# Systems with explicit rejections

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# Preliminaries

# A certain asymmetry

Grammar vs logic

- "*It is true that A*" corresponds to $True(A)$.
- "*It is false that A*" corresponds to $True(\neg A)$ as opposed to $False(A)$.

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*The Frege Point:*

We clearly need *assertion* and *negation* as primitives, thus primitive *rejection* is redundant.

The term is coined in Peter Geach (1965) *Assertion*.

# Who takes rejection seriously

Timothy Smiley (1996) *Rejection*.

Assertion and rejection as primitive notions.

Meta-linguistic notation $*A$ for "*A is rejected*" (not a connective).

Formula $A$ by itself is read as "*A is asserted*".

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A kind of natural deduction for classical logic.

Motivates *bilateralism*, see Ian Rumfitt (2000) *'Yes' and 'no'*

# A typical example

Nelson's logic N4 with *strong (constructible)* negation $\sim$.

D. Nelson (1949) *Constructible falsity*

A. Almukdad, D. Nelson (1984) *Constructible falsity and inexact predicates*

How does it take rejection seriously

i) relational semantics with two forcing relations;

ii) twist-structure algebraic semantics;

iii) some two-sorted sequent and display calculi;

iv) $\vdash_{\mathrm{N4}} A \leftrightarrow B$ is not a congruence but $\vdash_{\mathrm{N4}} (A \leftrightarrow B) \land (\sim A \leftrightarrow \sim B)$ is.

# 2-Intuitionistic logic

# Bi-intuitionistic logic

Bi-intuitionistic logic BiInt — a conservetive extension of Int with *co-implication* $\prec$.

C. Rauszer (1974) *Semi-boolean algebras and their applications to intuitionistic logic with dual operations*

Although BiInt is very natural semantically, proof theory is a problem:

- Most sequent calculi are either very non-standard or don't have cut elimination.
- There is no natural deduction system for BiInt (there is a non-standard one by Luca Tracnhini).
- Most natural proof theoretic framework for BiInt seems to be display calculi.

# 2-intuitionistic logic

2Int — a variant of bi-intuitionistic logic motivated by providing a natural deduction system for bi-intuitionistic connectives.

H. Wansing (2013) *Falsification, natural deduction and bi-intuitionistic logic*

The idea is to add rejection conditions for every connective as duals of assertion conditions for their duals.

Assertion/rejection of $\wedge, \vee, \rightarrow, \top, \bot$ can be treated as in N4.

# Natural deduction for 2Int

From proofs to refutations via dualization

$$\begin{array}{c} \\ \hline A \end{array} \qquad \mapsto \qquad \begin{array}{c} \\ \hline\hline A \end{array} .$$

Dualize all rules of intuitionistic natural deduction

$$\begin{array}{c} \\ \hline \top \end{array} \qquad \begin{array}{c} \\ \hline\hline \bot \end{array} \qquad \begin{array}{c} \\ \hline \bot \\ \hline A \end{array} \qquad \begin{array}{c} \\ \hline\hline \top \\ \hline\hline A \end{array}$$

$$\begin{array}{c} \begin{array}{c} \\ \hline A \end{array} \quad \begin{array}{c} \\ \hline B \end{array} \\ \hline A \wedge B \end{array} \qquad \begin{array}{c} \begin{array}{c} \\ \hline\hline A \end{array} \quad \begin{array}{c} \\ \hline\hline B \end{array} \\ \hline\hline A \vee B \end{array}$$

$$\begin{array}{c} \\ \hline A \wedge B \\ \hline A \end{array} \qquad \begin{array}{c} \\ \hline\hline A \vee B \\ \hline\hline A \end{array} \qquad \begin{array}{c} \\ \hline A \wedge B \\ \hline B \end{array} \qquad \begin{array}{c} \\ \hline\hline A \vee B \\ \hline\hline B \end{array}$$

$$\begin{array}{c} \\ \hline A \\ \hline A \vee B \end{array} \qquad \begin{array}{c} \\ \hline\hline A \\ \hline\hline A \wedge B \end{array} \qquad \begin{array}{c} \\ \hline B \\ \hline A \vee B \end{array} \qquad \begin{array}{c} \\ \hline\hline B \\ \hline\hline A \wedge B \end{array}$$

# Natural deduction for 2Int

$[A]$ is a discharged assumption about assertion,
$[\![A]\!]$ is a discharged assumption about rejection.

