Constructive and Non-Constructive Combinatorics

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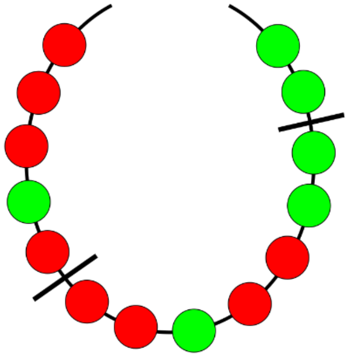


CSR 2018, Moscow, June 2018

I Constructive vs Non-constructive Combinatorics

Purely combinatorial proofs often provide efficient procedures for solving the corresponding algorithmic problems, even when dealing with NP-hard invariants

Examples: Dirac's Theorem: every graph with $n \geq 3$ vertices and minimum degree $\geq n/2$ is Hamiltonian.

Turán's Theorem: every graph with degrees d_i contains an independent set of size at least $\sum_i 1/(d_i + 1)$

Modern combinatorial techniques include topological, algebraic, geometric and probabilistic methods.

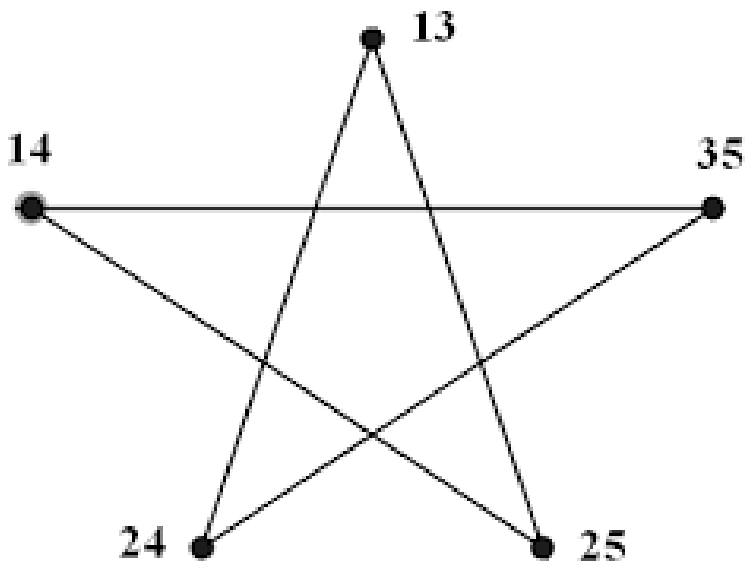
Proofs obtained using these methods (especially the first three) are often non-constructive, that is, provide no efficient algorithms for the corresponding problems.

II Topologoical methods: applying fixed point theorems

Thm (Lovász, 78): **In any coloring f of the k-subsets of an n-set by n-2k+1 colors, there are two disjoint k-subsets with the same color.**

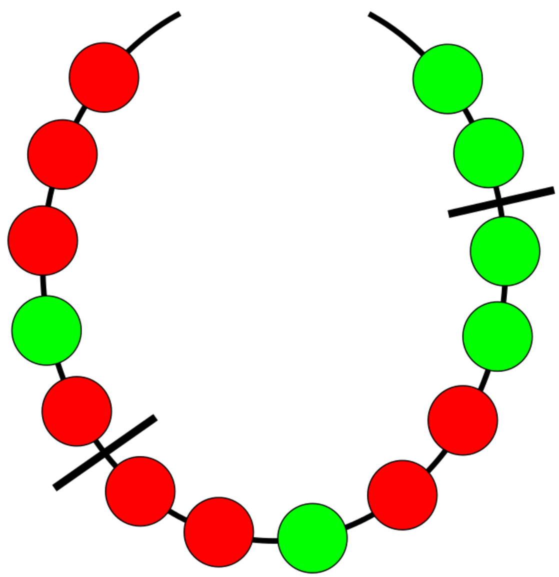
The shortest known proof (Greene 03) defines, using f, a coloring g of the sphere S^t with t=n-2k by t+1 colors, applies the Borsuk-Ulam Theorem to get two nearly antipodal points with the same color, and concludes, using the definition of g from f, that two disjoint k-sets have the same color.

