Boolean Constraint Satisfaction Problems

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or: When does Post's Lattice Help?

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Γ – a finite set of Boolean relations

Constraint: $R(x_1, \ldots, x_n)$ for $R \in \Gamma$, $x_1, \ldots, x_n$ propos. variables

Γ-formula: Conjunction of constraints over Γ

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Example: $\mathrm{R}_{\text{1-IN-3}} = \big\{(0,0,1),(0,1,0),(1,0,0)\big\}$.

Then: $\{\mathrm{R}_{\text{1-IN-3}}\}$-formulas = instances of 1-IN-3-SAT.

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Example: $\mathrm{R}_{\text{1-IN-3}} = \{(0,0,1),(0,1,0),(1,0,0)\}$.

Then: $\{\mathrm{R}_{\text{1-IN-3}}\}$-formulas $=$ instances of 1-IN-3-SAT.

CSP(Γ):

Input: a propositional Γ-formula F

Question: Is F satisfiable?

Comparing Complexities of CSPs

Goal: Determine the computational complexity of CSP(Γ) as a function of Γ!

- Determine Γ_0 such that CSP(Γ_0) is NP-complete and conclude that CSP(Γ) is NP-complete for all "harder" Γ as well.
- Determine Γ_1 such that CSP(Γ_1) is tractable and conclude that CSP(Γ) is tractable for all "easier" Γ as well.

Need a way to compare complexity of CSP(Γ) for different Γ.

Reductions

Question: When does $\mathrm{CSP}(\Gamma)$ reduce to $\mathrm{CSP}(\Gamma')$?

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Develop a reasonable notion of the class of relations that can be implemented by Γ'.

Let $\langle \Gamma \rangle$ be the relational clone (or co-clone) generated by Γ, i.e.,

- $\langle \Gamma \rangle$ contains the equality relation and all relations in Γ.
- $\langle \Gamma \rangle$ is closed under primitive positive definitions, i.e.,
  if ϕ is a $\langle \Gamma \rangle$-formula and
  $R(x_1, \ldots, x_n) \equiv \exists y_1 \ldots y_\ell \ \phi(x_1, \ldots, x_n, y_1, \ldots, y_\ell)$
  then $R \in \langle \Gamma \rangle$.
  (Such R are also called conjunctive queries over $\langle \Gamma \rangle$.)

$\langle \Gamma \rangle$ is called the expressive power of Γ.

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- Replace every constraint from Γ by its defining existentially quantified $\left(\Gamma' \cup \{=\}\right)$-formula.
- Delete existential quantifiers.
- Delete equality clauses and replace all variables that are connected via a chain of equality constraints by a common new variable (undirected graph accessibility problem).

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F' is a Γ'-formula.

Then: F is satisfiable iff F' is satisfiable.

- If $\Gamma \subseteq \langle \Gamma' \rangle$, then $\mathrm{CSP}(\Gamma) \leq_m^{\log} \mathrm{CSP}(\Gamma')$.
- If $\langle \Gamma \rangle = \langle \Gamma' \rangle$, then $\mathrm{CSP}(\Gamma) \equiv_m^{\log} \mathrm{CSP}(\Gamma')$,

  i.e., the complexity of $\mathrm{CSP}(\Gamma)$ depends only on $\langle \Gamma \rangle$.

We only have to study co-clones in order to obtain a full classification.

"Galois connection helps for satisfiability."

Relational Clones and CSPs

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What co-clones are there?

Closure Properties of Relations

Let $f\colon \{0,1\}^m \rightarrow \{0,1\}$, $R \subseteq \{0,1\}^n$.

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x_1 & = & x_{1,1} & x_{1,2} & x_{1,3} & \cdots & x_{1,n} & \in R \\
x_2 & = & x_{2,1} & x_{2,2} & x_{2,3} & \cdots & x_{2,n} & \in R \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \\
x_m & = & x_{m,1} & x_{m,2} & x_{m,3} & \cdots & x_{m,n} & \in R
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 & & \| & \| & \| & & \| & \\
 & & z_1 & z_2 & z_3 & \cdots & z_n &
\end{array}$$

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& \vdots & & \vdots & \vdots & \vdots & & \vdots & \\
& x_m & = & x_{m,1} & x_{m,2} & x_{m,3} & \cdots & x_{m,n} & \in R \\
& & & \| & \| & \| & & \| & \\
\text{then also} & z & = & z_1 & z_2 & z_3 & \cdots & z_n & \in R.
\end{array}
$$

R is invariant under f. f preserves R.

