The Tractability of Model-Checking for LTL: The Good, the Bad, and the Ugly Fragments

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The burning issue

Problem
The model-checking problem for full Linear Temporal Logic (LTL) is PSPACE-complete [Sistla, Clarke 1985]. That is, this problem is (most probably) intractable. 😞

Solution
Systematically restrict the propositional part of LTL.
〜 Many tractable (good) fragments 😊
〜 Many intractable (bad) fragments
What is Linear Temporal Logic?

LTL = propositional logic plus temporal operators, speaks about linear structures; for example:

A structure $P$

0 1 2 3 4 5 6 7 8

The tractability of LTL model-checking
The language and its interpretation

The structure $P$

0 1 2 3 4 5 6 7 8

$\text{we} \rightarrow \text{we} \rightarrow c \rightarrow \text{wc} \rightarrow \text{wc} \rightarrow \text{wc} \rightarrow \text{wc} \rightarrow c \rightarrow \ldots$

The following kinds of statements can be formulated in LTL.

- now $P, 2 \models (w \land \neg e) \lor c$
- at some time in the Future $P, 0 \models Fh$
- always Going to $P, 3 \models G\neg e$
- next time $P, 1 \models X(w \rightarrow e)$
- Until $P, 5 \models cU(\neg w)$
- Since $P, 3 \models cSw$
- $P, 0 \models F(c \land \neg e) \land G[(c \land \neg w) \rightarrow [X(w \land Xh) \land (h \rightarrow w \land c)Uc]]$
A model and a structure

A model (cf. Clarke et al. „Model Checking“):

Possible behaviour of a microwave oven

The tractability of LTL model-checking
A model and a structure

A structure:

*Actual* behaviour of a microwave oven

![Diagram of a microwave oven model](image-url)
Summing up: Models and structures

Model
A directed graph where every state has a successor. States are marked with assignments to propositional variables.

Structure
An infinite path in a model.
The model-checking problem

Model-Checking

Instance $\langle \varphi, M, a \rangle$

Question Does $M$ contain a structure $P$ with initial state $a$ such that $P, a \models \varphi$?

Theorem (Sistla, Clarke 1985)

Model-checking for LTL is PSPACE-complete.
When do LTL fragments suffice?

Example
Properties of “microwave oven runs” expressible in LTL fragments:

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<thead>
<tr>
<th>Property</th>
<th>Formula</th>
<th>Operators used</th>
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<td>An error never occurs.</td>
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LTL fragment

Let $T \subseteq \{F, G, X, U, S\}$ be a set of temporal operators and $B$ be a finite set of Boolean operators.\(^\star\)

$L(T, B) = \text{set of all LTL formulas with operators in } T \cup B.$

*For instance, $\{\land, \lor\}$ — monotone formulae.

Model-checking problem $\text{MC}(T, B)$ for LTL fragments

*Instance*: $\langle \varphi, M, a \rangle$ with $\varphi \in L(T, B)$

*Question*: Does $M$ contain a structure $P$ with initial state $a$ such that $P, a \vDash \varphi$?
Theorem ([Sistla, Clarke 1985] and [Markey 2004])

1. \( \text{MC}(\{G, X\}, \{\land, \lor, \neg\}) \) and \( \text{MC}(\{U\}, \{\land, \lor, \neg\}) \) are PSPACE-complete, even if negation is applied to atoms only.

2. \( \text{MC}(\{F\}, \{\land, \lor, \neg\}) \), \( \text{MC}(\{G\}, \{\land, \lor, \neg\}) \) and \( \text{MC}(\{X\}, \{\land, \lor, \neg\}) \) are NP-complete, even if negation is applied to atoms only.

3. \( \text{MC}(\{F, X\}, \{\land, \lor, \neg\}) \) in general is PSPACE-complete, but NP-complete if negation is applied to atoms only.
Consequences of results by [Sistla, Clarke 1985] and [Markey 2004]:

Hardness and completeness of $\text{MC}(T, B)$

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Bad fragments!
What we would like to know . . .

Goal

- classify the complexity of $MC(T, B)$ for all LTL fragments

- separate LTL fragments into
good (efficiently solvable) and bad (NP-hard)
What we would like to know . . .

