Matrix Multiplication and Graph Algorithms

Uri Zwick
Tel Aviv University

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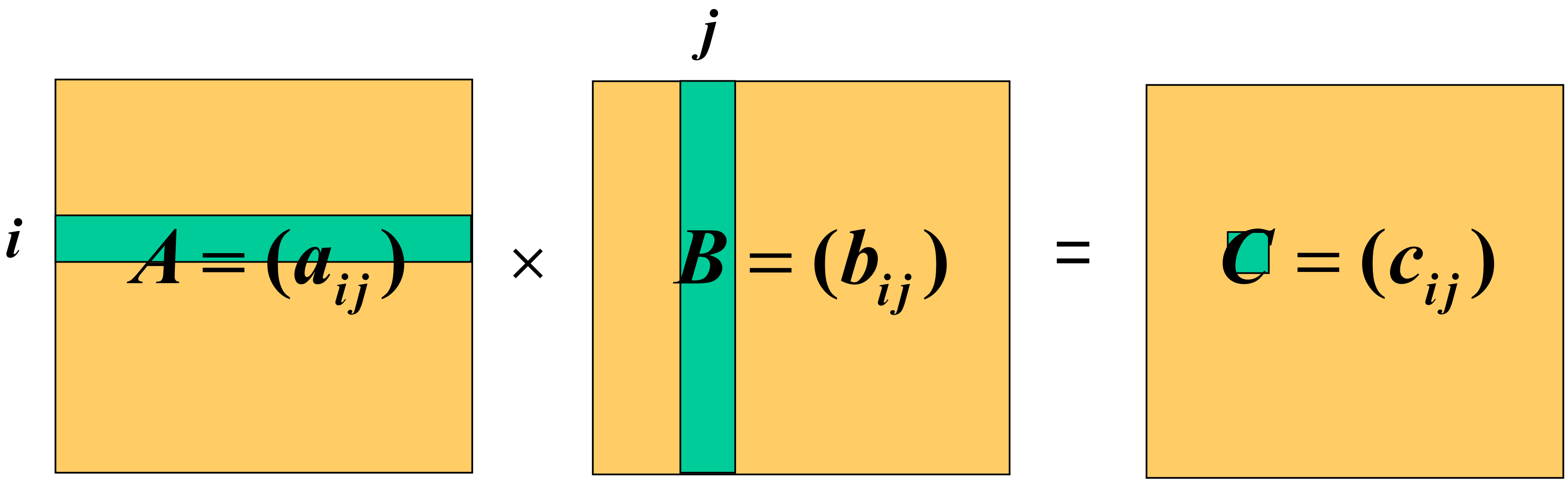
Outline

1. **Algebraic matrix multiplication**
   a. Strassen's algorithm
   b. Rectangular matrix multiplication
2. **Boolean matrix multiplication**
   a. Simple reduction to integer matrix multiplication
   b. Computing the transitive closure of a graph.
3. **Min-Plus matrix multiplication**
   a. Equivalence to the APSP problem
   b. Expensive reduction to algebraic products
   c. Fredman's trick

4. **APSP in undirected graphs**
    a. An $O(n^{2.38})$ algorithm for **unweighted** graphs (Seidel)
    b. An $O(Mn^{2.38})$ algorithm for **weighted** graphs (Shoshan-Zwick)
5. **APSP in directed graphs**
    1. An $O(M^{0.68}n^{2.58})$ algorithm (Zwick)
    2. An $O(Mn^{2.38})$ preprocessing / $O(n)$ query answering algorithm (Yuster-Zwick)
    3. An $O(n^{2.38}\log M)$ $(1+\varepsilon)$-approximation algorithm
6. **Summary and open problems**

Short introduction to Fast matrix multiplication

Algebraic Matrix Multiplication



$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Can be computed naively in $O(n^3)$ time.

Matrix multiplication algorithms

Complexity	Authors
n^3	—
$n^{2.81}$	**Strassen (1969)**
⋮	
$n^{2.38}$	**Coppersmith, Winograd (1990)**

Conjecture/Open problem: $n^{2+o(1)}$???

Multiplying 2×2 matrices

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$$\begin{aligned} C_{11} &= A_{11}B_{11} + A_{12}B_{21} \\ C_{12} &= A_{11}B_{12} + A_{12}B_{22} \\ C_{21} &= A_{21}B_{11} + A_{22}B_{21} \\ C_{22} &= A_{21}B_{12} + A_{22}B_{22} \end{aligned}$$

8 multiplications

4 additions

Works over any ring!

Multiplying $n \times n$ matrices

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$$\begin{aligned} C_{11} &= A_{11}B_{11} + A_{12}B_{21} \\ C_{12} &= A_{11}B_{12} + A_{12}B_{22} \\ C_{21} &= A_{21}B_{11} + A_{22}B_{21} \\ C_{22} &= A_{21}B_{12} + A_{22}B_{22} \end{aligned}$$

8 multiplications

4 additions

$$T(n) = 8\,T(n/2) + O(n^2)$$

$$T(n) = O(n^{\log 8/\log 2}) = O(n^3)$$

Strassen's 2×2 algorithm

$$C_{11} = A_{11}B_{11} + A_{12}B_{21}$$
$$C_{12} = A_{11}B_{12} + A_{12}B_{22}$$
$$C_{21} = A_{21}B_{11} + A_{22}B_{21}$$
$$C_{22} = A_{21}B_{12} + A_{22}B_{22}$$

$$C_{11} = M_1 + M_4 - M_5 + M_7$$
$$C_{12} = M_3 + M_5$$
$$C_{21} = M_2 + M_4$$
$$C_{22} = M_1 - M_2 + M_3 + M_6$$

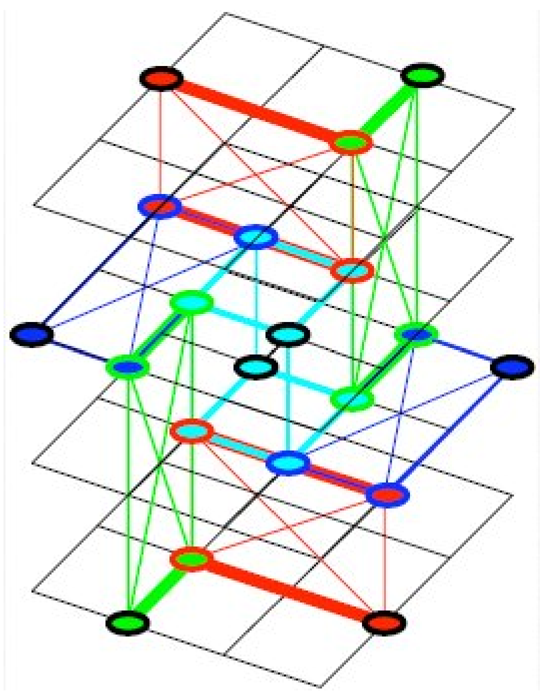
Subtraction!

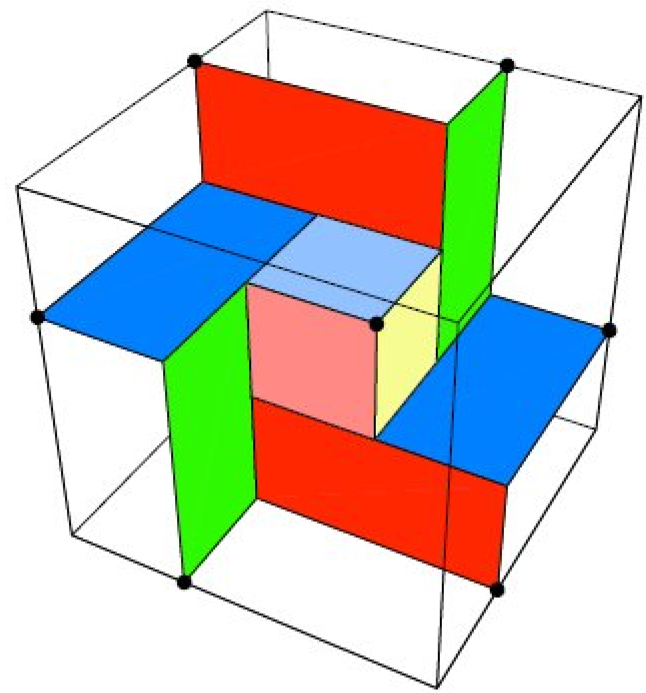
$$M_1 = ($$
$$M_2 = (A_{21} + \quad B_{11}$$
$$M_3 = A_{11}(B_{12} - B_{22})$$
$$M_4 = A_{22}(B_{21} - B_{11})$$
$$M_5 = (A_{11} + A_{12})B_{22}$$
$$M_6 = (A_{21} - A_{11})(B_{11} + B_{12})$$
$$M_7 = (A_{12} - A_{22})(B_{21} + B_{22})$$

7 multiplications
18 additions/subtractions

Works over any ring!

"Strassen Symmetry"
(by Mike Paterson)





Strassen's $n \times n$ algorithm

View each $n \times n$ matrix as a 2×2 matrix whose elements are $n/2 \times n/2$ matrices.

Apply the 2×2 algorithm recursively.

$$T(n) = 7\,T(n/2) + O(n^2)$$

$$T(n) = O(n^{\log 7/\log 2}) = O(n^{2.81})$$

Matrix multiplication algorithms

The $O(n^{2.81})$ bound of Strassen was improved by Pan, Bini-Capovani-Lotti-Romani, Schönhage and finally by Coppersmith and Winograd to $O(n^{2.38})$.

The algorithms are much more complicated…

New group theoretic approach [Cohn-Umans '03] [Cohn-Kleinberg-szegedy-Umans '05]

We let $2 \le \omega < 2.38$ be the exponent of matrix multiplication.

Many believe that $\omega=2+o(1)$.

Determinants / Inverses

The title of Strassen's 1969 paper is:
"Gaussian elimination is not optimal"

Other matrix operations that can be performed in $O(n^{\omega})$ time:

- Computing determinants: $\det A$
- Computing inverses: A^{-1}
- Computing **characteristic** polynomials

Matrix Multiplication
Determinants / Inverses

What is it good for?

Transitive closure

Shortest Paths

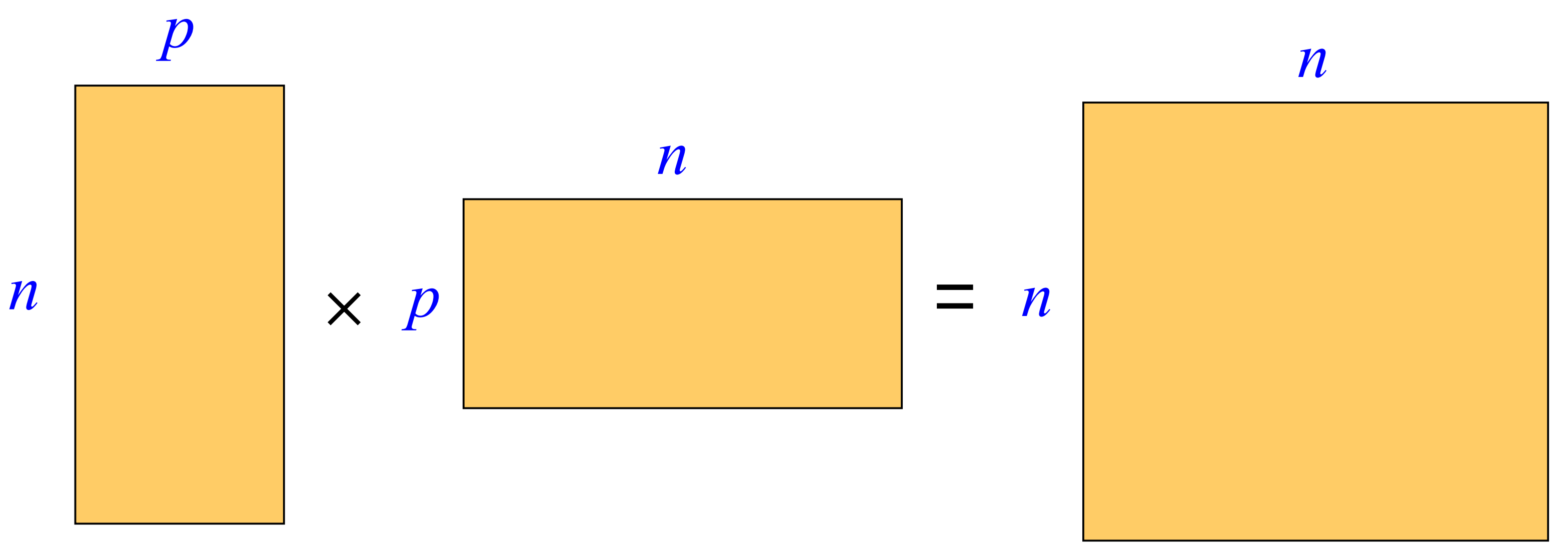
Perfect/Maximum matchings

Dynamic transitive closure

k-vertex connectivity

Counting spanning trees

Rectangular Matrix multiplication



Naïve complexity: n^2p

[Coppersmith '97]: $n^{1.85}p^{0.54}+n^{2+o(1)}$

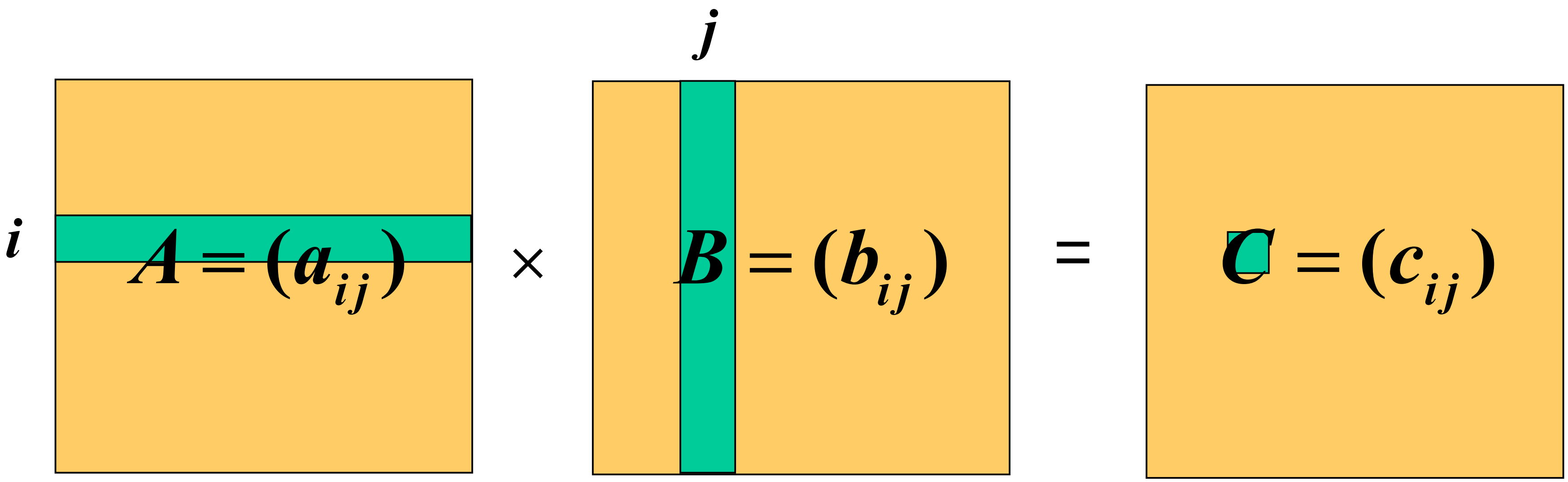
For $p \leq n^{0.29}$, complexity = $n^{2+o(1)}$!!!

BOOLEAN MATRIX MULTIPLICATION

and

TRANSIVE CLOSURE

Boolean Matrix Multiplication



$$c_{ij} = \bigvee_{k=1}^{n} a_{ik} \wedge b_{kj}$$

Can be computed naively in O(n^3) time.