Clones of Polymorphisms

Pol(Γ) is the set of all polymorphisms of Γ, i.e., the set of all Boolean functions that preserve every relation in Γ.

- Pol(Γ) is a clone, i.e., a set of Boolean functions that contains all projections and is closed under composition.

Post's lattice [Emil Post, 1921/1941]:

- List of all Boolean clones
- Inclusion structure among them
- Finite basis for each of them

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[Post 1941]:

Every clone B can be characterized by the set of its invariant constraints:

Let Γ_0 be a basis for the co-clone Inv(B). Then,

- A function belongs to B iff it preserves all relations in Γ_0.

The Galois Correspondence

- $\mathrm{Inv}\big(\mathrm{Pol}(\Gamma)\big) = \langle \Gamma \rangle$.
- $\mathrm{Pol}\big(\mathrm{Inv}(B)\big) = [B]$.

One-one correspondence between clones and co-clones;
obtain complete list of co-clones from Post's lattice.

Determine easy bases for relational clones!

If $\Gamma \subseteq \mathrm{Inv}(E_2)$ $(\wedge \approx \Gamma)$ then $\mathrm{CSP}(\Gamma) \in \mathrm{P}$ (Horn relations).

If $\Gamma \subseteq \mathrm{Inv}(V_2)$ $(\vee \approx \Gamma)$ then $\mathrm{CSP}(\Gamma) \in \mathrm{P}$ (anti-Horn relations).

If $\Gamma \subseteq \mathrm{Inv}(D_2)$ $(T_2^3 \approx \Gamma)$ then $\mathrm{CSP}(\Gamma) \in \mathrm{P}$ (2-CNF relations).

If $\Gamma \subseteq \mathrm{Inv}(L_2)$ $(\oplus^3 \approx \Gamma)$ then $\mathrm{CSP}(\Gamma) \in \mathrm{P}$ (affine relations).

If $\Gamma \subseteq \mathrm{Inv}(I_1)$ $(1 \approx \Gamma)$ then $\mathrm{CSP}(\Gamma) \in \mathrm{P}$ (1-valid relations).

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What remains?

Efficient SAT Algorithms

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What remains?

$\langle\Gamma\rangle \supseteq \mathrm{Inv}(N_2)$, i.e., only polymorphism is negation.

Schaefer's Theorem

$\mathsf{R}_{\mathsf{NAE}} = \{(0,0,1),(0,1,0),(0,1,1),(1,0,0),(1,0,1),(1,1,0)\}$.

$\mathsf{Pol}(\mathsf{R}_{\mathsf{NAE}}) = \mathsf{N}_2$.

Schaefer's Theorem

$R_{NAE} = \{(0,0,1),(0,1,0),(0,1,1),(1,0,0),(1,0,1),(1,1,0)\}$.
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But: $\mathrm{CSP}(\{R_{NAE}\}) =$ NOT-ALL-EQUAL-SAT, NP-complete.

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But: $\mathrm{CSP}(\{R_{NAE}\})$ = NOT-ALL-EQUAL-SAT, NP-complete.

- If $\langle\Gamma\rangle \supseteq \mathrm{Inv}(N_2)$ then $\mathrm{CSP}(\Gamma)$ is NP-complete, otherwise $\mathrm{CSP}(\Gamma)$ is in P. [Schaefer 1978]

Through "polynomial-time glasses", we observe dichotomy.

