Two Proof Methods for two Three-Valued Modal Logics (Propositional Case).

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Abstract

The necessity to combine many-valuedness and modality came from the problem of finding a logical framework for formal representation of expert reasoning. The main contribution of this paper is the series of calculi which are correct and complete for the three-valued **S4** and for another logic that can be viewed as a restriction of **S4** and is closely related to a formalism designed with the aim of adequate knowledge representation.

1 Motivations

The problem of finding a logical framework for an adequate formalization of the totality of an expert knowledge and reasoning is of a great importance now. An approach to the problem was outlined in [2], where some basic notions were introduced as well. Informally speaking it was suggested to represent the whole totality of knowledge as consisting of relatively independent pieces or "situations" (do not mix the notion with that of introduced by Barwise and Perry). Some relations on situations were defined which allow to consider a set of situations as a "situational structure". An important example of such a relation is the relation "to-be-more-general" defined on pairs of situations. Intuitively it means that every statement that is valid (or true) in a more general situation may be transferred in any less general one. As the relation should be reflexive and transitive, it can be treated as the accessibility relation for a kind of a Kripke structure with some restriction on the pairs of accessible worlds. So it appeared to be possible to introduce modalities and to use them for description

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of a "situational logic" instead of using explicitly, say, a new sort of variables for situations. On the other hand, as every statement must be considered w.r.t. a given situation rather that "absolutely", it is clear that a statement may be true, false and undefined (or senseless) in a situation. So we needed for our aims at least a three-valued logic rather than two-valued one in a combination with modalities and a corresponding proof theory to be developed.

The first step in this direction was done in [4] where a three-valued logic and a resolution method for it were proposed and substantiated but the problem of how to combine many-valuedness and modalities remained untouched. This paper continues the guide-line and contains the next step — an extension of Gentzen style and resolution style proof methods to some three valued modal logics.

The entire language **SSL** for situational structures description (see [2]) contains two kinds of modalities, a dynamic fragment aimed at actions representation, usual first order quantifiers and propositional connectives which were chosen specifically taking into account that the underlying logic is of three values. The simplest fragment $\mathbf{SSL_0}$ of **SSL** is that which does not contain neither modalities nor actions, it was considered in [4]. The fragment $\mathbf{SSL_1}$ is $\mathbf{SSL_0}$ augmented with the above mentioned modality, $\mathbf{SSL_1P}$ is the propositional fragment of $\mathbf{SSL_1}$. The modality under consideration looks rather similar to that of **S4** and indeed it will be shown that the problem of establishing validity of $\mathbf{SSL_1P}$-formulas may be reduced to a similar problem in three-valued **S4**. So we will begin with formulating and proving results for the three-valued **S4**.

2 Proof and model theory for three-valued S4

2.1 Language

We will follow the usual pattern for classical **S4**. The language $\mathbf{SSL_1P}$ will be the same, only the class of models will change. First of all there are countable many *propositional variables* which will be denoted P, Q (possibly with indexes). *Propositional formulas* (P-formulas for brevity) are constructed from variables by means of conjunction, disjunction, implication and two kinds of negation (denoted $\wedge$, $\vee$, $\Rightarrow$, $\neg$, $\sim$ resp.) In *modal formulas*, necessity ($\Box$) and possibility ($\Diamond$) are also allowed. Formulas of the form $*P$ where $*$ is arbitrary sequence of negations are called *literals*. Letters F, G (possibly indexed) denote arbitrary formulas.

As usual, *a sequent* is a syntactical object of the form $\Gamma \vdash \Delta$, where Γ, Δ are finite sets of formulas.

2.2 Semantics of possible worlds

Formulas are evaluated in the three-element lattice $\Omega = \{\bot, \Lambda, \top\}$, ordered as $\bot < \Lambda < \top$. *An P-evaluation* (for *propositional* evaluation) is a map from propositional variables into Ω. Any P-evaluation is extended in the usual way onto the set of all P-formulas; hereby minimum and maximum on Ω correspond to conjunction and disjunction respectively, functions corresponding to negations are defined in Table 1.

Table 1:

x	$\bot$	Λ	$\top$
$\neg x$	$\top$	$\bot$	$\bot$
$\sim x$	$\top$	Λ	$\bot$

Note. Among several possible candidates for implication we prefer that which is the pseudo-complement in Ω. The choice may be approved by an intended "dynamic meaning" of the implication but we do not need the arguments here. It can be easily verified that $F \Rightarrow G$ is semantically equivalent to $\neg F \vee G \vee (\neg\neg{\sim}F \wedge \neg\neg G)$, and we will not consider the implication anymore.