Goal

▶ classify the complexity of $MC(T, B)$ for all LTL fragments

▶ separate LTL fragments into
good (efficiently solvable) and bad (NP-hard)
Fragments of propositional logic: Clones

Post’s lattice
(est’d 1941 by Emil Post)

| X_2 | without constants |
| X_{0,1} with constant 0,1 |
| BF | all BF |
| M | monotone functions |
| S_1 | x \land \bar{y} |
| S_0 | x \rightarrow y |
| D | f(a_1, \ldots, a_n) = f(\bar{a_1}, \ldots, \bar{a_n}) |
| L | x \oplus y (xor) |
| V | x \lor y |
| E | x \land y |
| N | \neg x |
| I | identities |

The tractability of LTL model-checking
Clones with both constants

All relevant sets of Boolean operators

Every other set of Boolean op’s can be reduced to one of these.
Tractability of model-checking: Fragments with $F, G, X$

Hardness and completeness of $MC(T, B)$

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$(PS = \text{PSPACE})$
Hardness and completeness of $\text{MC}(T, B)$

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Tractability of model-checking: Fragments with $S$, $U$
Theorem (Sistla, Clarke 1985)

$MC(\{F\}, \{\land\})$ is NP-hard.

Proof sketch.

- Reduction from 3SAT
- From $(x_1 \lor \neg x_2 \lor \neg x_4) \land (\neg x_1 \lor x_3 \lor \neg x_4) \land (\neg x_2 \lor x_4)$ we obtain the model

```
q  b_1
\downarrow       \downarrow       \downarrow
x_1       x_2       x_3       x_4
\downarrow       \downarrow       \downarrow
\neg x_1     \neg x_2     \neg x_3     \neg x_4
\downarrow       \downarrow       \downarrow
b_2       b_1, b_3       b_1, b_2
s
```

and the $L(\{F\}, \{\land\})$-formula $Fb_1 \land Fb_2 \land Fb_3$. 