A Finer Classification w.r.t. Logspace-Reductions

- If $\langle\Gamma\rangle \in \{\mathrm{Inv}(\mathsf{I}_2), \mathrm{Inv}(\mathsf{N}_2)\}$, then CSP($\Gamma$) is NP-complete.
- If $\langle\Gamma\rangle \in \{\mathrm{Inv}(\mathsf{V}_2), \mathrm{Inv}(\mathsf{E}_2)\}$, then CSP($\Gamma$) is P-complete.
- If $\langle\Gamma\rangle \in \{\mathrm{Inv}(\mathsf{L}_2), \mathrm{Inv}(\mathsf{L}_3)\}$, then CSP($\Gamma$) is $\oplus$L-complete.
- If $\mathrm{Inv}(\mathsf{S}_{00}^2) \subseteq \langle\Gamma\rangle \subseteq \mathrm{Inv}(\mathsf{S}_{00})$ or $\mathrm{Inv}(\mathsf{S}_{10}^2) \subseteq \langle\Gamma\rangle \subseteq \mathrm{Inv}(\mathsf{S}_{10})$ or $\langle\Gamma\rangle \in \{\mathrm{Inv}(\mathsf{D}_2), \mathrm{Inv}(\mathsf{M}_2)\}$, then CSP($\Gamma$) is NL-complete.
- If $\langle\Gamma\rangle \in \{\mathrm{Inv}(\mathsf{D}_1), \mathrm{Inv}(\mathsf{D})\}$ or $\mathrm{Inv}(\mathsf{R}_2) \subseteq \langle\Gamma\rangle \subseteq \mathrm{Inv}(\mathsf{S}_{02}$ or $\mathrm{Inv}(\mathsf{R}_2) \subseteq \langle\Gamma\rangle \subseteq \mathrm{Inv}(\mathsf{S}_{12}$, then CSP($\Gamma$) is in L.
- If $\Gamma \subseteq \mathrm{Inv}(\mathsf{I}_0)$ or $\Gamma \subseteq \mathrm{Inv}(\mathsf{I}_1)$, then every constraint formula over Γ is satisfiable, and therefore CSP(Γ) is trivial.

[Allender-Bauland-Immerman-Schnoor-Vollmer 2005]

Through "logspace glasses", there are 5 complexity levels for CSP.

- QBF (determination of truth of a closed quantified Boolean formula) is PSPACE-complete. [Stockmeyer-Meyer 1973]
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- QCSP(Γ) (determination of truth of a closed quantified Γ-formula) is PSPACE-complete if $\langle\Gamma\rangle \supseteq \mathrm{Inv}(\mathrm{N})$, otherwise QCSP($\Gamma$) is tractable.
  [Schaefer 1978, Dalmau 2000, Creignou-Khanna-Sudan 2001]

- QBF_i (restriction of QBF to prenex normal-form with $i-1$ quantifier alternations, starting with existential) is complete for the class Σ_i^{p} of the polynomial-time hierarchy.
- For i odd, QCNF_i is Σ_i^{p}-complete.
- For i even, QDNF_i is Σ_i^{p}-complete.

[Wrathall, 1977]

Bounded Number of Alternations

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How to define QCSP_i?

$\mathrm{QCSP}_i(\Gamma)$:

For i odd, determine if a closed quantified Γ-formula with $i-1$ quantifier alternations starting with existential quantifier is true.

For i even, determine if a closed quantified Γ-formula with $i-1$ quantifier alternations starting with universal quantifier is false.

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- If $\Gamma \subseteq \langle \Gamma' \rangle$, then $\mathrm{QCSP}_i(\Gamma) \leq_m^{\log} \mathrm{QCSP}_i(\Gamma')$.

"Galois connection helps for quantified satisfiability."

- $\mathrm{QCSP}_i\big(\{R_{\text{1-IN-3}}\}\big)$ is Σ_i^p-complete,

  since $\mathrm{Inv}(R_{\text{1-IN-3}})$ is the class of all Boolean relations.

- $\mathrm{QCSP}_i\big(\{\mathrm{R}_{\text{1-IN-3}}\}\big)$ is Σ_i^{p}-complete,
  since $\mathrm{Inv}(\mathrm{R}_{\text{1-IN-3}})$ is the class of all Boolean relations.
- $\mathrm{QCSP}_i\big(\{\mathrm{R}_{\mathrm{NAE}}\}\big)$ is Σ_i^{p}-complete:
  Replace every constraint $\mathrm{R}_{\text{1-IN-3}}(x_1, x_2, x_3)$ by
  $\mathrm{R}_{\text{2-IN-4}}(x_1, x_2, x_3, t)$ for a (common) new variable t, and observe
  $\mathrm{R}_{\text{2-IN-4}}(x_1, x_2, x_3, t) = \bigwedge_{i \neq j} \mathrm{R}_{\mathrm{NAE}}(x_i, x_j, t) \wedge \mathrm{R}_{\mathrm{NAE}}(x_1, x_2, x_3)$.
  Quantify t in first quantifier block.