Note. One can consider yet another kind of negation (say, $\angle$) that permutes $\top$ and $\bot$ and maps Λ to $\top$, but it can be expressed via $\neg$ and $\sim$ as $\angle P = \neg\neg{\sim}P$. So, we do not need a separate connective for $\angle$ and will consider it as an abbreviation.

As one already has the semantics for P-formulas, it is possible to make use of Kripke's construction of possible worlds. In order to fix notations, we consider it in more details.

A system of worlds $\mathbf{S}$ is a set S (whose elements we call *possible worlds*) with transitive reflexive *accessibility relation* $\mathbf{Acc}$ on it. Given a system of worlds $\mathbf{S}$, *an evaluation in* $\mathbf{S}$ is a collection of P-evaluations σ_s indexed by elements of $\mathbf{S}$. We call a pair $\langle \mathbf{S}, \{\sigma_s\}\rangle$ an interpretation. Now, we define $\mathbf{ev}_s\, \sigma F$ *(the truth value of a formula F in the world s under σ)* as in the classical case, namely:

- if F is a variable, $\quad \mathbf{ev}_s\, \sigma F = \mathbf{ev}\, \sigma_s F$;
- if the main connective of F is propositional one, the value of F is calculated as before;
- if $F = \Box G$, $\quad \mathbf{ev}_s\, \sigma F = \min\{\mathbf{ev}\, \sigma_{s'} G \mid \mathbf{Acc}(s, s')\}$;
- if $F = \Diamond G$, $\quad \mathbf{ev}_s\, \sigma F = \max\{\mathbf{ev}\, \sigma_{s'} G \mid \mathbf{Acc}(s, s')\}$;

Two formulas F, G are said to be *semantically equivalent* (symbolically $F \Leftrightarrow G$) if for all interpretations $\langle \mathbf{S}, \{\sigma_s\}\rangle$, for all $s \in \mathbf{S}$ $\mathbf{ev}_s\, \sigma F = \mathbf{ev}_s\, \sigma G$. Further on, we need the following equivalences which can be easily established:

Table 2:

$$\begin{array}{rcl}
\sim(F \wedge G) & \Leftrightarrow & \sim F \vee \sim G \\
\sim(F \vee G) & \Leftrightarrow & \sim F \wedge \sim G \\
\neg(F \wedge G) & \Leftrightarrow & \neg F \vee \neg G \\
\neg(F \vee G) & \Leftrightarrow & \neg F \wedge \neg G \\
\sim\sim F & \Leftrightarrow & F \\
\neg\neg\neg F & \Leftrightarrow & \neg F \\
\sim\neg F & \Leftrightarrow & \neg\neg F \\
\sim\Box F & \Leftrightarrow & \Diamond\sim F \\
\sim\Diamond F & \Leftrightarrow & \Box\sim F \\
\neg\Box F & \Leftrightarrow & \Diamond\neg F \\
\neg\Diamond F & \Leftrightarrow & \Box\neg F
\end{array}$$

2.3 Validity

In order to define a notion of validity one needs first to choose a set of distinguished truth values $\mathcal{D}$. The natural candidates for it are $\{\top\}$ and $\{\top, \Lambda\}$. Corresponding notions are called $\top$- and $\top\Lambda$-validity. Since definitions do not depend on a particular choice of distinguished values, we will omit prefixes. A sequence of definitions is standard:
A formula F is said to be valid under an interpretation $\langle \mathbf{S}, \{\sigma_s\}\rangle$ in a world $s \in \mathbf{S}$ if $\mathbf{ev}_s\, \sigma F \in \mathcal{D}$.
A sequent $\Gamma \vdash \Delta$ is said to be valid under an interpretation $\langle \mathbf{S}, \{\sigma_s\}\rangle$ in a world $s \in \mathbf{S}$ if either Δ contains a formula that is valid under $\langle \mathbf{S}, \{\sigma_s\}\rangle$ in s or Γ contains a formula that is not valid under $\langle \mathbf{S}, \{\sigma_s\}\rangle$ in s.
A sequent is said to be valid under interpretation an $\langle \mathbf{S}, \{\sigma_s\}\rangle$ if it is valid under $\langle \mathbf{S}, \{\sigma_s\}\rangle$ in all worlds $s \in \mathbf{S}$.
A sequent is said to be valid if it is valid under all interpretations.

Note. We need further the following equivalencies, which hold respectively

$$F \Leftrightarrow_{\top} \neg\sim F \qquad\qquad \sim F \Leftrightarrow_{\top} \neg F \tag{1}$$

for $\top$-validity, and

$$F \Leftrightarrow_{\top\Lambda} \neg\neg F \qquad\qquad \sim F \Leftrightarrow_{\top\Lambda} \neg\neg\sim F \tag{2}$$

for $\top\Lambda$-validity.

