

Lower Bounds and Hardness Magnification for Sublinear-Time Shrinking Cellular Automata

Augusto Modanese

Karlsruhe Institute of Technology (KIT), Germany
modanese@kit.edu

Abstract The minimum circuit size problem (MCSP) is a string compression problem with a parameter s in which, given the truth table of a Boolean function over inputs of length n , one must answer whether it can be computed by a Boolean circuit of size at most $s(n) \geq n$. Recently, McKay, Murray, and Williams (STOC, 2019) proved a hardness magnification result for MCSP involving (one-pass) streaming algorithms: For any reasonable s , if there is no $\text{poly}(s(n))$ -space streaming algorithm with $\text{poly}(s(n))$ update time for $\text{MCSP}[s]$, then $\text{P} \neq \text{NP}$. We prove an analogous result for the (provably) strictly less capable model of shrinking cellular automata (SCAs), which are cellular automata whose cells can spontaneously delete themselves. We show every language accepted by an SCA can also be accepted by a streaming algorithm of similar complexity, and we identify two different aspects in which SCAs are more restricted than streaming algorithms. We also show there is a language which cannot be accepted by any SCA in $o(n/\log n)$ time, even though it admits an $O(\log n)$ -space streaming algorithm with $O(\log n)$ update time.

Keywords: Cellular automata · Hardness magnification · Minimum circuit size problem · Streaming algorithms · Sublinear-time computation

1 Introduction

The ongoing quest for lower bounds in complexity theory has been an arduous but by no means unfruitful one. Recent developments have brought to light a phenomenon dubbed *hardness magnification* [5, 6, 7, 17, 22, 23], giving several examples of natural problems for which even slightly non-trivial lower bounds are as hard to prove as major complexity class separations such as $\text{P} \neq \text{NP}$. Among these, the preeminent example appears to be the *minimum circuit size problem*:

Definition 1 (MCSP). For a Boolean function $f: \{0, 1\}^n \rightarrow \{0, 1\}$, let $\text{tt}(f)$ denote the truth table representation of f (as a binary string in $\{0, 1\}^+$ of length $|\text{tt}(f)| = 2^n$). For $s: \mathbb{N}_+ \rightarrow \mathbb{N}_+$, the minimum circuit size problem $\text{MCSP}[s]$ is the problem where, given such a truth table $\text{tt}(f)$, one must answer whether there is a Boolean circuit C on inputs of length n and size at most $s(n)$ that computes f , that is, $C(x) = f(x)$ for every input $x \in \{0, 1\}^n$.

It is a well-known fact that there is a constant $K > 0$ such that, for any function f on n variables as above, there is a circuit of size at most $K \cdot 2^n/n$ that computes f ; hence, $\text{MCSP}[s]$ is only non-trivial for $s(n) < K \cdot 2^n/n$. Furthermore, $\text{MCSP}[s] \in \text{NP}$ for any constructible s and, since every circuit of size at most $s(n)$ can be described by a binary string of $O(s(n) \log s(n))$ length, if $2^{O(s(n) \log s(n))} \subseteq \text{poly}(2^n)$ (e.g., $s(n) \in O(n/\log n)$), by enumerating all possibilities we have $\text{MCSP}[s] \in \text{P}$. (Of course, such a bound is hardly useful since $s(n) \in O(n/\log n)$ implies the circuit is degenerate and can only read a strict subset of its inputs.) For large enough $s(n) < K \cdot 2^n/n$ (e.g., $s(n) \geq n$), it is unclear whether $\text{MCSP}[s]$ is NP-complete (under polynomial-time many-one reductions); see also [13, 21]. Still, we remark there has been some recent progress regarding NP-completeness under *randomized* many-one reductions for *certain variants* of MCSP [12].

Oliveira and Santhanam [23] and Oliveira, Pich, and Santhanam [22] recently analyzed hardness magnification in the average-case as well as in the worst-case approximation (i.e., gap) settings of MCSP for various (uniform and non-uniform) computational models. Meanwhile, McKay, Murray, and Williams [17] showed similar results hold in the standard (i.e., exact or gapless) worst-case setting and proved the following magnification result for (single-pass) *streaming algorithms* (see Definition 6), which is a very restricted uniform model; indeed, as mentioned in [17], even string equality (i.e., the problem of recognizing $\{ww \mid w \in \{0, 1\}^+\}$) cannot be solved by streaming algorithms (with limited space).

Theorem 2 ([17]). *Let $s: \mathbb{N}_+ \rightarrow \mathbb{N}_+$ be time constructible and $s(n) \geq n$. If there is no $\text{poly}(s(n))$ -space streaming algorithm with $\text{poly}(s(n))$ update time for (the search version of) $\text{MCSP}[s]$, then $\text{P} \neq \text{NP}$.*

In this paper, we present the following hardness magnification result for a (uniform) computational model which is provably *even more restricted* than streaming algorithms: *shrinking cellular automata* (SCAs). Here, Block_b refers to a slightly modified presentation of $\text{MCSP}[s]$ that is only needed due to certain limitations of the model (see further discussion as well as Section 3.1).

Theorem 3. *For a certain $m \in \text{poly}(s(n))$, if $\text{Block}_b(\text{MCSP}[s]) \notin \text{SCA}[n \cdot f(m)]$ for every $f \in \text{poly}(m)$ and $b \in O(f)$, then $\text{P} \neq \text{NP}$.*

Furthermore, we show every language accepted by a sublinear-time SCA can also be accepted by a streaming algorithm of comparable complexity:

Theorem 4. *Let $t: \mathbb{N}_+ \rightarrow \mathbb{N}_+$ be computable by an $O(t)$ -space random access machine (as in Definition 6) in $O(t \log t)$ time. Then, if $L \in \text{SCA}[t]$, there is an $O(t)$ -space streaming algorithm for L with $O(t \log t)$ update and $O(t^2 \log t)$ reporting time.*

Finally, we identify and prove *two distinct limitations* of SCAs compared to streaming algorithms (under sublinear-time constraints):

1. They are insensitive to the length of long unary substrings in their input (Lemma 17), which means (standard versions of) fundamental problems such as parity, modulo, majority, and threshold cannot be solved in sublinear time (Proposition 18 and Corollary 20).

2. Only a limited amount of information can be transferred between cells which are far apart (in the sense of one-way communication complexity; see Lemma 23).

Both limitations are inherited from the underlying model of cellular automata. The first can be avoided by presenting the input in a special format (the previously mentioned Block_n) that is efficiently verifiable by SCAs, which we motivate and adopt as part of the model (see the discussion below). The second is more dramatic and results in lower bounds even for languages presented in this format:

Theorem 5. *There is a language L_1 for which $\text{Block}_n(L_1) \notin \text{SCA}[o(N/\log N)]$ (N being the instance length) can be accepted by an $O(\log N)$ -space streaming algorithm with $\tilde{O}(\log N)$ update time.*

From the above, it follows that any proof of $\text{P} \neq \text{NP}$ based on a lower bound for solving $\text{MCSP}[s]$ with streaming algorithms and Theorem 2 must implicitly contain a proof of a lower bound for solving $\text{MCSP}[s]$ with SCAs. From a more “optimistic” perspective (with an eventual proof of $\text{P} \neq \text{NP}$ in mind), although not as widely studied as streaming algorithms, SCAs are thus at least as good as a “target” for proving lower bounds against and, in fact, should be an easier one if we are able to exploit their aforementioned limitations. Refer to Section 6 for further discussion on this, where we take into account a recently proposed barrier [5] to existing techniques and which also applies to our proof of Theorem 5.

From the perspective of cellular automata theory, our work furthers knowledge in sublinear-time cellular automata models, a topic seemingly neglected by the community at large (as pointed out in, e.g., [19]). Although this is certainly not the first result in which complexity-theoretical results for cellular automata and their variants have consequences for classical models (see, e.g., [15, 24] for results in this sense), to the best of our knowledge said results address only *necessary* conditions for separating classical complexity classes. Hence, our result is also novel in providing an implication in the other direction, that is, a *sufficient* condition for said separations based on lower bounds for cellular automata models.

The Model. (One-dimensional) cellular automata (CAs) are a parallel computational model composed of identical *cells* arranged in an array. Each cell operates as a deterministic finite automaton (DFA) that is connected with its left and right neighbors and operates according to the same local rule. In classical CAs, the cell structure is immutable; *shrinking* CAs relax the model in that regard by allowing cells to spontaneously vanish (with their contents being irrecoverably lost). The array structure is conserved by reconnecting every cell with deleted neighbors to the nearest non-deleted ones in either direction.

SCAs were introduced by Rosenfeld, Wu, and Dubitzki in 1983 [25], but it was not until recent years that the model received greater attention by the CA community [16, 20]. SCAs are a natural and robust model of parallel computation which, unlike classical CAs, admit (non-trivial) sublinear-time computations.

We give a brief intuition as to how shrinking augments the classical CA model in a significant way. Intuitively speaking, any two cells in a CA can only

communicate by signals, which necessarily requires time proportional to the distance between them. Assuming the entire input is relevant towards acceptance, this imposes a linear lower bound on the time complexity of the CA. In SCAs, however, this distance can be shortened as the computation evolves, thus rendering acceptance in sublinear time possible. As a matter of fact, the more cells are deleted, the faster distant cells can communicate and the computation can evolve. This results in a trade-off between space (i.e., cells containing information) and time (i.e., amount of cells deleted).

Comparison with Related Models. Unlike other parallel models such as random access machines, SCAs are *incapable of random access* to their input. In a similar sense, SCAs are constrained by the *distance* between cells, which is an aspect usually disregarded in circuits and related models except perhaps for VLSI complexity [4, 28], for instance. In contrast to VLSI circuits, however, in SCAs distance is a fluid aspect, changing dynamically as the computation evolves. Also of note is that SCAs are a *local* computational model in a quite literal sense of locality that is coupled with the above concept of distance (instead of more abstract notions such as that from [30], for example).

These limitations hold not only for SCAs but also for standard CAs. Nevertheless, SCAs are more powerful than other CA models capable of sublinear-time computation such as ACAs [11, 19], which are CAs with their acceptance behavior such that the CA accepts if and only if all cells simultaneously accept. This is because SCAs can *efficiently aggregate results* computed in parallel (by combining them using some efficiently computable function); in ACAs any such form of aggregation is fairly limited as the underlying cell structure is static.

Block Words. As mentioned above, there is an input format which allows us to circumvent the first of the limitations of SCAs compared to streaming algorithms and which is essential in order to obtain a more serious computational model. In this format, the input is subdivided into *blocks* of the same size and which are separated by delimiters and numbered in ascending order from left to right. Words with this structure are dubbed *block words* accordingly, and a set of such words is a *block language*. There is a natural presentation of any (ordinary) word as a block word (by mapping every symbol to its own block), which means there is a block language version to any (ordinary) language. (See Section 3.1.)

The concept of block words seems to arise naturally in the context of sublinear-time (both shrinking and standard) CAs [11, 19]. The syntax of block words is very efficiently verifiable (more precisely, in time linear in the block length) by a CA (without need of shrinking). In addition, the translation of a language to its block version (and its inverse) is a very simple map; one may frame it, for instance, as an AC^0 reduction. Hence, the difference between a language and its block version is solely in presentation.