Thm (Schrijver (78)): **Given a cycle C of length n, for any coloring of the independent sets of size k of C by n-2k+1 colors, there are two disjoint independent sets with the same color.**



The Necklace Thm [A (87)]: Any open necklace with ka_i beads of type i ($1 \le i \le t$) can be partitioned into intervals using at most $(k-1)t$ cuts, so that the resulting intervals can be partitioned into k collections, each containing exactly a_i beads of type i, for all $1 \le i \le t$.

This is tight for all k and t.



Steps of proofs:

Show that the validity of the statement for k_1 and k_2 implies its validity for $k=k_1k_2$

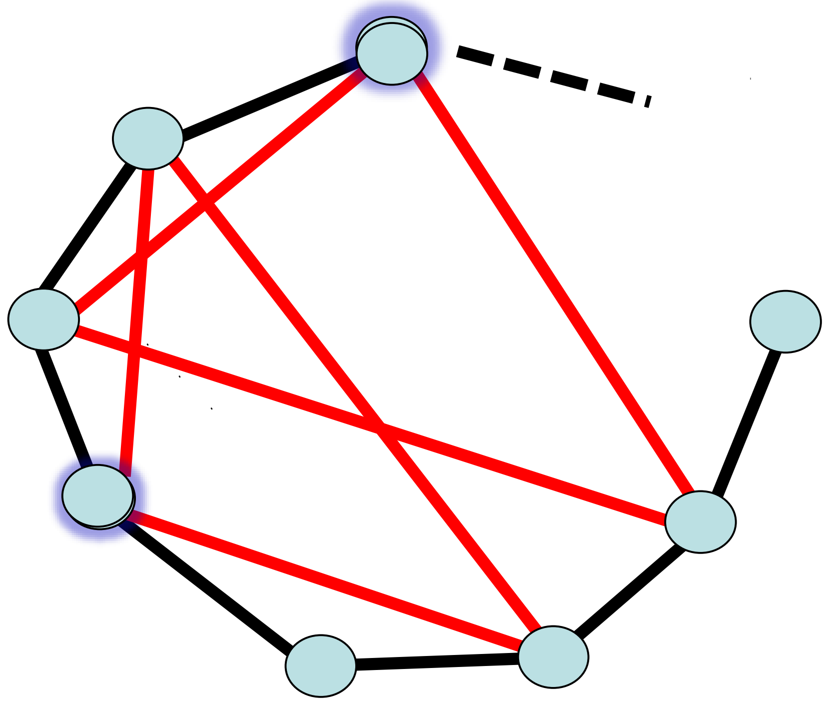
Consider a continuous version of the problem, in which the necklace is an interval colored by t colors

Apply a fixed-point theorem (Bárány,Shlosman, Szűcs (81)) to prove the statement for prime k.

Open: **Can we find the (k-1)t cuts efficiently, for a given input necklace ?**

The cycle+triangles conjecture (Du, Hsu, Hwang (90)):

Let G=(V,E) be a graph on 3n vertices whose edges are the union of a Hamilton cycle (of length 3n) and n pairwise vertex disjoint triangles. Then G contains an independent set of size n.



A stronger conjecture (Erdös (91)): Any such G is 3-colorable

Thm (Fleischner and Stiebitz (92)): Any such G is 3-choosable: for any assignment of a list of 3 colors to each vertex, there is a proper vertex coloring assigning to each vertex a color from its list.

The proof is based on the algebraic approach of A-Tarsi (92).

A new proof of the cycle + triangles original conjecture, based on Schrijver's Theorem (whose proof is based on the Borsuk-Ulam Theorem). [Extensions appear in Aharaoni, A, Berger, Chudnovsky, Kotlar, Loebl, Ziv(17)]

Schrijver(78): Any coloring of the independent sets of size k in a cycle of length m by m-2k+1 colors contains two disjoint independent sets of the same color.

Assume, for contradiction, that there is a graph G=(V,E) on a set V of 3n vertices whose edges are a Hamilton cycle C and n disjoint triangles, with no independent set of size n.