The tractability of LTL model-checking
An NP-hardness proof

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```
q \quad \text{\[}b_1\text{\[}b_2\text{\[}b_3
\text{\\[}x_1\quad \text{\[}x_2\quad \text{\[}x_3\quad \text{\[}x_4
\text{\[}b_2\quad \text{\[}b_1, b_3\quad \text{\[}b_1, b_2
```

and the $L(\{F\}, \{\land\})$-formula $Fb_1 \land Fb_2 \land Fb_3$.  

The tractability of LTL model-checking
Theorem

MC({U}, ∅) is NP-hard.

Proof sketch.

- Reduction from 3SAT
- From \((x_1 \lor \neg x_2 \lor \neg x_4) \land (\neg x_1 \lor x_3 \lor \neg x_4) \land (\neg x_2 \lor x_4)\)
we obtain the model

\[
q_1 \xrightarrow{a_1} q_2 \xrightarrow{a_2} q_3 \xrightarrow{a_3} b_1 \xrightarrow{a_3} x_1 \xrightarrow{a_2} x_2 \xrightarrow{a_2} x_3 \xrightarrow{a_3} x_4 \xrightarrow{a_3} b_3 \xrightarrow{a_1} b_1, b_3 \xrightarrow{a_2} b_1, b_2 \xrightarrow{a_1} s
\]

and the \(L({U}, ∅)\)-formula \(((a_1 \lor b_1) \lor (a_2 \lor b_2)) \lor (a_3 \lor b_3)\) \(\square\)
An NP-hardness proof

Theorem

$MC(\{U\}, \emptyset)$ is NP-hard.

Proof sketch.

- Reduction from 3SAT
- From $(x_1 \lor \neg x_2 \lor \neg x_4) \land (\neg x_1 \lor x_3 \lor \neg x_4) \land (\neg x_2 \lor x_4)$
  we obtain the model

$$L(\{U\}, \emptyset)$$-formula $((a_1 U b_1) U (a_2 U b_2)) U (a_3 U b_3)$. \(\square\)
An NP-hardness proof

Theorem

$MC(\{U\}, \emptyset)$ is NP-hard.

Proof sketch.

- Reduction from 3SAT
- From $(x_1 \lor \neg x_2 \lor \neg x_4) \land (\neg x_1 \lor x_3 \lor \neg x_4) \land (\neg x_2 \lor x_4)$ we obtain the model

and the $L(\{U\}, \emptyset)$-formula $((a_1 U b_1) U (a_2 U b_2)) U (a_3 U b_3)$. □
Theorem
MC(\{U\}, \emptyset) is NP-hard.

Proof sketch.
- Reduction from 3SAT
- From \((x_1 \lor \neg x_2 \lor \neg x_4) \land (\neg x_1 \lor x_3 \lor \neg x_4) \land (\neg x_2 \lor x_4)\)
  we obtain the model

\[
\begin{array}{c}
q_1 \to q_2 \to q_3 \\

\neg x_1 \to \neg x_2 \to \neg x_3 \to \neg x_4 \\

b_1 \to b_2 \to b_3 \\

a_1, a_2, a_3 \\

b_1, b_2, b_3 \\

s
\end{array}
\]

and the \(L(\{U\}, \emptyset)\)-formula \(((a_1 U b_1)(a_2 U b_2)(a_3 U b_3))\).
Tractability of model-checking: Fragments with $F, G, X$

Hardness and completeness of $\text{MC}(T, B)$

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Theorem
MC(\{F, X\}, \{\lor\}) is NL-complete.

Proof sketch.

- NL-hardness: Reduction from the Graph Accessibility Problem
- NL-membership via a logspace computable normal form

Given \(\langle \varphi, M, a \rangle\), transform \(\varphi\) into

\[ \varphi' = FX_{i_1}y_1 \lor \cdots \lor FX_{i_n}y_n \lor X_{i_{n+1}}y_{n+1} \lor \cdots \lor X_{i_m}y_m. \]

- Guess one of the disjuncts \((F)X^{i_j}\).
- Guess the initial section of a path in \(M\) from \(a\).
  (Its length is determined by \(i_j\).)
- Check the truth of \((F)X^{i_j}\) at \(a\).
An NL-completeness proof

Theorem
MC(\{F, X\}, \{\lor\}) is NL-complete.

Proof sketch.

- NL-hardness: Reduction from the Graph Accessibility Problem
- NL-membership via a logspace computable normal form

Given \langle \varphi, M, a \rangle, transform \varphi into

\[ \varphi' = F X^{i_1} y_1 \lor \cdots \lor F X^{i_n} y_n \lor X^{i_{n+1}} y_{n+1} \lor \cdots \lor X^{i_m} y_m. \]

- Guess one of the disjuncts (F)X^{i_j}.
- Guess the initial section of a path in M from a. (Its length is determined by \(i_j\).)
- Check the truth of (F)X^{i_j} at a.
Another good fragment

**Theorem**
$\text{MC}([G, X], \{\land\})$ is \text{NL}-complete.

**Proof sketch.**

- **NL-hardness**: as above
- **NL-membership**:

  Example:

  $$(xb \land GX(Ga \land XGXGXb)) \equiv xb \land XGa \land XXXGb$$

  goal: guess a path with following properties:

  - in state 0: nothing to check
  - in state 1: $b$ and $a$ hold
  - in state 2: $a$ holds
  - in state 3: $a$ and $b$ hold
  - in state 4: $a$ and $b$ hold
Duality

- $\text{MC}({\{F, X\}, \{\lor\}})$ is NL-complete.
- $\text{MC}({\{G, X\}, \{\land\}})$ is NL-complete.
- $\text{MC}({\{F\}, \{\land\}})$ is NP-complete.
- $\text{MC}({\{G\}, \{\lor\}})$ is NL-complete, even $\text{MC}({\{F, G\}, \{\lor\}})$. 

The tractability of LTL model-checking
Duality

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Theorem
For each finite $B$ with $[\mathbf{B}] \supseteq \mathbf{M}$: $\text{MC}(\{\mathbf{G}, \mathbf{X}\}, B)$ is PSPACE-hard.

Proof sketch.

- PSPACE-hardness of $\text{MC}(\{\mathbf{G}, \mathbf{X}\}, \{\land, \lor\})$ follows from [Markey 2004].
- Every operator in $B$ can be represented by a short $\land, \lor$-formula.
- Hence, $\text{MC}(\{\mathbf{G}, \mathbf{X}\}, \{\land, \lor\}) \leq^\log_m \text{MC}(\{\mathbf{G}, \mathbf{X}\}, B)$. 

\[\square\]
Lemma (lower bounds are inherited to larger clones)

Let $B \subseteq \{\land, \lor, \lnot\}$ and $B \subseteq [C]$. Then $\MC(T, B) \leq_{m} \log \MC(T, C)$.

specific: $\MC(\{G, X\}, \{\lor\})$ is NP-hard.

general: Let $C$ be a finite set of Boolean functions such that $\{\lor\} \subseteq [C]$. Then $\MC(\{G, X\}, C)$ is NP-hard.

specific: $\MC(\{G, X\}, \{\lor, \land\})$ is PSPACE-complete.

general: Let $C$ be a finite set of Boolean functions such that $\{\lor, \land\} \subseteq [C]$. Then $\MC(\{G, X\}, C)$ is PSPACE-hard.
General results: upper bounds

Fear
Upper bounds are not necessarily inherited to smaller clones. Does $MC(\{G, X\}, C) \in \text{PSPACE}$ hold for every $C$?

Some upper bounds can be generalized. For example:

specific: $MC(\{F, X\}, \{\lor\})$ is in $\text{NL}$.

general: Let $C$ be a finite set of Boolean functions such that $[C] \subseteq [\{\lor\}]$. Then $MC(\{F, X\}, C)$ in $\text{NL}$.
## Tractability of model-checking: Fragments with $F, G, X$

### Hardness and completeness of $MC(T, B)$

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The tractability of LTL model-checking
Tractability of model-checking: Fragments with $S$, $U$

Hardness and completeness of $MC(T, B)$

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The tractability of LTL model-checking
Conclusion

Achieved

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Related work

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Thank you!