2.4 Sequent calculi

One can describe two sequent calculi; one of them realizes $\top$-validity, the other — $\top\Lambda$-validity, i.e. correctness and completeness hold for them. Both calculi are almost the same, so proofs are given for $\top$-validity only.

It turns out that rules for all connectives except those for negations are the same as in classical **S4**. So, we need only to understand how to deal with negations. Note that the classical negation is involutive in two aspects:

- F is semantically equivalent to double-negated F
- negation of F is valid iff F is not valid

Roughly speaking, these properties entail Gentzen rules for negation.

In the three-valued case, the situation is different. The first property holds for $\sim$, but the second does not. In turn, the second property holds for $\angle$ (i.e. $\neg\neg\sim$) w.r.t. $\top$-validity, and for $\neg$ w.r.t. $\top\Lambda$-validity. There are also other difficulties with negation (like de Morgan law) which we need to take into account. So, we accept the following convention:

> *Formulas in sequents are considered up to semantical equivalencies listed in Table 2. In addition, equivalencies (1) (resp. (2)) are implicitly used everywhere when $\top$-validity (resp. $\top\Lambda$-validity) is concerned. All negations are supposed to occur only in literals.*

After this, we adopt usual rules for the "real" negation which is either $\angle$ or $\neg$ depending on the validity considered.

Now, we formulate the calculus $\top$**G** for $\top$-validity.

Axioms are $F \vdash F$ and $F \vdash \neg\neg F$, where F is a literal. The inference rules are as follows:

$$\frac{\Gamma \vdash \Delta, F_1, F_2}{\Gamma \vdash \Delta, F_1 \vee F_2}\ (\mathrm{RI}\vee) \qquad \frac{F_1, \Gamma_1 \vdash \Delta_1; F_2, \Gamma_2 \vdash \Delta_2}{F_1 \vee F_2, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ (\mathrm{LI}\vee)$$

$$\frac{\Gamma_1 \vdash \Delta_1, F_1; \Gamma_2 \vdash \Delta_2, F_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, F_1 \wedge F_2}\ (\mathrm{RI}\wedge) \qquad \frac{F_1, F_2, \Gamma \vdash \Delta}{F_1 \wedge F_2, \Gamma \vdash \Delta}\ (\mathrm{LI}\wedge)$$

$$\frac{F, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg\neg\sim F}\ (\mathrm{RI}\angle) \qquad \frac{\Gamma \vdash \Delta, F}{\neg\neg\sim F, \Gamma \vdash \Delta}\ (\mathrm{LI}\angle)$$

$$\frac{\Box\Gamma \vdash \Diamond\Delta, F}{\Box\Gamma \vdash \Diamond\Delta, \Box F}\ (\mathrm{RI}\Box) \qquad \frac{F, \Gamma \vdash \Delta}{\Box F, \Gamma \vdash \Delta}\ (\mathrm{LI}\Box)$$

$$\frac{\Gamma \vdash \Delta, F}{\Gamma \vdash \Delta, \Diamond F}\ (\mathrm{RI}\Diamond) \qquad \frac{F, \Box\Gamma \vdash \Diamond\Delta}{\Diamond F, \Box\Gamma \vdash \Diamond\Delta}\ (\mathrm{LI}\Diamond)$$

$$\frac{\Gamma \vdash \Delta}{\Gamma_1, \Gamma \vdash \Delta, \Delta_1}\ (\mathrm{Thin})$$

where $\Box\Gamma$ stands for $\{\Box F \mid F \in \Gamma\}$ (similarly for $\Diamond\Gamma$).

Note. To obtain the calculus $\top\Lambda$**G** for $\top\Lambda$-validity one must:

- take sequents $F \vdash F$, $\neg\sim F \vdash F$ where F is a literal as axioms.
- replace rules for $\angle$ with similar rules for $\neg$:

$$\frac{F, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg F} \ (\mathrm{RI}\neg) \qquad \frac{\Gamma \vdash \Delta, F}{\neg F, \Gamma \vdash \Delta} \ (\mathrm{LI}\neg)$$

Theorem 1 *If a sequent $\Gamma \vdash \Delta$ is derivable in $\top\mathbf{G}$(resp. $\top\Lambda\mathbf{G}$), then it is $\top$-valid (resp. $\top\Lambda$-valid).*

Proof. Use the standard induction on the structure of derivations.