Block words coupled with CAs form a computational paradigm that appears to be substantially diverse from linear- and real-time CA computation (see [19] for examples). Often we shall describe operations on a block (rather than on a cell) level and, by making use of block numbering, two blocks with distinct

numbers may operate differently even though their contents are the same; this would be impossible at a cell level due to the locality of CA rules. In combination with shrinking, certain block languages admit merging groups of blocks in parallel; this gives rise to a form of reduction we call *blockwise reductions* and which we employ in a manner akin to downward self-reducibility as in [1].

An additional technicality which arises is that the number of cells in a block is fixed at the start of the computation; this means a block cannot “allocate extra space” (beyond a constant multiple of the block length). This is the same limitation as that of linear bounded automata (LBAs) compared to Turing machines with unbounded space, for example. We cope with this limitation by increasing the block length in the problem instances as needed, that is, by padding each block so that enough space is available from the outset.¹ This is still in line with the considerations above; for instance, the resulting language is still AC^0 reducible to the original one (and vice-versa).

Techniques. We give a broad overview of the proof ideas behind our results.

Theorem 3 is a direct corollary of Theorem 24, proven in Section 5. The proof closely follows [17] (see the discussion in Section 5 for a comparison) and, as mentioned above, bases on a scheme similar to self-reducibility as in [1].

The lower bounds in Section 3.2 are established using Lemma 17, which is a generic technical limitation of sublinear-time models based on CAs (the first of the two aforementioned limitations of SCAs with respect to streaming algorithms) and which we also show to hold for SCAs.

One of the main technical highlights is the proof of Theorem 4, where we give a streaming algorithm to simulate an SCA with limited space. Our general approach bases on dynamic programming and is able to cope with the unpredictability of when, which, or even how many cells are deleted during the simulation. The space efficiency is achieved by keeping track of only as much information as needed as to determine the state of the SCA’s decision cell step for step.

A second technical contribution is the application of *one-way* communication complexity to obtain lower bounds for SCAs, which yields Theorem 5. Essentially, we split the input in some position i of our choice (which may even be non-uniformly dependent on the input length) and have A be given as input the symbols preceding i while B is given the rest, where A and B are (non-uniform) algorithms with unbounded computational resources. We show that, in this setting, A can determine the state of the SCA’s decision cell with only $O(1)$ information from B for every step of the SCA. Thus, an SCA with time complexity t for a language L yields a protocol with $O(t)$ one-way communication complexity for the above problem. Applying this in the contrapositive, Theorem 5 then follows from the existence of a language L_1 (in some contexts referred to as the indexing or memory access problem) that has nearly linear one-way communication complexity despite admitting an efficient streaming algorithm.

¹ An alternative solution is allowing the CA to “expand” by dynamically creating new cells between existing ones; however, this may result in a computational model which is dramatically more powerful than standard CAs [18, 20].

Organization. The rest of the paper is organized as follows: Section 2 presents the basic definitions. In Section 3 we introduce block words and related concepts and discuss the aforementioned limitations of sublinear-time SCAs. Following that, in Section 4 we address the proof of Theorem 4 and in Section 5 that of Theorem 3. Finally, Section 6 concludes the paper.

2 Preliminaries

We denote the set of integers by \mathbb{Z} , that of positive integers by \mathbb{N}_+ , and $\mathbb{N}_+ \cup \{0\}$ by \mathbb{N}_0 . For $a, b \in \mathbb{N}_0$, $[a, b] = \{x \in \mathbb{N}_0 \mid a \leq x \leq b\}$. For sets A and B , B^A is the set of functions $A \rightarrow B$.

We assume the reader is familiar with cellular automata as well as with the fundamentals of computational complexity theory (see, e.g., standard references [2, 8, 10]). Words are indexed starting with index zero. For a finite, non-empty set Σ , Σ^* denotes the set of words over Σ , and Σ^+ the set $\Sigma^* \setminus \{\varepsilon\}$. For $w \in \Sigma^*$, we write $w(i)$ for the i -th symbol of w (and, in general, w_i stands for another word altogether, *not* the i -th symbol of w). For $a, b \in \mathbb{N}_0$, $w[a, b]$ is the subword $w(a)w(a+1) \cdots w(b-1)w(b)$ of w (where $w[a, b] = \varepsilon$ for $a > b$). $|w|_a$ is the number of occurrences of $a \in \Sigma$ in w . $\text{bin}_n(x)$ stands for the binary representation of $x \in \mathbb{N}_0$, $x < 2^n$, of length $n \in \mathbb{N}_+$ (padded with leading zeros). $\text{poly}(n)$ is the class of functions polynomial in $n \in \mathbb{N}_0$. REG denotes the class of regular languages, and TISP[t, s] (resp., TIME[t]) that of problems decidable by a Turing machine (with one tape and one read-write head) in $O(t)$ time and $O(s)$ space (resp., unbounded space). Without restriction, we assume the empty word ε is not a member of any of the languages considered.

An ω -word is a map $\mathbb{N}_0 \rightarrow \Sigma$, and a $\omega\omega$ -word is a map $\mathbb{Z} \rightarrow \Sigma$. We write $\Sigma^\omega = \Sigma^{\mathbb{N}_0}$ for the set of ω -words over Σ . For $x \in \Sigma$, x^ω denotes the (unique) ω -word with $x^\omega(i) = x$ for every $i \in \mathbb{N}_0$. To each $\omega\omega$ -word w corresponds a unique pair (w_-, w_+) of ω -words $w_-, w_+ \in \Sigma^\omega$ with $w_+(i) = w(i)$ for $i \geq 0$ and $w_-(i) = w(-i-1)$ for $i < 0$. (Partial) ω -word homomorphisms are extendable to (partial) $\omega\omega$ -word homomorphisms as follows: Let $f: \Sigma^\omega \rightarrow \Sigma^\omega$ be an ω -word homomorphism; then there is a unique $f_{\omega\omega}: \Sigma^\mathbb{Z} \rightarrow \Sigma^\mathbb{Z}$ such that, for every $w \in \Sigma^\mathbb{Z}$, $w' = f_{\omega\omega}(w)$ is the $\omega\omega$ -word with $w'_+ = f(w_+)$ and $w'_- = f(w_-)$.

For a circuit C , $|C|$ denotes the *size* of C , that is, the total number of gates in C . It is well-known that any Boolean circuit C can be described by a binary string of $O(|C| \log |C|)$ length.

Definition 6 (Streaming algorithm). *Let $s, u, r: \mathbb{N}_+ \rightarrow \mathbb{N}_+$ be functions. An s -space streaming algorithm A is a random access machine which, on input w , works in $O(s(|w|))$ space and, on every step, can either perform an operation on a constant number of bits in memory or read the next symbol of w . A has u update time if, for every w , the number of operations it performs between reading $w(i)$ and $w(i+1)$ is at most $u(|w|)$. A has r reporting time if it performs at most $r(|w|)$ operations after having read $w(|w|-1)$ (until it terminates).*

Our interest lies in s -space streaming algorithms that, for an input w , have $\text{poly}(s(|w|))$ update and reporting time for sublinear s (i.e., $s(|w|) \in o(|w|)$).

Cellular Automata. We consider only CAs with the standard neighborhood. The symbols of an input w are provided from left to right in the cells 0 to $|w| - 1$ and are surrounded by inactive cells, which conserve their state during the entire computation (i.e., the CA is bounded). Acceptance is signaled by cell zero (i.e., the leftmost input cell).

Definition 7 (Cellular automaton). A cellular automaton (CA) C is a tuple $(Q, \delta, \Sigma, q, A)$ where: Q is a non-empty and finite set of states; $\delta: Q^3 \rightarrow Q$ is the local transition function; $\Sigma \subsetneq Q$ is the input alphabet of C ; $q \in Q \setminus \Sigma$ is the inactive state, that is, $\delta(q_1, q, q_2) = q$ for every $q_1, q_2 \in Q$; and $A \subseteq Q \setminus \{q\}$ is the set of accepting states of C . A cell which is not in the inactive state is said to be active. The elements of $Q^{\mathbb{Z}}$ are the (global) configurations of C . δ induces the global transition function $\Delta: Q^{\mathbb{Z}} \rightarrow Q^{\mathbb{Z}}$ of C by $\Delta(c)(i) = \delta(c(i-1), c(i), c(i+1))$ for every cell $i \in \mathbb{Z}$ and configuration $c \in Q^{\mathbb{Z}}$.

C accepts an input $w \in \Sigma^+$ if cell zero is eventually in an accepting state, that is, there is $t \in \mathbb{N}_0$ such that $(\Delta^t(c_0))(0) \in A$, where $c_0 = c_0(w)$ is the initial configuration (for w): $c_0(i) = w(i)$ for $i \in [0, |w| - 1]$, and $c_0(i) = q$ otherwise. For a minimal such t , we say C accepts w with time complexity t . $L(A) \subseteq \Sigma^+$ denotes the set of words accepted by C . For $t: \mathbb{N}_+ \rightarrow \mathbb{N}_0$, $\text{CA}[t]$ is the class of languages accepted by CAs with time complexity $O(t(n))$, n being the input length.

For convenience, we extend Δ in the obvious manner (i.e., as a map induced by δ) so it is also defined for every (finite) word $w \in Q^*$. For $|w| \leq 2$, we set $\Delta(w) = \varepsilon$; for longer words, $|\Delta(w)| = |w| - 2$ holds.

Some remarks concerning the classes $\text{CA}[t]$: $\text{CA}[\text{poly}] = \text{TISP}[\text{poly}, n]$ (i.e., the class of polynomial-time LBAs), and $\text{CA}[t] = \text{CA}[1] \subsetneq \text{REG}$ for every sublinear t . Furthermore, $\text{CA}[t] \subseteq \text{TISP}[t^2, n]$ (where $t^2(n) = (t(n))^2$) and $\text{TISP}[t, n] \subseteq \text{CA}[t]$.

Definition 8 (Shrinking CA). A shrinking CA (SCA) S is a CA with a delete state $\otimes \in Q \setminus (\Sigma \cup \{q\})$. The global transition function Δ_S of S is given by applying the standard CA global transition function Δ (as in Definition 7) followed by removing all cells in the state \otimes ; that is, $\Delta_S = \Phi \circ \Delta$, where $\Phi: Q^{\mathbb{Z}} \rightarrow Q^{\mathbb{Z}}$ is the (partial) $\omega\omega$ -word homomorphism resulting from the extension to $Q^{\mathbb{Z}}$ of the map $\varphi: Q \rightarrow Q$ with $\varphi(\otimes) = \varepsilon$ and $\varphi(x) = x$ for $x \in Q \setminus \{\otimes\}$. For $t: \mathbb{N}_+ \rightarrow \mathbb{N}_0$, $\text{SCA}[t]$ is the class of languages accepted by SCAs with time complexity $O(t(n))$, where n denotes the input length.

Note that Φ is only partial since, for instance, any $\omega\omega$ -word in $\otimes^\omega \cdot \Sigma^* \cdot \otimes^\omega$ has no proper image (as it is not mapped to a $\omega\omega$ -word). Hence, Δ_S is also only a partial function (on $Q^{\mathbb{Z}}$); nevertheless, Φ is total on the set of $\omega\omega$ -words in which \otimes occurs only finitely often and, in particular, Δ_S is total on the set of configurations arising from initial configurations for finite input words (which is the setting we are interested in).