Algebraic Product

$$C = AB$$

$$c_{ij} = \sum_k a_{ik} b_{kj}$$

$O(n^{2.38})$

algebraic operations

Boolean Product

$$C = A \cdot B$$

$$c_{ij} = \bigvee_k a_{ik} \wedge b_{kj}$$

$O(n^{2.38})$?

Logical or ($\vee$) has no inverse!

But we can work over the integers! (modulo $n+1$)

operations on $O(\log n)$ bit words

Transitive Closure

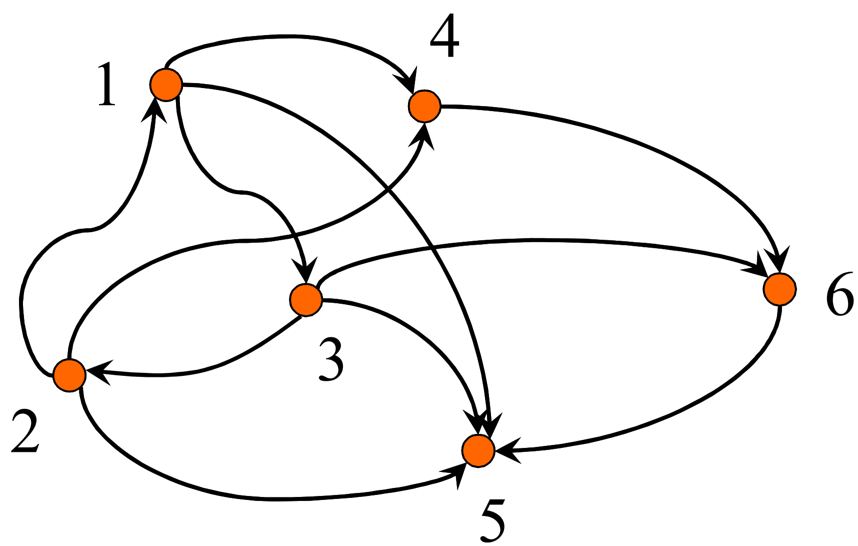
Let $G=(V,E)$ be a directed graph.

The transitive closure $G^*=(V,E^*)$ is the graph in which $(u,v)\in E^*$ iff there is a path from u to v.

Can be easily computed in $O(mn)$ time.

Can also be computed in $O(n^\omega)$ time.

Adjacency matrix of a directed graph



$$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Exercise 0: If A is the adjacency matrix of a graph, then $(A^k)_{ij}=1$ iff there is a path of length k from i to j.

Transitive Closure using matrix multiplication

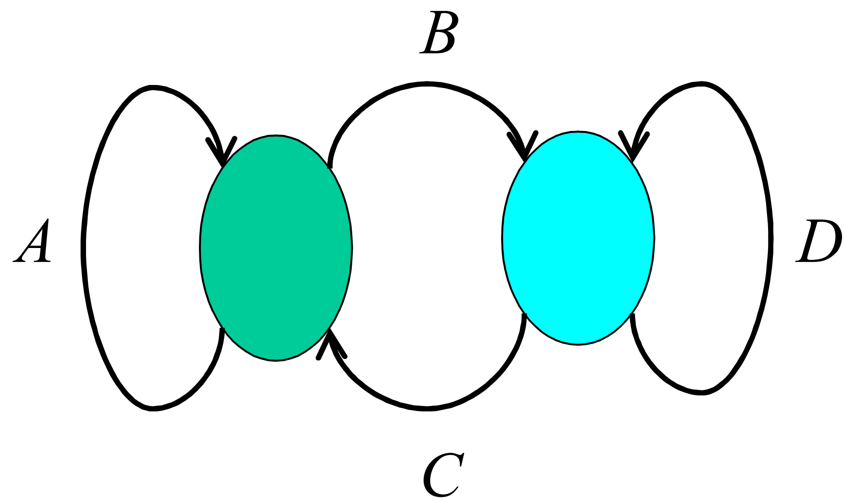
Let $G=(V,E)$ be a directed graph.

If A is the adjacency matrix of G,
then $(A\vee I)^{n-1}$ is the adjacency matrix of G^*.

The matrix $(A\vee I)^{n-1}$ can be computed by $\log n$ squaring operations in $O(n^\omega \log n)$ time.

It can also be computed in $O(n^\omega)$ time.

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$



$$X^* = \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

$$\mathbf{TC}(n) \le {\color{red}\mathbf{2}}\ \mathbf{TC}(n/2) + {\color{red}\mathbf{6}}\ \mathbf{BMM}(n/2) + \mathbf{O}(n^2)$$

Exercise 1: Give $O(n^\omega)$ algorithms for findning, in a directed graph,

a) a triangle

b) a simple quadrangle

c) a simple cycle of length k.

Hints:

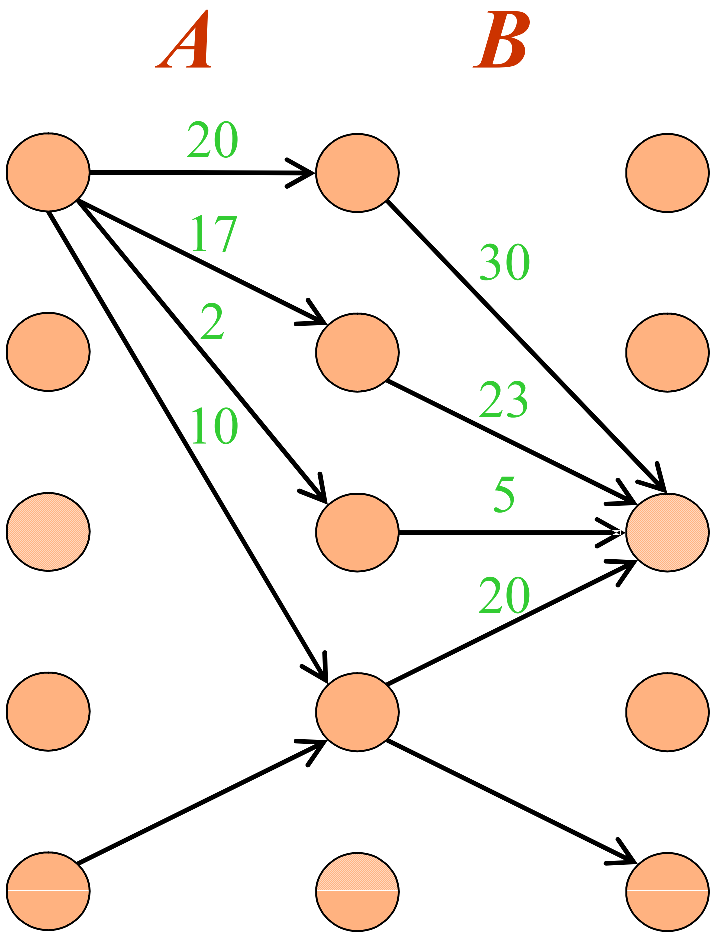
1. In an **acyclic** graph all paths are simple.
2. In c) running time may be **exponential** in k.
3. **Randomization** makes solution much easier.

MIN-PLUS MATRIX MULTIPLICATION

and

ALL-PAIRS SHORTEST PATHS (APSP)

An interesting special case of the APSP problem



$$C = A * B$$

$$c_{ij} = \min_k \{a_{ik} + b_{kj}\}$$

Min-Plus product

Min-Plus Products

$$C = A * B$$

$$c_{ij} = \min_k \{a_{ik} + b_{kj}\}$$

$$\begin{pmatrix} -6 & -3 & -10 \\ 2 & 5 & -2 \\ -1 & -7 & -5 \end{pmatrix} = \begin{pmatrix} 1 & -3 & 7 \\ +\infty & 5 & +\infty \\ 8 & 2 & -5 \end{pmatrix} * \begin{pmatrix} 8 & +\infty & -4 \\ -3 & 0 & -7 \\ 5 & -2 & 1 \end{pmatrix}$$

Solving APSP by repeated squaring

If W is an n by n matrix containing the edge weights of a graph. Then W^n is the distance matrix.

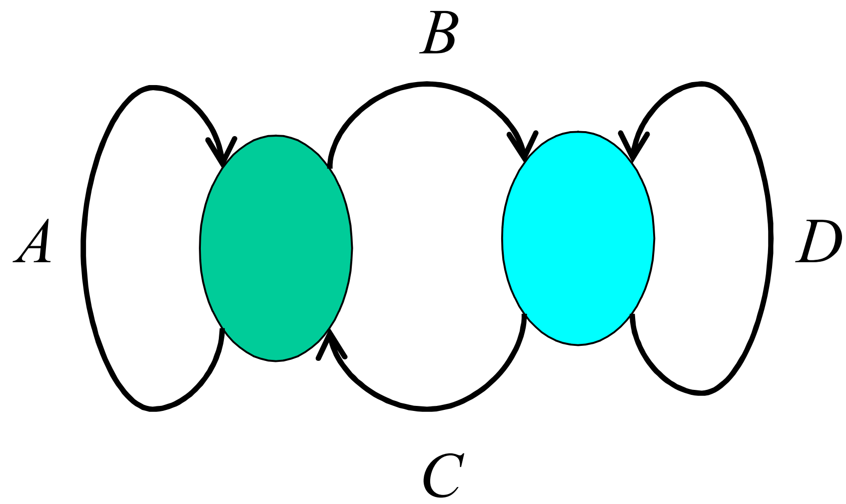
By induction, W^k gives the distances realized by paths that use at most k edges.

$$
\begin{array}{l}
D \leftarrow W \\
\text{for } i \leftarrow 1 \text{ to } \lceil \log_2 n \rceil \\
\text{do } D \leftarrow D * D
\end{array}
$$

Thus: $\mathrm{APSP}(n) \le \mathrm{MPP}(n) \log n$

Actually: $\mathrm{APSP}(n) = \mathrm{O}(\mathrm{MPP}(n))$

$$X = \begin{array}{|c|c|} \hline A & B \\ \hline C & D \\ \hline \end{array}$$



$$X^* = \begin{array}{|c|c|} \hline E & F \\ \hline G & H \\ \hline \end{array} = \begin{array}{|c|c|} \hline (A \vee BD^*C)^* & EBD^* \\ \hline D^*CE & D^* \vee GBD^* \\ \hline \end{array}$$

APSP(n) ≤ 2 APSP(n/2) + 6 MPP(n/2) + O(n^2)

Algebraic Product

$$C = A \cdot B$$

$$c_{ij} = \sum_k a_{ik} b_{kj}$$

$O(n^{2.38})$

Min-Plus Product

$$C = A * B$$

$$c_{ij} = \min_k \{a_{ik} + b_{kj}\}$$

min operation has no inverse!

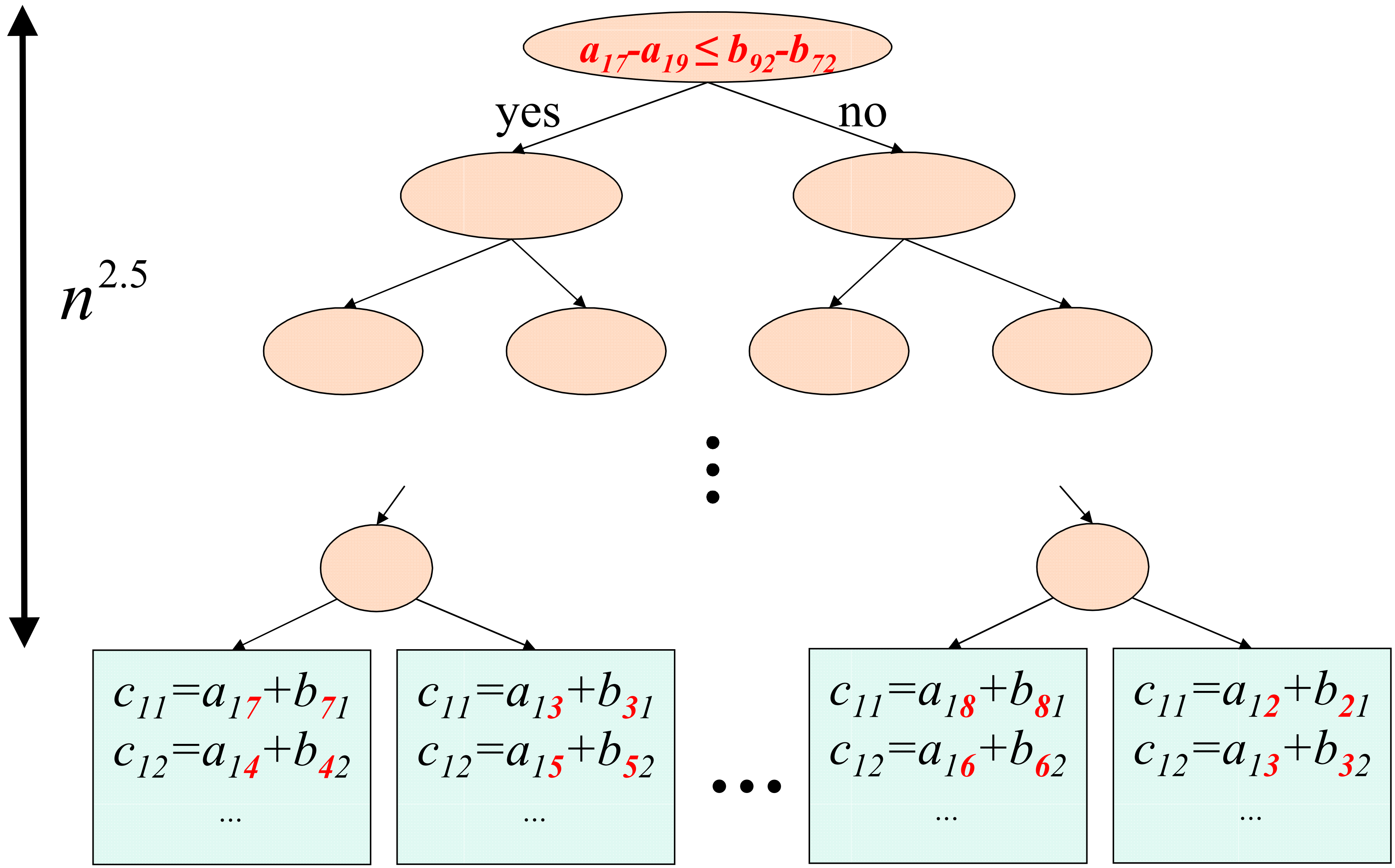
?

Fredman's trick

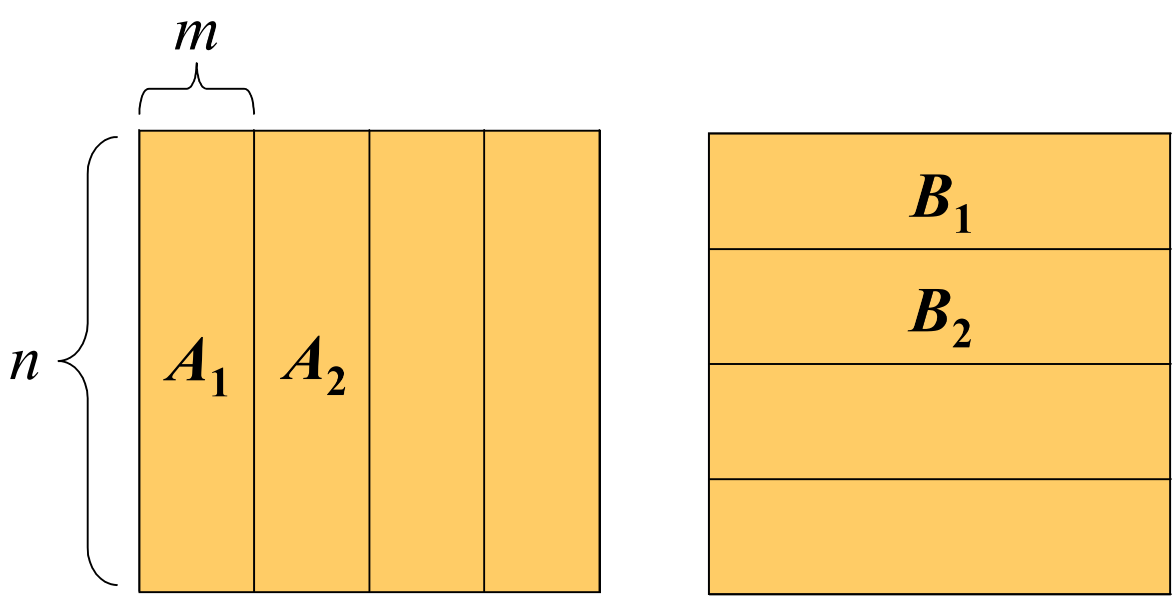
The **min-plus** product of two $n \times n$ matrices can be **deduced** after only $O(n^{2.5})$ additions and comparisons.

