Computer Science and the fine structure of Borel sets

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Abstract

I) Wadge defined a natural refinment of the Borel hierarchy, now called the Wadge hierarchy WH. The fundamental properties of WH follow from results of Kuratowski,Martin, Wadge and Louveau. We introduce a game theoretic variant of WH: the **conciliatory** hierarchy CH, which is defined over Borel subsets of $\Lambda^{\leq\omega}$ for **any** well ordered set Λ. We show that

(a) the fundamental properties of WH continue to holds in CH; but unlike WH, CH is in addition generated by canonical operations which make its structure transparent,

(b) for countable sets Λ, CH is isomorphic to WH.

Combining (b) with case $\Lambda = \omega$ of (a), gives a new proof of the mentioned results on WH. And the truth of (a) for uncountable Λ is surprising, because countability is crucial in the original proofs of Kuratowski, Wadge and Louveau. It shows that the methods we use have something new; and the extension to the uncountable raises intriguing problems.
II) The works of Büchi, Landweber, Wagner on Büchi automata (accepting words of length ω) proved to be a fruitful theoretical background for the design and study of processors in interaction with their environment. Büchi automata can also accept words of **trans**finite length ω^n, for any $n < \omega$; in this way they accept Borel sets of rank up to $2n + 1$. We conjecture that the whole work of Büchi, Landweber, Wagner has an extension able to manage these sets of transfinite words - still keeping an effective notion of processor working in ω steps. This should be useful in Computer Science, because of the latter feature. We obtain partial results in this direction; they rest on an effective version of the above work on CH. Thus the extreme finesse of WH, CH compared to the usual Borel hierarchy may be useful, in addition to being challenging...

1 Recalls: The Wadge hierarchy WH

All spaces Λ^ω are equipped with the product topology:

$$\text{basic open sets } V_s = \{x \in \Lambda^\omega : \ x \restriction n = s\}, \ n < \omega, \ s \in \Lambda^n.$$

Definition 1 *For $A, B \subseteq \omega^\omega$, $A \leq_W B$ iff A has a continuous reduction to A: there is a continuous map $f : \omega^\omega \longmapsto \omega^\omega$ such that "$x \in A$" reduces to "$f(x) \in B$". $A <_W B$ iff $A \leq_W B$ but not $B \leq_W A$; $A \equiv_W B$ iff $A \leq_W B \leq_W A$.*

WH is the class of Borel subsets of ω^ω, equipped with $\leq_W$ and $\equiv_W$. WH is a natural refinment of the Borel hierarchy BH; the latter classifies Borel sets A by their **rank** $r(A)$ ($r(A)$ is the ordinal counting the minimum number of nested countable unions or intersections needed to obtain A from open and closed sets). In fact the relation $A \leq_W B$ is not only finer, but also more natural that the relation $r(A) \leq r(B)$; so WH is more natural than BH! However the Wadge hierarchy is terribly refined compared to BH: see thm.2 and Rem.1(a) below. This has put severe limitations on the use of WH in mathematical practice; which are also limitations to the audience of the beautiful work on WH done by Martin, Wadge, Louveau, Saint Raymond and others (see the bibliography). We tried to add to their work a postscriptum - sketched below in §§2 to 4, and fully exposed in [D a], [D b]. Let us hope that it will help in making the subject more accessible.

We end §1 by recalling the fundamental results about WH. The complement $\omega^\omega \setminus A$ of a set $A \in$ WH is denoted ^-A.

Theorem 1 (Wadge,Martin) *Up to the complement and $\equiv_W$, WH is a well ordered hierarchy: there is an ordinal $|WH|$ (called the **length** of the hierarchy) and a map d_W° from WH onto $|WH|$, such that for all $A, B \in WH$*

$$d_W^\circ A < d_W^\circ B \longleftrightarrow A <_W B$$

$$d_W^\circ A = d_W^\circ B \longleftrightarrow A \equiv_W B \quad or \quad A \equiv_W {}^-B$$

After thm.1, a natural question is: determine the ordinal |WH|. The answer to this question is Wadge's main theorem:

Theorem 2 *$|WH|$ is $V^{\omega_1}(0)$, the **Veblen ordinal** of base ω_1.*

Definition 2 *Given a cardinal κ, the Veblen hierarchy of base κ consists of the functions $(V^\xi)_{\xi<\kappa}$, $V^\xi : On \longmapsto On$ defined by:*