Color the independent sets of size n in C as follows: each set I is colored by the index of the first triangle that contains at least 2 points of I.

By Schrijver, since 3n-2n+2>n there are two disjoint independent sets I_1, I_2 with the same color. This is impossible, as it ,means that the same triangle contains 2 points of each of them. ■

Open: **Given a graph G on 3n vertices whose edges are the union of a Hamilton cycle and n disjoint triangles, can one find efficiently an independent set of size n in G ?**

III Algebraic Methods

Hilbert's Nullstellensatz (1893):

If F is an algebraically closed field, f, g_1 , …, g_m polynomials in F[x_1, x_2, …, x_n] and f vanishes whenever all g_i do, then there is k ≥ 1 and polynomials h_i so that

$$f^k = \sum_i h_i g_i$$

Combinatorial Nullstellensatz [CN1] (A-99):

Let F be a field, $f(x_1, x_2, \ldots, x_n)$ a polynomial over F, let $S_1, S_2, \ldots, S_n$ be subsets of F, and put

$$g_i(x_i) = \prod_{s \in S_i} (x - s)$$

If f vanishes whenever all g_i do, then there are polynomials h_i with $\deg(h_i) \leq \deg(f) - \deg(g_i)$ and

$$f = \sum_i h_i g_i$$

Combinatorial Nullstellensatz [CN2] (A-99):

Let F be a field, $f(x_1, x_2, \ldots, x_n)$ a polynomial over F, and $t_1, t_2, \ldots, t_n$ positive integers. If the degree of f is $t_1+t_2+\ldots+t_n$, and the coefficient of

$$\prod_{i=1}^{n} x_i^{t_i}$$

in f is nonzero, then for any subsets $S_1,\ldots,S_n$ of F, where $|S_i| \geq t_i+1$ for all i, there are s_i in S_i so that $f(s_1,\ldots,s_n)$ is not 0.

The choice number ch(G) (or list chromatic number) of a graph G=(V,E) is the minimum k so that for any assignment of a list L_v of k colors to each vertex v, there is a proper coloring f of G with f(v) in L_v for each v.

This was defined independently by Vizing(76) and by Erdős, Rubin and Taylor (79).

Clearly $ch(G) \geq \chi(G)$ for every G.

(Very) strict inequality is possible.

Sylvester (1878), Petersen (1891): The graph polynomial of a graph G=(V,E) on the set of vertices V={1,2,..,n} is

$$f_G(x_1, \dots, x_n) = \prod_{ij \in E, i<j} (x_i - x_j)$$

If S_1, S_2 , … ,S_n are finite lists of colors (represented by real or complex numbers) then there are s_i in S_i so that f_G (s_1 , … ,s_n) $\neq 0$ iff there is a proper coloring of G assigning to each vertex i a color from its list S_i .

By CN1, a graph G is not 3-colorable iff there are polynomials h_i so that

$$f_G = \sum_i h_i\,(x_i{}^3 - 1)$$

Exercise: use this fact to prove that K_4 is not 3-colorable.

Note: this does not prove that NP=coNP

By CN2, if G has kn edges and the coefficient of $\prod x_i^{k}$ in f_G is nonzero, then ch(G) ≤ k+1

In A-Tarsi(92) this coefficient is interpreted combinatorially in terms of Eulerian orientations of G.

Using this interpretation, Fleishner and Stiebitz(92) proved that the relevant coefficient is nonzero for any 4-regular graph G consisting of a Hamilton cycle+triangles, hence ch(G) ≤ 3.

Open: Given a graph G on 3n vertices whose edges are the union of a Hamilton cycle and n disjoint triangles, can one find **efficiently** an **independent set** of size n in G ?

Can we find efficiently a **proper 3-coloring** of the vertices?

Given lists of size 3 for the vertices, can we find efficiently a **proper vertex coloring** assigning to each vertex a color from its **list** ?