2.5 Semantic tableaux

We established above the *correctness* theorem for $\top\mathbf{G}$ and $\top\Lambda\mathbf{G}$. The proof of the *completeness* theorem is carried out with the help of the semantic tableaux method.

Every *tableau structure* corresponds to a set of Kripke worlds and is presented by a set of *individual* tableaux and accessibility relation on it that is, of course, reflexive and transitive. Any individual tableau represents the "state of affairs" in the given world and is a pair $\{\Gamma|\Delta\}$ of two sets of modal formulas. Again, formulas are considered up to equivalencies listed in Table 2. The naive semantics of $\{\Gamma|\Delta\}$ is as follows: *in the given world every element of* Γ *is valid and every element of* Δ *is not.*

A tableau $\{\Gamma|\Delta\}$ is *closed* if there exists $F \in \Gamma$ and $G \in \Delta$ such that G is either F or $\neg\neg F$. A tableau structure is *closed* if it contains a closed tableau.

Given a sequent $\Gamma \vdash \Delta$, one can obtain *a complete set of tableau structures* as a result of a process described below.

At any stage of the construction we will consider a set of the tableau structures. The initial set of tableau structures contains the only element which in turn consists of the only tableau $\{\Gamma|\Delta\}$. Then we will generate new sets according to the syntactic form of elements of Γ, Δ. Further on, let a set $\mathcal{S}$ of tableau structures were generated on the previous stage. Assume that there is $\mathbf{S} \in \mathcal{S}$ and $T = \{\Gamma|\Delta\} \in \mathbf{S}$ such that:

1. $\Gamma = \Gamma', \neg\neg\sim F$; then replace T in $\mathbf{S}$ with $\{\Gamma|\Delta, F\}$;
2. $\Delta = \Delta', \neg\neg\sim F$; then replace T in $\mathbf{S}$ with $\{F, \Gamma|\Delta\}$;
3. $\Gamma = \Gamma', F \vee G$; then duplicate $\mathbf{S}$, replace T with $\{F, \Gamma|\Delta\}$ in one copy and with $\{G, \Gamma|\Delta\}$ in the other;
4. $\Delta = \Delta', F \vee G$; then replace T in $\mathbf{S}$ with $\{\Gamma|\Delta, F, G\}$;
5. $\Gamma = \Gamma', F \wedge G$; then replace T in $\mathbf{S}$ with $\{F, G, \Gamma|\Delta\}$;

6. $\Delta = \Delta', F \wedge G$; then duplicate **S**, replace T with $\{\Gamma|\Delta, F\}$ in one copy and with $\{\Gamma|\Delta, G\}$ in the other;

7. $\Gamma = \Gamma', \Box F$; then replace T in **S** with $\{F, \Gamma|\Delta\}$;

8. $\Delta = \Delta', \Box F$; then construct a new tableau $\{\Gamma'|\Delta'', F\}$ that is accessible from T, where Γ' (resp. Δ'') consist of all $\Box G \in \Gamma$ ($\Diamond G \in \Delta'$).

9. $\Gamma = \Gamma', \Diamond F$; then construct a new tableau $\{\Gamma'|\Delta' F\}$ that is accessible from T, where Γ' (resp. Δ') consist of all $\Box G \in \Gamma$ ($\Diamond G \in \Delta$).

10. $\Delta = \Delta', \Diamond F$; then replace T in **S** with $\{\Gamma|\Delta, F\}$;

We will consider any "rule" of 1–10 as being inapplicable to an individual tableau T if either T is closed or the tableau produced from it coincides with another tableau T' of the same structure. In the latter case we extend the accessibility relation with the pair (T, T')

It is easy to show that the process of the tableau structures generation always terminates, so the definition of a complete set of tableau structures is correct.

Note. Tableau semantics given above corresponds to $\top$-validity; to obtain the semantics for $\top\Lambda$-validity, one has to do the following changes:

- give the new definition of a closed tableau:

  A tableau $\{\Gamma|\Delta\}$ is *closed* if there exist $F \in \Gamma$ and $G \in \Delta$ such that F is either G or $\neg{\sim}G$.

- replace rules 1–2 with the following ones:

  1. $\Gamma = \Gamma', \neg F$; then replace in **S** $\{\Gamma|\Delta\}$ with $\{\Gamma|\Delta, F\}$;
  2. $\Delta = \Delta', \neg F$; then replace in **S** $\{\Gamma|\Delta\}$ with $\{F, \Gamma|\Delta\}$.

2.6 Completeness

We will prove the following theorems for the $\top$-validity, proofs for $\top\Lambda$-validity are practically the same.