- *V^0 is the ordinal addition*
- *for $\nu > 0$ V^ν is the function that increasingly enumerates all ordinals of cofinality κ that are closed under each function V^μ, $\mu < \nu$.*

Remarks 1 (a) The function V^1 is exponentiation of base $\kappa : \alpha \longmapsto \kappa^\alpha$, **except** that for each limit ordinal α of cofinality $\leq \kappa$, there is a shift to avoid κ^α. It is clear that the functions V^ν $(\nu > 1)$ are fast growing; so that $|WH|$ is rather large...

(b) Veblen provided for each ordinal $\alpha < V^\kappa(0)$ a "Veblen normal form" of α, which is a finite term with parameters $< \kappa$, with value α and chosen in a canonical way. In particular, the V.N.F. of α is defined uniquely without use of axiom of choice. Using this normal form of α, one can choose a **unique** decomposition of α into smaller ordinals which takes the form

$$\alpha = \beta + \gamma \quad \text{or} \quad \alpha = V^\beta(\gamma) \quad \text{or} \quad \alpha = \sup_{i<\lambda} \alpha_i$$

with β, $\gamma < \alpha$ **or** λ ordinal limit $< \kappa$ and $\alpha_i < \alpha$ $\forall i$. (The best way to prove it is indeed to use the **functorial** construction of the Veblen hierarchy due to Girard and Vauzeilles [G V 84]).

2 The conciliatory hierarchy

Definition 3

*(a) We call **Sets** (with **capital** S) all sets A of finite or infinite strings: $A \subseteq \Lambda_A^{\leq\omega}$, where Λ_A is any set - called the **alphabet** of A. We fix an infinite regular cardinal κ and restrict Sets to alphabets of cardinal $\leq \kappa$.*

(b) *Given Sets A and B, $\boldsymbol{C}(A,B)$ denotes the following infinite game between two players, I and II.*

(i) *I chooses $x \in \Lambda_A^{\leq\omega}$ and II chooses $y \in \Lambda_B^{\leq\omega}$: at move $p < \omega$, I chooses $x_p \in \Lambda_A$, or chooses to* ***skip****, in which case x_p is the empty sequence $<>$. And II replies with $y_p \in \Lambda_B \cup \{<>\}$; after ω moves, the* ***play*** *is (x,y), where $x = x_0{}^\frown x_1{}^\frown \ldots$ and $y = y_0{}^\frown y_1{}^\frown \ldots$.*
(This is the usual notion of infinite game ***except*** *that* ***both*** *players may skip as much as they want to, so that the plays x, y may be finite).*

(ii) *II wins play (x,y) iff $(x \in A \longleftrightarrow y \in B)$.*

(c) *We set:*

$$\begin{array}{lcl} A \leq_c B & \Longleftrightarrow & \textit{II has a winning strategy (w.s.) in } \boldsymbol{C}(A,B) \\ A <_c B & \Longleftrightarrow & A \leq_c B \textit{ but not conversely} \\ A \equiv_c B & \Longleftrightarrow & A \leq_c B \leq_c A. \end{array}$$

The conciliatory hierarchy CH is the class of all sets $A \subseteq \Lambda_A^{\leq\omega}$ such that $A \cap \Lambda_A^{\omega}$ is Borel, equipped with $\leq_c$ and $\equiv_c$.

Nota Bene 1

(a) If the reader discovered the Wadge hierarchy in the above §1, he may not see the relation of CH with WH. This is because in §1 we skipped **Wadge's lemma**:
For $A, B \in \Lambda^\omega$ $A \leq_W B$ iff II has a winning strategy in the ***Wadge game*** *$\boldsymbol{W}(A,B)$ which is defined like $\boldsymbol{C}(A,B)$ except that*

- *I has no right to skip*
- *II has this right, but only provided he ensures that his final play is* ***infinite****.*

(b) Our definition of $\mathbf{C}(A,B)$ of course derives from $\mathbf{W}(A,B)$: it is $\mathbf{W}(A,B)$ made symmetric w.r. to the players, so that **it has no importance** which player starts the first move.