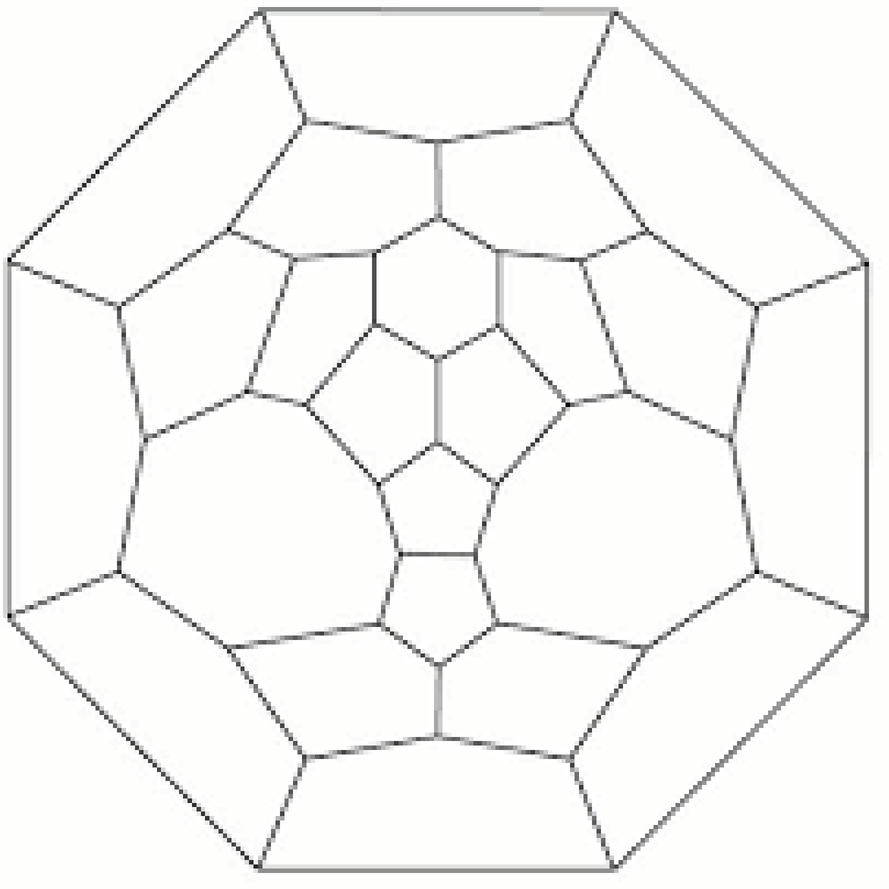
A similar reasoning provides a strengthening of the Four Color Theorem (4CT).

By Tait, the 4CT (Appel and Haken (76), Robertson,Sanders,Seymour and Thomas (96)) is equivalent to the fact that the chromatic number of the line graph of any cubic, bridgeless planar graph is 3.

A-Jaeger-Tarsi (same + extension by Ellingham-Goddyn): The choice number of the line graph of any cubic, bridgeless, planar graph is 3.

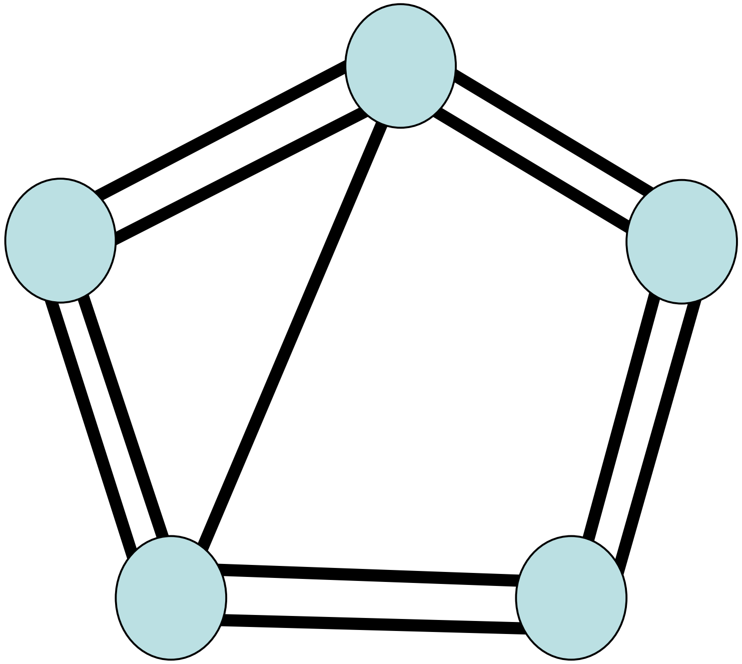
This is proved using CN2, by showing that the relevant coefficient of the graph polynomial is the number of proper 3 colorings of this line graph, which is nonzero, by 4CT

Open: Given a cubic, bridgeless, planar graph with a list of 3 colors for every edge, can one fine efficiently a proper coloring of the edges assigning to each edge a color from its list ?