$$\dfrac{\overline{A \vee B} \qquad \dfrac{\begin{matrix}[A]\\ \vdots\end{matrix}}{C} \qquad \dfrac{\begin{matrix}[B]\\ \vdots\end{matrix}}{C}}{C} \qquad\qquad \dfrac{\overline{A \wedge B} \qquad \dfrac{\begin{matrix}[\![A]\!]\\ \vdots\end{matrix}}{C} \qquad \dfrac{\begin{matrix}[\![B]\!]\\ \vdots\end{matrix}}{C}}{C}$$

$$\dfrac{\overline{A} \qquad \overline{A \to B}}{B} \qquad \dfrac{\overline{A} \qquad \overline{B \prec A}}{B} \qquad \dfrac{\dfrac{\begin{matrix}[A]\\ \vdots\end{matrix}}{B}}{A \to B} \qquad \dfrac{\dfrac{\begin{matrix}[\![A]\!]\\ \vdots\end{matrix}}{B}}{B \prec A}$$

# Natural deduction for 2Int

*Q:* how do we refute implicative formulas?

*A:* like in Nelson's logics.

$$\begin{array}{c} \overline{A} \qquad \overline{\overline{B}} \\ \hline\hline A \to B \end{array} \qquad\qquad \begin{array}{c} \overline{\overline{A \to B}} \\ \hline A \end{array} \qquad\qquad \begin{array}{c} \overline{\overline{A \to B}} \\ \hline\hline B \end{array}$$

*Q:* how do we assert co-implicative formulas?

*A:* dualize.

$$\begin{array}{c} \overline{A} \qquad \overline{\overline{B}} \\ \hline A \prec B \end{array} \qquad\qquad \begin{array}{c} \overline{A \prec B} \\ \hline A \end{array} \qquad\qquad \begin{array}{c} \overline{A \prec B} \\ \hline\hline B \end{array}$$

# Two consequence relations of 2Int

Assertion-based consequence $\Gamma : \Delta \vdash^{+}_{\mathsf{N2Int}} A$:

$$\dfrac{\dfrac{\overline{B} \quad B \in \Gamma}{\vdots} \qquad \dfrac{\overline{\overline{C}} \quad C \in \Delta}{\vdots}}{A}$$

*Intuitively:* *"if all formulas in $\Gamma$ are proved and all formulas in $\Delta$ are refuted, then $A$ is proved".*

# Two consequence relations of 2Int

Assertion-based consequence $\Gamma : \Delta \vdash^{+}_{\mathsf{N2Int}} A$:

$$
\begin{array}{c}
\underline{\overline{B} \qquad B \in \Gamma} \qquad \underline{\overline{\overline{C}} \qquad C \in \Delta} \\
\vdots \qquad\qquad\qquad \vdots \\
\hline
A
\end{array}
$$

*Intuitively: "if all formulas in $\Gamma$ are proved and all formulas in $\Delta$ are refuted, then $A$ is proved".*

Rejection-based consequence $\Gamma : \Delta \vdash^{-}_{\mathsf{N2Int}} A$:

$$
\begin{array}{c}
\underline{\overline{B} \qquad B \in \Gamma} \qquad \underline{\overline{\overline{C}} \qquad C \in \Delta} \\
\vdots \qquad\qquad\qquad \vdots \\
\hline\hline
A
\end{array}
$$

*Intuitively: "if all formulas in $\Gamma$ are proved and all formulas in $\Delta$ are refuted, then $A$ is refuted".*

# Semantics for 2Int

# 2Int-models

A 2Int-*frame* is a partially ordered set $\mathcal{W} = \langle W, \leq \rangle$.

A 2Int-*model* $\mu = \langle \mathcal{W}, v^+, v^- \rangle$ is a 2Int-frame together with two valutations satisfying *intuitionistic heredity*:

$$x \in v^\delta(p) \text{ and } x \leq y \text{ implies } y \in v^\delta(p), \quad \delta \in \{+, -\}.$$

*Remark:* these models are exactly the same as N4-models, except...

# Two forcing relations

For a 2Int-model $\mu = \langle W, \leq, v^+, v^- \rangle$ and $x \in W$ put

$$\mu, x \models^+ A \to B \iff \forall y \geq x\, (\mu, y \models^+ A \Rightarrow \mu, y \models^+ B);$$
$$\mu, x \models^- A \to B \iff \mu, x \models^+ A \text{ and } \mu, x \models^- B;$$
$$\mu, x \models^+ A \prec B \iff \mu, x \models^+ A \text{ and } \mu, x \models^- B;$$
$$\mu, x \models^- A \prec B \iff \forall y \geq x\ (\mu, y \models^- B \Rightarrow \mu, y \models^- A);$$