(c) The **complement** of a Set A, denoted ^{-}A, is $\Lambda_A^{\leq\omega} \setminus A$. Using (b) it is easy to see that $A \equiv_c {}^{-}A$, **never** holds; so the hierarchy CH has **no** "self dual" Sets.

Theorem A

(a) *There is an ordinal $|CH|$ and a map d_c° from CH onto $|CH|$ such that*

$$d_c^\circ A < d_c^\circ B \longleftrightarrow A <_c B$$

$$d_c^\circ A = d_c^\circ B \longleftrightarrow A \equiv_c B \quad \textit{or} \quad A \equiv_c {}^{-}B$$

(b) *$|CH|$ is the ordinal $\sup_{\nu<\omega_1} V^\nu(0)$.*

(In particular, $|CH| = |WH|$ when $\kappa = \omega_1$; and case $\kappa = \omega_1$ of thm.A is easily deduced from Wadge's main theorem. However, Wadge's unpublished proof is some 300 pages long. So the direct proof given in [D a] is perhaps preferable - it takes some 30 pages and its ideas are explained below). Note that in case $\kappa = \omega_1$ $|CH| = \sup_{\nu<\omega_1} V^\nu(0) = V^{\omega_1}(0)$, the Veblen ordinal . But this fails as soon as $\kappa > \omega_1$: $\sup_{\nu<\omega_1}$ is of cofinality $\omega_1 < \kappa$ so $|CH| = \sup_{\nu<\omega_1} V^\nu(0) < V^{\omega_1}(0)$, which is itself much smaller than $V^\kappa(0)$, the Veblen ordinal of base κ.
We outline the proof of thm.A.

1) The ordinals $\alpha < \kappa$ are by Rem.1(b) obtained in a unique way from 0 by use of the operations: $+$ ($=V^0$), V^ν (for $0 < \nu < \omega_1$) and $\sup_{i<\lambda}$ (for λ limit$< \kappa$). We shall define **Borel counterparts** $+_c$, V_c^ν, sup_c of these operations: namely simple and natural Borel functions defined on all Sets, and such that

   (*) the structure $(\mathrm{CH}, <_c, +_c, sup_c, V_c^\nu)/\equiv_c$ is isomorphic to $(|\mathrm{CH}|, <, +, \sup, V^\nu)$

   Since $|\mathrm{CH}|$ is an ordinal the isomorphism is unique, it is the map d_c° proving thm.A.

2) We first define $V_c^1(A)$ for all sets A. Consider a game $\mathbf{C}(A,B)$ or $\mathbf{C}(B,A)$ and let J denote the player in charge of A. Imagine a "priviledged" player J^1 in charge of A too, but allowed to **cheat** his ennemy: at each move $p < \omega$, J^1 can erase any final segment $y_i{}^\frown \ldots {}^\frown y_{p-1}$ of his previous play, so that y_p becomes the new y_i. After ω moves of this kind the final play y of J^1 is the concatenation $y_0{}^\frown y_1 \ldots {}^\frown y_n{}^\frown$ of all moves y_i which he never erased **after** some stage $p_i < \omega$.

   Lemma 1 *: there is a Set $V_c^1(A)$ such that (for any B):*

$$I \textit{ wins } \boldsymbol{C}(V_c^1(A), B) \quad \textit{iff} \quad I^1 \textit{ wins } \boldsymbol{C}(A,B)$$
$$I\!I \textit{ wins } \boldsymbol{C}(B, V_c^1(A)) \quad \textit{iff} \quad I\!I^1 \textit{ wins } \boldsymbol{C}(B,A).$$

   Moreover we can choose $V_c^1(A)$ so that the map : $A \longmapsto V_c^1(A)$ on all Sets is Borel.

   □ **lem. 1** - Set $\Lambda = \Lambda_A \cup \{\leftarrow\}$, where $\{\leftarrow\}$ is a new letter called "eraser"; for $x \in \Lambda^{\leq\omega}$ let $x^{\leftarrow}$ denote the string x, once every "$\leftarrow$" occurring in x has been "evaluated" to the **back Space** operation (the one familiar to your computer!) - proceeding from left to right inside x. Set

$$V_c^1(A) = \{x : x^{\leftarrow} \in A\}$$

   1□

Remarks 2

(a) Replacing J by J^1 is a quite natural and **unique** operation on games $\mathbf{C}(A,B)$ or $\mathbf{C}(B,A)$; on the other hand, Set $V_c^1(A)$ is by far not the unique way to encode this operation. But this is not important since the only property of the Set $V_c^1(A)$ used in proofs is its correspondance with J^1 stated by the lemma. So since by (*) and Rmk.1(a) the V_c^1 operation corresponds to the ordinal exponentiation, it appears that V_c^1 is a kind of **canonical exponentiation of Sets...**