Theorem 2 *If the complete set of tableau structures of a given sequent $\Gamma \vdash \Delta$ contains a non-closed structure* **S***, then there exists a countermodel for this sequent (i.e. an interpretation in which it is not valid).*

Proof. In fact, **S** gives this countermodel. More formally, a system of worlds of the interpretation we are looking for consist of all the tableaux in **S**, the accessibility relation remains the same, and evaluation σ is defined as follows: Consider arbitrary $T = \{\Gamma'|\Delta'\} \in$ **S** and propositional variable P. Choose a value for $\sigma T\, P$ so that for any literal of the form $*P$, $\sigma T *P = \top$ if $*P \in \Gamma'$ and $\sigma T *P \in \{\Lambda, \bot\}$ if $*P \in \Delta'$.

It can be done because T is non-closed. In order to conclude the proof it is sufficient to note that, for every $T = \{\Gamma'|\Delta'\} \in \mathbf{S}$, all formulas from Γ' are valid under σ and all from Δ' are not.

Theorem 3 *If a complete set of tableau structures of a sequent consist of closed structures, then there exists an inference of this sequent in the sequent calculus.*

Proof. Use a simple induction on the number of steps required to construct a complete set of tableau structures. Note that each step is, in fact, a counter-application of the corresponding rule in $\top\mathbf{G}$.

3 Resolution-like calculi for three-valued S4

Following the classical paper [3], we will formulate calculi $\top\mathbf{RR}$ and $\top\Lambda\mathbf{RR}$ of the resolutional kind (which realize $\top$- and $\top\Lambda$-validity respectively) and prove correctness and completeness for them. Again, we consider only the case of $\top$-validity in details, while giving formulations for $\top\Lambda$-validity in remarks.

3.1 Clauses and translation formulas into clauses

Recall that *a literal* is a formula of the form $*P$ where $*$ is arbitrary sequence of negations. We will denote literals as L_i. Using equivalencies from Table 2 and equations (1), one can consider literals only of one of the following form:

$$P \qquad \neg\neg P \qquad \sim P \qquad \neg\neg\sim P$$

As usual, we identify $\angle\angle L$ with L (it follows from $\neg\sim P \Leftrightarrow_\top P$).

Modal literals is one of the form L, $\Box L$, or $\Diamond L$. We will denote them as M_i. *A clause* is a disjunction of modal literals and is considered up to the order and the number of occurrences of its members. Sometimes we consider a set of modal literals as a clause. *Modal clause* is a formula $\Box D$, where D is a clause.

Let F be a formula. For every non-atomic subformula G of F a new propositional variable X_G is introduced. We set $X_P \equiv P$ for propositional variables. Let $G \equiv G_1 * G_2$ (where $*$ is a binary connective). Then $\mathbf{T}_F$ stands for $\Box(X_G \Leftrightarrow (X_{G_1} * X_{G_2}))$. Likewise if $G \equiv *G_1$, then T_G stands for $\Box(X_G \Leftrightarrow (*X_{G_1}))$.

It is easy to see that any formula T_G is equivalent to some set of modal clauses. Let $\mathbf{TR}(F)$ denote the union of modal clause sets equivalent to T_G for all non-atomic subformulas G of F.

Theorem 4 *A formula F is valid iff a sequent $\mathbf{TR}(F) \vdash X_F$ is valid.*

Proof.

($\Rightarrow$) It is sufficient to prove that the sequent $F, \mathbf{TR}(F) \vdash X_F$ is valid; that can be easy established by replacement of equivalents.

($\Leftarrow$) If one substitutes G for all variables X_G corresponding to non-atomic subformulas of F into $\mathbf{TR}(F) \vdash X_F$, one obtains the sequent of the form $\Box T_1, \ldots \Box T_n \vdash F$, where all T_i are valid.

This concludes the proof.

Note. In case of $\top\Lambda$-validity,

- literals are
$$P \qquad \sim P \qquad \neg P \qquad \neg\sim P$$
- $\neg\neg L$ and L are identified.
- we use $\neg\neg\,\mathbf{TR}(F)$ instead of $\mathbf{TR}(F)$.

3.2 Sequent calculus

We need only a fragment of full $\top\mathbf{G}$ here, and we formulate it in a slightly different manner than above. The equivalence of both formulations can be easily proved.

We consider only connectives $\neg$, $\sim$, $\vee$, and $\Box$ as basic, the others are treated as abbreviations, using equivalencies from Table 2. Again, we consider formulas up to these equivalencies from Table 2 and (1). Moreover, disjuntions are treated up to number and order of occurrences of their components.