The acceptance condition of SCAs is the same as in Definition 7 (i.e., acceptance is dictated by cell zero). Unlike in standard CAs, the index of one same cell can differ from one configuration to the next; that is, a cell index does not uniquely determine a cell on its own (rather, only in conjunction with a time step).

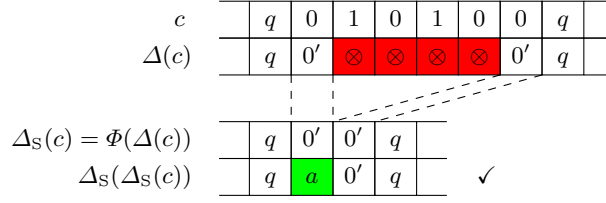


Figure 1. Computation of an SCA that recognizes $L = \{w \in \{0, 1\}^+ \mid w(0) = w(|w| - 1)\}$ in $O(1)$ time. Here, the input word is $010100 \in L$.

This is a consequence of applying Φ , which contracts the global configuration towards cell zero. More precisely, for a configuration $c \in Q^{\mathbb{Z}}$, the cell with index $i \geq 0$ in $\Delta(c)$ corresponds to that with index $i + d_i$ in c , where d_i is the number of cells with index $\leq i$ in c that were deleted in the transition to $\Delta(c)$. This also implies the cell with index zero in $\Delta(c)$ is the same as that in c with minimal positive index that was not deleted in the transition to $\Delta(c)$; thus, in any time step, cell zero is the leftmost active cell (unless all cells are inactive; in fact, cell zero is inactive if and only if all other cells are inactive). Granted, what indices a cell has is of little importance when one is interested only in the configurations of an SCA and their evolution; nevertheless, they are relevant when simulating an SCA with another machine model (as we do in Sections 3.3 and 4).

Naturally, $\text{CA}[t] \subseteq \text{SCA}[t]$ for every function t , and $\text{SCA}[\text{poly}] = \text{CA}[\text{poly}]$. For sublinear t , $\text{SCA}[t]$ contains non-regular languages if, for instance, $t \in \Omega(\log n)$ (see below); hence, the inclusion of $\text{CA}[t]$ in $\text{SCA}[t]$ is strict. In fact, this is the case even if we consider only regular languages. One simple example is $L = \{w \in \{0, 1\}^+ \mid w(0) = w(|w| - 1)\}$, which is in $\text{SCA}[1]$ and regular but not in $\text{CA}[o(n)] = \text{CA}[O(1)]$. One obtains an SCA for L by having all cells whose both neighbors are active delete themselves in the first step; the two remaining cells then compare their states, and cell zero accepts if and only if this comparison succeeds or if the input has length 1 (which it can notice immediately since it is only for such words that it has two inactive neighbors). Formally, the local transition function δ is such that, for $z_1, z_3 \in \{0, 1, q\}$ and $z_2 \in \{0, 1\}$, $\delta(z_1, z_2, z_3) = \otimes$ if both z_1 and z_3 are in $\{0, 1\}$, $\delta(z_1, z_2, z_3) = z_2'$ if $z_1 = q$ or $z_3 = q$, and $\delta(q, z_2', z_2') = \delta(q, z_2', q) = a$; in all other cases, δ simply conserves the cell's state. See Figure 1 for an example.

Using a textbook technique to simulate a (bounded) CA with an LBA (simply skipping deleted cells), we have:

Proposition 9. *For every function $t: \mathbb{N}_+ \rightarrow \mathbb{N}_+$ computable by an LBA in $O(n \cdot t(n))$ time, $\text{SCA}[t] \subseteq \text{TISP}[n \cdot t(n), n]$.*

The inclusion is actually proper (see Corollary 19). Using the well-known result $\text{TIME}[o(n \log n)] = \text{REG}$ [14], it follows that at least a logarithmic time bound is needed for SCAs to recognize languages which are not regular:

Corollary 10. $\text{SCA}[o(\log)] \subsetneq \text{REG}$.

This bound is tight: It is relatively easy to show that any language accepted by ACAs in $t(n)$ time can also be accepted by an SCA in $t(n) + O(1)$ time. Since there is a non-regular language recognizable by ACAs [11] in $O(\log n)$ time, the same language is recognizable by an SCA in $O(\log n)$ time.

For any finite, non-empty set Σ , we say a function $f: \Sigma^+ \rightarrow \Sigma^+$ is *computable in place* by an (S)CA if there is an (S)CA S which, given $x \in \Sigma^+$ as input (surrounded by inactive cells), produces $f(x)$. Additionally, $g: \mathbb{N}_+ \rightarrow \mathbb{N}_+$ is *constructible in place* by an (S)CA if $g(n) \leq 2^n$ and there is an (S)CA S which, given $n \in \mathbb{N}_0$ in unary, produces $\text{bin}_n(g(n) - 1)$ (i.e., $g(n) - 1$ in binary). Note the set of functions computable or constructible in place by an (S)CA in at most $t(n)$ time, where n is the input length and $t: \mathbb{N}_+ \rightarrow \mathbb{N}_+$ is some function, includes (but is not limited to) all functions computable by an LBA in at most $t(n)$ time.

3 Capabilities and Limitations of Sublinear-Time SCAs

3.1 Block Languages

Let Σ be a finite, non-empty set. For $\Sigma_\varepsilon = \Sigma \cup \{\varepsilon\}$ and $x, y \in \Sigma^+$, $\binom{x}{y}$ denotes the (unique) word in $(\Sigma_\varepsilon \times \Sigma_\varepsilon)^+$ of length $\max\{|x|, |y|\}$ for which $\binom{x}{y}(i) = (x(i), y(i))$, where $x(i) = y(j) = \varepsilon$ for $i \geq |x|$ and $j \geq |y|$.

Definition 11 (Block word). *Let $n, m, b \in \mathbb{N}_+$ be such that $b \geq n$ and $m \leq 2^n$. A word w is said to be an (n, m, b) -block word (over Σ) if it is of the form $w = w_0 \# w_1 \# \cdots \# w_{m-1}$ and $w_i = \binom{\text{bin}_n(x_i)}{y_i}$, where $x_0 \geq 0$, $x_{i+1} = x_i + 1$ for every i , $x_{m-1} < 2^n$, and $y_i \in \Sigma^b$. In this context, w_i is the i -th block of w .*

Hence, every (n, m, b) -block word w has m many blocks of length b , and its total length is $|w| = (b + 1) \cdot m - 1 \in \Theta(bm)$. For example,

$$w = \binom{01}{0100} \# \binom{10}{1100} \# \binom{11}{1000}$$

is a $(2, 3, 4)$ -block word with $x_0 = 1$, $y_0 = 0100$, $y_1 = 1100$, and $y_2 = 1000$. n is implicitly encoded by the entries in the upper track (i.e., the x_i) and we shall see m and b as parameters depending on n (see Definition 12 below), so the structure of each block can be verified locally (i.e., by inspecting the immediate neighborhood of every block). Note the block numbering starts with an arbitrary x_0 ; this is intended so that, for $m' < m$, an (n, m, b) -block word admits (n, m', b) -block words as infixes (which would not be the case if we required, say, $x_0 = 0$).

When referring to block words, we use N for the block word length $|w|$ and reserve n for indexing block words of different block length, overall length, or total number of blocks (or any combinations thereof). With m and b as parameters depending on n , we obtain sets of block words:

Definition 12 (Block language). *Let $m, b: \mathbb{N}_+ \rightarrow \mathbb{N}_+$ be non-decreasing and constructible in place by a CA in $O(m(n) + b(n))$ time. Furthermore, let $b(n) \geq n$ and $m(n) \leq 2^n$. Then, \mathfrak{B}_b^m denotes the set of all $(n, m(n), b(n))$ -block words for $n \in \mathbb{N}_+$, and every subset $L \subseteq \mathfrak{B}_b^m$ is an $((n, m, b)$ -)block language (over Σ).*

An SCA can *verify* its input is a valid block word in $O(b(n))$ time, that is, locally check that the structure and contents of the blocks are consistent (i.e., as in Definition 11). This can be realized using standard CA techniques without need of shrinking (see [11, 19] for constructions). Recall Definition 8 does not require an SCA S to explicitly reject inputs not in $L(S)$, that is, the time complexity of S on an input w is only defined for $w \in L(S)$. As a result, when $L(S)$ is a block language, the time spent verifying that w is a block word is only relevant if $w \in L(S)$ and, in particular, if w is a (valid) block word. Provided the state of every cell in S eventually impacts its decision to accept (which is the case for all constructions we describe), it suffices to have a cell mark itself with an error flag whenever a violation in w is detected (even if other cells continue their operation as normal); since every cell is relevant towards acceptance, this eventually prevents S from accepting (and, since $w \notin L(S)$, it is irrelevant how long it takes for this to occur). Thus, for the rest of this paper, when describing an SCA for a block language, we implicitly require that the SCA checks its input is a valid block word beforehand.

As stated in the introduction, our interest in block words is as a special input format. There is a natural bijection between any language and a block version of it, namely by mapping each word z to a block word w in which each block w_i contains a symbol $z(i)$ of z (padded up to the block length b) and the blocks are numbered from 0 to $|z| - 1$:

Definition 13 (Block version of a language). *Let $L \subseteq \Sigma^+$ be a language and b as in Definition 12. The block version $\text{Block}_b(L)$ of L (with blocks of length b) is the block language for which, for every $z \in \Sigma^+$, $z \in L$ holds if and only if we have $w \in \text{Block}_b(L)$ where w is the $(n, m, b(n))$ -block word (as in Definition 11) with $m = |z|$, $n = \lceil \log m \rceil$, $x_0 = 0$, and $y_i = z(i)0^{b(n)-1}$ for every $i \in [0, m - 1]$.*

Note that, for any such language L , $\text{Block}_b(L) \notin \text{REG}$ for any b (since $b(n) \geq n$ is not constant); hence, $\text{Block}_b(L) \in \text{SCA}[t]$ only for $t \in \Omega(\log n)$ (and constructible b). For $b(n) = n$, $\text{Block}_n(L)$ is the block version with minimal padding.

For any two finite, non-empty sets Σ_1 and Σ_2 , say a function $f: \Sigma_1^+ \rightarrow \Sigma_2^+$ is *non-stretching* if $|f(x)| \leq |x|$ for every $x \in \Sigma_1^+$. We now define k -blockwise maps, which are maps that operate on block words by grouping $k(n)$ many blocks together and mapping each such group (in a non-stretching manner) to a single block of length at most $(b(n) + 1) \cdot k(n) - 1$.