It is not known how to implement the algorithm in $O(n^{2.5})$ time.

Algebraic Decision Trees



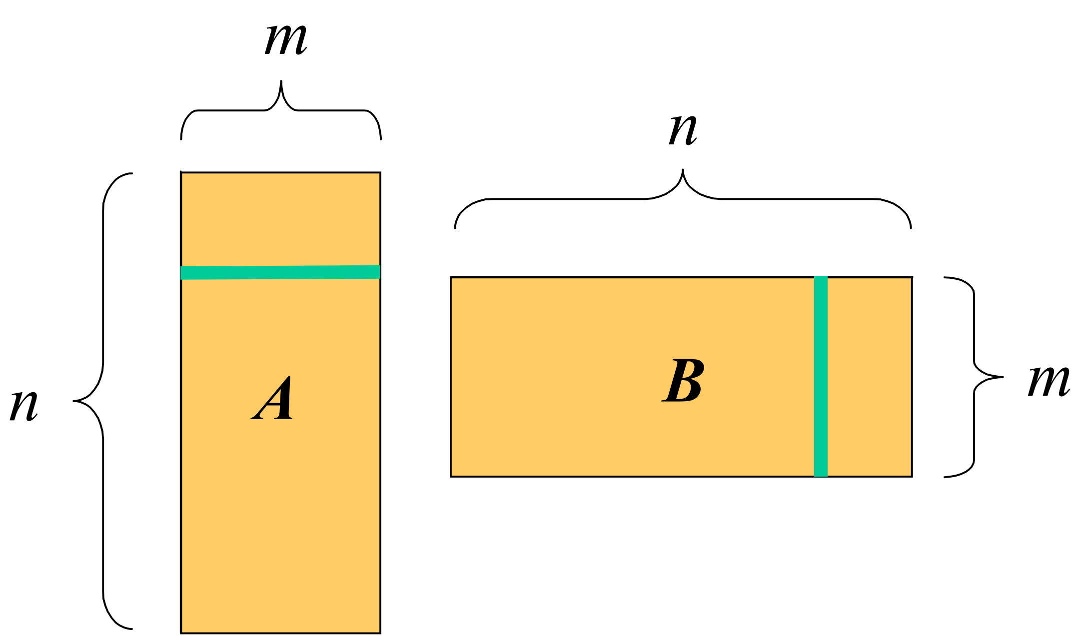
Breaking a square product into several rectangular products



$$A * B = \min_i A_i * B_i$$

MPP(n) ≤ (n/m) (MPP(n,m,n) + n^2)

Fredman's trick



$$a_{ir}+b_{rj} \leq a_{is}+b_{sj}$$
$$\Updownarrow$$
$$a_{ir} - a_{is} \leq b_{sj} - b_{rj}$$

Naïve calculation requires $\boldsymbol{n^2m}$ operations

Fredman observed that the result can be inferred after performing only $O(\boldsymbol{nm^2})$ operations

Fredman's trick (cont.)

$$a_{ir}+b_{rj} \le a_{is}+b_{sj} \quad \Leftrightarrow \quad a_{ir} - a_{is} \le b_{sj} - b_{rj}$$

- **Generate** all the differences $a_{ir} - a_{is}$ and $b_{sj} - b_{rj}$.
- **Sort** them using $O(nm^2)$ comparisons. (Non-trivial!)
- **Merge** the two sorted lists using $O(nm^2)$ comparisons.

The ordering of the elements in the sorted list determines the result of the min-plus product !!!

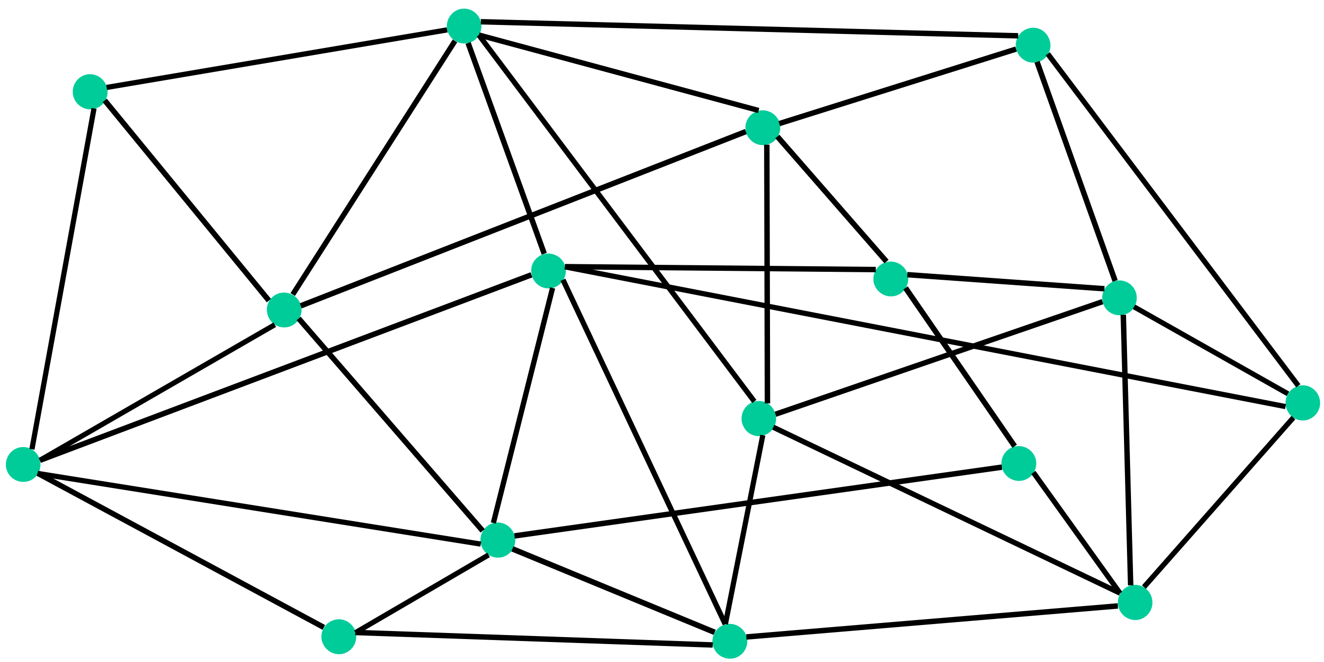
All-Pairs Shortest Paths

in directed graphs with “real” edge weights

Running time	**Authors**
n^3	[Floyd ’62] [Warshall ’62]
$n^3 (\log \log n / \log n)^{1/3}$	[Fredman ’76]
$n^3 (\log \log n / \log n)^{1/2}$	[Takaoka ’92]
$n^3 / (\log n)^{1/2}$	[Dobosiewicz ’90]
$n^3 (\log \log n / \log n)^{5/7}$	[Han ’04]
$n^3 \log \log n / \log n$	[Takaoka ’04]
$n^3 (\log \log n)^{1/2} / \log n$	[Zwick ’04]
$n^3 / \log n$	[Chan ’05]
$n^3 (\log \log n / \log n)^{5/4}$	[Han ’06]
$n^3 (\log \log n)^3 / (\log n)^2$	[Chan ’07]

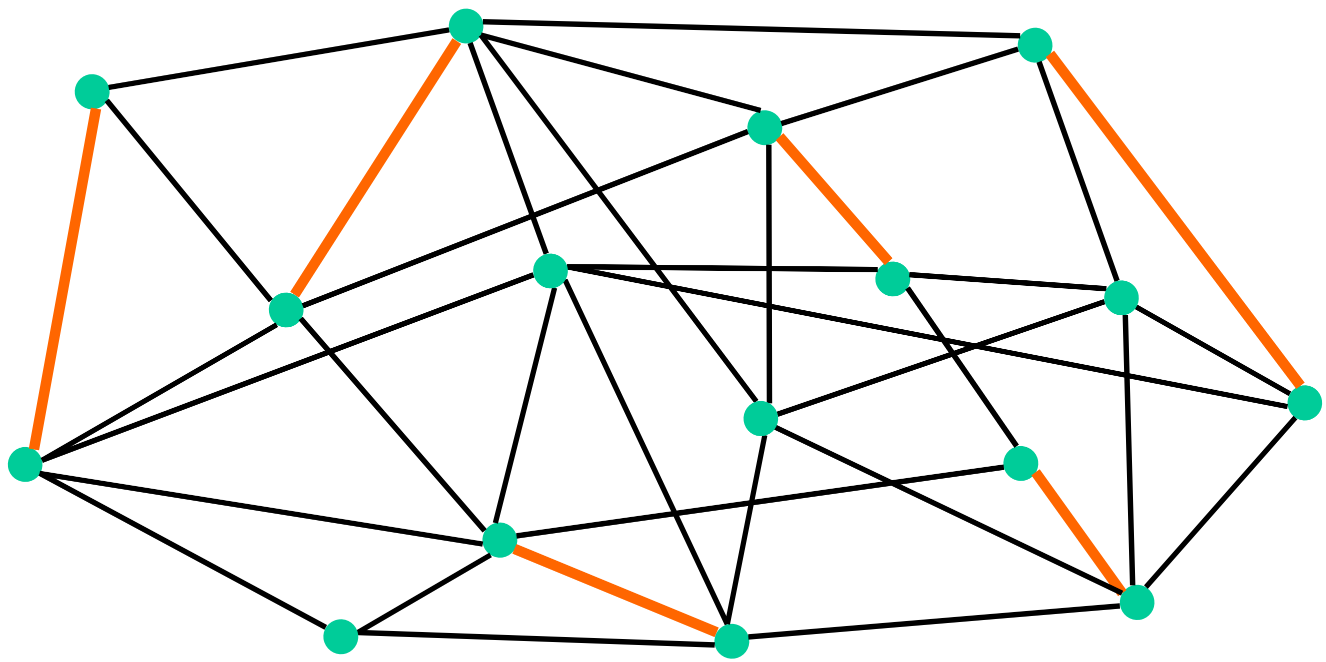
PERFECT MATCHINGS

Matchings



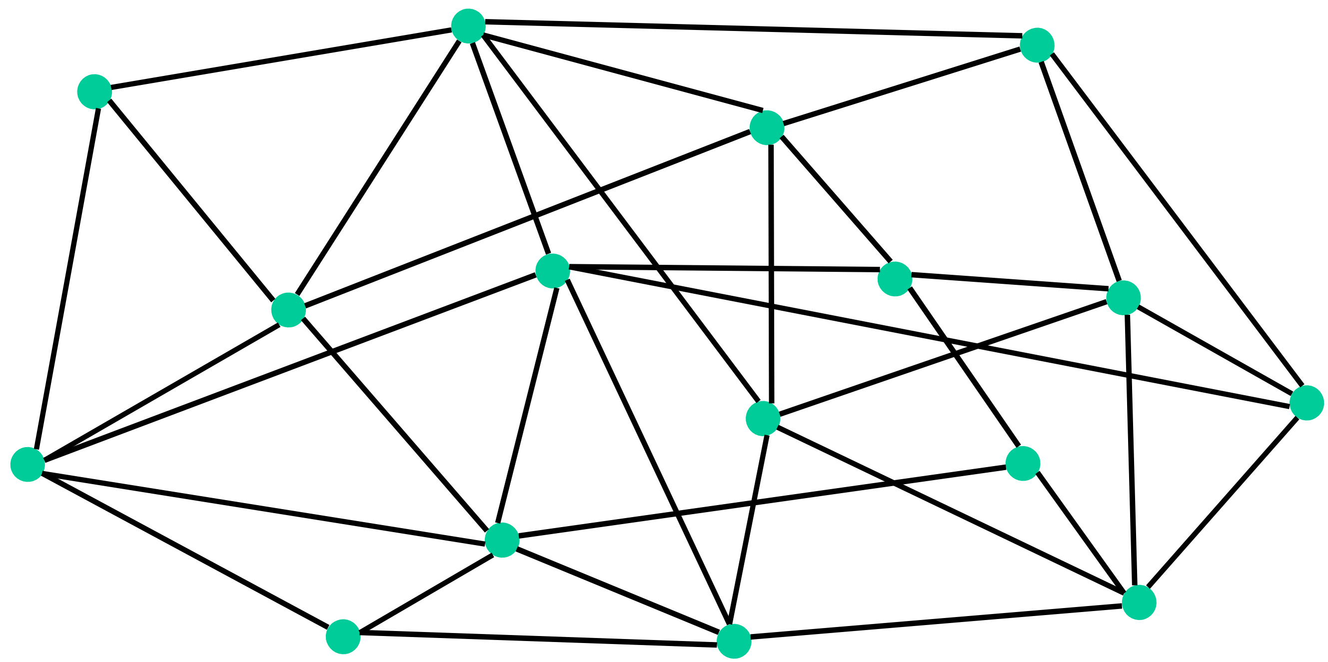
A matching is a subset of edges that do not touch one another.

Matchings



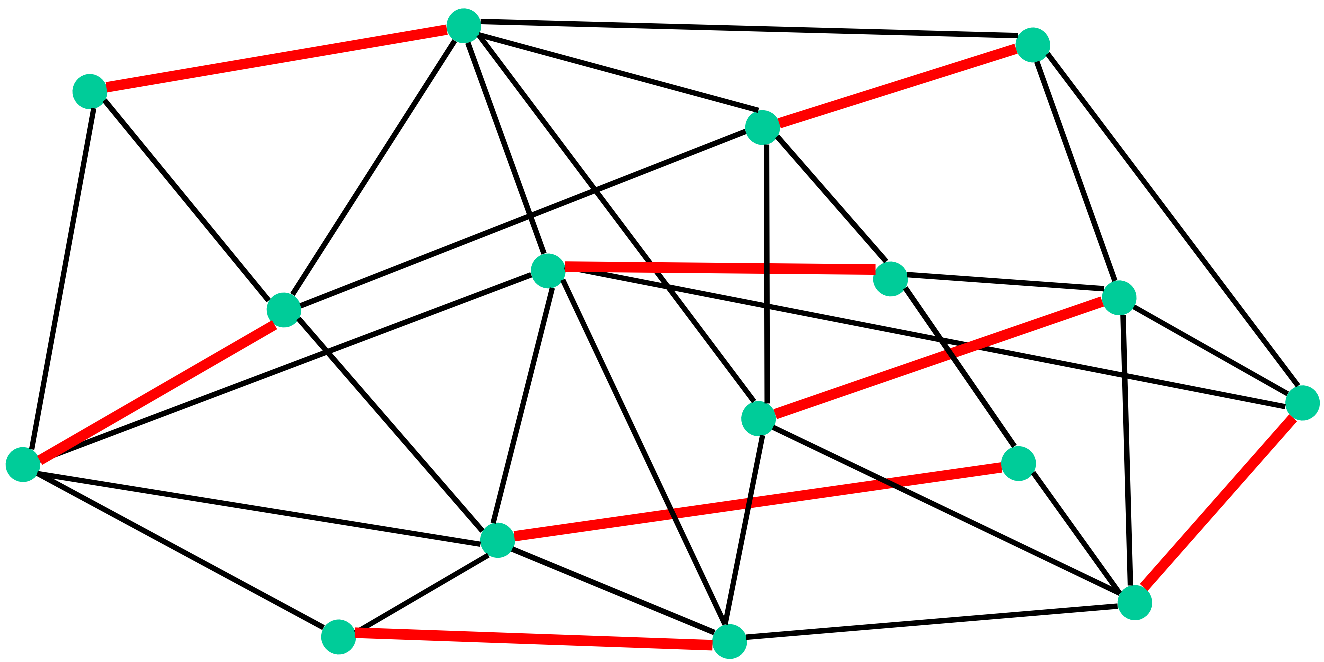
A matching is a subset of edges that do not touch one another.