(b) The $+_c$, sup_c operations are defined by a similar use of "priviledged" players J^+, J^{sup}:

   ⋄ assume $\Lambda_A \subseteq \Lambda_B$ and $|\Lambda_B \setminus \Lambda_A| \geq 2$ with $\{\Lambda_+, \Lambda_-\}$ a partition of $\Lambda_B \setminus \Lambda_A$ in two non empty sets, let $A \subseteq \Lambda_A^{\leq\omega}$, $B \subseteq \Lambda_B^{\leq\omega}$,

$$B{+}A = A \cup \{u^\frown < a > {}^\frown \beta : \ u \in \Lambda_A^{<\omega},\ (a \in \Lambda_+ \text{ and } \beta \in B) \textit{ or } (a \in \Lambda_- \text{ and } \beta \notin B)\}$$

   A player J in charge of $B + A$ is like a player J^+ that starts the game being in charge of A but which at any moment of the run, may restart, being in charge this time of B or of ${}^-B$. He can choose between B and ${}^-B$ only once, for then, until the game is over, he remains in charge of the selected set. This choice is made by J^+ with the first letter not in Λ_A he plays, depending on this letter is in Λ_+ or in Λ_-.

$\diamond$ sup_c is a generalization of $+_c$: let $I \subseteq \Lambda_A$, $\forall i \in I$ $A_i \subseteq \Lambda_A^{\leq\omega}$,

$$sup_c(A_i)_{i\in I} = \{< i > {}^\frown \alpha : \ i \in I, \alpha \in A_i\}$$

a player J in charge of $sup_c(A_i)_{i\in I}$ is like a player J^{sup} allowed to choose different sets $A - i$ the one he wants to be in charge of (as sup_c is defined on conciliatory sets, J^{sup} is also allowed to make no choice at all, by skipping every move).

(c) The proof of (*) for V_c^1 essentially defines an **inverse** operation V_c^{-1}: so that $V_c^{-1}(V_c^1(A)) \equiv_c A$; thus V_c^{-1} is a kind of **logarithm** on Sets! But this time, V_c^{-1} is not a canonical operation: it needs the axiom of choice to be made functional. However (*) does not depend on Choice, because the functional character of V_c^{-1} is not needed in its proof.

(d) The operation V_c^ν $(1 < \nu < \omega_1)$ is defined by iterating ν times the above operation V_c^1 - in the way that corresponds to the iterations needed to obtain V^ν from V^1.

3) Once $+_c$, V_c^ν, sup_c are defined (and before (*) is proved) we can use the unique decomposition of the ordinals $\alpha < |\mathrm{CH}|$ given by Rem.1(b) to define by induction on α a Set $\Omega(\alpha)$: we only have to set

$$\Omega(0) = \emptyset, \ \ \Omega(\beta+\gamma) = \Omega(\beta) +_c \Omega(\gamma), \ \ \Omega(V^\nu(\beta)) = V_c^\nu(\Omega(\beta)), \ \ \Omega\left(\sup_{i<\lambda} \alpha_i\right) = sup_c(\Omega(\alpha_i))_{i<\lambda}$$

Nota Bene 2 We asserted that $+_c$ and V_c^1 are isomorphic via d_c° to the functions $+$ and V^1 on ordinals; for instance $d_c^\circ(A +_c B) = d_c^\circ A + d_c^\circ B$ for all sets A, B. This is cheating a little because there are two initial exceptions which force us to shift a bit the inductive definition of $\Omega(\alpha)$ in order to satisfy (*):

(a) $d_c^\circ(\emptyset) = 0$ but it is easy to see that $d_c^\circ(\emptyset +_c \emptyset) = 1$, (while 0+0=0!)

(b) $V_c^1(\emptyset) = \emptyset$, so $d_c^\circ(V_c^1(\emptyset)) = 0$, (while $V^1(0) = \kappa$).

But making this shift is a simple technicality, and the essence of the proof of thm.A remains as outlined below.