For a set of formulas, $\Gamma$, put:

$$\mu, x \models^+ \Gamma \iff \mu, x \models^+ A \text{ for all } A \in \Gamma;$$
$$\mu, x \models^- \Gamma \iff \mu, x \models^- A \text{ for all } A \in \Gamma;$$

# Two negations

We can define *intuitionistic negation* $\neg A := A \to \bot$

$$\mu, x \models^{+} \neg A \iff \forall y \geq x : \ \mu, y \not\models^{+} A;$$
$$\mu, x \models^{-} \neg A \iff \mu, x \models^{+} A;$$

and *dual intuitionistic negation* $\urcorner A := \top \prec A$

$$\mu, x \models^{+} \urcorner A \iff \mu, x \models^{-} A;$$
$$\mu, x \models^{-} \urcorner A \iff \forall y \geq x : \ \mu, x \not\models^{-} A.$$

Observe that

i) dual negation $\urcorner$ acts as a switch from assertion to rejection;

ii) negation $\neg$ acts as a switch from rejection to assertion.

# Semantics for 2Int

Two semantic consequence relations

$\Gamma : \Delta \models^{+}_{\text{N2Int}} A$ if for any 2Int-model $\mu = \langle W, \leq, v^{+}, v^{-} \rangle$

$$\forall x \in W\ (\mu, x \models^{+} \Gamma \text{ and } \mu, x \models^{-} \Delta \implies \mu, x \models^{+} A).$$

$\Gamma : \Delta \models^{-}_{\text{N2Int}} A$ if for any 2Int-model $\mu = \langle W, \leq, v^{+}, v^{-} \rangle$:

$$\forall x \in W\ (\mu, x \models^{-} \Gamma \text{ and } \mu, x \models^{-} \Delta \implies \mu, x \models^{-} A).$$

Completeness [Wansing2013]

$$\Gamma : \Delta \vdash^{+}_{\text{N2Int}} A \iff \Gamma : \Delta \models^{+}_{\text{N2Int}} A;$$
$$\Gamma : \Delta \vdash^{-}_{\text{N2Int}} A \iff \Gamma : \Delta \models^{-}_{\text{N2Int}} A.$$

# Replacement for 2Int

*Remark:* 2Int shares N4's problems with replacement.

*Weak replacement* for 2Int:

$$\frac{\overline{A \leftrightarrow B} \qquad \overline{\neg A \leftrightarrow \neg B}}{C[A] \leftrightarrow C[B]},$$

*Positive replacement* for 2Int:

$$\frac{\overline{A \leftrightarrow B}}{C[A] \leftrightarrow C[B]}$$, where $C$ is $\prec$-free. ·

# Replacement for 2Int

Put $A \succ\!\!\prec B := (A \prec B) \vee (B \prec A)$.

*Dual weak replacement* for 2Int:

$$\dfrac{\overline{\overline{A \succ\!\!\prec B}} \qquad \overline{\overline{\neg A \succ\!\!\prec \neg B}}}{\overline{C[A] \succ\!\!\prec C[B]}} \, ,$$

*Dual positive replacement* for 2Int:

$$\dfrac{\overline{\overline{A \succ\!\!\prec B}}}{\overline{C[A] \succ\!\!\prec C[B]}} \, , \text{ where } C \text{ is } \to\text{-free.} \cdot$$

# Change of perspective

# Internalizing attitudes

A *signed formula* is just $A^{+}$, $A^{-}$, where $A$ is a formula.

$A^{+}$ corresponds to *"A is asserted"*.

$A^{-}$ corresponds to *"A is rejected"*.

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Use $\bar{A}$, $\bar{B}$, $\bar{C}$ for signed formulas;

Use $\bar{\Gamma}$, $\bar{\Delta}$ for sets of signed formulas.

# A simple correspondence

For a *set of formulas*, $\Gamma$, put

$$\Gamma^{+} = \{A^{+} \mid A \in \Gamma\} \qquad \Gamma^{-} = \{A^{-} \mid A \in \Gamma\}.$$

For a *set of signed formulas*, $\bar{\Gamma}$, put

$$\bar{\Gamma}_{+} := \{A \mid A^{+} \in \bar{\Gamma}\} \qquad \bar{\Gamma}_{-} := \{A \mid A^{-} \in \bar{\Gamma}\}.$$

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From *pairs of sets of formulas* to *sets of signed formulas*:

$$\Gamma : \Delta \quad \mapsto \quad \Gamma^{+} \cup \Delta^{-}.$$

From *sets of signed formulas* to *pairs of sets of formulas*:

$$\bar{\Gamma} \quad \mapsto \quad \bar{\Gamma}_{+} : \bar{\Gamma}_{-}.$$

# Rewriting consequence relations of 2Int

*Step 1:* identify antecedent with a set of signed formulas;