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- $\mathrm{QCSP}_i(\{R_0\})$ is Σ_i^{p}-complete,
  where $R_0(u, v, x_1, x_2, x_3) \equiv u = v \vee \mathrm{NAE}(x_1, x_2, x_3)$:
  Replace every constraint $\mathrm{NAE}(x_1, x_2, x_3)$ by $R_0(u, v, x_1, x_2, x_3)$.
  Quantify u, v in last universal quantifier block.

$\boxed{\mathrm{R}_{\text{1-IN-3}} \in \mathrm{Inv}(\mathrm{I}_2)}$

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  Replace every constraint $\mathrm{NAE}(x_1, x_2, x_3)$ by $R_0(u, v, x_1, x_2, x_3)$.
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Classification of $\mathrm{QCSP}_i(\Gamma)$

- $\mathrm{QCSP}_i(\{\mathrm{R}_{\text{1-IN-3}}\})$ is Σ_i^{p}-complete, $\boxed{\mathrm{R}_{\text{1-IN-3}} \in \mathrm{Inv}(\mathrm{I}_2)}$
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Hemaspaandra's Theorem

- QCSP(Γ) is tractable if Γ is Horn, anti-Horn, bijunctive, or affine. [Schaefer 1978, Creignou-Khanna-Sudan 2001]

If Γ is not in one of these cases, then $\langle\Gamma\rangle \supseteq \mathrm{Inv}(\mathrm{N}) \ni R_0$.

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If Γ is not in one of these cases, then $\langle\Gamma\rangle \supseteq \mathrm{Inv}(\mathrm{N}) \ni R_0$. Hence:

- If $\langle\Gamma\rangle \supseteq \mathrm{Inv}(\mathrm{N})$ then QCSP$_i$(Γ) is Σ_i^{p}-complete and QCSP(Γ) is PSPACE-complete; otherwise QCSP$_i$(Γ) and QCSP(Γ) are tractable. [Hemaspaandra 2004]

$\#\mathrm{QCSP}_i(\Gamma)$:

For i odd, determine number of satisfying assignments of a quantified Γ-formula with $i-1$ quantifier alternations starting with existential quantifier.

For i even, determine number of unsatisfying assignments of a quantified Γ-formula with $i-1$ quantifier alternations starting with universal quantifier.

- If $\Gamma \subseteq \langle \Gamma' \rangle$, then $\#\mathrm{QCSP}_i(\Gamma) \leq_m^{\mathrm{p}} \#\mathrm{QCSP}_i(\Gamma')$.

"Galois connection helps for $\#\mathrm{QCSP}_i$."

A – binary relation s.t. $(x, y) \in A \implies |y|$ is polynomial in $|x|$

$A(x) = \{\, y \mid (x, y) \in A \,\}$, $\#A(x) = |A(x)|$.

$\#A \leq^{\mathrm{p}}_{m} \#B$ if there is polynomial-time computable function f s.t. for all x, $\#A(x) = \#B(f(x))$. [Valiant 1979]

("parsimonious reductions")

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("parsimonious reductions")

#SAT is $\leq^{\mathrm{p}}_{m}$-complete for #P, but not many further complete problems are known.

Reductions for Counting Problems

$\#A \leq^{\mathrm{p}}_{cnt} \#B$ if there are polynomial-time computable function f, g s.t. for all x, $\#A(x) = g\big(\#B(f(x))\big)$. [Zankó 1991]

("counting reductions")

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Look for a reduction powerful enough to prove completeness results but strict enough to distinguish among levels of the $\#\Sigma^{\mathrm{p}}_k$-hierarchy.

$\#A \leq^{\mathrm{p}}_{ssub} \#B$ if there are polynomial-time computable function f, g s.t. for all x,

- $B(g(x)) \subseteq B(f(x))$.
- $\#A(x) = \#B(f(x)) - \#B(g(x))$.

"Subtractive reduction" $\leq^{\mathrm{p}}_{sub}$ is the transitive closure of strong subtractive reduction $\leq^{\mathrm{p}}_{ssub}$. [Durand-Hermann-Kolaitis 2000]

- #P and all classes $\#\Pi^{\mathrm{p}}_{k}$ for $k > 1$ are closed under subtractive reductions, but $\leq^{\mathrm{p}}_{sub}(\#\Sigma^{\mathrm{p}}_{k}) = \#\Pi^{\mathrm{p}}_{k}$.