An even older result:

Thm (A-Friedland-Kalai (84)): Any (multi)graph with average degree > 4 and maximum degree at most 5 contains a 3-regular subgraph.



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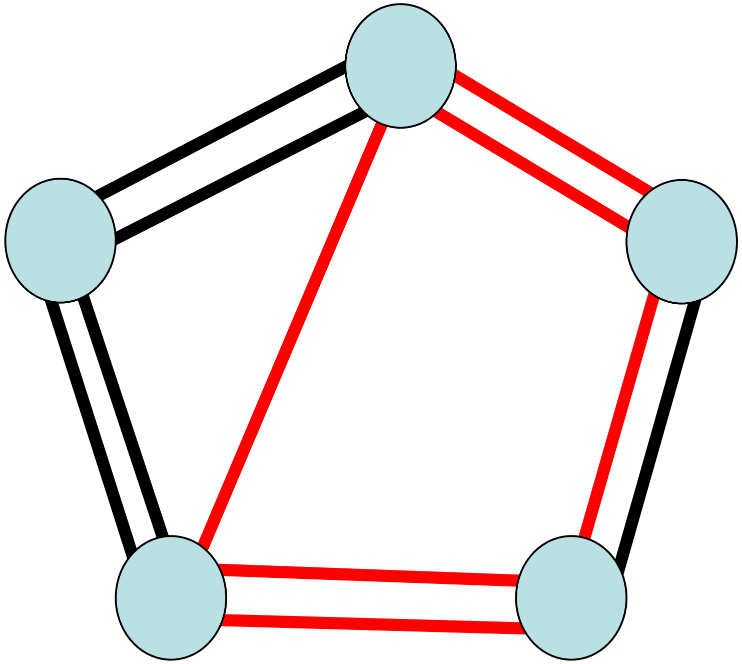


Proof using **CN2**: Let G=(V,E) be such a graph, and put $a_{v,e}$=1 if v lies in e, 0 otherwise.

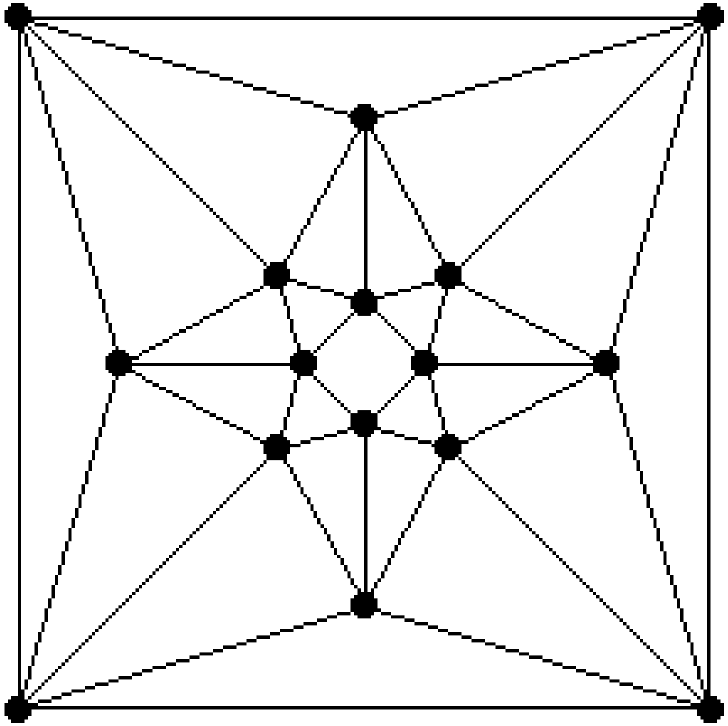
Apply **CN** to the following **polynomial** in the variables x_e over Z_3:

$$\prod_{v \in V} \left[1 - \left(\sum_{e, v \in e} a_{v,e} x_e \right)^2 \right] - \prod_{e \in E} (1 - x_e)$$

with S_e = {0,1} for all e.