Perfect Matchings



A matching is perfect if there are no unmatched vertices

Perfect Matchings



A matching is perfect if there are no unmatched vertices

Algorithms for finding perfect or maximum matchings

Combinatorial approach:

A matching M is a maximum matching iff it admits no augmenting paths

Algorithms for finding perfect or maximum matchings

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A matching M is a maximum matching iff it admits no augmenting paths

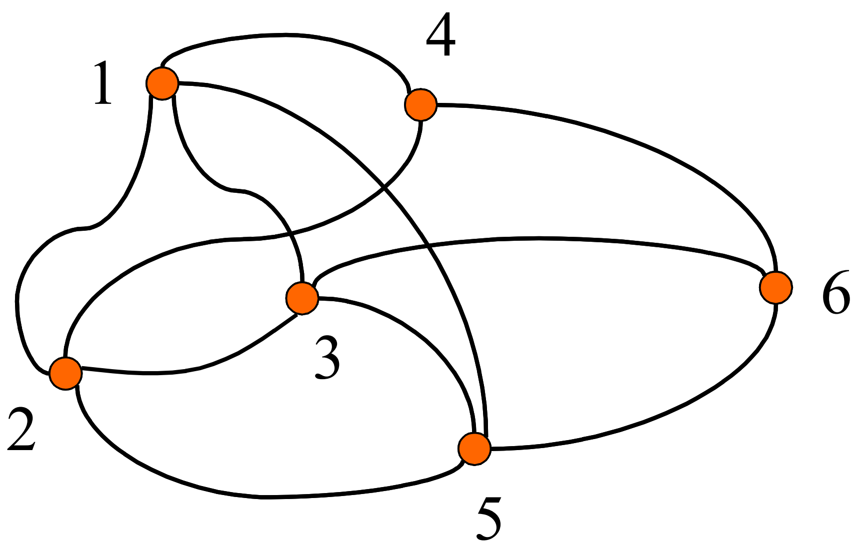
Combinatorial algorithms for finding perfect or maximum matchings

In bipartite graphs, augmenting paths can be found quite easily, and maximum matchings can be used using max flow techniques.

In non-bipartite the problem is much harder. (Edmonds' Blossom shrinking techniques)

Fastest running time (in both cases):
$O(mn^{1/2})$ [Hopcroft-Karp] [Micali-Vazirani]

Adjacency matrix
of a undirected graph



$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

The adjacency matrix of an undirected graph is symmetric.

Matchings, Permanents, Determinants

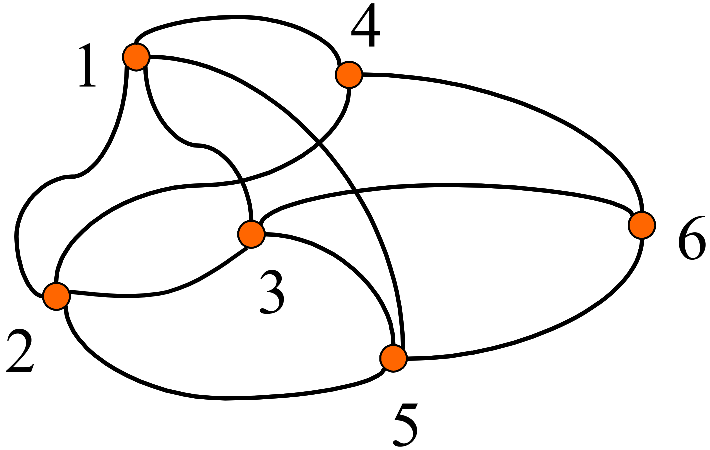
$$\det(A) \;=\; \sum_{\pi \in S_n} sign(\pi) \prod_{i=1}^{n} a_{i\pi(i)}$$

$$\mathrm{per}(A) \;=\; \sum_{\pi \in S_n} \prod_{i=1}^{n} a_{i\pi(i)}$$

Exercise 2: Show that if A is the adjacency matrix of a bipartite graph G, then per(A) is the number of perfect matchings in G.

Unfortunately computing the permanent is **#P-complete**…

Tutte's matrix
(Skew-symmetric symbolic adjacency matrix)



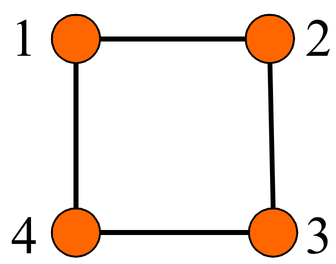
$$\begin{pmatrix} 0 & x_{12} & x_{13} & x_{14} & x_{15} & 0 \\ -x_{12} & 0 & x_{23} & x_{24} & x_{25} & 0 \\ -x_{13} & -x_{23} & 0 & 0 & x_{35} & x_{36} \\ -x_{14} & -x_{24} & 0 & 0 & 0 & x_{46} \\ -x_{15} & -x_{25} & -x_{35} & 0 & 0 & x_{56} \\ 0 & 0 & -x_{36} & -x_{46} & -x_{56} & 0 \end{pmatrix}$$

$$a_{ij} = \begin{cases} x_{ij} & \text{if } \{i,j\} \in E \text{ and } i < j, \\ -x_{ji} & \text{if } \{i,j\} \in E \text{ and } i > j, \\ 0 & \text{otherwise} \end{cases}$$

$$A^T = -A$$

Tutte's theorem

Let $G=(V,E)$ be a graph and let A be its Tutte matrix. Then, G has a perfect matching iff $\det A \neq 0$.



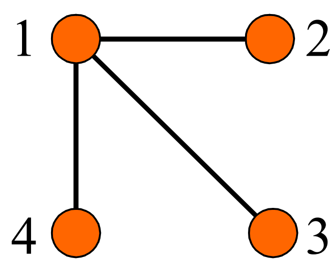
$$A = \begin{pmatrix} 0 & x_{12} & 0 & x_{14} \\ -x_{12} & 0 & x_{23} & 0 \\ 0 & -x_{23} & 0 & -x_{34} \\ -x_{14} & 0 & -x_{34} & 0 \end{pmatrix}$$

$$\det A = x_{12}^2 x_{34}^2 + x_{14}^2 x_{23}^2 + 2x_{12}x_{23}x_{34}x_{41} \neq 0$$

There are perfect matchings

Tutte's theorem

Let ***G*=(*V*,*E*)** be a graph and let ***A*** be its Tutte matrix. Then, ***G*** has a perfect matching iff det ***A*** ≠0.



$$A = \begin{pmatrix} 0 & x_{12} & x_{13} & x_{14} \\ -x_{12} & 0 & 0 & 0 \\ -x_{13} & 0 & 0 & 0 \\ -x_{14} & 0 & 0 & 0 \end{pmatrix}$$

$$\det A = 0$$

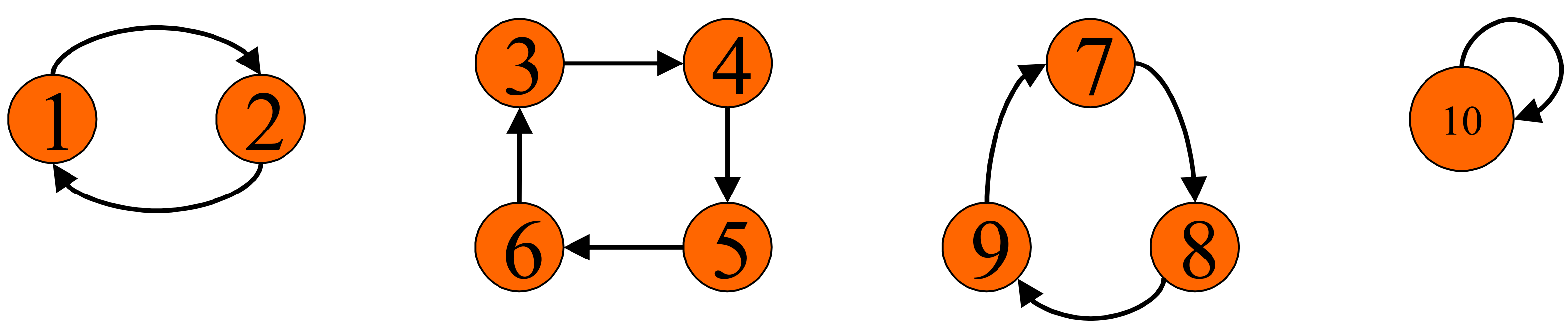
No perfect matchings

Proof of Tutte's theorem

$$\det A = \sum_{\pi \in S_n} sign(\pi) \prod_{i=1}^{n} a_{i\pi(i)}$$

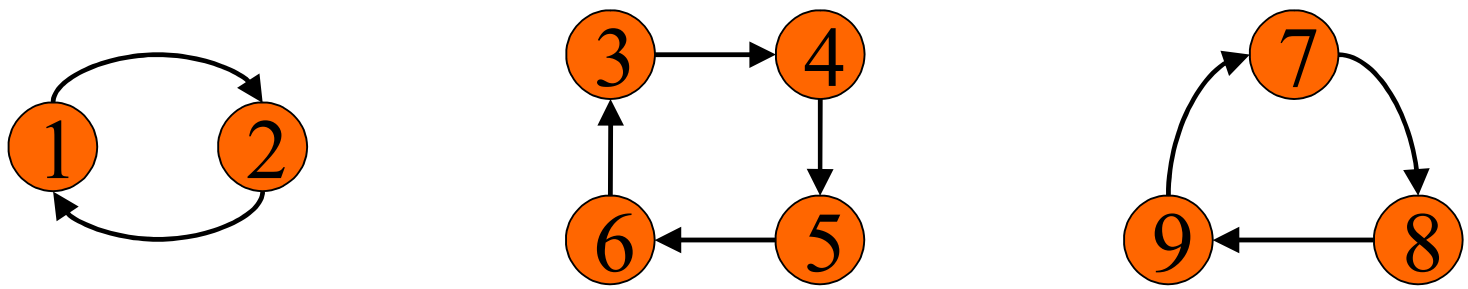
Every permutation $\pi \in S_n$ defines a cycle collection

$$\pi = (2\ 1\ 4\ 5\ 6\ 3\ 8\ 9\ 7\ 10)$$



Cycle covers

A permutation $\pi \in S_n$ for which $\{i, \pi(i)\} \in E$, for $1 \leq i \leq k$, defines a cycle cover of the graph.

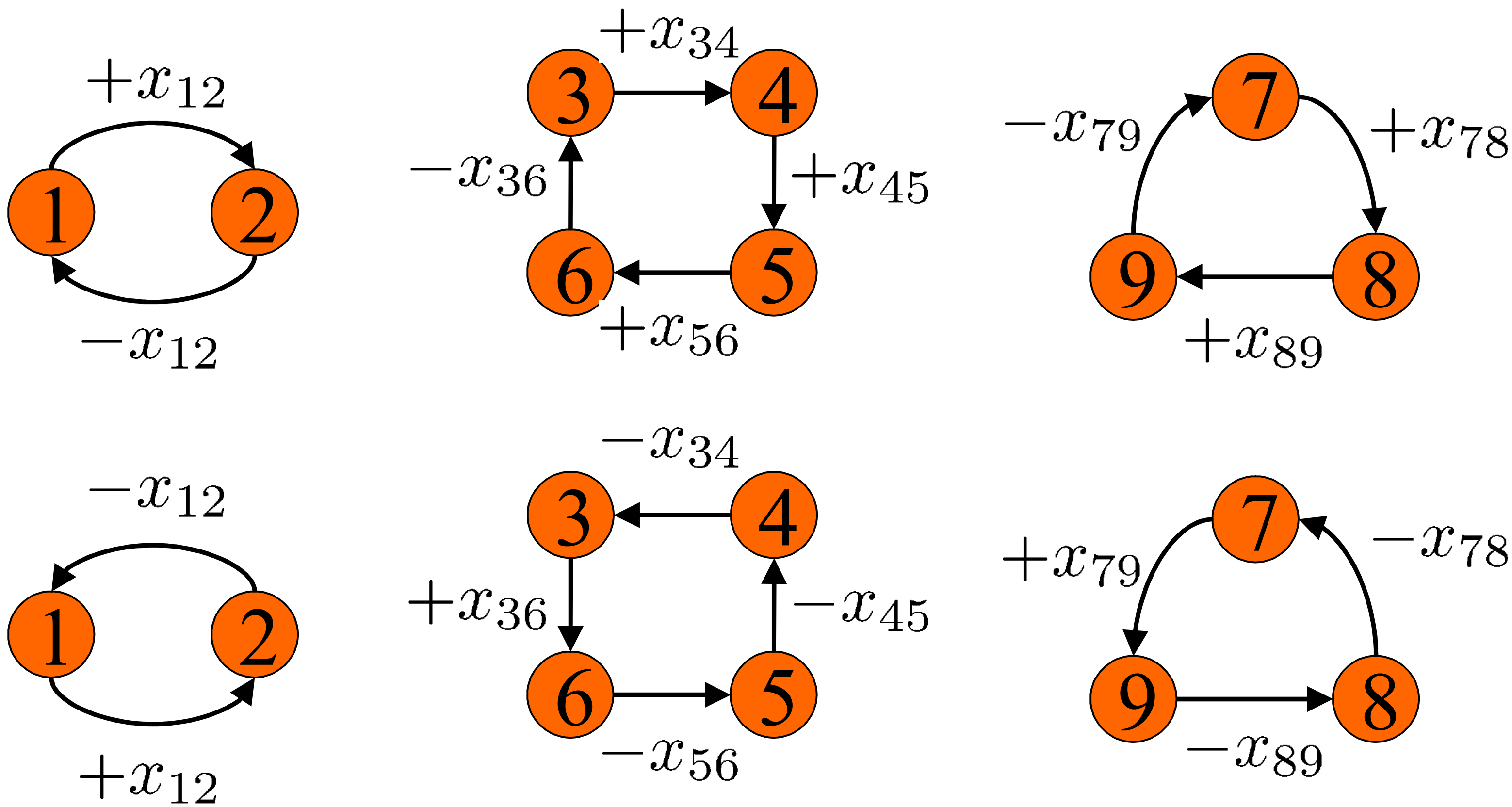


Exercise 3: If π' is obtained from π by reversing the direction of a cycle, then $sign(\pi') = sign(\pi)$.

$$\prod_{i=1}^{n} a_{i\pi'(i)} = \pm \prod_{i=1}^{n} a_{i\pi(i)}$$

Depending on the parity of the cycle!

Reversing Cycles



$$\prod_{i=1}^{n} a_{i\pi'(i)} = \pm \prod_{i=1}^{n} a_{i\pi(i)}$$

Depending on the parity of the cycle!

Proof of Tutte's theorem (cont.)

$$\det A = \sum_{\pi \in S_n} sign(\pi) \prod_{i=1}^{n} a_{i\pi(i)}$$

The permutations $\pi \in S_n$ that contain an **odd** cycle cancel each other!