Definition 14 (Blockwise map). *Let $k: \mathbb{N}_+ \rightarrow \mathbb{N}_+$, $k(n) \geq 2$, be a non-decreasing function and constructible in place by a CA in $O(k(n))$ time. A map $g: \mathfrak{B}_b^{km} \rightarrow \mathfrak{B}_b^m$ is a k -blockwise map if there is a non-stretching $g': \mathfrak{B}_b^k \rightarrow \Sigma^+$ such that, for every $w \in \mathfrak{B}_b^{km}$ (as in Definition 11) and $w'_i = w_{ik} \# \cdots \# w_{(i+1)k-1}$:*

$$g(w) = \left(\binom{\text{bin}_n(x_0)}{g'(w'_0)} \right) \# \cdots \# \left(\binom{\text{bin}_n(x_{m-1})}{g'(w'_{m-1})} \right).$$

Using blockwise maps, we obtain a very natural form of reduction operating on block words and which is highly compatible with sublinear-time SCAs as a

computational model. The reduction divides an (n, km, b) -block word in m many groups of k many contiguous blocks and, as a k -blockwise map, maps each such group to a single block (of length b):

Definition 15 (Blockwise reducible). *For block languages L and L' , L is (k) -blockwise reducible to L' if there is a computable k -blockwise map $g: \mathfrak{B}_b^{km} \rightarrow \mathfrak{B}_b^m$ such that, for every $w \in \mathfrak{B}_b^{km}$, we have $w \in L$ if and only if $g(w) \in L'$.*

Since every application of the reduction reduces the instance length by a factor of approximately k , logarithmically many applications suffice to produce a trivial instance (i.e., an instance consisting of a single block). This gives us the following computational paradigm of chaining blockwise reductions together:

Lemma 16. *Let $k, r: \mathbb{N}_+ \rightarrow \mathbb{N}_0$ be functions, and let $L \subseteq \mathfrak{B}_b^{k^r}$ be such that there is a series $L = L_0, L_1, \dots, L_{r(n)}$ of languages with $L_i \subseteq \mathfrak{B}_b^{k^{r-i}}$ and such that L_i is $k(n)$ -blockwise reducible to L_{i+1} via the (same) blockwise reduction g . Furthermore, let g' be as in Definition 14, and let $t_{g'}: \mathbb{N}_+ \rightarrow \mathbb{N}_+$ be non-decreasing and such that, for every $w' \in \mathfrak{B}_b^r$, $g'(w')$ is computable in place by an SCA in $O(t_{g'}(|w'|))$ time. Finally, let $L_{r(n)} \in \text{SCA}[t]$ for some function $t: \mathbb{N}_+ \rightarrow \mathbb{N}_+$. Then, $L \in \text{SCA}[r(n) \cdot t_{g'}(O(k(n) \cdot b(n))) + O(b(n)) + t(b(n))]$.*

Proof. We consider the SCA S which, given $w \in \mathfrak{B}_b^{k^r}$, repeatedly applies the reduction g , where each application of g is computed by applying g' on each group of relevant blocks (i.e., the w'_i from Definition 14) in parallel.

One detail to note is that this results in the same procedure P being applied to different groups of blocks in parallel, but it may be so that P requires more time for one group of blocks than for the other. Thus, we allow the entire process to be carried out asynchronously but require that, for each group of blocks, the respective results be present before each execution of P is started. (One way of realizing this, for instance, is having the first block in the group send a signal across the whole group to ensure all inputs are available and, when it arrives at the last block in the group, another signal is sent to trigger the start of P .)

Using that $t_{g'}$ is non-decreasing and that g' is non-stretching, the time needed for each execution of P is $t_{g'}(|w'_i|) \in t_{g'}(O(k(n) \cdot b(n)))$ (which is not impacted by the considerations above) and, since there are $r(n)$ reductions in total, we have $r(n) \cdot t_{g'}(O(k(n) \cdot b(n)))$ time in total. Once a single block is left, the cells in this block synchronize themselves and then behave as in the SCA S' for $L_{r(n)}$ guaranteed by the assumption; using a standard synchronization algorithm, this requires $O(b(n))$ for the synchronization, plus $t(b(n))$ time for emulating S' . \square

3.2 Block Languages and Parallel Computation

In this section, we prove the first limitation of SCAs discussed in the introduction (Lemma 17) and which renders them unable of accepting the languages PAR , MOD_q , MAJ , and THR_k (defined next) in sublinear time. Nevertheless, as is shown in Proposition 21, the *block versions* of these languages can be accepted quite efficiently. This motivates the block word presentation for inputs; that is,

this first limitation concerns only the *presentation* of instances (and, hence, is not a *computational* limitation of SCAs).

Let $q > 2$ and let $k: \mathbb{N}_+ \rightarrow \mathbb{N}_+$ be constructible in place by a CA in at most $t_k(n)$ time for some $t_k: \mathbb{N}_+ \rightarrow \mathbb{N}_+$. Additionally, let PAR (resp., MOD $_q$; resp., MAJ; resp., THR $_k$) be the language consisting of every word $w \in \{0, 1\}^+$ for which $|w|_1$ is even (resp., $|w|_1 = 0 \pmod{q}$; resp., $|w|_1 \geq |w|_0$; resp., $|w|_1 \geq k(|w|)$).

The following is a simple limitation of sublinear-time CA models such as ACAs (see also [26]) which we show also to hold for SCAs.

Lemma 17. *Let S be an SCA with input alphabet Σ , and let $x \in \Sigma$ be such that there is a minimal $t \in \mathbb{N}_+$ for which $\Delta_S^t(y) = \varepsilon$, where $y = x^{2t+1}$ (i.e., the symbol x concatenated $2t+1$ times with itself). Then, for every $z_1, z_2 \in \Sigma^+$, $w = z_1 y z_2 \in L(S)$ holds if and only if for every $i \in \mathbb{N}_0$ we have $w_i = z_1 y x^i z_2 \in L(S)$.*

Proof. Given w and i as above, we show $w_i \in L(S)$; the converse is trivial. Since w and w_i both have $z_1 y$ as prefix and $\Delta_S^{t'}(y) \neq \varepsilon$ for $t' < t$, if S accepts w in t' steps, then it also accepts w_i in t' steps. Thus, assume S accepts w in $t' \geq t$ steps, in which case it suffices to show $\Delta_S^t(w) = \Delta_S^t(w_i)$. To this end, let α_j for $j \in [0, t]$ be such that $\alpha_0 = x$ and $\alpha_{j+1} = \delta(\alpha_j, \alpha_j, \alpha_j)$. Hence, $\Delta(\alpha_j^{k+2}) = \alpha_{j+1}^k$ holds for every $k \in \mathbb{N}_+$ (and $j < t$) and, by an inductive argument as well as by the assumption on y (i.e., $\alpha_t = \otimes$), $\Delta_S^t(yx^i) = \Delta_S^t(\alpha_0^{2t+i+1}) = \varepsilon$. Using this along with $|y| \geq t$ and $y \in \{x\}^+$, we have $\Delta_S^t(q^t z_1 y x^i) = \Delta_S^t(q^t z_1 y)$ and $\Delta_S^t(yx^i z_2 q^t) = \Delta_S^t(x^i y z_2 q^t) = \Delta_S^t(y z_2 q^t)$; hence, $\Delta_S^t(w) = \Delta_S^t(w_i)$ follows. \square

An implication of Lemma 17 is that every unary language $U \in \text{SCA}[o(n)]$ is either finite or cofinite. As $\text{PAR} \cap \{1\}^+$ is neither finite nor cofinite, we can prove:

Proposition 18. $\text{PAR} \notin \text{SCA}[o(n)]$ (where n is the input length).

Proof. Let S be an SCA with $L(S) = \text{PAR}$. We show S must have $\Omega(n)$ time complexity on inputs from the infinite set $U = \{1^{2^m} \mid m \in \mathbb{N}_+\} \subset \text{PAR}$. If $\Delta_S^t(1^{2^{t+1}}) = \varepsilon$ for some $t \in \mathbb{N}_0$, then, by Lemma 17, $L(S) \cap \{1\}^+$ is either finite or cofinite, which contradicts $L(S) = \text{PAR}$. Hence, $\Delta_S^t(1^{2^{t+1}}) \neq \varepsilon$ for every $t \in \mathbb{N}_0$. In this case, the trace of cell zero on input $w = 11^{2^{t+1}}1$ in the first t steps is the same as that on input $w' = 11^{2^{t+1}}11$. Since $w \in \text{PAR}$ if and only if $w' \notin \text{PAR}$, it follows that S has $\Omega(t) = \Omega(n)$ time complexity on U . \square

Corollary 19. $\text{REG} \not\subseteq \text{SCA}[o(n)]$.

The argument above generalizes to MOD $_q$, MAJ, and THR $_k$ with $k \in \omega(1)$. For MOD $_q$, consider $U = \{1^{qm} \mid m \in \mathbb{N}_+\}$. For MAJ and THR $_k$, set $U = \{0^m 1^m \mid m \in \mathbb{N}_+\}$ and $U = \{0^{n-k(n)} 1^{k(n)} \mid n \in \mathbb{N}_+\}$, respectively; in this case, U is not unary, but the argument easily extends to the unary suffixes of the words in U .

Corollary 20. MOD $_q$, MAJ $\notin \text{SCA}[o(n)]$. Also, THR $_k \in \text{SCA}[o(n)]$ if and only if $k \in O(1)$.

The *block versions* of these languages, however, are not subject to the limitation above:

Proposition 21. For $L \in \{\text{PAR}, \text{MOD}_q, \text{MAJ}\}$, $\text{Block}_n(L) \in \text{SCA}[(\log N)^2]$, where $N = N(n)$ is the input length. Also, $\text{Block}_n(\text{THR}_k) \in \text{SCA}[(\log N)^2 + t_k(n)]$.

Proof. Given $L \in \{\text{PAR}, \text{MOD}_q, \text{MAJ}, \text{THR}_k\}$, we construct an SCA S for $L' = \text{Block}_n(L)$ with the purported time complexity. Let $w \in \mathfrak{B}_n^m$ be an input of S . For simplicity, we assume that, for every such w , $m = m(n) = 2^n$ is a power of two; the argument extends to the general case in a simple manner. Hence, we have $N = |w| = n \cdot m$ and $n = \log m \in \Theta(\log N)$.

Let $L_0 \subset \mathfrak{B}_n^m$ be the language containing every such block word $w \in \mathfrak{B}_n^m$ for which, for y_i as in Definition 11 and $y = \sum_{i=0}^{m-1} y_i$, we have $f_L(y) = f_{L,n}(y) = 0$, where $f_{\text{PAR}}(y) = y \bmod 2$, $f_{\text{MOD}_q}(y) = y \bmod q$, $f_{\text{MAJ}}(y) = 0$ if and only if $y \geq 2^{n-1}$, and $f_{\text{THR}_k}(y) = 0$ if and only if $y \geq k(n)$. Thus, (under the previous assumption) we have $L_0 = L'$ (and, in the general case, $L_0 = L' \cap \mathfrak{B}_n^{2^n}$).

Then, L_0 is 2-blockwise reducible to a language $L_1 \subseteq \mathfrak{B}_n^{m/2}$ by mapping every $(n, 2, n)$ -block word of the form $\binom{\text{bin}_n(2x)}{y_{2x}} \# \binom{\text{bin}_n(2x+1)}{y_{2x+1}}$ with $x \in [0, 2^{n-1} - 1]$ to $\binom{\text{bin}_n(x)}{y_{2x} + y_{2x+1}}$. To do so, it suffices to compute $\text{bin}_n(x)$ from $\text{bin}_n(2x)$ and add the y_{2x} and y_{2x+1} values in the lower track; using basic CA arithmetic and cell communication techniques, this is realizable in $O(n)$ time. Repeating this procedure, we obtain a chain of languages L_0, \dots, L_n such that L_i is 2-blockwise reducible to L_{i+1} in $O(n)$ time. By Lemma 16, $L' \in \text{SCA}[n^2 + t(n)]$ follows, where $t: \mathbb{N}_+ \rightarrow \mathbb{N}_0$ is such that $L_n \in \text{SCA}[t]$. For $L \in \{\text{PAR}, \text{MOD}_q, \text{MAJ}\}$, checking the above condition on $f_L(y)$ can be done in $t(n) \in O(n)$ time; as for $L = \text{THR}_k$, we must also compute k , so we have $t(n) \in O(n + t_k(n))$.