*Step 2:* shift the sign from turnstile onto formula in the consequent.

$$\Gamma : \Delta \vdash^{+}_{\mathsf{N2Int}} A \qquad\qquad \Gamma : \Delta \vdash^{-}_{\mathsf{N2Int}} A$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\Gamma^{+} \cup \Delta^{-} \vdash^{s}_{\mathsf{N2Int}} A^{+} \qquad\qquad \Gamma^{+} \cup \Delta^{-} \vdash^{s}_{\mathsf{N2Int}} A^{-}$$

$$\searrow \qquad\qquad \swarrow$$

$$\bar{\Gamma} \vdash^{s}_{\mathsf{N2Int}} \bar{A}$$

*Result:* a single consequence relation on signed formulas.

*Remark:* can do the same with semantic consequence.

# Some familiar looking properties

*Reflexivity:*
If $\bar{A} \in \bar{\Gamma}$, then $\bar{\Gamma} \vdash^{s}_{\mathrm{N2Int}} \bar{A}$.

*Monotonicity:*
If $\bar{\Gamma} \vdash^{s}_{\mathrm{N2Int}} \bar{A}$ and $\bar{\Gamma} \subseteq \bar{\Delta}$ then $\bar{\Delta} \vdash^{s}_{\mathrm{N2Int}} \bar{A}$.

*Transitivity:*
If $\bar{\Gamma} \vdash^{s}_{\mathrm{N2Int}} \bar{B}$ for all $\bar{B} \in \bar{\Delta}$ and $\bar{\Delta} \vdash^{s}_{\mathrm{N2Int}} \bar{A}$ then $\bar{\Gamma} \vdash^{s}_{\mathrm{N2Int}} \bar{A}$.

*Compactness:*
If $\bar{\Gamma} \vdash^{s}_{\mathrm{N2Int}} \bar{A}$ then $\bar{\Delta} \vdash^{s}_{\mathrm{N2Int}} \bar{A}$ for some finite $\bar{\Delta} \subseteq \bar{\Gamma}$.

*Structurality:*
If $\bar{\Gamma} \vdash^{s}_{\mathrm{N2Int}} \bar{A}$ then $\{s(\bar{B}) \mid \bar{B} \in \bar{\Gamma}\} \vdash^{s}_{\mathrm{N2Int}} s(\bar{A})$
for any substitution $s$.

Here, $s(A^{\delta}) := (s(A))^{\delta}$.

# Replacement theorems

# Signed equivalences and subformulas

*Equivalence* of signed formulas

$$\bar{A} \equiv \bar{B} \iff \bar{A} \vdash^{s}_{\mathrm{N2Int}} \bar{B} \text{ and } \bar{B} \vdash^{s}_{\mathrm{N2Int}} \bar{A}.$$

Define $\bar{B} \preceq \bar{A}$ — *"$\bar{B}$ is an occurrence of a signed subformula in $\bar{A}$"*:

i) $\bar{A} \preceq \bar{A}$;

ii) if $(B \circ C)^{\delta} \preceq \bar{A}$, then $B^{\delta}, C^{\delta} \preceq \bar{A} \quad \circ \in \{\wedge, \vee\}, \delta \in \{+, -\}$;

iii) if $(B \to C)^{+} \preceq \bar{A}$, then $B^{+}, C^{+} \preceq \bar{A}$;

iv) if $(B \to C)^{-} \preceq \bar{A}$, then $B^{+}, C^{-} \preceq \bar{A}$;

v) if $(B \prec C)^{+} \preceq \bar{A}$, then $B^{+}, C^{-} \preceq \bar{A}$;

vi) if $(B \prec C)^{-} \preceq \bar{A}$, then $B^{-}, C^{-} \preceq \bar{A}$.

# Signed replacement

**Theorem.**

Suppose $\epsilon \in \{+, -\}$ and $p^\epsilon \preceq \bar{A}$, then if $B^\epsilon$ and $C^\epsilon$ are equivalent, then so are $\bar{A}(B^\epsilon)$ and $\bar{A}(C^\epsilon)$:

$$\frac{B^\epsilon \equiv C^\epsilon}{\bar{A}(B^\epsilon) \equiv \bar{A}(C^\epsilon)} .$$

$\bar{A}(B)$ is the result of replacing corresponding $p$ with $B$.

$\bar{A}(C)$ is the result of replacing corresponding $p$ with $C$.