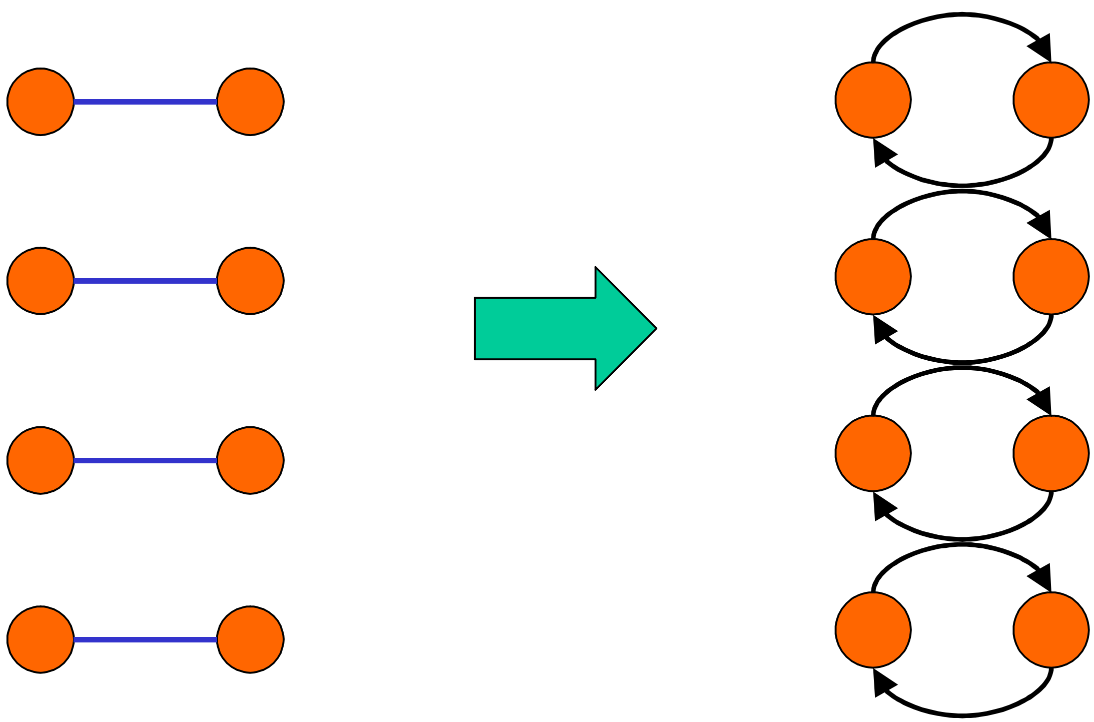
We effectively sum only over **even cycle covers**.

A graph contains a perfect matching iff it contains an **even cycle cover**.

Proof of Tutte's theorem (cont.)

A graph contains a perfect matching iff it contains an **even cycle cover**.

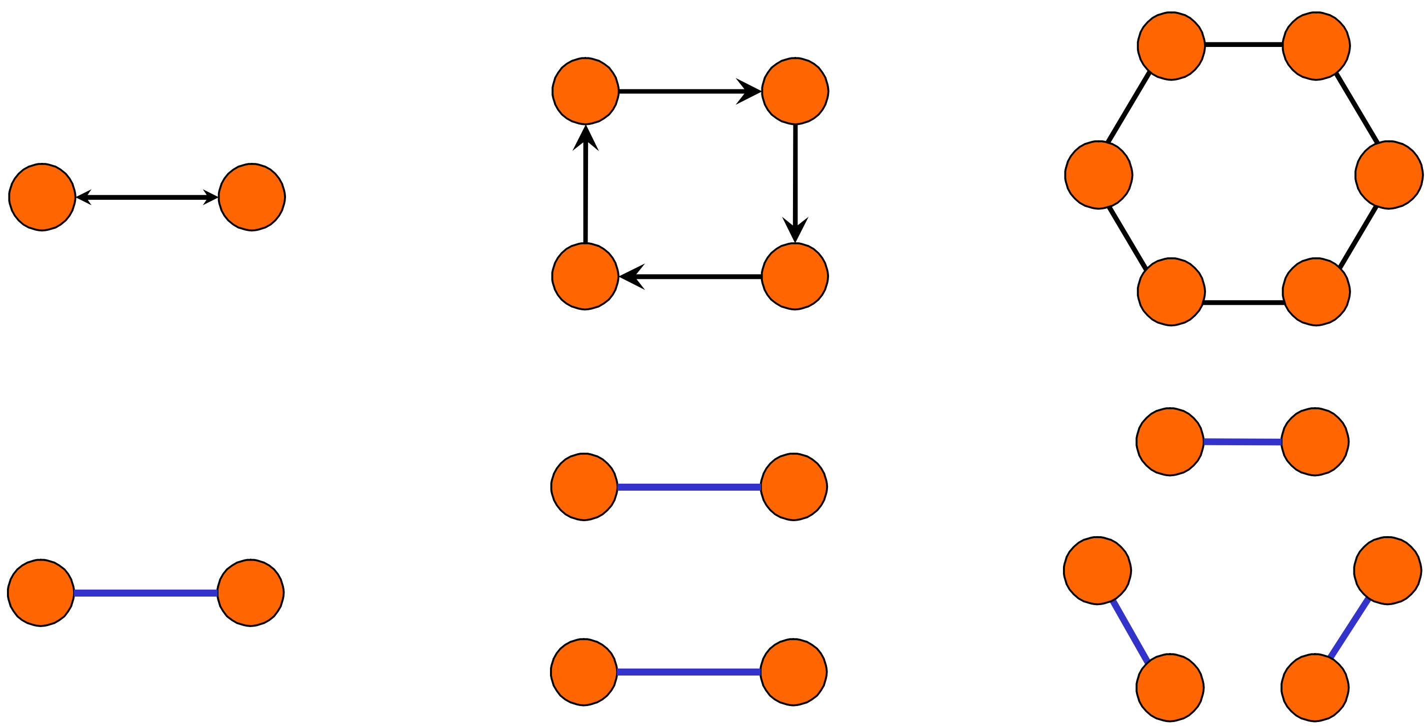
Perfect Matching → Even cycle cover



Proof of Tutte's theorem (cont.)

A graph contains a perfect matching
iff it contains an **even cycle cover**.

Even cycle cover → Perfect matching



An algorithm for perfect matchings?

- Construct the Tutte matrix A.
- Compute $\det A$.
- If $\det A \neq 0$, say 'yes', otherwise 'no'.

Problem: $\det A$ is a symbolic expression that may be of exponential size!

Lovasz's solution: Replace each variable x_{ij} by a random element of Z_p, where $p = \Theta(n^2)$ is a prime number

The Schwartz-Zippel lemma

Let $P(x_1,x_2,\ldots,x_n)$ be a polynomial of degree d over a field F. Let $S \subseteq F$. If $P(x_1,x_2,\ldots,x_n)\neq 0$ and $a_1,a_2,\ldots,a_n$ are chosen randomly and independently from S, then

$$\Pr[\,P(a_1,a_2,\ldots,a_n)=0\,] \;\le\; \frac{d}{|S|}$$

Proof by induction on n.

For $n=1$, follows from the fact that polynomial of degree d over a field has at most d roots

Lovasz's algorithm for existence of perfect matchings

- Construct the Tutte matrix $\boldsymbol{A}$.
- Replace each variable x_{ij} by a random element of Z_p, where $p = O(n^2)$ is prime.
- Compute $\det \boldsymbol{A}$.
- If $\det \boldsymbol{A} \neq 0$, say 'yes', otherwise 'no'.

If algorithm says 'yes', then
the graph contains a perfect matching.

If the graph contains a perfect matching, then
the probability that the algorithm says 'no',
is at most $O(1/n)$.

Parallel algorithms

Determinants can be computed
very quickly in parallel

$\text{DET} \in \text{NC}^2$

Perfect matchings can be detected
very quickly in parallel (using randomization)

$\text{PERFECT-MATCH} \in \text{RNC}^2$

Open problem:

??? $\text{PERFECT-MATCH} \in \text{NC}$???

Finding perfect matchings

Self Reducibility

Delete an edge and check
whether there is still a perfect matching

Needs $O(n^2)$ determinant computations

Running time $O(n^{\omega+2})$

Fairly slow…

Not parallelizable!

Finding perfect matchings

Rabin-Vazirani (1986): An edge $\{i,j\} \in E$ is contained in a perfect matching iff $(A^{-1})_{ij} \neq 0$.

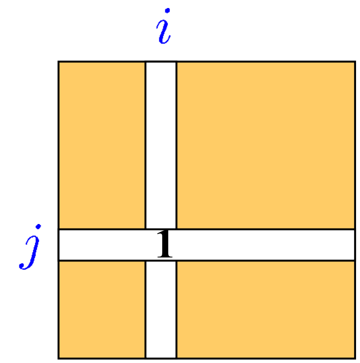
Leads immediately to an $O(n^{\omega+1})$ algorithm: Find an allowed edge $\{i,j\} \in E$, delete it and its vertices from the graph, and recompute A^{-1}.

Mucha-Sankowski (2004): Recomputing A^{-1} from scratch is very wasteful. Running time can be reduced to $O(n^{\omega})$!

Harvey (2006): A simpler $O(n^{\omega})$ algorithm.

Adjoint and Cramer's rule

$$(\mathrm{adj}(A))_{ij} = (-1)^{i+j}\det(A^{j,i}) = \det$$



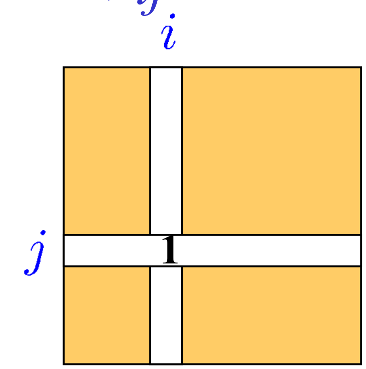
A with the j-th row
and i-th column deleted

Cramer's rule: $$A^{-1} = \frac{\mathrm{adj}(A)}{\det(A)}$$

Finding perfect matchings

Rabin-Vazirani (1986): An edge $\{i,j\} \in E$ is contained in a perfect matching iff $(A^{-1})_{ij} \neq 0$.

$$(\mathrm{adj}(A))_{ij} = (-1)^{i+j} \det(A^{j,i}) = \det$$



Leads immediately to an $O(n^{\omega+1})$ algorithm: Find an allowed edge $\{i,j\} \in E$, delete it and its vertices from the graph, and recompute A^{-1}.

Still not parallelizable

Finding unique minimum weight perfect matchings
[Mulmuley-Vazirani-Vazirani (1987)]

Suppose that edge $\{i,j\} \in E$ has integer weight w_{ij}

Suppose that there is a unique minimum weight perfect matching M of total weight W

Replace x_{ij} by $2^{w_{ij}}$

Then, $2^{2W} \mid \det(A)$ but $2^{2W+1} \nmid \det(A)$

Furthermore, $\{i,j\} \in M$ iff $\frac{2^{w_{ij}} \det(A^{ij})}{2^{2W}}$ is odd

Isolating lemma
[Mulmuley-Vazirani-Vazirani (1987)]

Suppose that G has a perfect matching

Assign each edge $\{i,j\} \in E$
a random integer weight $w_{ij} \in [1, 2m]$

With probability of at least ½, the minimum weight perfect matching of G is unique

Lemma holds for general collecitons of sets, not just perfect matchings

Proof of Isolating lemma
[Mulmuley-Vazirani-Vazirani (1987)]

An edge $\{i,j\}$ is ambivalent if there is a minimum weight perfect matching that contains it and another that does not

Suppose that weights were assigned to all edges except for $\{i,j\}$

Let a_{ij} be the largest weight for which $\{i,j\}$ participates in some minimum weight perfect matchings

If $w_{ij}<a_{ij}$, then $\{i,j\}$ participates in all minimum weight perfect matchings

The probability that $\{i,j\}$ is ambivalent is at most $1/(2m)$!

Finding perfect matchings

[Mulmuley-Vazirani-Vazirani (1987)]

Choose random weights in $[1,2m]$

Compute determinant and adjoint

Read of a perfect matching (w.h.p.)

Is using m-bit integers **cheating**?

Not if we are willing to pay for it!

Complexity is $O(mn^{\omega}) \leq O(n^{\omega+2})$

Finding perfect matchings in RNC^2

Improves an RNC^3 algorithm by

[Karp-Upfal-Wigderson (1986)]

Multiplying two N-bit numbers

``School method''

$$N^2$$

[Schöonhage-Strassen (1971)]

$$N \log N \log \log N$$

[Fürer (2007)]

[De-Kurur-Saha-Saptharishi (2008)]

$$N \log N \, 2^{O(\log^* N)}$$

For our purposes… $\tilde{O}(N)$

Finding perfect matchings

We are not over yet…

[Mucha-Sankowski (2004)]
Recomputing A^{-1} from scratch is wasteful.
Running time can be reduced to $O(n^{\omega})$!

[Harvey (2006)]
A simpler $O(n^{\omega})$ algorithm.

Using matrix multiplication to compute min-plus products

$$\begin{pmatrix} c_{11} & c_{12} & \\ c_{21} & c_{22} & \\ & & \ddots \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \\ a_{21} & a_{22} & \\ & & \ddots \end{pmatrix} * \begin{pmatrix} b_{11} & b_{12} & \\ b_{21} & b_{22} & \\ & & \ddots \end{pmatrix}$$

$$c_{ij} = \min_k \{a_{ik} + b_{kj}\}$$

$$\begin{pmatrix} c'_{11} & c'_{12} & \\ c'_{21} & c'_{22} & \\ & & \ddots \end{pmatrix} = \begin{pmatrix} x^{a_{11}} & x^{a_{12}} & \\ x^{a_{21}} & x^{a_{22}} & \\ & & \ddots \end{pmatrix} \times \begin{pmatrix} x^{b_{11}} & x^{b_{12}} & \\ x^{b_{21}} & x^{b_{22}} & \\ & & \ddots \end{pmatrix}$$

$$c'_{ij} = \sum_k x^{a_{ik}+b_{kj}} \qquad c_{ij} = \mathit{first}(c'_{ij})$$

Using matrix multiplication to compute min-plus products

Assume: $0 \le a_{ij}, b_{ij} \le M$

$$\begin{pmatrix} c'_{11} & c'_{12} & \\ c'_{21} & c'_{22} & \\ & & \ddots \end{pmatrix} = \begin{pmatrix} x^{a_{11}} & x^{a_{12}} & \\ x^{a_{21}} & x^{a_{22}} & \\ & & \ddots \end{pmatrix} * \begin{pmatrix} x^{b_{11}} & x^{b_{12}} & \\ x^{b_{21}} & x^{b_{22}} & \\ & & \ddots \end{pmatrix}$$

n^{ω} polynomial products $\times$ M operations per polynomial product $=$ Mn^{ω} operations per max-plus product

SHORTEST PATHS

APSP – All-Pairs Shortest Paths

SSSP – Single-Source Shortest Paths

UNWEIGHTED
UNDIRECTED
SHORTEST PATHS

4. **APSP in undirected graphs**

   a. An $O(n^{2.38})$ algorithm for **unweighted** graphs (Seidel)

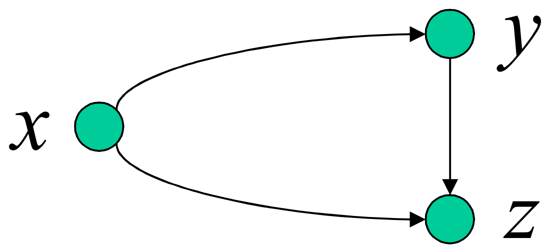
   b. An $O(Mn^{2.38})$ algorithm for **weighted** graphs (Shoshan-Zwick)