Axioms remain the same, the rules are as follows:

$$\frac{\Gamma \vdash \Delta, F_1, \ldots, F_n}{\Gamma \vdash \Delta, F_1 \vee \ldots \vee F_n}\ (\mathrm{RI}\vee) \qquad \frac{F_1, \Gamma_1 \vdash \Delta_1 \ldots F_n, \Gamma_n \vdash \Delta_n}{F_1 \vee \ldots \vee F_n, \Gamma \vdash \Delta}\ (\mathrm{LI}\vee)$$

$$\frac{F, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg\neg\sim F}\ (\mathrm{RI}\angle) \qquad \frac{\Gamma \vdash \Delta, F}{\neg\neg\sim F, \Gamma \vdash \Delta}\ (\mathrm{LI}\angle)$$

$$\frac{\Box\Gamma \vdash F}{\Box\Gamma \vdash \Box F}\ (\mathrm{RI}\Box) \qquad \frac{F, \Gamma \vdash \Delta}{\Box F, \Gamma \vdash \Delta}\ (\mathrm{LI}\Box)$$

$$\frac{\Gamma \vdash \Delta}{\Gamma_1, \Gamma \vdash \Delta, \Delta_1}\ (\mathrm{Thin})$$

($\Gamma = \bigcup \Gamma_i$, $\Delta = \bigcup \Delta_i$ in (LI$\vee$)).

A pruned derivation is one containing (Thin) only at the very end. It is obvious that any derivation can be transformed into the pruned derivation of the same sequent (by moving thinnings down).

3.3 Calculus $\top\mathbf{RR}$

Describe now the $\top\mathbf{RR}$ itself. Derivable objects of this calculus are modal clauses, inference rules are resolution rule and several additional rules which are applied in conjunction with the resolution rule.

Pairs of *contrary literals* are:

$$(L, \neg\neg\sim L), \qquad (\neg\neg L, \sim L), \qquad (\sim L, L).$$

In addition, if L_1 and L_2 are contrary literals, then modal literals $\Box L_1$ and $\Diamond L_2$ are also contrary.

The resolution rule is defined as

$$\frac{\Box(M \vee D_1) \quad \Box(\overline{M}_n \vee D_2)}{\Box(D_1 \vee D_2)}(\mathrm{Rp})$$

where M and $\overline{M}$ are contrary modal literals.

Additional rules mentioned above are

$$\frac{\Box(D \vee L)}{\Box(D \vee \Diamond L)}(\Diamond) \qquad \frac{\Box(\Diamond D \vee L)}{\Box(\Diamond D \vee \Box L)}(\Box)$$

Note. Calculus $\top\Lambda\mathbf{RR}$ differs from $\top\mathbf{RR}$ only in the notion of contrarity. Namely, $\top\Lambda\mathbf{RR}$-contrary literals are defined to be

$$(L, \neg L), \qquad (\sim L, \neg\sim L), \qquad (\neg\sim L, \sim L).$$

Theorem 5 *Let Γ be a set of modal clauses, D be a modal clause. If D is derivable from Γ in $\top\mathbf{RR}$, then $\Gamma \vdash D$ is derivable in $\top\mathbf{G}$*

Proof. One can use a direct translation of $\top\mathbf{RR}$-derivations into $\top\mathbf{G}$-derivations which is similar to that of Mints'. But we prefer a simple semantical argument: all $\top\mathbf{RR}$-rules are correct w.r.t. $\top$-validity.

Theorem 6 *Let Γ be a set of modal clauses, D be modal clause. If $\Gamma \vdash D$ is derivable in $\top\mathbf{G}$ then D' is derivable from Γ in $\top\mathbf{RR}$for some subset D' of D.*

Proof. The proof of the theorem can be done with the help of the standard technic developed by Mints (see [3]). The main idea is to translate $\top\mathbf{G}$-derivations into $\top\mathbf{RR}$-derivations.

1. Let D be $\Box(L_1 \vee \ldots \vee L_k)$. Then $\Gamma \vdash D$ is derivable iff $\Gamma \vdash L_1, \ldots, L_k$ is derivable. In a pruned derivation of the last sequent, the premise of the final thinning (if any) is of the form $\Gamma' \vdash L_{i_1}, \ldots, L_{i_l}$, where $\Gamma' \subseteq \Gamma$. So, it is sufficient to prove that we can transform the last derivation into $\top\mathbf{RR}$. This will be done in two steps. First we describe an intermediate calculus $\top\mathbf{RR}'$, and then we define translations from $\top\mathbf{G}$ into $\top\mathbf{RR}'$ and from $\top\mathbf{RR}'$ into $\top\mathbf{RR}$.