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*Intuition:* we can replace signed formulas by equivalent signed formulas as long as we respect the attitudes (signs).

*Remark:* weak replacement, positive replacement and their duals all follow from signed replacement.

# A Hilbert-style calculus that takes rejection seriously

# Idea

Natural deduction for 2Int consists of

- natural deduction rules for intuitionistic logic (assertion);
- their duals (rejection);
- interplay rules.

*Q.* Can we replace first two with Hilbert-style calculi for intuitionistic and dual intuitionistic logic to get Hilbert-style calculus for both assertion and rejection?

*A.* Kind of.

# Signed Hilbert-style calculus H2Int

Initial axioms of H2Int:

- intuitionistic axioms with plus sign;
- duals of intuitionistic axioms with minus sign.

Modus ponens and its dual:

$$\frac{A^+ \qquad (A \to B)^+}{B^+}\,, \qquad \frac{(B \prec A)^- \qquad A^-}{B^-}\,.$$

Interplay rules:

$$\frac{A^+ \qquad B^-}{(A \prec B)^+}\,, \qquad \frac{(A \prec B)^+}{B^-}\,,$$

$$\frac{A^+ \qquad B^-}{(A \to B)^-}\,, \qquad \frac{(A \to B)^-}{A^+}\,.$$

# Signed Hilbert-style calculus H2Int

Additional axioms of H2Int:

$$(A \prec B) \leftrightarrow (A \wedge \invneg B)^{+}, \qquad (A \to B) \succ\!\!\prec (B \vee \neg A)^{-},$$
$$\invneg (A \to B) \leftrightarrow (A \wedge \invneg B)^{+}, \qquad \neg(A \prec B) \succ\!\!\prec (B \vee \neg A)^{-},$$
$$\invneg (A \prec B) \to (\invneg B \to \invneg A)^{+}, \qquad (\neg A \prec \neg B) \prec \neg(B \to A)^{-}.$$

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A kind of signed canonical models method gives us

**Theorem.**

$$\bar{\Gamma} \vdash^{s}_{\mathrm{H2Int}} \bar{A} \iff \bar{\Gamma} \models^{s}_{\mathrm{2Int}} \bar{A}.$$

# General framework

# Signed consequence relations

A *signed consequence relation* is a relation

$$\vdash^s \subseteq P(\mathrm{For}^s\mathcal{L}) \times \mathrm{For}^s\mathcal{L}$$

where $\mathrm{For}^s\mathcal{L}$ are all signed $\mathcal{L}$-formulas, satisfying

*Reflexivity:* if $\bar{A} \in \bar{\Gamma}$, then $\bar{\Gamma} \vdash^s \bar{A}$.

*Monotonicity:* if $\bar{\Gamma} \vdash^s \bar{A}$ and $\bar{\Gamma} \subseteq \bar{\Delta}$ then $\bar{\Delta} \vdash^s \bar{A}$.

*Transitivity:* if $\bar{\Gamma} \vdash^s \bar{B}$ for all $\bar{B} \in \bar{\Delta}$ and $\bar{\Delta} \vdash^s \bar{A}$ then $\bar{\Gamma} \vdash^s \bar{A}$.

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It is *compact*, if $\bar{\Gamma} \vdash^s \bar{A}$ then $\bar{\Delta} \vdash^s \bar{A}$ for some finite $\bar{\Delta} \subseteq \bar{\Gamma}$;

and *structural*, if $\bar{\Gamma} \vdash^s \bar{A}$ implies $s(\bar{\Gamma}) \vdash^s s(\bar{A})$ for any substitution $s$.

# Wansing's approach

Wansing develops two-consequence relations approach to taking rejection seriously, which

> *leads us to understanding a logic not as a pair* $(\mathcal{L}, \vdash)$ *consisting of a language and a consequence relation, but as a triple* $(\mathcal{L}, \vdash, \vdash^d)$ *consisting of a language, a consequence relation, and a dual consequence relation [...]*

where $\vdash$ corresponds to assertion and $\vdash^d$ to rejection.

H. Wansing (2017) *A more general general proof theory.*

Signed consequences generalize this approach since

$$\Gamma \vdash A :\iff \Gamma^+ \vdash^s A^+; \qquad \Gamma \vdash^d A :\iff \Gamma^- \vdash^s A^-.$$

# Bochman's biconsequences

*Biconsequences* are relations $\vdash^b \subseteq (\mathrm{For}\,\mathcal{L})^4$, satisfying some properties, where

$$\Gamma_1 : \Gamma_2 \vdash^b \Delta_1 : \Delta_2$$

holds *"if all propositions from $\Gamma_1$ are true and all proposition from $\Gamma_2$ are false, then either one of the proposition from $\Delta_1$ is true or one of the propositions from $\Delta_2$ is false"*.