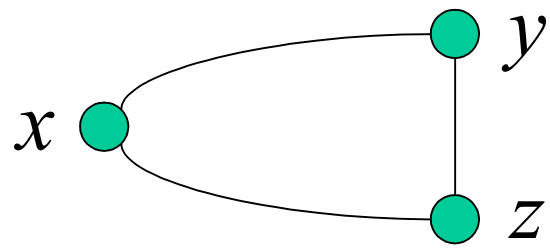
5. **APSP in directed graphs**

   1. An $O(M^{0.68}n^{2.58})$ algorithm (Zwick)
   2. An $O(Mn^{2.38})$ preprocessing / $O(n)$ query answering algorithm (Yuster-Zwick)
   3. An $O(n^{2.38}\log M)$ $(1+\varepsilon)$-approximation algorithm

6. **Summary and open problems**

Directed versus undirected graphs





$\delta(x,z) \le \delta(x,y) + \delta(y,z)$

Triangle inequality

$\delta(x,z) \le \delta(x,y) + \delta(y,z)$

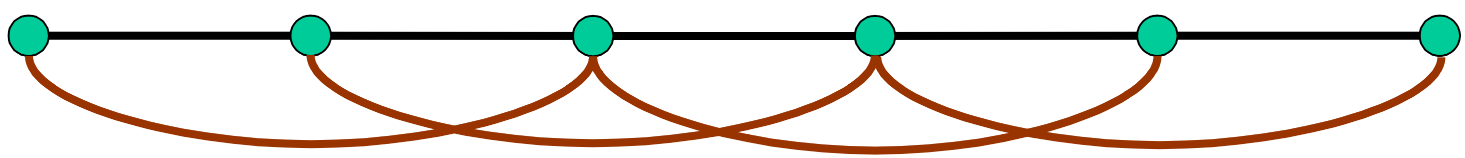
$\delta(x,y) \le \delta(x,z) + \delta(z,y)$

$\delta(x,z) \ge \delta(x,y) - \delta(y,z)$

Inverse triangle inequality

Distances in G and its square G^2

Let $G=(V,E)$. Then $G^2=(V,E^2)$, where $(u,v)\in E^2$ if and only if $(u,v)\in E$ or there exists $w\in V$ such that $(u,w),(w,v)\in E$

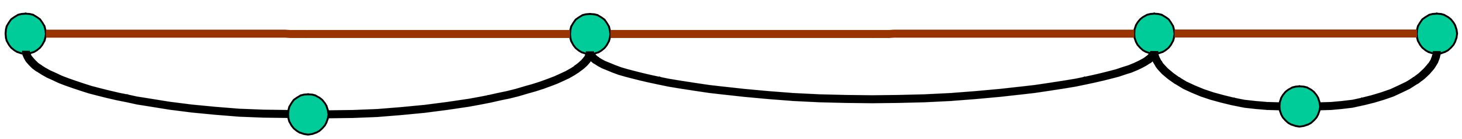


Let $\delta(u,v)$ be the distance from u to v in G.
Let $\delta^2(u,v)$ be the distance from u to v in G^2.

Distances in G and its square G^2 (cont.)

Lemma: $\delta^2(u,v) = \lceil \delta(u,v)/2 \rceil$, for every $u,v \in V$.

$$\delta^2(u,v) \leq \lceil \delta(u,v)/2 \rceil$$



$$\delta(u,v) \leq 2\delta^2(u,v)$$

Thus: $\delta(u,v) = 2\delta^2(u,v)$ or

$\delta(u,v) = 2\delta^2(u,v) - 1$

Distances in G and its square G^2 (cont.)

Lemma: If $\delta(u,v)=2\delta^2(u,v)$ then for every neighbor w of v we have $\delta^2(u,w) \geq \delta^2(u,v)$.

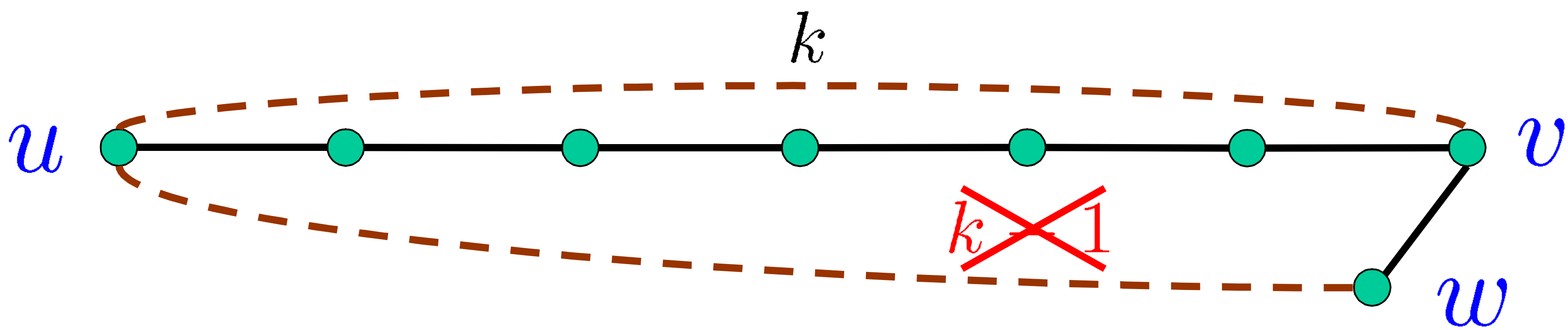
Lemma: If $\delta(u,v)=2\delta^2(u,v)-1$ then for every neighbor w of v we have $\delta^2(u,w) \leq \delta^2(u,v)$ and for at least one neighbor $\delta^2(u,w) < \delta^2(u,v)$.

Let A be the adjacency matrix of the G.
Let C be the distance matrix of G^2

$$\sum_{(v,w)\in E} c_{uw} = \sum_{w\in V} c_{uw} a_{wv} = (CA)_{uv} \geq \deg(v) c_{uv}$$

Even distances

Lemma: If $\delta(u,v)=2\delta^2(u,v)$ then for every neighbor w of v we have $\delta^2(u,w) \geq \delta^2(u,v)$.



Let A be the adjacency matrix of the G.

Let C be the distance matrix of G^2

$$\sum_{(v,w)\in E} c_{uw} = \sum_{w\in V} c_{uw} a_{wv} = (CA)_{uv} \geq \deg(v) c_{uv}$$

Odd distances

Lemma: If $\delta(u,v)=2\delta^2(u,v)-1$ then for every neighbor w of v we have $\delta^2(u,w) \leq \delta^2(u,v)$ and for at least one neighbor $\delta^2(u,w) < \delta^2(u,v)$.

Exercise 4: Prove the lemma.

Let A be the adjacency matrix of the G.
Let C be the distance matrix of G^2

$$\sum_{(v,w)\in E} c_{uw} = \sum_{w\in V} c_{uw} a_{wv} = (CA)_{uv} < \deg(v) c_{uv}$$

Seide

1. If A is an all one matrix, then all distances are 1.
2. Compute A^2, the adjacency matrix of the squared graph.
3. Find, recursively, the distances in the squared graph.
4. Decide, using one integer matrix multiplication, for every two vertices u,v, whether their distance is **twice** the distance in the square, or **twice minus 1**.

Assume that A has 1's on the diagonal.

Boolean matrix multiplicaion

$A)$

else

$C \leftarrow \mathrm{APD}(A^2)$

$X \leftarrow CA$, $\deg \leftarrow Ae-1$

$\deg_j]$

Integer matrix multiplicaion

Complexity:

$O(n^{\omega} \log n)$

Exercise 5: (*) Obtain a version of Seidel’s algorithm that uses only **Boolean** matrix multiplications.

Hint: Look at distances also modulo 3.

Distances vs. Shortest Paths

We described an algorithm for computing all **distances**.

How do we get a representation of the **shortest paths**?

We need **witnesses** for the Boolean matrix multiplication.

Witnesses for Boolean Matrix Multiplication

$$C = AB$$

$$c_{ij} = \bigvee_{k=1}^{n} a_{ik} \wedge b_{kj}$$

A matrix W is a matrix of **witnesses** iff

If $c_{ij} = 0$ then $w_{ij} = 0$

If $c_{ij} = 1$ then $w_{ij} = k$ where $a_{ik} = b_{kj} = 1$

Can be computed naively in $O(n^3)$ time.

Can also be computed in $O(n^\omega \log n)$ time.

Exercise 6:

a) Obtain a deterministic $O(n^\omega)$-time algorithm for finding **unique** witnesses.

b) Let $1 \leq d \leq n$ be an integer. Obtain a randomized $O(n^\omega)$-time algorithm for finding witnesses for all positions that have between d and $2d$ witnesses.

c) Obtain an $O(n^\omega \log n)$-time algorithm for finding all witnesses.

Hint: In b) use **sampling**.

All-Pairs Shortest Paths
in graphs with small integer weights

Undirected graphs.
Edge weights in $\{0,1,\ldots M\}$

Running time	**Authors**
Mn^{ω}	[Shoshan-Zwick '99]

Improves results of
[Alon-Galil-Margalit '91] [Seidel '95]

DIRECTED

SHORTEST PATHS

Exercise 7:

Obtain an $O(n^{\omega} \log n)$ time algorithm for computing the **diameter** of an unweighted directed graph.

Using matrix multiplication to compute min-plus products

$$\begin{pmatrix} c_{11} & c_{12} & \\ c_{21} & c_{22} & \\ & & \text{O} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \\ a_{21} & a_{22} & \\ & & \text{O} \end{pmatrix} * \begin{pmatrix} b_{11} & b_{12} & \\ b_{21} & b_{22} & \\ & & \text{O} \end{pmatrix}$$

$$c_{ij} = \min_k \{a_{ik} + b_{kj}\}$$

$$\begin{pmatrix} c'_{11} & c'_{12} & \\ c'_{21} & c'_{22} & \\ & & \text{O} \end{pmatrix} = \begin{pmatrix} x^{a_{11}} & x^{a_{12}} & \\ x^{a_{21}} & x^{a_{22}} & \\ & & \text{O} \end{pmatrix} \times \begin{pmatrix} x^{b_{11}} & x^{b_{12}} & \\ x^{b_{21}} & x^{b_{22}} & \\ & & \text{O} \end{pmatrix}$$

$$c'_{ij} = \sum_k x^{a_{ik}+b_{kj}} \qquad c_{ij} = \mathit{first}(c'_{ij})$$

Using matrix multiplication to compute min-plus products

Assume: $0 \le a_{ij}, b_{ij} \le M$

$$\begin{pmatrix} c'_{11} & c'_{12} & \\ c'_{21} & c'_{22} & \\ & & \ddots \end{pmatrix} = \begin{pmatrix} x^{a_{11}} & x^{a_{12}} & \\ x^{a_{21}} & x^{a_{22}} & \\ & & \ddots \end{pmatrix} * \begin{pmatrix} x^{b_{11}} & x^{b_{12}} & \\ x^{b_{21}} & x^{b_{22}} & \\ & & \ddots \end{pmatrix}$$

n^{ω} polynomial products $\times$ M operations per polynomial product $=$ Mn^{ω} operations per max-plus product

Trying to implement the repeated squaring algorithm

$D \leftarrow W$
for $i \leftarrow 1$ to $\log_2 n$
do $D \leftarrow D * D$

Consider an easy case: all weights are 1.

After the i-th iteration, the finite elements in D are in the range $\{1,\ldots,2^i\}$.

The cost of the min-plus product is $2^i n^\omega$

The cost of the last product is $n^{\omega+1}$!!!

Sampled Repeated Squaring (Z '98)

$D \leftarrow W$
for $i \leftarrow 1$ **to** $\log_{3/2} n$ **do**
{
$\quad s \leftarrow (3/2)^{i+1}$
$\quad B \leftarrow \text{rand}(\,V, (9n \ln n)/s\,)$
$\quad D \leftarrow \min\{\,D\,,\, D[V,B]*D[B,V]\,\}$
}

Choose a subset of V of size $\approx n/s$

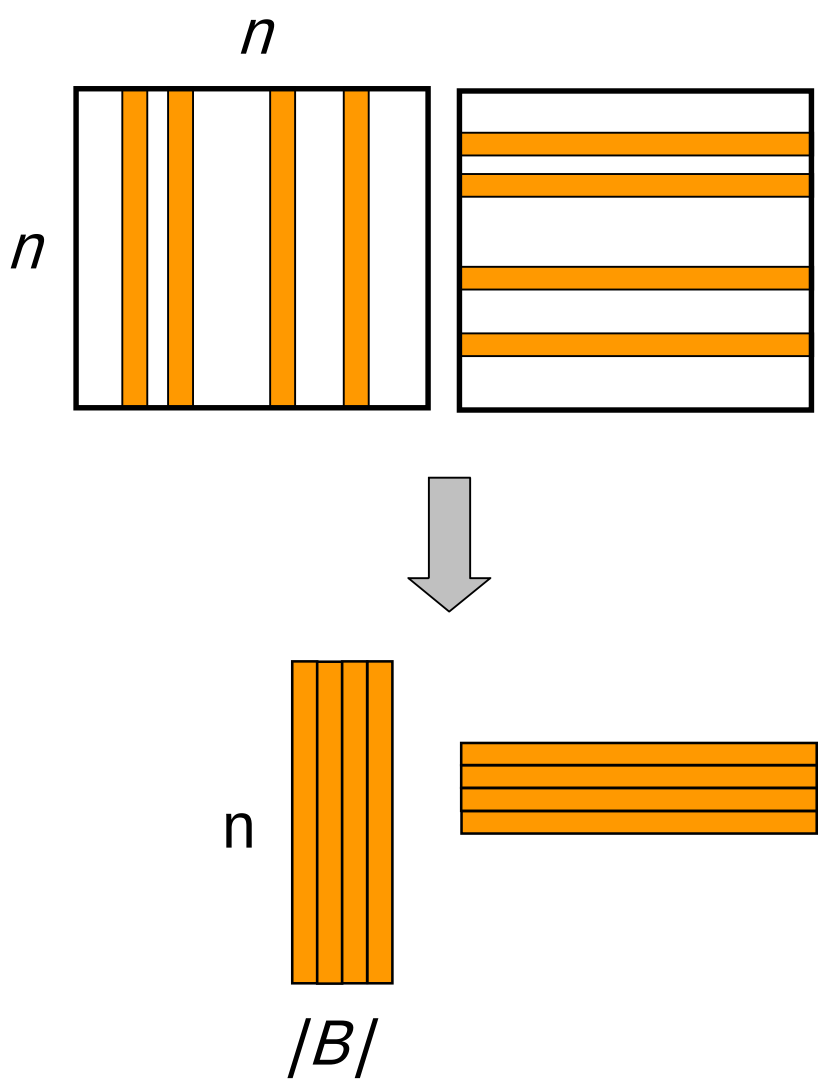
Select the **columns** of D whose indices are in B

Select the **rows** of D whose indices are in B

With high probability, all distances are correct!