The general case follows from adapting the above reductions so that words with an odd number of blocks are also accounted for (e.g., by ignoring the last block of w and applying the reduction on the first $m - 1$ blocks). \square

3.3 An Optimal SCA Lower Bound for a Block Language

Corollary 19 already states SCAs are strictly less capable than streaming algorithms. However, the argument bases exclusively on long unary subwords in the input (i.e., Lemma 17) and, therefore, does not apply to block languages. Hence Theorem 5, which shows SCAs are more limited than streaming algorithms *even considering only block languages*:

Theorem 5. *There is a language L_1 for which $\text{Block}_n(L_1) \notin \text{SCA}[o(N/\log N)]$ (N being the instance length) can be accepted by an $O(\log N)$ -space streaming algorithm with $\tilde{O}(\log N)$ update time.*

Let L_1 be the language of words $w \in \{0, 1\}^+$ such that $|w| = 2^n$ is a power of two and, for $i = w(0)w(1) \cdots w(n-1)$ (seen as an n -bit binary integer), $w(i) = 1$. It is not hard to show that its block version $\text{Block}_n(L_1)$ can be accepted by an $O(\log m)$ -space streaming algorithm with $\tilde{O}(\log m)$ update time.

The $O(N/\log N)$ upper bound for $\text{Block}_n(L_1)$ is optimal since there is an $O(N/\log N)$ time SCA for it: Shrink every block to its respective bit (i.e., the y_i from Definition 11), reducing the input to a word w' of $O(N/\log N)$ length;

while doing so, mark the bit corresponding to the n -th block. Then shift the contents of the first n bits as a counter that decrements itself every new cell it visits and, when it reaches zero, signals acceptance if the cell it is currently at contains a 1. Using counter techniques as in [27, 29], this requires $O(|w'|)$ time.

The proof of Theorem 5 bases on communication complexity. The basic setting is a game with two players A and B (both with unlimited computational resources) which receive inputs w_A and w_B , respectively, and must produce an answer to the problem at hand while exchanging a limited amount of bits. We are interested in the case where the concatenation $w = w_A w_B$ of the inputs of A and B is an input to an SCA and A must output whether the SCA accepts w . More importantly, we analyze the case where *only B is allowed to send messages*, that is, the case of *one-way* communication.²

Definition 22 (One-way communication complexity). Let $m, f: \mathbb{N}_+ \rightarrow \mathbb{N}_+$ be functions with $0 < m(N) \leq N$. A language $L \subseteq \Sigma^+$ is said to have (m -)one-way communication complexity f if there are families of algorithms (with unlimited computational resources) $(A_N)_{N \in \mathbb{N}_+}$ and $(B_N)_{N \in \mathbb{N}_+}$ such that the following holds for every $w \in \Sigma^*$ of length $|w| = N$, where $w_A = w[0, m(N) - 1]$ and $w_B = w[m(N), N - 1]$:

1. $|B_N(w_B)| \leq f(N)$; and
2. $A_N(w_A, B(w_B)) = 1$ (i.e., accept) if and only if $w \in L$.

$\mathfrak{C}_{\text{ow}}^m(L)$ indicates the (pointwise) minimum over all such functions f .

Note that A_N and B_N are nonuniform, so the length N of the (complete) input w is known implicitly by both algorithms.

Lemma 23. For any computable $t: \mathbb{N}_+ \rightarrow \mathbb{N}_+$ and m as in Definition 22, if $L \in \text{SCA}[t]$, then $\mathfrak{C}_{\text{ow}}^m(L)(N) \in O(t(N))$.

The proof idea is to have A and B simulate the SCA for L simultaneously, with A maintaining the first half c_A of the SCA configuration and B the second half c_B . (Hence, A is aware of the leftmost active state in the SCA and can detect whether the SCA accepts or not.) The main difficulty is guaranteeing that A and B can determine the states of the cells on the right (resp., left) end of c_A (resp., c_B) despite the respective local configurations “overstepping the boundary” between c_A and c_B . Hence, for each step in the simulation, B communicates the states of the two leftmost cells in c_B ; with this, A can compute the states of all cells of c_A in the next configuration as well as that of the leftmost cell α of c_B , which is added to c_A . (See Figure 2 for an illustration.) This last technicality is needed due to one-way communication, which renders it impossible for B to determine the next state of α (since its left neighbor is in c_A and B cannot receive messages from A). As the simulation requires at most $t(N)$ steps and B sends $O(1)$ information at each step, this yields the purported $O(t(N))$ upper bound.

² One-way communication complexity can also be defined as the maximum over *both* communication directions (i.e., B to A and A to B ; see [9] for an example in the setting of CAs). Since our goal is to prove a *lower bound* on communication complexity, it suffices to consider a single (arbitrary) direction (in this case B to A).

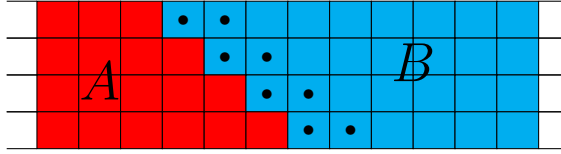


Figure 2. Simulating an SCA with low one-way communication complexity. (For simplicity, in this example the SCA does not shrink.) B communicates the states of the cells marked with “•”. The colors indicate which states are computed by each player.

The attentive reader may have noticed this discussion does not address the fact that the SCA may shrink; indeed, we shall also prove that shrinking does not interfere with this strategy.

Proof. Let S be an SCA for L with time complexity $O(t)$. Furthermore, let Q be the state set of S and $q \in Q$ its inactive state. We construct algorithms A_N and B_N as in Definition 22 and such that $|B_N(w_B)| \leq 2 \log(|Q|) \cdot t(N)$.

Fix $N \in \mathbb{N}_+$ and an input $w \in \Sigma^N$. For $w_B^0 = w_B q^{2t(N)+2}$ and $w_B^{i+1} = \Delta_S(w_B^i)$ for $i \in \mathbb{N}_0$, B_N computes and outputs the concatenation

$$B_N(w_B) = w_B^0(0)w_B^0(1)w_B^1(0)w_B^1(1) \cdots w_B^{t(N)}(0)w_B^{t(N)}(1).$$

Similarly, let $w_A^0 = q^{2t(N)+2}w_A$ and $w_A^{i+1} = \Delta_S(w_A^i w_B^i(0)w_B^i(1))$ for $i \in \mathbb{N}_0$. A computes $t(N)$ and w_A^i for $i \in [0, t(N)]$ and accepts if there is any j such that $w_A^i(j)$ is an accept state of S and $w_A^i(j') = q$ for all $j' < j$; otherwise, A rejects.

To prove the correctness of A , we show by induction on $i \in \mathbb{N}_0$: $w_A^i w_B^i = \Delta_S^i(q^{2t(n)+2}w q^{2t(n)+2})$. Hence, the $w_A^i(j)$ of above corresponds to the state of cell zero in step i of S , and it follows that A accepts if and only if S does. The induction basis is trivial. For the induction step, let $w' = \Delta_S(w_A^i w_B^i)$. Using the induction hypothesis, it suffices to prove $w_A^{i+1} w_B^{i+1} = w'$. Note first that, due to the definition of w_A^{i+1} and w_B^{i+1} , we have $w' = \Delta_S(w_A^i) \alpha \beta \Delta_S(w_B^i)$, where $\alpha, \beta \in Q \cup \{\varepsilon\}$. Let $\alpha_1 = w_A^i(|w_A^i| - 2)$, $\alpha_2 = w_A^i(|w_A^i| - 1)$, and $\alpha_3 = w_B^i(0)$ and notice $\alpha = \delta(\alpha_1, \alpha_2, \alpha_3)$; the same is true for β and $\beta_1 = \alpha_2$, $\beta_2 = \alpha_3$, and $\beta_3 = w_B^i(1)$. Hence, we have $w_A^{i+1} = \Delta_S(w_A^i) \alpha \beta$, and the claim follows. \square

We are now in position to prove Theorem 5.

Proof of Theorem 5. We prove that, for our language L_1 of before and $m(n) = n(n+1)$ (i.e., A_N receives the first n input blocks), $\mathfrak{C}_{\text{ow}}^m(\text{Block}_n(L_1))(N) \geq 2^n - n$. Since the input length is $N \in \Theta(n \cdot 2^n)$, the claim then follows from the contrapositive of Lemma 23.

The proof is by a counting argument. Let A_N and B_N be as in Definition 22, and let $Y = \{0, 1\}^{2^n - n}$. The basic idea is that, for the same input w_A , if B_N is given different inputs w_B and w'_B but $B_N(w_B) = B_N(w'_B)$, then $w = w_A w_B$ is accepted if and only if $w' = w_A w'_B$ is accepted. Hence, for any $y, y' \in Y$ with $y \neq y'$, we must have $B_N(w_B) \neq B_N(w'_B)$, where $w_B, w'_B \in \mathfrak{B}_n^{2^n - n}$ are the block word versions of y and y' , respectively; this is because, letting $j \in [0, 2^n - n]$

be such that $y(j) \neq y'(j)$ and $z = \text{bin}_n(n + j)$, precisely one of the words zy and zy' is in L_1 (and the other not). Finally, note there is a bijection between Y and the set Y' of block words in $\mathfrak{B}_n^{2^n - n}$ whose block numbering starts with $n + 1$ (i.e., $x_0 = n + 1$, where x_0 is as in Definition 11) and with block entries of the form $a0^{n-1}$ where $a \in \{0, 1\}$ (i.e., Y' is essentially the block version of Y as in Definition 13 but where we set $x_0 = n + 1$ instead of $x_0 = 0$). We conclude $\mathfrak{C}_{\text{ow}}^n(\text{Block}_n(L_1))(N) \geq |Y'| = |Y| = 2^n - n$, and the claim follows. \square

4 Simulation of an SCA by a Streaming Algorithm

In this section, we recall and prove:

Theorem 4. *Let $t: \mathbb{N}_+ \rightarrow \mathbb{N}_+$ be computable by an $O(t)$ -space random access machine (as in Definition 6) in $O(t \log t)$ time. Then, if $L \in \text{SCA}[t]$, there is an $O(t)$ -space streaming algorithm for L with $O(t \log t)$ update and $O(t^2 \log t)$ reporting time.*

Before we state the proof, we first introduce some notation. Having fixed an input w , let $c_t(i)$ denote the state of cell i in step t on input w . Note that here we explicitly allow $c_t(i)$ to be the state \otimes and also disregard any changes in indices caused by cell deletion; that is, $c_t(i)$ refers to the *same cell* i as in the initial configuration c_0 (of Definition 7; see also the discussion following Definition 8). For a finite, non-empty $I = [a, b] \subseteq \mathbb{Z}$ and $t \in \mathbb{N}_0$, let $\text{nndcl}_t(I) = \max\{i \mid i < a, c_t(i) \neq \otimes\}$ denote the nearest non-deleted cell to the left of I ; similarly, $\text{nndcr}_t(I) = \min\{i \mid i > b, c_t(i) \neq \otimes\}$ is the nearest such cell to the right of I .