The edges of the required subgraph are all e with x_e =1. ■

Open: **Given a graph with average degree > 4 and maximum degree 5, can we find efficiently a 3-regular subgraph ?**



The Permanent Lemma

If A is an n by n matrix over a field, Per(A)≠0 and b is a vector in F^n then there is a 0/1 vector x so that $(Ax)_i \neq b_i$ in all coordinates.

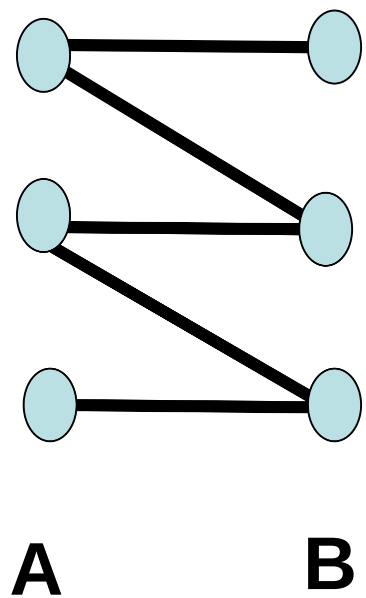
Proof: Apply CN2 to

$$f = \prod_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} x_j - b_i \right)$$

with $t_1 = t_2 = \ldots = t_n = 1$, $S_i = \{0,1\}$ for all i.

Corollary: If G is a bipartite graph with classes of vertices A,B, |A|=|B|=n, $B=\{b_1,b_2,\ldots,b_n\}$ which contains a perfect matching, then for any integers $d_1,\ldots,d_n$ there is a subset X of A so that for each i the number of neighbors of b_i in X is not d_i

Example:



$d_1 = 0$

$d_2 = 1$

$d_3 = 2$

Corollary: If G is a bipartite graph with classes of vertices A,B, |A|=|B|=n, B={b_1,b_2,…,b_n} which contains a perfect matching, then for any integers d_1,…,d_n there is a subset X of A so that for each i the number of neighbors of b_i in X is not d_i

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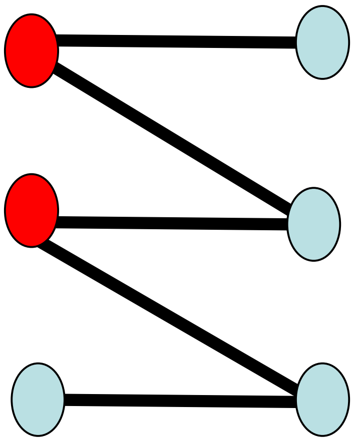

$d_1 = 0$

$d_2 = 1$

$d_3 = 2$

Problem: **Given a bipartite graph with a perfect matching on the vertex classes A and $B=\{b_1,..,b_n\}$, and given integers $d_1,..,d_n$, can one find efficiently a subset X of A so that the number of neighbors of each b_i in X is not d_i ?**

IV Hardness

Are these algorithmic problems complete for some natural complexity classes (like PPAD)?

Prop: **The following algorithmic problem is at least as hard as inverting one-way permutations (e.g., computing discrete logarithm in Z_p^*) :**

Given an arithmetic circuit computing an f in $F[x_1 , \ldots ,x_n]$ with $\deg(f)=\sum_i t_i$ and coefficient of

$$\prod_i x_i^{t_i}$$

being nonzero, and given S_i in F of size $t_i +1$, find s_i in S_i with $f(s_1,\ldots,s_n) \neq 0$.

The algorithmic versions of Borsuk-type fixed point theorems are also hard in general.

However, the problems discussed here (necklace, cycle+triangles, choice 4CT, 3-regular subgraph) and additional similar ones may be simpler. Are they ?