2. The calculus $\top\mathbf{RR}'$ deals with modal clauses. It has axioms of the form $\Box(\angle L \vee L)$ and $\Box(\angle L \vee \neg\neg L)$ where L is a literal. Its rules are $(\Diamond)$, $(\Box)$, and (Rp')

$$\frac{\Box(M_1 \vee \ldots \vee M_n); \quad \Box(\overline{M}_1 \vee D_1); \ldots; \Box(\overline{M}_n \vee D_n)}{\Box(D_1 \vee \ldots \vee D_n)}(\mathrm{Rp}')$$

where $(M_i, \overline{M}_i)$ are pairs of contrary modal literals of the form $(L, \angle L)$.

Arbitrary $\top\mathbf{RR}'$-derivation can be translated into the $\top\mathbf{RR}$-derivation in two steps. First, every application of rule (Rp') is translated into the sequence of (Rp) rules. Then, resolutions with axioms are dropped. It is easy to see that we will obtain the correct $\top\mathbf{RR}$-derivation.

3. Consider arbitrary pruned thinningless $\top\mathbf{G}$-inference d of a sequent $\Gamma, \Delta \vdash \Delta'$, where Γ is a set of modal clauses, Δ and Δ' are sets of modal literals. It can be easily seen that each sequent in d has the same form. Now, it is sufficient to show how $\top\mathbf{G}$-derivation of such a sequent can be transformed into $\top\mathbf{RR}'$-derivation of the clause $\Box(\angle\Delta \vee \Delta')$ from the set of clauses Γ.

We define the transformation by induction on d.

- d is an axiom $L \vdash L$.

  Then d is transformed into an axiom $\Box(\angle L \vee L)$

- d is an axiom $L \vdash \neg\neg L$.

  Then d is transformed into an axiom $\Box(\angle L \vee \neg\neg L)$

- The last rule of d is (RI$\vee$).

  This is not possible.

- The last rule of d is (LI$\vee$).

$$\frac{L_1, \Gamma'_1, \Delta_1 \vdash \Delta'_1; \ldots; L_n, \Gamma'_n, \Delta_n \vdash \Delta'_n}{L_1 \vee \ldots \vee L_n, \Gamma', \Delta \vdash \Delta';}$$

  is transformed to

$$\frac{\Box(L_1 \vee \ldots \vee L_n); \quad \Box(\angle L_1 \vee \angle\Delta_1 \vee \Delta'_1); \ldots; \Box(\angle L_n \vee \angle\Delta_n \vee \Delta'_n); \ldots;}{\Box(\angle\Delta \vee \Delta')}$$

- The last rule of d is (RI$\angle$).

  This is transformed into the identity rule.

- The last rule of d is (LI$\angle$).

  If the rule is applied to a formula in Γ', it is transformed into the repetition. In the other case,

$$\frac{\Gamma', \Delta \vdash \Delta', L}{\angle L, \Gamma', \Delta \vdash \Delta'} \quad \text{is transformed into} \quad \frac{\Box\angle L; \quad \Box(\angle\Delta \vee \Delta' \vee L)}{\Box(\Delta \vee \Delta')}$$

- The last rule of d is (RI$\Box$).

$$\frac{\Box\Gamma', \Box\Delta \vdash L}{\Box\Gamma', \Box\Delta \vdash \Box L} \quad \text{is transformed into} \quad \frac{\Box(\Diamond\angle\Delta \vee L)}{\Box(\Diamond\angle\Delta \vee \Box L)}$$

- The last rule of d is (LI$\Box$).

  If the rule is applied to a formula in Γ', it is transformed into the identity rule. In the other case,

$$\frac{L, \Gamma', \Delta \vdash \Delta'}{\Box L, \Gamma', \Delta \vdash \Delta'} \quad \text{is transformed into} \quad \frac{\Box(\angle L \vee \angle\Delta \vee \Delta')}{\Box(\Diamond\angle L, \vee\Delta \vee \Delta')}$$

This concludes the proof.

3.4 Main theorem

Theorem 7 *Formula F is $\top$-valid iff X_F is derivable from* $\mathbf{TR}(F)$ *in* $\top\mathbf{RR}$.

4 Situational logic

As it was already said, the language $\mathbf{SSL_1P}$ remains the same as above. We describe now the intended class of models. The worlds in a model will be called *situations* and the accessibility relation $\leq$ should correspond to the intended meaning of the relation "to be more general". Further on, we will suppose that any situation is completely determined by the set of all statements which are true in it. It follows that any situation S can be defined with the help of its diagram $\mathcal{D}[S]$ — two disjoint sets of atoms $\mathcal{D}^+[S]$ and $\mathcal{D}^-[S]$, which correspond to the positive and negative part of the diagram respectively.

