Bounding the existential variables after the elimination of a bounded universal quantifier

Yuri Matiyasevich

In 1961 Martin Davis, Hilary Putnam and Julia Robinson published selebrated paper [1] where they described a method for transforming a formula of the form

$$\forall y_{\leq b} \exists z_1 \ldots z_m [P(a_1, \ldots, a_n, y, z_1, \ldots, z_m) = 0], \tag{1}$$

where P is a polynomial with integer coefficients, into an equivalent formula of the form

$$\exists x_1 \ldots x_l [E_{\mathrm{L}}(a_1, \ldots, a_n, b, c, x_1, \ldots, x_l) = E_{\mathrm{R}}(a_1, \ldots, a_n, b, c, x_1, \ldots, x_l)], \tag{2}$$

where E_{L} and E_{R} are so called *exponential polynomials*, i.e., some expressions constructed using the conventional rules from particular natural numbers and variables $a_1, \ldots, a_n$ and $x_1, \ldots, x_l$ by addition, multiplication and exponentiation.

In 1970 the author [3] improved this result by showing that exponentiation is not necessary in (2), i.e., that one can find a formula which is equivalent to (1) and which has the form

$$\exists x_1 \ldots x_l [Q(a_1, \ldots, a_n, b, c, x_1, \ldots, x_l) = 0], \tag{3}$$

where Q is a polynomial with integer coefficients.

This result evidently has a counterpart for formulas with bounded existential quantifiers: a formula of the form

$$\forall y_{\leq b} \exists z_{1 \leq c} \ldots z_{m \leq c} [P(a_1, \ldots, a_n, y, z_1, \ldots, z_m) = 0] \tag{4}$$

can be transformed into an equivalent formula of the form

$$\exists x_{1 \leq X(a_1, \ldots, a_n, b, c)} \ldots x_{l \leq X(a_1, \ldots, a_n, b, c)} [Q(a_1, \ldots, a_n, b, c, x_1, \ldots, x_l) = 0]. \tag{5}$$

The original technique for eliminating the single universal quantifier was modified by several authors (see [4, 2]). An essentially different technique was introduced in [5]. The technique presented in [6] for Diophantine simulation of partial recursive functions can be also viewed as a method for eliminating a bounded

universal quantifier because such a quantifier can be easily modelled by primitive recursion. Yet another method for eliminating bounded universal quantifiers was recently proposed by the author in [7].

However, all these techniques suffer from the same shortcoming: being applied to a formula (4) with bounded existential quantifiers they produce an equivalent formula (5) in which the bound X actually depends on the parameters $a_1, \ldots, a_n$. In 1992 Zofia Adamovicz asked the question whether this is inevitable or not. It turned out that the abovementioned shortcoming can be overcome.

Theorem *For every formula of the form* (4) *one can effectively find an equivalent formula of the form*

$$\exists x_1 \leq X(b,c) \ldots x_l \leq X(b,c)[Q(a_1, \ldots, a_n, b, c, x_1, \ldots, x_l) = 0], \tag{6}$$

where Q is a polynomial with integer coefficients.

References

[1] Davis M., Putnam H., Robinson J. The decision problem for exponential Diophantine equations. *Annals of Mathematics, Second Series*, 74(3):425–436, 1961.

[2] Hirose K., Shigeaki I. A proof of negative answer to Hilbert's 10th Problem. *Proceedings of the Japan Academy*, 49(1):10–12,1973.

[3] Matiyasevich Yu. V. Diofantovost perechislimykh mnozhestv. *Dokl. AN SSSR*, 191(2):278–282, 1970. Translated in: *Soviet Math. Doklady*, 11(2):354-358, 1970.

[4] Matiyasevich Yu. V. Diofantovy mnozhestva. *Uspekhi Mat. Nauk*, 27:5(167), 185–222, 1972. Translated in: *Russian Mathematical Surveys*, 27(5):124–164, 1972.

[5] Matiyasevich Yu.V. *Desyataya Problema Gilberta.* Moscow, Fizmatlit, 1993. English translation: Hilbert's tenth problem. MIT Press, 1993.

[6] Matiysevich Yu. A direct method for simulating partial recursive functions by Diophantine equations. *Annals Pure Appl. Logic*, 67, 325–348, 1994.

[7] Matiyasevich Yu.V. A new Technique for Obtaining Diophantine Representations via Elimination of bounded universal Quantifiers. *Zap. nauchn. seminar. Leningr. otd. Mat. in-ta RAN*, 220, 83–92, 1995.