$\#A \leq^{\mathrm{p}}_{scom} \#B$ if there are polynomial-time computable function f, g and a bipartite permutation π on the alphabet underlying B s.t. for all x,

- $B(g(x)) \subseteq B(f(x))$.
- $y \in B(x) \iff \pi(y) \in B(x)$
- $2 \cdot \#A(x) = \#B(f(x)) - \#B(g(x))$.

"Complementive reduction" $\leq^{\mathrm{p}}_{com}$ is the transitive closure of strong complementive reduction $\leq^{\mathrm{p}}_{scom}$ and parsimonious reduction $\leq^{\mathrm{p}}_{m}$.

[Bauland-Chapdelaine-Creignou-Hermann-Vollmer 2004]

- $\#$P and all classes $\#\Pi^{\mathrm{p}}_k$ for $k > 1$ are closed under complementive reductions, but $\leq^{\mathrm{p}}_{com}(\#\Sigma^{\mathrm{p}}_k) = \#\Pi^{\mathrm{p}}_k$.

Classification of #QCSP

For every $i \geq 1$,

- if $\Gamma \subseteq \mathrm{Inv}(\mathrm{L}_2)$ then $\#\mathrm{QCSP}_i(\Gamma)$ and $\#\mathrm{QCSP}(\Gamma)$ are tractable,
- else if $\Gamma \subseteq \mathrm{Inv}(\mathrm{E}_2)$ or $\Gamma \subseteq \mathrm{Inv}(\mathrm{V}_2)$ or $\Gamma \subseteq \mathrm{Inv}(\mathrm{D}_2)$ then $\#\mathrm{QCSP}_i(\Gamma)$ and $\#\mathrm{QCSP}(\Gamma)$ are $\leq^{\mathrm{p}}_{cnt}$-complete for $\#\mathrm{P}$,
- else (note: $\langle\Gamma\rangle \supseteq \mathrm{Inv}(\mathrm{N})$) $\#\mathrm{QCSP}_i(\Gamma)$ is $\leq^{\mathrm{p}}_{com}$-complete for $\#\Sigma^{\mathrm{p}}_i$ and $\#\mathrm{QCSP}(\Gamma)$ is $\leq^{\mathrm{p}}_{com}$-complete for $\#\mathrm{PSPACE}$.

[Bauland-Böhler-Creignou-Reith-Schnoor-Vollmer 2006]

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[Bauland-Böhler-Creignou-Reith-Schnoor-Vollmer 2006]

- In 2nd case, $\#\mathrm{QCSP}_i(\Gamma)$ is not tractable unless $\mathrm{FP} = \#\mathrm{P}$.
- In 3rd case, $\#\mathrm{QCSP}_i(\Gamma)$ is not in $\#\Sigma^{\mathrm{p}}_{i-1}$ unless $\#\Sigma^{\mathrm{p}}_i = \#\Pi^{\mathrm{p}}_{i-1}$.

The Galois connection holds *a priori* for a computational problem Π, if we can prove

- If $\Gamma \subseteq \langle \Gamma' \rangle$ then $\Pi(\Gamma) \leq_m^{\log} \Pi(\Gamma')$

and use this to obtain a complexity theoretic classification.

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For problems above, the Galois connection holds *a priori*.

Let F be a Γ-formula. Construct F' as follows:

- Replace every constraint from Γ by equivalent $\big(\Gamma' \cup \{=\}\big)$-formula.
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Problem: Introduction of new existentially quantified variables.

Preserves satisfiability, but does not preserve number of solutions, etc.

When does the Galois Connection Hold?

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Examples:

- Satisfiability
- Several computational problems for quantified constraints

Positive Examples

– Circumscription: [Nordh-Jonsson 2004]

Given formula F, subset M of variables, clause C, determine if C holds in every satisfying assignment of F that is minimal on M in componentwise order.

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– Frozen variables: [Jonsson-Krokhin 2003] [Bauland-Chapdelaine-Creignou-Hermann-Vollmer 2004]

  Given formula F, subset M of variables, check if there is a variable $x \in M$ that has the same value in every satisfying assignment of F.