The is also a slightly more complicated deterministic algorithm

Sampled Distance Products (Z '98)



In the i-th iteration, the set B is of size $\approx n/s$, where $s = (3/2)^{i+1}$

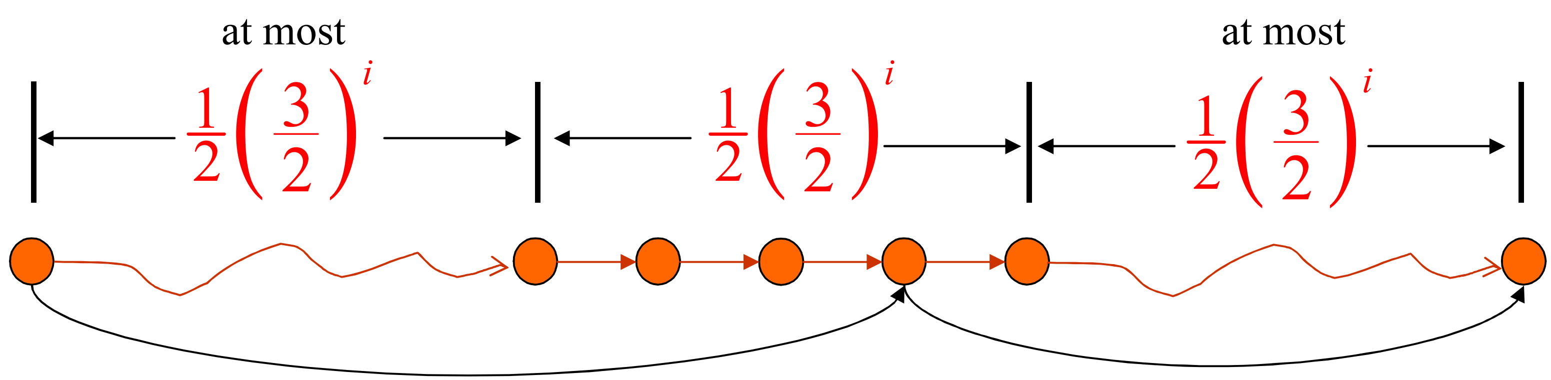
The matrices get smaller and smaller but the elements get larger and larger

Sampled Repeated Squaring - Correctness

```
D ← W
for i ←1 to log_{3/2} n do
{
      s ← (3/2)^{i+1}
      B ← rand(V,(9n ln n)/s)
      D ← min{ D , D[V,B]*D[B,V] }
}
```

Invariant: After the i-th iteration, distances that are attained using at most $(3/2)^i$ edges are correct.

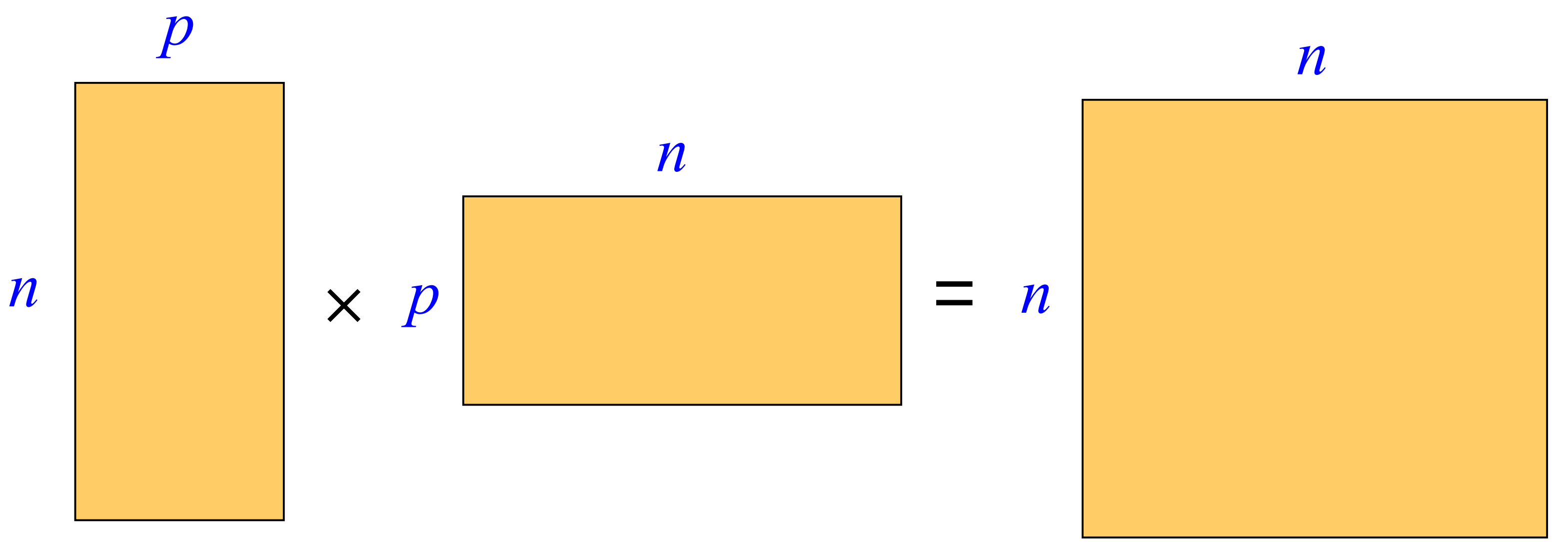
Consider a shortest path that uses at most $(3/2)^{i+1}$ edges



Let $s = (3/2)^{i+1}$

Failure probability : $\left(1-\frac{9\ln n}{s}\right)^{s/3} < n^{-3}$

Rectangular Matrix multiplication



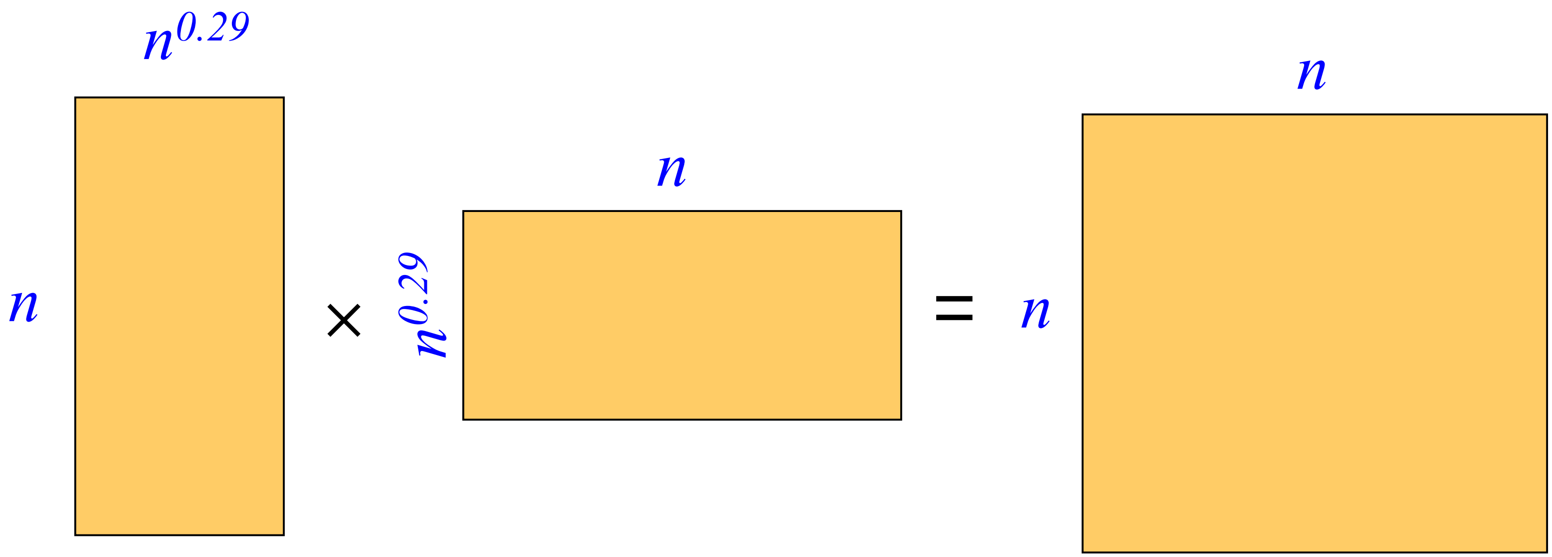
Naïve complexity: $n^2 p$

[Coppersmith (1997)] [Huang-Pan (1998)]

$$n^{1.85} p^{0.54} + n^{2+o(1)}$$

For $p \leq n^{0.29}$, complexity = $n^{2+o(1)}$!!!

Rectangular Matrix multiplication



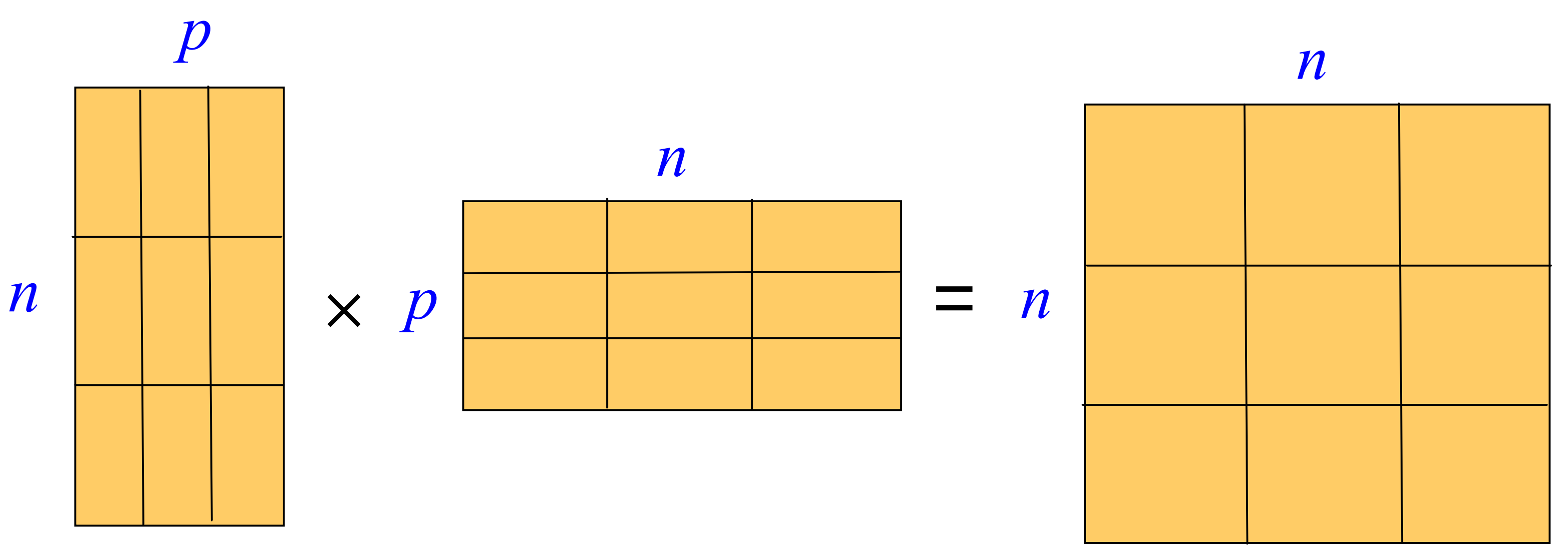
[Coppersmith (1997)]

$n \times n^{0.29}$ by $n^{0.29} \times n$

$n^{2+o(1)}$ operations!

$\alpha = 0.29\ldots$

Rectangular Matrix multiplication



[Huang-Pan (1998)]

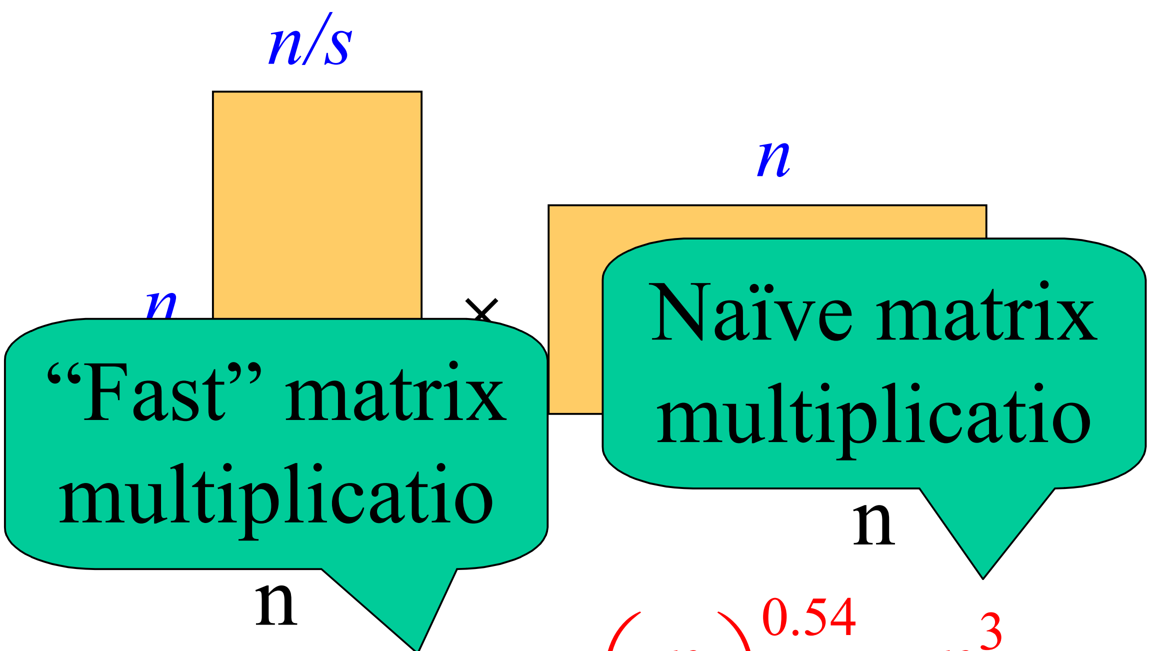
Break into $q \times q^{\alpha}$ and $q^{\alpha} \times q$ sub-matrices

$$q = \left(\frac{n}{p}\right)^{\frac{1}{1-\alpha}} \qquad \left(\frac{n}{q}\right)^{\omega} \cdot q^2 = n^{\omega - \frac{\omega-2}{1-\alpha}} \cdot p^{\frac{\omega-2}{1-\alpha}}$$

$$\approx n^{1.85} p^{0.54}$$

Complexity of APSP algorithm

The i-th iteration:



$s = (3/2)^{i+1}$

The elements are of absolute value at most Ms

$$\min\left\{ Ms \cdot n^{1.85} \left(\frac{n}{s}\right)^{0.54}, \frac{n^3}{s} \right\} \le M^{0.68} n^{2.58}$$

All-Pairs Shortest Paths
in graphs with small integer weights

Undirected graphs.
Edge weights in $\{0,1,\ldots M\}$

Running time	**Authors**
$Mn^{2.38}$	[Shoshan-Zwick '99]

Improves results of
[Alon-Galil-Margalit '91] [Seidel '95]

All-Pairs Shortest Paths
in graphs with small integer weights

Directed graphs.
Edge weights in $\{-M,\ldots,0,\ldots M\}$

Running time	**Authors**
$M^{0.68} n^{2.58}$	[Zwick '98]

Improves results of
[Alon-Galil-Margalit '91] [Takaoka '98]

Open problem:

Can APSP in directed graphs be solved in $O(n^{\omega})$ time?

[Yuster-Z (2005)]

A directed graphs can be processed in $O(n^{\omega})$ time so that any distance query can be answered in $O(n)$ time.

Corollary:

SSSP in directed graphs in $O(n^{\omega})$ time.

