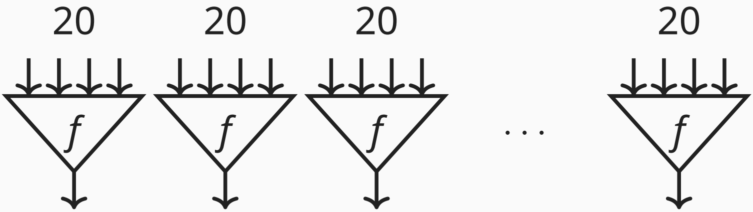

- But we don't know how to prove this!

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Open problem: What are the functions avoiding mass production effect?

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Can we at least prove superlinear lower bounds on circuits of logarithmic depth?

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Open problem: Improve $2^{\sqrt{n}}$ lower bound for depth three circuits.

Constant Depth Circuits

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Open problem: Can one convert a circuit with s gates into a, say, $\mathrm{OR}_{2^{\frac{s}{4}}} \circ \mathrm{AND} \circ \mathrm{OR}_2$ formula?

Summary of Open Problems

1. Prove that there exists an affine disperser of linear circuit size!
2. Construct an explicit quadratic disperser!
3. Prove a $5n$ lower bound for an n-to-n function!
4. Prove $3n$ lower bound without gate elimination!
5. Find $C(\mathrm{SUM}_n)$!
6. Prove that faster than brute force SAT algorithm for circuits of size cn imply cn circuit lower bounds!
7. Construct functions avoiding mass production effect!
8. Convert lower bounds for depth-3 circuits to lower bounds for unrestricted circuits!

Thank you!