– Abduction: [Creignou-Zanuttini 2006]

Given formula F, subset M of variables, variable $x \notin M$, check if there is a set E of literals over M such that $F \wedge \bigwedge E$ is satisfiable but $F \wedge \bigwedge E \wedge \neg x$ is not? (E is "explanation" of x.)

A posteriori

The Galois connection holds *a posteriori* for a computational problem Π, if we obtain a complexity classification "by hand" that speaks only of co-clones, and we can read the implication

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For many problems, the Galois connection holds *a posteriori*, e.g.

- Counting [Creignou-Hermann 1996]
- Enumeration [Creignou-Hébrard 1997]
- Equivalence and isomorphism [Böhler-Hemaspaandra-Reith-Vollmer 2002,4]

Negative Examples

The Galois connection does not hold for

- MaxSAT
- Fixed parameter tractability
- Approximation

When does the Galois Connection Hold?

Open Problem: Determine properties of computational problems Π that imply that the Galois connection holds for Π.

Different Galois Connections

Problems arise from existentially quantified variables in definition of relational clone.

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- $\langle\Gamma\rangle'$ contains the equality relation and all relations in Γ.
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Road map: Look for Galois connection between lattice of classes $\langle\Gamma\rangle'$ and suitable refinement of Post's lattice.

$\rightsquigarrow$ Talk by Ilka Schnoor.

If $\Gamma \subseteq \langle \Gamma' \rangle$ then $\mathbf{CSP}(\Gamma) \leq_m^{\log} \mathbf{CSP}(\Gamma')$

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Can we do better than logspace-reductions?

Example 1: $\Gamma_1 = \{x, \overline{x}\}$:

A Γ_1-formula F is unsatisfiable iff it contains clauses x and $\overline{x}$ for some x, hence $\mathrm{CSP}(\Gamma_1) \in \mathrm{AC}^0$.

The Equality Constraint

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Example 2: $\Gamma_2 = \{x, \overline{x}, =\}$:

Then $\mathrm{CSP}(\Gamma_2)$ can express undirected graph reachability as follows: Given G, s, t, construct F to consist of clauses $\overline{s}$, t, and $u = v$ for every edge $(u, v) \in G$.

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Thus: Provably different complexity: $\mathrm{CSP}(\Gamma_2) \not\leq_m^{\mathrm{AC}^0} \mathrm{CSP}(\Gamma_1)$,

but $\mathrm{Pol}(\Gamma_1) = \mathrm{Pol}(\Gamma_2) \quad (= \mathrm{R}_2)$.

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- If Γ can express equality then $\mathrm{CSP}(\Gamma)$ is hard for L, otherwise $\mathrm{CSP}(\Gamma) \in \mathrm{AC}^0$.

Inside LOGSPACE

Two remaining cases: $\mathsf{Pol}(\Gamma) \in \{\mathsf{D}_1, \mathsf{D}\}$ and $\mathsf{S}_{02} \subseteq \mathsf{Pol}(\Gamma) \subseteq \mathsf{R}_2$ or $\mathsf{S}_{12} \subseteq \mathsf{Pol}(\Gamma) \subseteq \mathsf{R}_2$.

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- If $\mathrm{Pol}(\Gamma) \in \{\mathrm{D}_1, \mathrm{D}\}$, then $\mathrm{CSP}(\Gamma)$ is L-complete.

Proof: $x \oplus y \in \mathrm{Inv}(\Gamma)$, i.e., there is conjunctive query over $\Gamma \cup \{=\}$ that defines $x \oplus y$. Equality clauses here appear only between existentially quantified new variables and can be removed locally.

Hence, Γ can express $x \oplus y$.

Now, $(\exists z)\big((x \oplus z) \wedge (z \oplus y)\big)$ expresses equality.

Inside LOGSPACE

- If $S_{02} \subseteq \mathrm{Pol}(\Gamma) \subseteq R_2$ or $S_{12} \subseteq \mathrm{Pol}(\Gamma) \subseteq R_2$, then either $\mathrm{CSP}(\Gamma)$ is in AC^0, or $\mathrm{CSP}(\Gamma)$ is L-complete.

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Proof: Logspace upper bound:

If $\Gamma \subseteq \mathrm{Inv}(\mathrm{S}_{02}) = \bigcup_m \mathrm{Inv}(S_{02}^m) = \bigcup_m \langle \{\vee^m, =, x, \overline{x}\} \rangle$,

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Given Γ-formula F is satisfiable iff

- for each clause $x_1 \vee \cdots \vee x_k$
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$\Gamma \subseteq \mathsf{Inv}(\mathsf{S}_{12})$: analogously with NAND^m.