Given a situation S and a propositional formula F, we will call F completely defined in S iff no subformula of F (including F itself) takes the value Λ in S. The intuitive meaning of $\leq$ leads then to the following condition:

> for any $\mathbf{SSL_1}$-structure and for any two situations S_1, S_2 in it, if $S_1 \leq S_2$ and a formula F is completely defined and valid in S_1, then F is completely defined and valid in S_2 as well.

In other words, any completely defined formula should preserve its logical value while "moving" from a more general situation to a less general one. (Note that this property is used rather often in "natural" mathematical proofs).

We consider the condition above as a formal definition that distinguishes the $\mathbf{SSL_1P}$-structures. To express it syntactically on the axiomatic level, we note that:

- it is sufficient to consider atoms only;
- there are only two opportunities for an atom A to violate this condition:

$$S_1 \leq S \text{ and } S_2 \leq S \text{ and } A \in \mathcal{D}^+[S_1] \text{ and } A \in \mathcal{D}^-[S_2];$$
$$S_1 \leq S_2 \text{ and } A \in \mathcal{D}^+[S_1] \cup \mathcal{D}^-[S_1] \text{ and } A \notin \mathcal{D}^+[S_2] \cup \mathcal{D}^-[S_2]$$

  where S, S_1, S_2 are arbitrary situations.

It results immediately in the following axiom schemata (*correctness axioms* for propositional variable P):

$$\mathrm{Corr}_1(P) \qquad \Box(\neg P \vee \neg{\sim}P \vee \Box(\neg\neg P \wedge \neg\neg{\sim}P))$$
$$\mathrm{Corr}_2(P) \qquad \Box(\Box\neg\neg{\sim}P \vee \Box\neg\neg P)$$

$\mathrm{CORR}(F)$ denotes the collection of all correctness axioms for all propositional variables of the formula F.

The definitions lead directly to the following theorem that allows to use resolution methods for proving statements in $\mathbf{SSL_1P}$.

Theorem 8 *A formula F is valid in* $\mathbf{SSL_1P}$ *iff a sequent* $CORR(F) \vdash F$ *is* $\top$*-valid.*

5 Concluding remarks

It would be in some sense more naturally to consider *three*-sided sequents for *three*-valued logic as it ws proposed in [1]. Indeed, there is nice semantics for this kind of sequents which is analogous to that for two-sided sequents. Moreover, there exists a cutfree Gentzen-style calculus for propositional fragment of our logic, and this calculus is complete w.r.t. semantics mentioned above. Unfortunately, we do not know how to extend these results to the full modal logic. Though it is quite clear how to translate three-sided sequents into two-sided ones and vice versa. Indeed, let a sequent $\Gamma|\Delta|\Sigma$ be given. It easy to see that it is valid (according to the semantics described in [1]) iff the sequent $\vdash \bigvee \chi_\perp(\Gamma) \vee \bigvee \chi_\Lambda(\Delta) \vee \bigvee \chi_\top(\Sigma)$ is $\top$-valid (and also $\top\Lambda$-valid as it never takes the value of Λ). Here $\chi_\perp$, χ_Λ, $\chi_\top$ are characteristic functions of corresponding logical values:

$$\chi_\perp(F) = \neg F, \quad \chi_\Lambda(F) = \neg\neg F \wedge \neg\neg{\sim}F, \quad \chi_\top(F) = \neg{\sim}F,$$

$\chi(\Gamma)$ stands for the set $\{\chi(F) \,|\, F \in \Gamma\}$ and $\bigvee \Gamma$ designates the disjunction of all elements of Γ. On the other hand, $\top$-validity ($\top\Lambda$-validity) of $\Gamma \vdash \Delta$ is semantically equivalent to the validity of $\Gamma|\Gamma|\Delta$ (resp. $\Gamma|\Delta|\Delta$). Nevertheless we do not see a direct way to rise up the inference rules of the calculi represented here to the calculus of three-sided sequents.

The results of this paper can be evidently generalized to the case when the set Ω of truth values is a finite chain. All we need to this aim is the sufficient number of "negations" (antimonotonic functions which permute top and bottom elements of Ω) among the logical connectives. We can consider in this case several notions of validity (in fact one less than the number of elements in Ω) depending on the place in Ω where the turnsign is put to. One has to provide only the opportunity to move a formula from the left hand side of a sequent to the right hand side and vice versa by means of combining of negations in the prefix.

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