Since $+_c$, V_c^ν are Borel, and so is sup_c when restricted to sequences of sets in $\mathbf{\Sigma}_\xi^\circ$ for some $\xi < \omega_1$, induction on α shows that each Set $\Omega(\alpha)$ is Borel. And the definition of these operations allows to prove

Lemma 2

(a) $\alpha < \beta \iff \Omega(\alpha) <_c \Omega(\beta)$

(b) $|CH| = \{A : \ A \leq_c V_c^\nu(\emptyset +_c \emptyset) \ \textit{for some} \ \nu < \omega_1\}$

$\square$ **lem. 2** - The proof of (a) is easy: when $\alpha < \beta$ the definition of $\Omega(\alpha)$ and $\Omega(\beta)$ gives a clear advantage to the player J in charge of $\Omega(\beta)$, in both games $\mathbf{C}(\Omega(\alpha), \Omega(\beta))$ and $\mathbf{C}(\Omega(\beta), \Omega(\alpha))$. So that if J is *II*, he easily finds a w.s. in $\mathbf{C}(\Omega(\alpha), \Omega(\beta))$, showing $\Omega(\alpha) \leq_c \Omega(\beta)$. And if J is *I*, he has the same w.s. in $\mathbf{C}(\Omega(\beta), \Omega(\alpha))$, showing $\Omega(\alpha) <_c \Omega(\beta)$.
Proof of lem.(b) is easy too. 2$\square$

4) Next comes a more difficult "weak separation lemma":

Lemma 3 *for every Set A,*

$$A \leq_c \Omega(\alpha) \textit{ and } A \leq_c {}^-\Omega(\alpha) \Longrightarrow A \leq_c \Omega(\beta) \textit{ or } A \leq_c {}^-\Omega(\beta), \textit{ for some } \beta < \alpha$$

Then given $A \in$ CH, let α be the least ordinal such that $A \leq_c \Omega(\alpha)$ **or** $A \leq_c {}^-\Omega(\alpha)$ - say $A \leq_c \Omega(\alpha)$ (α exists by lem.2(b)). Then there remains to prove $A \leq_c \Omega(\alpha)$; to that end let us play $\mathbf{C}(\Omega(\alpha), A)$. If I had a w.s. τ, then τ would provide a w.s. for II in $\mathbf{C}(A, {}^-\Omega(\alpha))$ (this easily follows from the definition of the game). Then $A \leq_c {}^-\Omega(\alpha)$ would imply $A \leq_c \Omega(\beta)$ for some $\beta < \alpha$, by lem.3. But this contradicts the choice of α; so I has no w.s. in game $\mathbf{C}(\Omega(\alpha), A)$. Then it is II which has a w.s.; for the determinacy of all Borel infinite games (which is Martin's theorem) easily implies the determinacy of $\mathbf{C}(\Omega(\alpha), A)$. Thus $A \leq_c \Omega(\alpha)$, hence $A \equiv_c \Omega(\alpha)$. We proved that $A \in$ CH iff $A \equiv_c \Omega(\alpha)$ or $A \equiv_c {}^-\Omega(\alpha)$, for some α. With the help of lem.2 it easily implies thm.A.

5) There remains to outline the proof of lem.3

Case 1: $\Omega(\alpha)$ is of the form $B +_c C$. We assume that II is given w.s. σ_+ and σ_- in the games $\mathbf{C}(A, \Omega(\alpha))$ and $\mathbf{C}(A, {}^-\Omega(\alpha))$. Then II can build from σ_+ and σ_- a third strategy τ, to win $\mathbf{C}(A, \Omega(\beta))$ for some $\beta < \alpha$. (This only requires some ingenuity...)

Case 2: $\Omega(\alpha)$ is of the form $sup_c(B_i)_{i<\alpha}$. This case is similar to case 1, only simpler. There only remains

Case 3: $\Omega(\alpha)$ is of the form $V_c^\nu(\Omega(\beta))$, where V_c^ν denotes the operation V_c^1 iterated ν times ($0 < \nu < \omega_1$). By taking ν maximum, we can assume that $\Omega(\beta)$ is in case 1 or 2 (or $\Omega(\beta) = \emptyset$ which is trivial). So lem.3 is **already proved** for $\Omega(\beta)$. Then we use our "secret weapon" $V_c^{-\nu}$: this ν^{th} iterate of the operation V_c^{-1} is a sufficiently good homomorphism from CH into itself, to allow the **reduction** of lem.3 for $V_c^\nu(\Omega(\beta))$ **to** lem.3 for $V_c^{-\nu}(V_c^\nu(\Omega(\beta)))$, which is $\equiv_c \Omega(\beta)$. Thus lem.3 for $\Omega(\alpha)$ is reduced to lem.3 for $\Omega(\beta)$, which is already proved.
This ends the proofs.