Can We Express Equality?

Let $R \in \mathsf{Inv}(\mathsf{S}_{02}^m)$, i.e., R is defined by conjunctive query ϕ over $\{\vee^m, =, x, \overline{x}\}$.

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- For all clauses $x_1 = x_2$:

  If x_1 or x_2 occur in literals in ϕ, delete $x_1 = x_2$ and insert corresponding literal for the other variable.
- For all clauses $x_1 \vee \cdots \vee_k$:

  If there is a literal $\overline{x_i}$, delete x_i in this clause.
- For all clauses $x_1 \vee \cdots \vee_k$:

  If occuring variables are connected by $=$-path, delete all of them except one.

Case 1: No clause $x_1 = x_2$ remains. Then $\mathrm{CSP}\big(\{R, \vee^m, x, \overline{x}\}\big) \in \mathrm{AC}^0$.

(Satisfiable iff no contradictory literals and every disjunction has variable that does not occcur in negative literal.)

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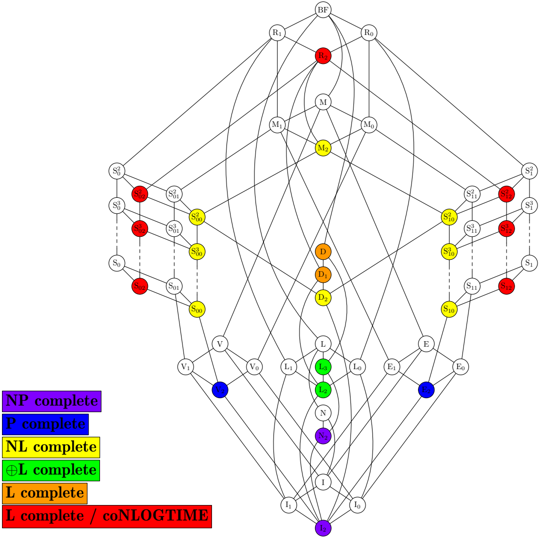
Analogous argument with NAND^m for $\Gamma \subseteq \mathrm{Inv}(\mathrm{S}_{12})$.

Classification of CSP-Satisfiability



Post's lattice: $\mathrm{L}_2 \subseteq \mathrm{R}_2$, hence $\mathrm{Inv}(\mathrm{R}_2) \subseteq \mathrm{Inv}(\mathrm{L}_2)$.

Hence:

- Undirected graph accessibility is in $\oplus$L, in other words:
  $\mathrm{SL} \subseteq \oplus\mathrm{L}$. [Karchmer, Wigderson, 1993]

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(Today we even know SL $\subseteq$ L.)

Isomorphism Theorem holds for $\leq_m^{\mathrm{AC}^0}$-reducibility:

- For every constraint language Γ, CSP(Γ) is AC^0-isomorphic either to $0\Sigma^\star$ or to the standard complete set for one of the complexity classes NP, P, $\oplus$L, NL, or L.

Through FO glasses, there are only six different CSP-problems!

Why study Boolean CSP?

Provide a reasonably accurate bird's eye view of complexity theory:

[Creignou-Khanna-Sudan 2001]

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- structure of complete problems

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- playground for the study of many issues related to counting classes
- CSP isomorphism problems yield good candidates for "intermediate problems"

Why study Boolean CSP?

Classifications of problems for Boolean CSPs provide a guidepost for study of general CSPs:

- If Galois connection holds *a priori*, then usually for arbitrary CSPs.
- Hard cases translate from Boolean to general case, sometimes in nontrivial way: #QCSP

  [Bauland-Böhler-Creignou-Reith-Schnoor-Vollmer 2006]
- Issues from Post's lattice show direction for general classification:

  Non-FO CSPs are logspace-hard: $\rightsquigarrow$ Talk by Benoît Larose

Open Questions for Boolean CSP

- Obtain fine classification for Boolean counting problem.

- Study different Galois connections.

- Uniform Boolean CSP?

Open Questions for General CSP

- Study different Galois connections.

- Obtain fine classification for satisfiability over 3-element domain.

- Study different computational problems (besides satisfiability) for general CSPs.