Proof. Let S be an $O(t)$ -time SCA for L . Using S , we construct a streaming algorithm A (Algorithm 1) for L and prove it has the purported complexities.

Construction. Let w be an input to A . To decide L , A computes the states of the cells of S in the time steps up to $t(|w|)$. In particular, A sequentially determines the state of the leftmost active cell in each of these time steps (starting from the initial configuration) and accepts if and only if at least one of these states is accepting. To compute these states efficiently, we use an approach based on dynamic programming, reusing space as the computation evolves.

A maintains lists `leftIndex`, `leftState`, `centerIndex`, and `centerState` and which are indexed by every step j starting with step zero and up to the current step τ . The lists `leftIndex` and `centerIndex` store cell indices while `leftState` and `centerState` store the states of the respective cells, that is, `leftState[j] = c_j(leftIndex[j])` and `centerState[j] = c_j(centerIndex[j])`.

Recall the state $c_{j+1}(y)$ of a cell y in step $j + 1$ is determined exclusively by the previous state $c_j(y)$ of y as well as the states $c_j(x)$ and $c_j(z)$ of the left and right neighbors x and z (respectively) of y in the previous step j (i.e., $x = \text{nndcl}_j(y)$ and $z = \text{nndcr}_j(y)$). In the variables maintained by A , x and $c_j(x)$ correspond to `leftIndex[j]` and `leftState[j]`, respectively, and y and

Algorithm 1: Streaming algorithm A

```
Compute  $t(|w|)$ ;  
Initialize lists leftIndex, centerIndex, leftState, and centerState;  
leftIndex[0]  $\leftarrow -1$ ; leftState[0]  $\leftarrow q$ ;  
centerIndex[0]  $\leftarrow 0$ ; centerState[0]  $\leftarrow w(0)$ ;  
next  $\leftarrow 1$ ;  
 $j_0 \leftarrow 0$ ;  
for  $\tau \leftarrow 0, \dots, t(|w|) - 1$  do  
A   |  $j \leftarrow j_0$ ;  
B   | if next  $< |w|$  then  
C   |   | rightIndex  $\leftarrow$  next; rightState  $\leftarrow w(\text{next})$ ;  
   |   | next  $\leftarrow$  next + 1;  
   |   |  
   |   | else  
D   |   | rightIndex  $\leftarrow |w|$ ; rightState  $\leftarrow q$ ;  
   |   |  $j_0 \leftarrow j_0 + 1$ ;  
   |   | end  
   |   | while  $j \leq \tau$  do  
E   |   |   | newRightIndex  $\leftarrow$  centerIndex[ $j$ ];  
   |   |   |   | newRightState  $\leftarrow \delta(\text{leftState}[j], \text{centerState}[j], \text{rightState})$ ;  
   |   |   |   | leftIndex[ $j$ ]  $\leftarrow$  centerIndex[ $j$ ]; leftState[ $j$ ]  $\leftarrow$  centerState[ $j$ ];  
   |   |   |   | centerIndex[ $j$ ]  $\leftarrow$  rightIndex; centerState[ $j$ ]  $\leftarrow$  rightState;  
   |   |   |   | rightIndex  $\leftarrow$  newRightIndex; rightState  $\leftarrow$  newRightState;  
F   |   |   | if rightState =  $\otimes$  then goto A;  
   |   |   |  $j \leftarrow j + 1$ ;  
   |   | end  
   |   | leftIndex[ $\tau + 1$ ]  $\leftarrow -1$ ; leftState[ $\tau + 1$ ]  $\leftarrow q$ ;  
   |   | centerIndex[ $\tau + 1$ ]  $\leftarrow$  rightIndex; centerState[ $\tau + 1$ ]  $\leftarrow$  rightState;  
G   |   | if centerState[ $\tau + 1$ ] =  $a$  then accept;  
   | end  
reject;
```

$c_j(y)$ to **centerIndex**[j] and **centerState**[j], respectively. z and $c_j(z)$ are not stored in lists but, rather, in the variables **rightIndex** and **rightState** (and are determined dynamically). The cell indices computed (i.e., the contents of the lists **leftIndex** and **centerIndex** and the variables **rightIndex** and **newRightIndex**) are not actually used by A to compute states and are inessential to the algorithm itself; we use them only to simplify the proof of correctness below (and, hence, do not count them towards the space complexity of A).

In each iteration of the **for** loop, A determines $c_{\tau+1}(z_0^\tau)$, where z_0^τ is the leftmost active cell of S in step τ , and stores it **centerState**[$\tau + 1$]. **next** is the index of the next symbol of w to be read (or $|w|$ once every symbol has been read), and j_0 is the minimal time step containing a cell whose state must be known to determine $c_{\tau+1}(z_0^\tau)$ and remains 0 as long as **next** $< |w|$. Hence, the termination of A is guaranteed by the finiteness of w , that is, **next** can only be increased a finite number of times and, once all symbols of w have been read (i.e., the condition in line B no longer holds), by the increment of j_0 in line D.

In each iteration of the **while** loop, the algorithm starts from a local configuration in step j of a cell $y = \text{centerIndex}[j]$ with left neighbor $x = \text{leftIndex}[j] = \text{nndcl}_j(y)$ and right neighbor $z = \text{rightIndex}[j] = \text{nndcl}_j(y)$. It then computes the next state $c_{j+1}(y)$ of y and sets y as the new left cell and z as the new center cell for step j . As long as it is not deleted (i.e., $c_{j+1}(y) \neq \otimes$), y then becomes the right cell for step $j + 1$. In fact, this is the only place (line F) in the algorithm where we need to take into consideration that S is a shrinking (and not just a regular) CA. The strategy we follow here is to continue computing states of cells to the right of the current center cell (i.e., $y = \text{centerIndex}[j]$) until the first cell to its right which has not deleted itself (i.e., $\text{nndcr}_j(y)$) is found. With this non-deleted cell we can then proceed with the computation of the state of $\text{centerIndex}[j + 1]$ in step $j + 1$. Hence, if y has deleted itself, to compute the state of the next cell to its right we must either read the next symbol of w or, if there are no symbols left, use quiescent cell number $|w|$ as right neighbor in step j_0 , computing states up until we are at step j again (hence the **goto** instruction).

Correctness. The following invariants hold for both loops in A :

1. $\text{centerIndex}[\tau] = \min\{z \in \mathbb{N}_0 \mid c_\tau(z) \neq \otimes\}$, that is, $\text{centerIndex}[\tau]$ is the leftmost active cell of S in step j .
2. If $j \leq \tau$, then $\text{rightIndex} = \text{nndcr}_j(\text{centerIndex}[j])$ and $\text{rightState} = c_j(\text{rightIndex})$.
3. For every $j' \in [j_0, \tau]$:
 - $\text{leftIndex}[j'] = \text{nndcl}_{j'}(\text{centerIndex}[j'])$,
 - $\text{leftState}[j'] = c_{j'}(\text{leftIndex}[j'])$; and
 - $\text{centerState}[j'] = c_{j'}(\text{centerIndex}[j'])$.

These can be shown together with the observation that, following the assignment of newRightIndex and newRightState in line E, we have $\text{newRightState} = c_{j+1}(\text{newRightIndex})$ and, in case $\text{newRightState} \neq \otimes$ and $j < \tau$, then also $\text{newRightIndex} = \text{nndcr}_j(\text{centerIndex}[j + 1])$. Using the above, it follows that after the execution of the **while** loop we have $j = \tau + 1$, $\text{rightState} \neq \otimes$, and $\text{rightState} = c_{\tau+1}(\text{rightIndex})$. Since then $\text{rightIndex} = \text{centerIndex}[j - 1] = \text{centerIndex}[\tau]$, we obtain $\text{rightIndex} = \min\{z \in \mathbb{N}_0 \mid c_{\tau+1}(z) \neq \otimes\}$. Hence, as $\text{centerState}[\tau + 1] = \text{rightState} = c_{\tau+1}(\text{rightIndex})$ holds in line G, if A then accepts, so does S accept w in step τ . Conversely, if A rejects, then S does not accept w in any step $\tau \leq t(|w|)$.

Complexity. The space complexity of A is dominated by the lists leftState and centerState , which has $O(t(|w|))$ many entries of $O(1)$ size. As mentioned above, we ignore the space used by the lists leftIndex and centerIndex and the variables rightIndex and newRightIndex since they are inessential (i.e., if we remove them as well as all instructions in which they appear, the algorithm obtained is equivalent to A).

As for the update time, note each list access or arithmetic operation costs $O(\log t(|w|))$ time (since $t(|w|)$ upper bounds all numeric variables). Every execution of the **while** loop body requires then $O(\log t(|w|))$ time and, since, there

are at most $O(t(|w|))$ executions between any two subsequent reads (i.e., line C), this gives us the purported $O(t(|w|) \log t(|w|))$ update time.

Finally, for the reporting time of A , as soon as $i = |w|$ holds after execution of line C (i.e., A has completed reading its input) we have that the **while** loop body is executed at most $\tau - j + 1$ times before line C is reached again. Every time this occurs (depending on whether line C is reached by the **goto** instruction or not), either j_0 or both j_0 and τ are incremented. Hence, since $\tau \leq t(|w|)$, we have an upper bound of $O(t(|w|)^2)$ executions of the **while** loop body, resulting (as above) in an $O(t(|w|)^2 \log t(|w|))$ reporting time in total. \square

5 Hardness Magnification for Sublinear-Time SCAs

Let $K > 0$ be constant such that, for any function $s: \mathbb{N}_+ \rightarrow \mathbb{N}_+$, every circuit of size at most $s(n)$ can be described by a binary string of length at most $\ell(n) = Ks(n) \log s(n)$. In addition, let \perp denote a string (of length at most $\ell(n)$) such that no circuit of size at most $s(n)$ has \perp as its description. Furthermore, let $\text{Merge}[s]$ denote the following search problem (adapted from [17]):

Given: the binary representation of $n \in \mathbb{N}_+$, the respective descriptions (padded to length $\ell(n)$) of circuits C_0 and C_1 such that $|C_i| \leq s(n)$, and $\alpha, \beta, \gamma \in \{0, 1\}^n$ with $\alpha \leq \beta \leq \gamma < 2^n$.

Find: the description of a circuit C with $|C| \leq s(n)$ and such that $\forall x \in [\alpha, \beta - 1] : C(x) = C_0(x)$ and $\forall x \in [\beta, \gamma - 1] : C(x) = C_1(x)$; if no such C exists or $C_i = \perp$ for any i , answer with \perp .