Remarks 3

(a) The proof shows more than thm.A; namely it defines in ZF a map: $\alpha \longmapsto \Omega(\alpha)$ such that $d_c^\circ \Omega(\alpha) = \alpha$.

(b) One should think of $\Omega(\alpha)$ as *the Veblen Normal Form of α,* ***once*** *in this normal form the ordinal 0 is evaluated to the Set $\emptyset$ and each operation on ordinals is evaluated to its Borel counterpart.* For this is the essence of $\Omega(\alpha)$, even if technically it would be cumbersome to make it explicit.

3 A new acccess to WH

Here CH means CH **in case $\kappa = \omega$**
Theorem B *There is an isomorphism between CH and WH:*

(a) there is a very simple map : $A \longmapsto A^b$, from Sets $A \subseteq \Lambda^{\leq\omega}$ to sets $A^b \subseteq (\Lambda \cup \{b\})^\omega$, such that $d_c^\circ A = d_W^\circ A^b$

(b) conversely, for every non self dual Borel set $A \subseteq \omega^\omega$ we can define a set $s(A) \subseteq \omega^{<\omega}$ so that $d_W^\circ A = d_c^\circ(A \cup s(A))$

Thus the main results of Martin and Wadge follow from thm.A+B. Moreover they are thus proved in a sharp and simple form as appear in Rmk.3, and in the simplicity of the definition of V_c^1. And the new proof is simple relatively speaking.

The proof of thm.B is a refinment of the proof of thm.A.

Remarks 4

(a) It follows from a result of Harrington [H 70] that in ZF+DC (with no full axiom of choice) we cannot define a function which to every ordinal $\alpha < |WH|$ associates a Borel Set $W(\alpha)$ of wadge degree α. On the other hand the above map: $\alpha \longmapsto \Omega(\alpha)$ is defined in ZF (then it only takes **dependent** choice to show that every Borel Set A satisfies $A \equiv_c \Omega(\alpha)$ or ${}^{-}\Omega(\alpha)$ for some $\alpha < |\mathrm{WH}|$. Let us discuss this surprising phenomenon.

(b) The negative result on WH depends on the fact that WH is a hierarchy on ω^ω **only**: as soon as WH is extended to all spaces α^ω with $\alpha < \omega_1$, it is a corollary of thm.A+B that a map: $\alpha < |\mathrm{WH}| \longmapsto W(\alpha)$ with $d_W^\circ(W(\alpha)) = \alpha$ is definable in ZF.

(c) On the other hand, here is a difference between WH and CH which persists even if we allow WH on α^ω, $(\alpha \in On)$: we noticed that the map: $\alpha \longmapsto \Omega(\alpha)$ is not only defined in ZF, but also canonical. This is not true of the map: $\alpha \longmapsto W(\alpha)$. For Louveau [L 83] defined a set of operations which from $\emptyset$ generate all $\equiv_W$classes of WH, just as $+_c$, sup_c, V_c^ν generate all classes of CH. And in [D c] we computed how Louveau's operations act on the Wadge degrees of the classes they produce: the action is not simple like in the case of CH. This seems inherent to WH and not due to Louveau's particular choice of operations. In other words the conciliatory version CH really simplifies the Wadge hierarchy...

(d) For the definition in ZF of the maps $\alpha \longmapsto \Omega(\alpha)$ and $\alpha \longmapsto W(\alpha)$, the application of Veblen's work recalled in the above Rmk. 1(b) is crucial. Thus "proof theoretic ordinals" turn out to have a bearing in Descriptive Set Theory; and Wadge's computation of the ordinal |WH| appears not only as a "gageure" but also as leading to a deepening of our understanding of Borel sets...