A. Bochman (1998) *Biconsequence relations.*

Since we know how to encode a pair of sets of formulas into a set of signed formulas, biconsequences are to signed consequence what Scott consequence relations are to Tarskian consequence relations.

# Unilateral components

With any signed consequence $\vdash^s$ we associate its

*positive component* $\vdash^+$:

$$\Gamma \vdash^+ A :\iff \Gamma^+ \vdash^s A^+;$$

*negative component* $\vdash^-$:

$$\Gamma \vdash^- A :\iff \Gamma^- \vdash^s A^-.$$

Both components are Tarskian consequence relations.

# Nelson’s logic bilaterally

# Axiomatics

N4 is the positive fragment of intuitionistic logic +

$$\sim(A \wedge B) \leftrightarrow \sim A \vee \sim B; \qquad \sim\sim A \leftrightarrow A;$$
$$\sim(A \vee B) \leftrightarrow \sim A \wedge \sim B; \qquad \sim(A \to B) \leftrightarrow A \wedge \sim B.$$

**Unilateraly** its positive fragment coincides with the positive fragment of intuitionsitic logic.

One can think of $\sim$ as internalizing rejection:

$$(\sim A)^+ \equiv A^- \quad \text{and} \quad (\sim A)^- \equiv A^+.$$

# Bilateral natural deduction for N4 ($\wedge$)

$$(i\wedge^+)\ \dfrac{A^+ \qquad B^+}{(A \wedge B)^+} \qquad (e\wedge^+)\ \dfrac{(A \wedge B)^+}{A^+} \qquad (e\wedge^+)\ \dfrac{(A \wedge B)^+}{B^+}$$

$$(i\wedge^-)\ \dfrac{A^-}{(A \wedge B)^-} \qquad (i\wedge^-)\ \dfrac{B^-}{(A \wedge B)^-}$$

$$(e\wedge^-)\ \dfrac{(A \wedge B)^- \qquad \begin{array}{c} [A^-] \\ \hline \vdots \\ \hline \bar{C} \end{array} \qquad \begin{array}{c} [B^-] \\ \hline \vdots \\ \hline \bar{C} \end{array}}{\bar{C}}$$

# Bilateral natural deduction for N4 ($\rightarrow$ and $\sim$)

$$(i \rightarrow^{+})\ \frac{\begin{array}{c}\underline{[A^{+}]}\\ \vdots \\ \overline{B^{+}}\end{array}}{(A \rightarrow B)^{+}} \qquad (e \rightarrow^{+})\ \frac{A^{+} \qquad (A \rightarrow B)^{+}}{B^{+}}$$

$$(i \rightarrow^{-})\ \frac{A^{+} \qquad B^{-}}{(A \rightarrow B)^{-}} \qquad (e \rightarrow^{-})\ \frac{(A \rightarrow B)^{-}}{A^{+}} \qquad (e \rightarrow^{-})\ \frac{(A \rightarrow B)^{-}}{B^{-}}$$

$$(i \sim^{+})\ \frac{A^{-}}{\sim A^{+}} \qquad (e \sim^{+})\ \frac{\sim A^{+}}{A^{-}}$$

$$(i \sim^{-})\ \frac{A^{+}}{\sim A^{-}} \qquad (e \sim^{-})\ \frac{\sim A^{-}}{A^{+}}$$

# Positive fragment of Nelson's logic

Denote this system by $\mathsf{N4^s}$. Then we can naturally define $\vdash^{s}_{\mathsf{N4}}$.

The positive component of $\vdash^{s}_{\mathsf{N4}}$ is the usual consequence of N4.

Let $\mathsf{PN4^s}$ be $\mathsf{N4^s}$ minus rules for $\sim$ (a bilateral positive fragment).

Then, e.g.,

$$A^{+}, B^{-} \vdash^{s}_{\mathsf{PN4}} (A \to B)^{+}.$$

**Bilaterally**, positive fragment of N4 still has meaningful rejection.

Compositionality and definitional equivalence

# Compositionality

*Q.* For an n-ary connective $f$ what does

$$\text{assertion } \mathcal{A}(f(p_1, \ldots, p_n)) \quad \text{and} \quad \text{rejection } \mathcal{R}(f(p_1, \ldots, p_n))$$

depend upon?