Note that the decision version of $\text{Merge}[s]$, that is, the problem of determining whether a solution to an instance $\text{Merge}[s]$ exists is in Σ_2^P . Moreover, $\text{Merge}[s]$ is Turing-reducible (in polynomial time) to a decision problem very similar to $\text{Merge}[s]$ and which is also in Σ_2^P , namely the decision version of $\text{Merge}[s]$ but with the additional requirement that the description of C admits a given string v of length $|v| \leq s(n)$ as a prefix.³

We now formulate our main theorem concerning SCAs and MCSP:

Theorem 24. *Let $s: \mathbb{N}_+ \rightarrow \mathbb{N}_+$ be constructible in place by a CA in $O(s(n))$ time. Furthermore, let $m = m(n)$ denote the maximum instance length of $\text{Merge}[s]$, and let $f, g: \mathbb{N}_+ \rightarrow \mathbb{N}_+$ with $f(m) \geq g(m) \geq m$ be constructible in place by a CA in $O(f(m))$ time and $O(g(m))$ space. Then, for $b(n) = \lfloor g(m)/2 \rfloor$, if $\text{Merge}[s]$ is computable in place by a CA in at most $f(m)$ time and $g(m)$ space, then the search version of $\text{Block}_b(\text{MCSP}[s])$ is computable by an SCA in $O(n \cdot f(m))$ time, where the instance size of the latter is in $\Theta(2^n \cdot b(n))$.*

We are particularly interested in the repercussions of Theorem 24 *taken in the contrapositive*. Since $\text{P} = \text{NP}$ implies $\text{P} = \Sigma_2^P$, it also implies there is a poly-time Turing machine for $\text{Merge}[s]$; since a CA can simulate a Turing machine with no time loss, for m as above we obtain:

³ This is a fairly common construction in complexity theory for reducing search to decision problems; refer to [10] for the same idea applied in other contexts.

Theorem 3. For a certain $m \in \text{poly}(s(n))$, if $\text{Block}_b(\text{MCSP}[s]) \notin \text{SCA}[n \cdot f(m)]$ for every $f \in \text{poly}(m)$ and $b \in O(f)$, then $\text{P} \neq \text{NP}$.

We now turn to the proof of Theorem 24, which follows [17] closely. First, we generalize blockwise reductions (see Definition 15) to search problems:

Definition 25 (Blockwise reducible (for search problems)). Let L and L' be block languages that correspond to search problems S and S' , respectively. Also, for an instance x , let $S(x)$ (resp., $S'(x)$) denote the set of solutions for x under the problem S (resp., S'). Then L is said to be (k) -blockwise reducible to L' if there is a computable k -blockwise map $g: \mathfrak{B}_b^{km} \rightarrow \mathfrak{B}_b^m$ such that, for every $w \in \mathfrak{B}_b^{km}$, we have $S(w) = S'(g(w))$.

Notice Lemma 16 readily generalizes to blockwise reductions in this sense.

Next, we describe the set of problems that we shall reduce $\text{Block}_b(\text{MCSP}[s])$ to. Let $r: \mathbb{N}_+ \rightarrow \mathbb{N}_+$ be a function. There is a straightforward 1-blockwise reduction from $\text{Block}_b(\text{MCSP}[s])$ to (a suitable block version of) the following search problem $\text{Merge}_r[s]$:

Given: the binary representation of $n \in \mathbb{N}_+$ and the respective descriptions (padded to length $\ell(n)$) of circuits C_1, \dots, C_r , where $|C_i| \leq s(n)$ for every i and $r = r(n)$.

Find: (the description of) a circuit C with $|C| \leq s(n)$ and such that, for every i and every $x \in [(i-1) \cdot 2^n/r, i \cdot 2^n/r - 1]$, $C(x) = C_i(x)$; if no such C exists or $C_i = \perp$ for any i , answer with \perp .

In particular, for the reduction mentioned above, we shall use $r = 2^n$. Evidently, $\text{Merge}_r[s]$ is a generalization of the problem $\text{Merge}[s]$ defined previously and, more importantly, every instance of $\text{Merge}_r[s]$ is simply a concatenation of $r/2$ many $\text{Merge}[s]$ instances where α , β , and γ are given implicitly. Using the assumption that $\text{Merge}[s]$ is computable by a CA in at most $f(m)$ time and $g(m)$ space, we can solve each such instance in parallel, thus producing an instance of $\text{Merge}_{r/2}[s]$ (i.e., halving r). This yields a 2-blockwise reduction from (the respective block versions of) $\text{Merge}_r[s]$ to $\text{Merge}_{r/2}[s]$ (cnf. the proof of Proposition 21). Using Lemma 16 and that $\text{Merge}_1[s]$ is trivial, we obtain the purported SCA for $\text{Block}_b(\text{MCSP}[s])$.

Proof. Let n be fixed, and let $r = 2^n$. First, we describe the 1-blockwise reduction from $\text{Block}_b(\text{MCSP}[s])$ to a block version of $\text{Merge}_r[s]$ (which we shall describe along with the reduction). Let T_a denote the (description of the) trivial circuit that is constant $a \in \{0, 1\}$, that is, $T_a(x) = a$ for every $x \in \{0, 1\}^n$. Then we map each block $\binom{\text{bin}_n(x)}{y 0^{b(n)-1}}$ with $y \in \{0, 1\}$ to the block $\binom{\text{bin}_n(x)}{T_y \pi}$, where $\pi \in \{0\}^*$ is a padding string so that the block length $b(n)$ is preserved. (This is needed to ensure enough space is available for the construction; see the details further below.) It is evident this can be done in time $O(b(n))$ and (since we just translate the truth-table 0 and 1 entries to the respective trivial circuits) that the reduction is correct, that is, that every solution to the original $\text{Block}_b(\text{MCSP}[s])$ instance must also be a solution of the produced instance of (the resulting block version of) $\text{Merge}_r[s]$ and vice-versa.

Next, maintaining the block representation described above, we construct the 2-blockwise reduction from the respective block versions of $\text{Merge}_\rho[s]$ to $\text{Merge}_{\rho/2}[s]$, where $\rho = 2^k$ for some $k \in [1, n]$. Let A denote the CA that, by assumption, computes a solution to an instance of $\text{Merge}[s]$ in place in at most $f(m)$ time and $g(m)$ space. Then, for $j \in [0, \rho/2 - 1]$, we map each pair $\binom{\text{bin}_n(2j)}{C_0 \pi_0} \# \binom{\text{bin}_n(2j+1)}{C_1 \pi_1}$ of blocks (where $\pi_0, \pi_1 \in \{0\}^*$ again are padding strings) to $\binom{\text{bin}_n(j)}{C \pi}$, where $\pi \in \{0\}^*$ is a padding string (as above) and C is the circuit produced by A for $\alpha = 2j \cdot 2^n / \rho$, $\beta = (2j + 1) \cdot 2^n / \rho$, and $\gamma = (2j + 2) \cdot 2^n / \rho$.

To actually execute A , we need $g(m)$ space (which is guaranteed by the block length $b(n)$) and, in addition, to prepare the input so it is in the format expected by A (i.e., eliminating the padding between the two circuit descriptions and writing the representations of α , β , and γ), which can be performed in $O(b(n)) \subseteq O(g(m)) \subseteq O(f(m))$ time. For the correctness, suppose the above reduces an instance of $\text{Merge}_\rho[s]$ with circuits C_1, \dots, C_ρ to an instance of $\text{Merge}_{\rho/2}[s]$ with circuits $D_1, \dots, D_{\rho/2}$ (and no \perp was produced). Then, a circuit E is a solution to the latter if and only if $E(x) = D_i(x)$ for every i and $x \in [(i - 1) \cdot 2^n / (\rho/2), i \cdot 2^n / (\rho/2) - 1]$. Using the definition of $\text{Merge}[s]$, every D_i must satisfy $D_i(x) = C_{2i-1}(x)$ and $D_i(y) = C_{2i}(y)$ for $x \in [(2i - 2) \cdot 2^n / \rho, (2i - 1) \cdot 2^n / \rho - 1]$ and $y \in [(2i - 1) \cdot 2^n / \rho, 2i \cdot 2^n / \rho - 1]$. Hence, E agrees with C_1, \dots, C_ρ if and only if it agrees with $D_1, \dots, D_{\rho/2}$ (on the respective intervals).

Since $s(n) \geq n$ and $\text{Merge}_1[s]$ is trivial (i.e., it can be accepted in $O(b(n))$ time), applying the generalization of Lemma 16 to blockwise reductions for search problems completes the proof. \square

Comparison with [17]. We conclude this section with a comparison of our result and proof with [17]. The most evident difference between the statements of Theorems 3 and 24 and the related result from [17] (i.e., Theorem 2) is that our results concern CAs (instead of Turing machines) and relate more explicitly to the time and space complexities of $\text{Merge}[s]$; in particular, the choice of the block length is tightly related with the space complexity of computing $\text{Merge}[s]$. As for the proof, notice that we only merge two circuits at a time, which makes for a smaller instance size m (of $\text{Merge}[s]$); this not only simplifies the proof but also minimizes the resulting time complexity of the SCA (as $f(m)$ is then smaller). Also, in our case, we make no additional assumptions regarding the first reduction from $\text{Block}_b(\text{MCSP}[s])$ to $\text{Merge}_r[s]$; in fact, this step can be performed unconditionally. Finally, we note that our proof renders all blockwise reductions explicit and the connection to the self-reductions of [1] more evident. Despite these simplifications, the argument extends to generalizations of MCSP with similar structure and instance size (e.g., MCSP in the setting of Boolean circuits with oracle gates as in [17] or MCSP for multi-output functions as in [12]).

6 Concluding Remarks

Proving SCA Lower Bounds for MCSP[s]. Recalling the language L_1 from the proof of Theorem 5, consider the intersection $L_1[s] = L_1 \cap \text{MCSP}[s]$. Evidently,

$L_1[s]$ is comparable in hardness to $\text{MCSP}[s]$ (e.g., it is solvable in polynomial time using a single adaptive query to $\text{MCSP}[s]$). By adapting the construction from the proof of Theorem 24 so the SCA additionally checks the L_1 property at the end in $\text{poly}(s(n))$ time (e.g., using the circuit C produced to check whether $C(x) = 1$ for $x = C(0) \cdots C(n-1)$), we can derive a hardness magnification result for $L_1[s]$ too: If $\text{Block}_b(L_1[s]) \notin \text{SCA}[\text{poly}(s(n))]$ (for every $b \in \text{poly}(s(n))$), then $\text{P} \neq \text{NP}$. Using the methods from Section 3.3 and that there are $2^{\Omega(s(n))}$ many (unique) circuits of size $s(n)$ or less⁴, this means that, if $\text{Block}_b(L_1[s]) \in \text{SCA}[t(n)]$ for some $b \in \text{poly}(n)$ and $t: \mathbb{N}_+ \rightarrow \mathbb{N}_+$, then $t \in \Omega(s(n))$. Hence, for an eventual proof of $\text{P} \neq \text{NP}$ based on Theorem 3, one would need to develop new techniques (see also the discussion below) to raise this bound at the very least beyond $\text{poly}(s(n))$.

Seen from another angle, this demonstrates that, although we can prove a tight SCA worst-case lower bound for L_1 (Theorem 5), establishing similar lower bounds on instances of L_1 with low circuit complexity (i.e., instances which are also in $\text{MCSP}[s]$) is at least as hard as showing $\text{P} \neq \text{NP}$. In other words, it is straightforward to establish a lower bound for L_1 using arbitrary instances, but it is absolutely non-trivial to establish similar lower bounds for *easy* instances of L_1 where instance hardness is measured in terms of circuit complexity.