4 CH extended to the uncountable

Here we consider the case $\kappa > \omega$ of CH. The Wadge hierarchy has an obvious extension to Borel subsets of Λ^ω for any Λ, and the isomorphism of thm.B remains true when WH is extended this way. So the theorems A to D essentially imply that **the results of Martin and Wadge for WH remain true** (in a simple and sharp form) **in the uncountable alphabet case**. But there are not many people who are aware of WH in the original case, so who should care for WH in case $\kappa > \omega$, still less effective!? Let us give two reasons of interest of this case $\kappa > \omega$.

Reason 1: a missing part of WH?

The length of CH and WH is $V^\kappa(0)$ if $\kappa = \omega_1$, but as remarked above it is much smaller than $V^\kappa(0)$, if $\kappa > \omega_1$. This raises the question: is there an extension WH^+ of WH in case $K > \omega$, with $|\mathrm{WH}^+| = V^\kappa(0)$? We conjecture that the answer is **yes, in ZF+DC**. This would mean an extension beyond Borel sets of the determinacy results **proved in ZF + DC**...

Reason 2: the extension of WH from $\kappa = \omega_1$ to $\kappa > \omega_1$ goes in the direction of **more** effectivity!

For Girard-Vauzeille [G V 84] extended the Veblen function $\kappa \longmapsto V^\kappa(0)$ to a **functor** V of ordinals, **which is effective** in the following sense: the restriction of V to **finite** objects (integers and morphisms between them) is primitive recursive and this restriction uniquely determines all of V, in an utterly direct way. Now theorems A+B show that we also have a function: $\kappa \longmapsto$ CH(κ) where CH(κ) is CH for $card(\Lambda) < \kappa$; and it is isomorphic to a suitable restriction of V. This may be the first indication that Borel sets have a **finitary** effective content...

5 The effective Conciliatory hierarchy

The refined character of the Wadge hierarchy is so far not very useful in Analysis; but this character may be welcome for the needs of Computer Science. And the simplification and canonical character brought in by the conciliatory version may be crucial in this context. Therefore we are starting to develop an effective version cH of CH.

Let us come back to the weak separation lemma (lem.3): its above mentioned proof is non effective, due to the use of the V_c^{-1} operation. But we also gave in [D 95] a (harder) constructive proof, from which it is possible to extract a class $\mathcal{S}$ of recursive strategies satisfying following version of lem.3.
Effective weak separation: if $A \leq_S \Omega(\alpha)$ **and** $^-\Omega(\alpha)$, then $A \leq_S \Omega(\beta)$ **or** $^-\Omega(\beta)$ for some $\beta < \alpha$; where $A \leq_S B$ means that in $\mathbf{C}(A,B)$ player *II* has a winning strategy σ **which belongs to $\mathcal{S}$**.
(Thus $A \leq_c B \iff A \leq_{S'} B$ when $\mathcal{S}'$ is the set of **all** strategies that is all functions $\sigma : (\Lambda_A \cup \{<>\})^{<\omega} \longmapsto \Lambda_B \cup \{<>\}$).

Definition 4

(a) $A \equiv_S B \longleftrightarrow A \leq_S B \leq_S A$

(b) cH is the closure of $\{\emptyset\}$ *under the operations* $+_c$, V_c^1 *and:* $(A, \sigma) \longmapsto A \circ \sigma$*; where* σ *ranges* ***over*** $\mathcal{S}$ *and* $A \circ \sigma = \{x : \sigma * x \in A\}$*. On cH we consider* $\leq_S$ *and* $\equiv_S$*.*

We plan to study cH in two steps. The first step would be to prove
effective determinacy: for any Sets $A,\ B \in$ cH, the player who wins $\mathbf{C}(A,B)$ has a strategy **in $\mathcal{S}$**.
Together with lem.2 and effective weak separation which are already proved, this step would yield **the analog for cH of thm.A** - by arguing just as in the above §2. Among others, one would thus prove that $|\text{cH}| = \epsilon_0$. More generally, our program is to provide a full analog for cH of the works of Büchi, Landweber and Wagner (see [T 90]) on **Büchi automata**. Let us recall this work and its wide application in Computer Science.
A Büchi automaton A is an automaton reading **infinite** words $x \in \Lambda_A^\omega$, and having a "Büchi acceptance condition" which tells wether or not it accepts x. A **transducer** is an automaton σ which while it reads $x \in \Lambda_A^{\leq\omega}$ writes a word $\sigma * x \in \Lambda_B^{\leq\omega}$ on an input tape (henceforth all alphabets are **finite**). The **Wagner hierarchy** wH consists of all Büchi automata equipped with the relations $\leq, \equiv$ defined by $A \leq B$ iff in the game $\mathbf{W}(A,B)$ player *II* has a w.s. σ **which is a transducer**, $A \equiv B$ iff $A \leq B \leq A$.

