Weak theory of linear inequalities with integer coefficients and variables

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EXTENDED ABSTRACT

Upper bounds for decidability algorithms for universal elementary theories of inequalities with integer coefficients and variables is presented. Proofs are based on reduction of Khachiyan's result to integer-valued systems of linear inequalities and one $LIN - SPACE - POLY - TIME$ algorithm for those systems with two addends per inequality with only coefficients 1 and -1.

Three polynomial solvability algorithms for systems of linear inequalities with integer coefficients and variables

Theorem 1 *Solvability problem of homogeneous systems of linear inequalities with integer coefficients and variables is recognizable by some deterministic Turing machine in polynomial number of steps.*

Proof. Given integer $m \times n$ matrices A and B it's needed to define if

$$\exists x \in Z^n (Ax > 0 \ \& \ Bx \geq 0).$$

This is equal to

$$\exists x \in Q^n (Ax > 0 \ \& \ Bx \geq 0),$$

or

$$\exists x \in Q^n \exists t \in Q (Ax - It \geq 0 \ \& \ t > 0 \ \& \ Bx \geq 0),$$

where I is vector of numbers 1. At last we have

$$\exists x \in Q^n (Ax \geq I \ \& \ Bx \geq 0).$$

Now it is sufficient to take use of Khachiyan's algorithm [1]. ∎

Corollary 1 *There exists a deterministic Turing machine which solves the problem*

$$\exists x \in Z^n (Ax \geq d)$$

where A is a given $m \times n$ matrix and $d \geq 0$ is a given n-ary vector in polynomial number of steps.

Proof. The initial problem is equal to

$$\exists x \in Z^n \exists u \in Z(Ax \geq du \ \& \ u \geq 1).$$

because $du \geq d$ where $u \geq 1$ and $du = d$ if $u = 1$. Since $u \geq 1$ implies $u > 0$ the last formula is equal to solvability problem for homogeneous integer systems of linear inequalities. So we can apply the previous theorem. ∎

In case of an arbitrary d we have general integer linear programming problem which is NP-complete. If we replace the condition $d \geq 0$ in the theorem formulation by $d \geq -1$ then the problem becomes NP-complete.

Consider integer system of linear inequalities S1

$$\begin{cases} x_{11} \prec_1 x_{12} \\ x_{21} \prec_2 x_{22} \\ \dots \\ x_{n1} \prec_n x_{n2}, \end{cases}$$

where x is a variable, or variable with minus, or integer constant where the sign "$\prec_i$" is used instead of strict or nonstrict inequality sign.

An equivalent system S2 will be considered simultaneously with S1 :

$$\begin{cases} x_{11} \prec_1 x_{12} \\ -x_{12} \prec_1 -x_{11} \\ x_{21} \prec_2 x_{22} \\ -x_{22} \prec_2 -x_{21} \\ \dots \\ x_{n1} \prec_n x_{n2} \\ -x_{n2} \prec_n -x_{n1}, \end{cases}$$

where double minuses are eliminated.

Lemma 1 *S2 is unsolvable iff at least one of the following conditions is satisfied.*

1. *S2 contains a cycle $x_1 \prec_{i_1} x_2 \prec_{i_2} \dots \prec_{i_r} x_1$ for some $r \geq 1$ and at least one of inequalities is strict.*

2. *Let S2 contains a cycle $x_1 \leq x_2 \leq \dots \leq x_r \leq x_1$.*

   2.1. If $r \geq 2$ and for some i, $(2 \leq i \leq r)$ $x_i = 0$ or $x_i = -x_1$ then the system with zeroes instead of each cycle member is unsolvable.

   2.2. If condition 2.1 is not satisfied and the cycle contains only one constant then the system with this constant instead of each cycle member is unsolvable.

   2.3. If conditions 2.1 and 2.2 are not satisfied then the system with a new variable instead of cycle members is unsolvable.

3. *S2 contains a chain* $x_1 \prec_{i_1} x_2 \prec_{i_2} \ldots \prec_{i_{r-1}} x_r$, *where only* x_1 *and* x_r *are constants and* $x_r < x_1$.

4. *S2 contains a chain* $x_1 \prec_{i_1} x_2 \prec_{i_2} \ldots \prec_{i_{r-1}} x_r$, *where* x_i *are different, only* x_1 *and* x_r *are constants and the length of chain calculated as a number of* ' $<$ ' *is not less then* r.

Idea of proof. It is easy to check that the conditions 2.1 — 2.3 are equivalent and the conditions 1, 3, and 4 imply unsolvability of S2 .

On the other hand, the negation of 1 provides cycles absence, the negation of 3 excludes such cases as $5 < x < 2$, and the negation of 4 does not allow S2 to contain chains like $1 < x < 2$ or $1 < x < y < 3$. This means that S2 defines a partially ordered structure procreated by "$<$" and "$\leq$" which can be mapped into integers. In this case the system is solvable. ∎

Call deterministic Turing machine $LIN-SPACE-POLY-TIME$ if its upper time bound is polynomial and upper space bound is linear in the length of input.

Theorem 2 *The solvability problem for S1 can be done by* $LIN-SPACE-POLY-TIME$ *algorithm.*

Universal elementary theories of linear inequalities
with integer coefficients and variables

Call deterministic Turing machine $EXP-LIN-TIME$ if its upper time bound is

$$2^{Cn},$$

where C is a constant and n is the length of an input.

Theorem 3 *Universal elementary theory of integer linear inequalities without addition and with only coefficients 1 and* -1 *is decidable by* $EXP-LIN-TIME$ *and simultaneously* $LIN-SPACE$ *algorithm on deterministic Turing machine.*

Theorem 4 *Universal elementary theory of integer linear homogeneous inequalities is decidable by* $EXP-TIME$ *and simultaneously* $POLY-SPACE$ *algorithm on deterministic Turing machine.*

Theorem 5 *Universal elementary theory of integer linear inequalities* $a_1x_1 + \ldots + a_nx_n \geq b$, *where* $b \geq 0$, *is decidable by* $EXP-TIME$ *and simultaneously* $POLY-SPACE$ *algorithm on deterministic Turing machine.*

Theorem 6 *Universal elementary theory of integer linear inequalities is decidable by* $EXP-TIME$ *algorithm on deterministic Turing machine.*

References

[1] Khachiyan L.G. Polynomial algorithms in linear programming, U.S.S.R. Computational mathematics and Mathematical Physics 20 (1980), 52-72.