*General compositionality*: on all of the

$$\mathcal{A}(p_1),\ \mathcal{R}(p_1),\ \ldots,\ \mathcal{A}(p_n),\ \mathcal{R}(p_n).$$

*Polarized compositionality*: for each $p_i$ chose one of

$$\mathcal{A}(p_i) \quad \text{or} \quad \mathcal{R}(p_i).$$

according to a *polarity* function.

# Polarity

For instance, in N4
$\mathcal{A}(A \to B)$ depends on $\mathcal{A}(A)$ and $\mathcal{A}(B)$;
$\mathcal{R}(A \to B)$ depends on $\mathcal{R}(A)$ and $\mathcal{A}(B)$.

*Polarity* $\alpha$ maps n-ary connective $f$ and a sign $\delta \in \{+, -\}$ into

$$\alpha(f, \delta) = \langle \alpha(f, \delta, 1), \ldots, \alpha(f, \delta, n) \rangle,$$

where $\alpha(f, \delta, i) \in \{+, -\}$.

Intuitively, say,

$$\alpha(f, +, 1) = -$$

means that to assert $f(p_1, \ldots, p_n)$ we need to know how to reject $p_1$.

# Polarity for N4

Polarity can be naturally defined for all systems with strong negation and for 2Int.

For instance, for N4 one can put:

$$
\begin{array}{ll}
\alpha(\wedge,+) := \langle +,+\rangle; & \alpha(\wedge,-) := \langle -,-\rangle; \\
\alpha(\vee,+) := \langle +,+\rangle; & \alpha(\vee,-) := \langle -,-\rangle; \\
\alpha(\rightarrow,+) := \langle +,+\rangle; & \alpha(\rightarrow,-) := \langle +,-\rangle; \\
\alpha(\sim,+) := \langle -\rangle; & \alpha(\sim,-) := \langle +\rangle.
\end{array}
$$

# On the way to definitional equivalence

Let us fix two language-polarity-signed consequence triples:

$$\langle \mathcal{L}_1, \alpha_1, \vdash_1^s \rangle, \qquad \langle \mathcal{L}_2, \alpha_2, \vdash_2^s \rangle$$

A *general base $(\mathcal{L}_1, \mathcal{L}_2)$-translation* $\theta$ maps any n-ary connective $f \in \mathcal{L}_1$ and a sign $\delta \in \{+, -\}$ to a $\mathcal{L}_2$-formula

$$\theta^\delta(f)(p_1, \ldots, p_{2n}).$$

A *polarized base $(\mathcal{L}_1, \mathcal{L}_2)$-translation* $\theta$ maps any n-ary connective $f \in \mathcal{L}_1$ and a sign $\delta \in \{+, -\}$ to a $\mathcal{L}_2$-formula

$$\theta^\delta(f)(p_1, \ldots, p_n).$$

# General structural translations

Let $\theta$ be a general base $(\mathcal{L}_1, \mathcal{L}_2)$-translation. let

For a sign $\delta \in \{+, -\}$ and an $\mathcal{L}_1$ formula $A$ define a $\mathcal{L}_2$-formula $\Theta^\delta(A)$:

- $\Theta^\delta(p) := p$ and
- $\Theta^\delta(f(A_1, \dots, A_n)) :=$

$$\theta^\delta(f)(\Theta^+(A_1),\ \Theta^-(A_1),\ \dots,\ \Theta^+(A_n),\ \Theta^-(A_n)).$$

Finally, $\Theta^s(A^\delta) := (\Theta^\delta(A))^\delta$. Then

$$\Theta^s : \mathrm{For}^s\mathcal{L}_1 \to \mathrm{For}^s\mathcal{L}_2$$

is a *general (structural signed) $(\mathcal{L}_1, \mathcal{L}_2)$-translation*.

# Polarized structural translations

Let $\theta$ be a polarized base $(\mathcal{L}_1, \mathcal{L}_2)$-translation. let

For a sign $\delta \in \{+, -\}$ and an $\mathcal{L}_1$ formula $A$ define a $\mathcal{L}_2$-formula $\Theta^\delta(A)$:

- $\Theta^\delta(p) := p$ and
- $\Theta^\delta(f(A_1, \dots, A_n)) :=$

$$\theta^\delta(f)(\Theta^{\alpha_1(f,\delta,1)}(A_1), \dots, \Theta^{\alpha_1(f,\delta,n)}(A_n)).$$

Finally, $\Theta^s(A^\delta) := (\Theta^\delta(A))^\delta$. Then

$$\Theta^s : \mathrm{For}^s\mathcal{L}_1 \to \mathrm{For}^s\mathcal{L}_2$$

is a *polarized (structural signed) $(\mathcal{L}_1, \mathcal{L}_2)$-translation*.