Wagner's theorem provides the structure of wH in a thorough and effective way; in particular it shows that $|\text{wH}| = \omega^\omega$.

Nota Bene 3

(a) *The above step 1 would realize the extension to cH of Wagner's theorem.*

(b) *Using Wagner's theorem it is easy to give a natural embedding of wH into cH. Thus cH stands in between wH and WH. But it is much richer that wH: Büchi automata have Borel rank up to 3. While cH has Borel sets of any finite rank.*

Given any set $Z \subseteq \Lambda_Z^\omega$ the **Gale-Stewart** game $\mathbf{G}(Z)$ lets two players choose $z \in \Lambda_Z^\omega$ (at move $p < \omega$, I chooses $z(2p)$ and II replies with $z(2p+1)$. The winner of this play z is II iff $z \in Z$. (The games $\mathbf{W}(A, B)$ and $\mathbf{C}(A, B)$ are easily reduced to special cases of $\mathbf{G}(Z)$, but not conversely).

The **Büchi-Landweber theorem** is the effective determinacy of Gale-Stewart games refereed by Büchi automata: for every Büchi automaton A, the winner of $\mathbf{G}(A)$ has a w.s. **which is a transducer.**

Present days industry raises a large number of problems of the form: design a processor working in real time interaction with its environment, which satisfies for a certain specification. The above theorem of Büchi-Landweber is the theoretical background of a successful modelization of this problem.

(a) One imagines an infinite game in which player I is the environment: its possible moves are all stimuli which the environment might send at once to the processor. Player II is the processor: its possible moves are all the reactions which the processor might have to make at once. The plays are all infinite sequences of alternate movees of I and II - each move coded by a letter of a suitable alphabet Λ.

(b) The specification is then represented by the set A of all plays $z \in \Lambda^\omega$ such that the moves of II are a satisfactory response to the moves of I, according to the specification; thus our problem becomes: find a transducer σ which is a w.s. for II in the game $\mathbf{G}(A)$.

(c) If A happens to be accepted by a Büchi automaton, then by the Büchi-Landweber theorem one of the player has a w.s. σ for $\mathbf{G}(A)$ which is a transducer. If this player is II, then σ is the desired processor; and if it is I, then no transducer can solve the problem. Moreover by another theorem of Büchi we can effectively determine which player has the w.s., and find out the transducer which realizes it.

The infinite length of $\mathbf{G}(A)$ is an imaginary feature; nevertheless the above model of processor design is accurate in a very large class of applications - for instance, in the design of processors used in modern planes. Which means that the degree of accuracy of this model can be very high.

We come back to our program concerning cH; its secund step would be to show that in every game $\mathbf{G}(A)$ where $A \in$ cH, the winner has a strategy **which belongs to $\mathcal{S}$.** In other words the secund step would extend to cH the Büchi-Landweber theorem, just as the first step consist in extending Wagner's theorem. The secund step is more difficult than the first; for instance, let A and B be recursive open sets; then it is easy to provide the winner of $\mathbf{C}(A, B)$ with a recursive w.s. and the winner of $\mathbf{W}(A, B)$ with a strategy which is Π_2^0. But this may be impossible for $\mathbf{G}(A)$: the winner may even have no **arithmetical** or even no "hyperarithmetical" w.s.. Thus fine tuning of the class of strategies $\mathcal{S}$ used to define cH will be necessary to realize our secund step. This is where the canonical and simple character of the basic operations of CH (hence of cH) is most useful...

The completion of our program is a challenge which we are uncertain to win. On the other hand we are convinced that success would have a bearing on Computer Science (just as the Büchi Landweber theorem has; but in a different way, since the strategies of $\mathcal{S}$ are no longer real time).
So we hope that the present Note is of interest not only for Set theorists, but also for "Logicians in Computer Science".

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