The Proof of Theorem 24 and the Locality Barrier. In a recent paper [5], Chen et al. propose the concept of a *locality barrier* to explain why current lower bound proof techniques (for a variety of non-uniform computational models) do not suffice to show the lower bounds needed for separating complexity classes in conjunction with hardness magnification (i.e., in our case above a $\text{poly}(s(n))$ lower bound that proves $\text{P} \neq \text{NP}$). In a nutshell, the barrier arises from proof techniques relativizing with respect to *local aspects* of the computational model at hand (in [5], concretely speaking, oracle gates of small fan-in), whereas it is known that a proof of $\text{P} \neq \text{NP}$ must not relativize [3].

The proof of Theorem 24 confirms the presence of such a barrier also in the uniform setting and concerning the separation of P from NP . Indeed, the proof mostly concerns the construction of an SCA where the overall computational paradigm of blockwise reductions (using Lemma 16) is unconditionally compatible with the SCA model (as exemplified in Proposition 21); the $\text{P} = \text{NP}$ assumption is needed exclusively so that the local algorithm for $\text{Merge}[s]$ in the statement of the theorem exists. Hence, the result also holds *unconditionally* for SCAs that are, say, augmented with oracle access (in a plausible manner, e.g., by using an additional oracle query track and special oracle query states) to $\text{Merge}[s]$. (Incidentally, the same argument also applies to the proof of the hardness magnification result for streaming algorithms (i.e., Theorem 2) in [17], which also builds on the existence of a similar locally computable function.) In particular, this means the lower

⁴ Let $K > 0$ be constant such that every Boolean function on m variables admits a circuit of size at most $K \cdot 2^m/m$. Setting $m = \lfloor \log s(n) \rfloor$, notice that, for sufficiently large n (and $s(n) \in \omega(1) \cap O(2^n/n)$), this gives us $s(n) \geq K \cdot 2^m/m$, thus implying that every Boolean function on $m \leq n$ variables admits a circuit of size at most $s(n)$. Since there are 2^{2^m} many such (unique) functions, it follows there are $2^{\Omega(s(n))}$ (unique) circuits of size at most $s(n)$.

bound techniques from the proof of Theorem 5 do not suffice since they extend to SCAs having oracle access to any computable function.

Open Questions. We conclude with a few open questions:

- By weakening SCAs in some aspect, certainly we can establish an unconditional MCSP lower bound for the weakened model which, were it to hold for SCAs, would imply the separation $P \neq NP$ (using Theorem 3). *What forms of weakening* (conceptually speaking) are needed for these lower bounds? How are these related to the locality barrier discussed above?
- Secondly, we saw SCAs are strictly more limited than streaming algorithms. Proceeding further in this direction, can we identify *further (natural) models of computation that are more restricted than SCAs* (whether CA-based or not) and for which we can prove results similar to Theorem 24?
- Finally, besides MCSP, what other (natural) problems admit similar SCA hardness magnification results? More importantly, can we identify some *essential property* of these problems that would explain these results? For instance, in the case of MCSP there appears to be some connection to the length of (minimal) witnesses being much smaller than the instance length. Indeed, one sufficient condition in this sense (disregarding SCAs) is sparsity [6]; nevertheless, it seems rather implausible that this would be the sole property responsible for all hardness magnification phenomena.

Acknowledgments. I would like to thank Thomas Worsch for the helpful discussions and feedback.

References

- [1] Eric Allender and Michal Koucký. “Amplifying lower bounds by means of self-reducibility.” In: *J. ACM* 57.3 (2010), 14:1–14:36.
- [2] Sanjeev Arora and Boaz Barak. *Computational Complexity: A Modern Approach*. Cambridge: Cambridge University Press, 2009.
- [3] Theodore P. Baker, John Gill, and Robert Solovay. “Relativizations of the $P = ? NP$ Question.” In: *SIAM J. Comput.* 4.4 (1975), pp. 431–442.
- [4] Bernard Chazelle and Louis Monier. “A Model of Computation for VLSI with Related Complexity Results.” In: *J. ACM* 32.3 (1985), pp. 573–588.
- [5] Lijie Chen, Shuichi Hirahara, Igor Carboni Oliveira, Ján Pich, Ninad Rajgopal, and Rahul Santhanam. “Beyond Natural Proofs: Hardness Magnification and Locality.” In: *11th Innovations in Theoretical Computer Science Conference, ITCS 2020, January 12–14, 2020, Seattle, Washington, USA*. Ed. by Thomas Vidick. Vol. 151. LIPIcs. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020, 70:1–70:48.
- [6] Lijie Chen, Ce Jin, and R. Ryan Williams. “Hardness Magnification for all Sparse NP Languages.” In: *60th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2019, Baltimore, Maryland, USA, November 9–12, 2019*. Ed. by David Zuckerman. IEEE Computer Society, 2019, pp. 1240–1255.

- [7] Mahdi Cheraghchi, Shuichi Hirahara, Dimitrios Myrisiotis, and Yuichi Yoshida. “One-Tape Turing Machine and Branching Program Lower Bounds for MCSP.” In: *Electronic Colloquium on Computational Complexity (ECCC)* 103 (2020).
- [8] M. Delorme and J. Mazoyer, eds. *Cellular Automata. A Parallel Model*. Mathematics and Its Applications 460. Dordrecht: Springer Netherlands, 1999.
- [9] Christoph Dürr, Ivan Rapaport, and Guillaume Theyssier. “Cellular automata and communication complexity.” In: *Theor. Comput. Sci.* 322.2 (2004), pp. 355–368.
- [10] Oded Goldreich. *Computational Complexity: A Conceptual Perspective*. Cambridge: Cambridge University Press, 2008.
- [11] Oscar H. Ibarra, Michael A. Palis, and Sam M. Kim. “Fast Parallel Language Recognition by Cellular Automata.” In: *Theor. Comput. Sci.* 41 (1985), pp. 231–246.
- [12] Rahul Ilango, Bruno Loff, and Igor Carboni Oliveira. “NP-Hardness of Circuit Minimization for Multi-Output Functions.” In: *35th Computational Complexity Conference, CCC 2020, July 28-31, 2020, Saarbrücken, Germany (Virtual Conference)*. Ed. by Shubhangi Saraf. Vol. 169. LIPIcs. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020, 22:1–22:36.
- [13] Valentine Kabanets and Jin-yi Cai. “Circuit minimization problem.” In: *Proceedings of the Thirty-Second Annual ACM Symposium on Theory of Computing, May 21-23, 2000, Portland, OR, USA*. Ed. by F. Frances Yao and Eugene M. Luks. ACM, 2000, pp. 73–79.
- [14] Kojiro Kobayashi. “On the structure of one-tape nondeterministic turing machine time hierarchy.” In: *Theor. Comput. Sci.* 40 (1985), pp. 175–193.
- [15] Martin Kutrib. “Complexity of One-Way Cellular Automata.” In: *Cellular Automata and Discrete Complex Systems - 20th International Workshop, AUTOMATA 2014, Himeji, Japan, July 7-9, 2014, Revised Selected Papers*. Ed. by Teijiro Isokawa, Katsunobu Imai, Nobuyuki Matsui, Ferdinand Peper, and Hiroshi Umeo. Vol. 8996. Lecture Notes in Computer Science. Springer, 2014, pp. 3–18.
- [16] Martin Kutrib, Andreas Malcher, and Matthias Wendlandt. “Shrinking One-Way Cellular Automata.” In: *Cellular Automata and Discrete Complex Systems - 21st IFIP WG 1.5 International Workshop, AUTOMATA 2015, Turku, Finland, June 8-10, 2015. Proceedings*. 2015, pp. 141–154.
- [17] Dylan M. McKay, Cody D. Murray, and R. Ryan Williams. “Weak lower bounds on resource-bounded compression imply strong separations of complexity classes.” In: *Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing, STOC 2019, Phoenix, AZ, USA, June 23-26, 2019*. 2019, pp. 1215–1225.
- [18] Augusto Modanese. “Complexity-Theoretic Aspects of Expanding Cellular Automata.” In: *Cellular Automata and Discrete Complex Systems - 25th IFIP WG 1.5 International Workshop, AUTOMATA 2019, Guadalajara, Mexico, June 26-28, 2019, Proceedings*. 2019, pp. 20–34.

- [19] Augusto Modanese. “Sublinear-Time Language Recognition and Decision by One-Dimensional Cellular Automata.” In: *Developments in Language Theory - 24th International Conference, DLT 2020, Tampa, FL, USA, May 11-15, 2020, Proceedings*. Ed. by Natasa Jonoska and Dmytro Savchuk. Vol. 12086. Lecture Notes in Computer Science. Springer, 2020, pp. 251–265.
- [20] Augusto Modanese and Thomas Worsch. “Shrinking and Expanding Cellular Automata.” In: *Cellular Automata and Discrete Complex Systems - 22nd IFIP WG 1.5 International Workshop, AUTOMATA 2016, Zurich, Switzerland, June 15-17, 2016, Proceedings*. 2016, pp. 159–169.
- [21] Cody D. Murray and R. Ryan Williams. “On the (Non) NP-Hardness of Computing Circuit Complexity.” In: *Theory of Computing* 13.1 (2017), pp. 1–22.
- [22] Igor Carboni Oliveira, Ján Pich, and Rahul Santhanam. “Hardness Magnification near State-Of-The-Art Lower Bounds.” In: *34th Computational Complexity Conference, CCC 2019, July 18-20, 2019, New Brunswick, NJ, USA*. Ed. by Amir Shpilka. Vol. 137. LIPIcs. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2019, 27:1–27:29.
- [23] Igor Carboni Oliveira and Rahul Santhanam. “Hardness Magnification for Natural Problems.” In: *59th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2018, Paris, France, October 7-9, 2018*. 2018, pp. 65–76.
- [24] Victor Poupet. “A Padding Technique on Cellular Automata to Transfer Inclusions of Complexity Classes.” In: *Computer Science - Theory and Applications, Second International Symposium on Computer Science in Russia, CSR 2007, Ekaterinburg, Russia, September 3-7, 2007, Proceedings*. 2007, pp. 337–348.
- [25] Azriel Rosenfeld, Angela Y. Wu, and Tsvi Dubitzki. “Fast language acceptance by shrinking cellular automata.” In: *Inf. Sci.* 30.1 (1983), pp. 47–53.
- [26] Rudolph Sommerhalder and S. Christian van Westrhenen. “Parallel Language Recognition in Constant Time by Cellular Automata.” In: *Acta Inf.* 19 (1983), pp. 397–407.
- [27] Michael Stratmann and Thomas Worsch. “Leader election in d -dimensional CA in time $\text{diam} \log(\text{diam})$.” In: *Future Gener. Comput. Syst.* 18.7 (2002), pp. 939–950.
- [28] C. D. Thompson. “A Complexity Theory for VLSI.” PhD thesis. Department of Computer Science, Carnegie-Mellon University, Aug. 1980.
- [29] Roland Vollmar. “On two modified problems of synchronization in cellular automata.” In: *Acta Cybern.* 3.4 (1977), pp. 293–300.
- [30] Andrew Chi-Chih Yao. “Circuits and Local Computation.” In: *Proceedings of the 21st Annual ACM Symposium on Theory of Computing, May 14-17, 1989, Seattle, Washington, USA*. Ed. by David S. Johnson. ACM, 1989, pp. 186–196.