# Definitional equivalence

Signed consequences $\vdash^s_1$ and $\vdash^s_2$ are *definitionally equivivalent w.r.t. general/polarized translations*, if

- there is a general/polarized $(\mathcal{L}_1, \mathcal{L}_2)$-translation $\Theta^s$;
- there is a general/polarized $(\mathcal{L}_2, \mathcal{L}_1)$-translation $\Lambda^s$;
- $\bar{\Gamma} \vdash^s_1 \bar{A} \iff \Theta^s(\bar{\Gamma}) \vdash^s_2 \Theta^s(\bar{A})$;
- $\bar{\Delta} \vdash^s_2 \bar{B} \iff \Lambda^s(\bar{\Delta}) \vdash^s_1 \Lambda^s(\bar{B})$;
- $\bar{A} \dashv\vdash^s_1 \Lambda^s\Theta^s(\bar{A})$;
- $\bar{B} \dashv\vdash^s_2 \Theta^s\Lambda^s(\bar{B})$.

# Slightly informal facts

Fact 1: both notions generalize usual definitional equivalence.

Fact 2: general is more general than polarized.

Fact 3: both come with their own problems.

# One example

Bilattice connective $\otimes$

$$A \otimes B \leftrightarrow A \wedge B; \qquad \sim (A \otimes B) \leftrightarrow \sim (A \vee B);$$

is definable in ($\{\wedge, \vee\}$-fragment of) N4.

Polarity for $\otimes$:

$$\alpha(+, \otimes) = \langle +, + \rangle; \qquad \alpha(-, \otimes) = \langle -, - \rangle.$$

Then the polarized definition is:

$$\Theta^{+}(A \otimes B) := \Theta^{+}(A) \wedge \Theta^{+}(B);$$

$$\Theta^{-}(A \otimes B) := \Theta^{-}(A) \vee \Theta^{-}(B).$$

**Unilaterally**, one can needs additional constants *neither* and *both* to define $\otimes$ in N4.

# N4 and 2Int are definitionally equivalent

Defining $\prec$ in N4:

$$\Theta^{+}(A \prec B) := \Theta^{+}(A) \wedge \Theta^{-}(B);$$
$$\Theta^{-}(A \prec B) := \sim (\sim \Theta^{-}(B) \rightarrow \sim \Theta^{-}(A)).$$

Defining $\sim$ in 2Int:

$$\Lambda^{+}(\sim A) := \top \prec \Lambda^{-}(A);$$
$$\Lambda^{-}(\sim A) := \Lambda^{+}(A) \rightarrow \bot.$$

# Polarized problems

In practice, polarized definition covers most natural cases.

But, what if there is a connective $f(p_1, \ldots, p_n)$ such that, say,

$$\mathcal{A}(f(p_1, \ldots, p_n))$$

depends both on $\mathcal{A}(p_1)$ and on $\mathcal{R}(p_1)$?

*Strong implication* $A \Rightarrow B := (A \to B) \wedge (\sim B \to \sim A)$ is such a connective.

Strong implication can be defined

- unilaterally;
- bilaterally wrt general definitions;
- but not bilaterally wrt polarized definitions.

# Trivial definitions

Suppose we want to keep a connective in place by giving it a trivial definition.

In the *polarized* setting that is easy:

$$\Theta^{\delta}(f(A_1, \ldots, A_n) = f(\Theta^{\alpha(\delta,f,1)}(A_1), \ldots, \Theta^{\alpha(\delta,f,n)}(A_n)).$$

But in the *general* setting it is entirely unclear.

The definition of an n-ary connective is a formula of 2n variables.

So, for instance,

$$\theta^{+}(\rightarrow)(p_1, p_2, p_3, p_4) = p_1 \rightarrow p_2;$$

$$\theta^{-}(\rightarrow)(p_1, p_2, p_3, p_4) = p_1 \rightarrow p_4.$$

# Defining strong negation

Clearly, in polarized setting one can define strong negation $\sim$ s.t.

$$\sim A^{+} \dashv\vdash^{s} A^{-}; \qquad \sim A^{-} \dashv\vdash^{s} A^{+}.$$

As long as we have formulas $B$ and $C$ s.t.

$$B(A)^{+} \dashv\vdash^{s} A^{-}; \qquad C(A)^{-} \dashv\vdash^{s} A^{+}.$$

Moreover, under some (semantically phrased) conditions concerning compositionality signed consequences can be *conservatively* expanded by the strong negation.

Thank you!