Also obtained, using a different technique, by Sankowski (2005)

The preprocessing algorithm (YZ '05)

$D \leftarrow W$; $B \leftarrow V$
for $i \leftarrow 1$ to $\log_{3/2} n$ do
{
    $s \leftarrow (3/2)^{i+1}$
    $B \leftarrow \mathrm{rand}(B,(9n \ln n)/s)$
    $D[V,B] \leftarrow \min\{ D[V,B] , D[V,B]*D[B,B]$
    }
    $D[B,V] \leftarrow \min\{ D[B,V] , D[B,B]*D[B,V]$
    }
}

The APSP algorithm

$D \leftarrow W$

for $i \leftarrow 1$ **to** $\log_{3/2} n$ **do**

{

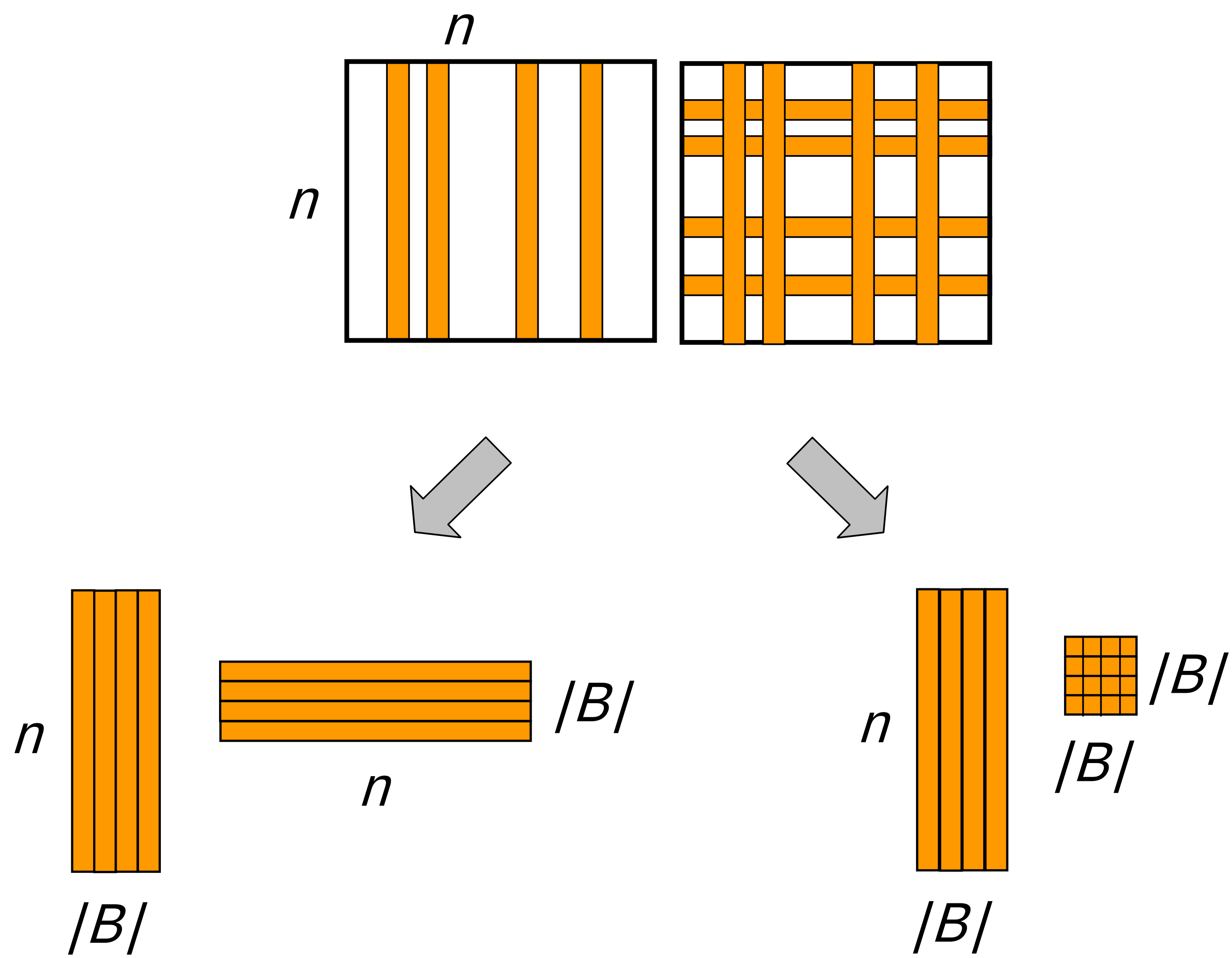
$\quad s \leftarrow (3/2)^{i+1}$

$\quad B \leftarrow \mathrm{rand}(V, (9n \ln n)/s)$

$\quad D \leftarrow \min\{ D , D[V,B]*D[B,V] \}$

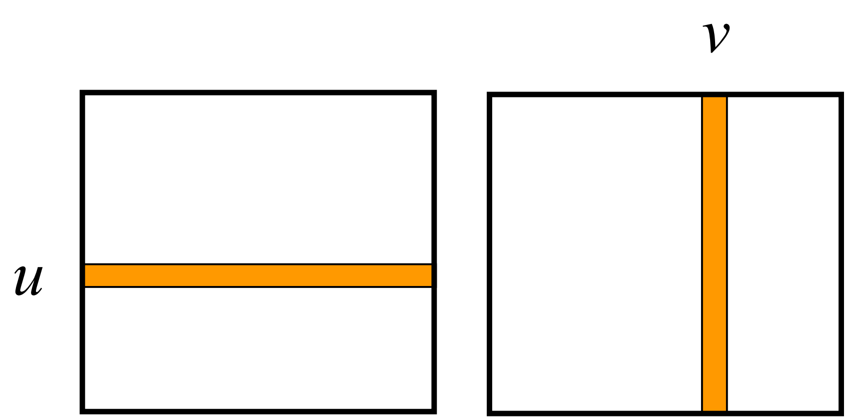
}

Twice Sampled Distance Products



The query answering algorithm

$$\boldsymbol{\delta(u,v) \leftarrow D[\{u\},V]*D[V,\{v\}]}$$



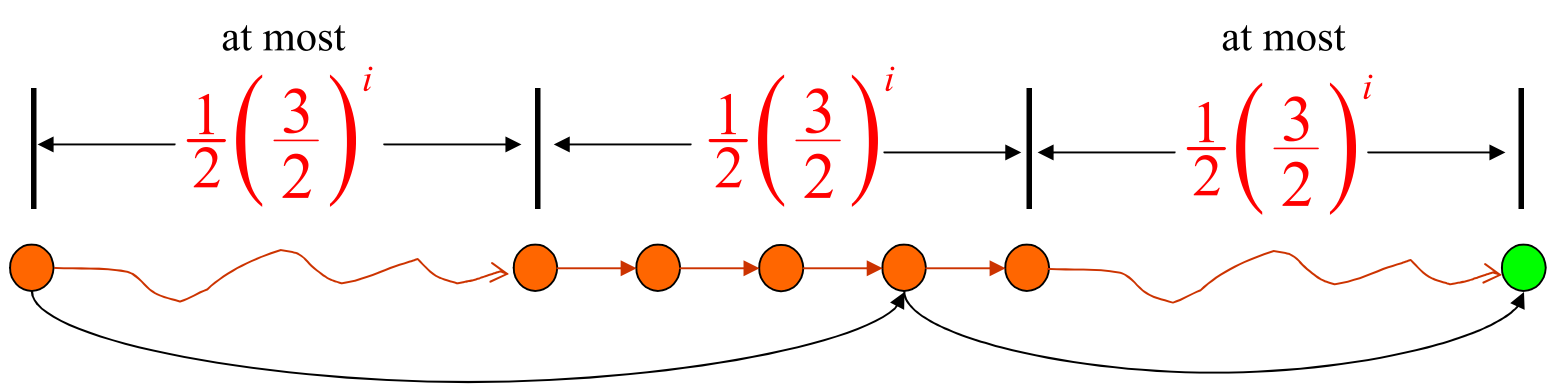
Query time: O(n)

The preprocessing algorithm: Correctness

Let B_i be the i-th sample. $B_1 \supseteq B_2 \supseteq B_3 \supseteq \ldots$

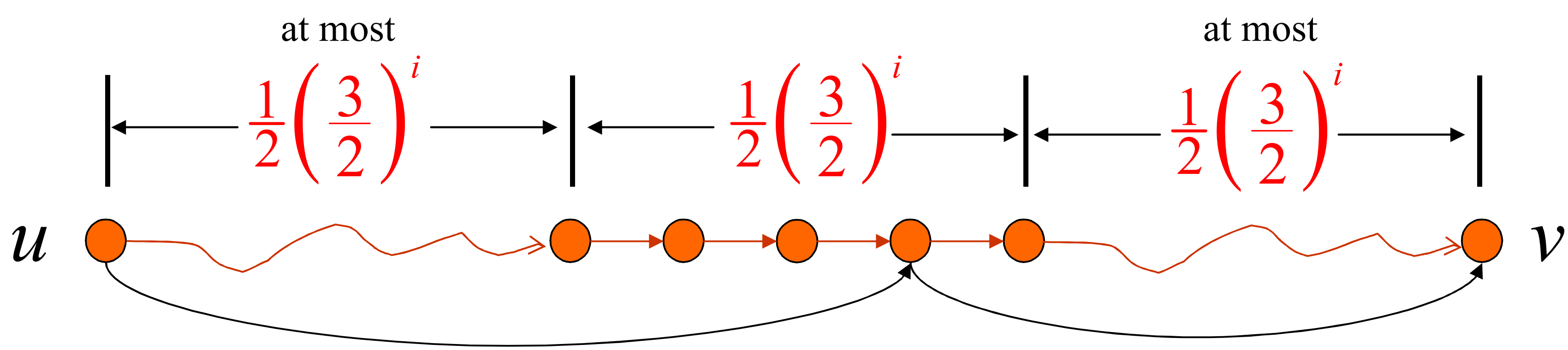
Invariant: After the i-th iteration, if $u \in B_i$ or $v \in B_i$ and there is a shortest path from u to v that uses at most $(3/2)^i$ edges, then $D(u,v)=\delta(u,v)$.

Consider a shortest path that uses at most $(3/2)^{i+1}$ edges



The query answering algorithm: Correctness

Suppose that the shortest path from u to v uses between $(3/2)^i$ and $(3/2)^{i+1}$ edges



1. **Algebraic matrix multiplication**
    a. Strassen's algorithm
    b. Rectangular matrix multiplication
2. **Min-Plus matrix multiplication**
    a. Equivalence to the APSP problem
    b. Expensive reduction to algebraic products
    c. Fredman's trick
3. **APSP in undirected graphs**
    a. An $O(n^{2.38})$ anlgorithm for unweighted graphs (Seidel)
    b. An $O(Mn^{2.38})$ algorithm for weighted graphs (Shoshan-Zwick)
4. **APSP in directed graphs**
    1. An $O(M^{0.68}n^{2.58})$ algorithm (Zwick)
    2. An $O(Mn^{2.38})$ preprocessing / $O(n)$ query answering alg. (Yuster-Z)
    3. An $O(n^{2.38}\log M)$ $(1+\varepsilon)$-approximation algorithm
5. **Summary and open problems**

Approximate min-plus products

Obvious idea: scaling

SCALE(A,M,R): $$a'_{ij} \leftarrow \begin{cases} \lceil R a_{ij} / M \rceil & , \quad \text{if } 0 \le a_{ij} \le M \\ +\infty & , \quad \text{otherwise} \end{cases}$$

APX-MPP(A,B,M,R) :
A'←SCALE(A,M,R)
B'←SCALE(B,M,R)
return MPP(A',B')

Complexity is $Rn^{2.38}$, instead of $Mn^{2.38}$, but small values can be greatly distorted.

Addaptive Scaling

APX-MPP(A,B,M,R) :

$C' \leftarrow \infty$

for $r \leftarrow \log_2 R$ to $\log_2 M$ do

$A' \leftarrow \text{SCALE}(A, 2^r, R)$

$B' \leftarrow \text{SCALE}(B, 2^r, R)$

$C' \leftarrow \min\{C', \text{MPP}(A', B')\}$

end

Complexity is $Rn^{2.38} \log M$

Stretch at most $1+4/R$

1. **Algebraic matrix multiplication**
    a. Strassen's algorithm
    b. Rectangular matrix multiplication
2. **Min-Plus matrix multiplication**
    a. Equivalence to the APSP problem
    b. Expensive reduction to algebraic products
    c. Fredman's trick
3. **APSP in undirected graphs**
    a. An $O(n^{2.38})$ anlgorithm for unweighted graphs (Seidel)
    b. An $O(Mn^{2.38})$ algorithm for weighted graphs (Shoshan-Zwick)
4. **APSP in directed graphs**
    1. An $O(M^{0.68}n^{2.58})$ algorithm (Zwick)
    2. An $O(Mn^{2.38})$ preprocessing / $O(n)$ query answering alg. (Yuster-Z)
    3. An $O(n^{2.38}\log M)$ (1+ε)-approximation algorithm
5. **Summary and open problems**

Answering distance queries

Directed graphs. Edge weights in $\{-M,\ldots,0,\ldots M\}$

Preprocessing time	Query time	Authors
$Mn^{2.38}$	n	[Yuster-Zwick '05]

In particular, any $Mn^{1.38}$ distances can be computed in $Mn^{2.38}$ time.

For dense enough graphs with small enough edge weights, this improves on Goldberg's SSSP algorithm.
$Mn^{2.38}$ vs. $mn^{0.5}\log M$

Approximate All-Pairs Shortest Paths
in graphs with non-negative integer weights

Directed graphs.
Edge weights in $\{0,1,\ldots M\}$

$(1+\varepsilon)$-approximate distances

Running time	**Authors**
$(n^{2.38} \log M)/\varepsilon$	[Zwick '98]

Open problems

An $O(n^{\omega})$ algorithm for the directed unweighted APSP problem?

An $O(n^{3-\varepsilon})$ algorithm for the APSP problem with edge weights in $\{1,2,\ldots,n\}$?

An $O(n^{2.5-\varepsilon})$ algorithm for the SSSP problem with edge weights in $\{-1,0,1,2,\ldots, n\}$?

DYNAMIC
TRANSITIVE CLOSURE

Dynamic transitive closure

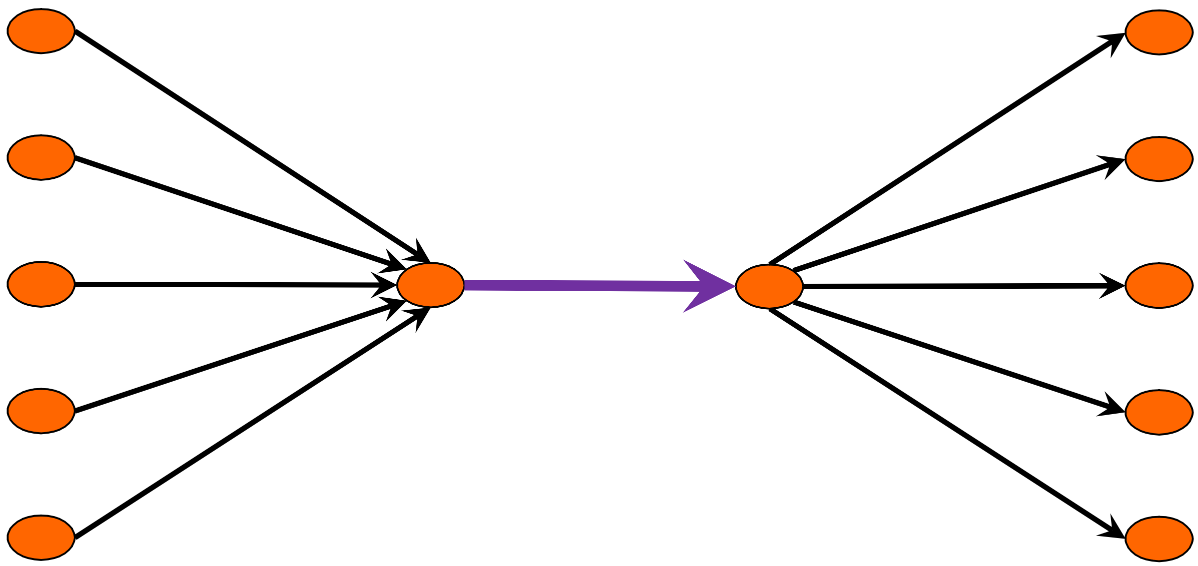
- **Edge-Update**(e) – add/remove an edge e
- **Vertex-Update**(v) – add/remove edges touching v.
- **Query**(u,v) – is there are directed path from u to v?

[Sankowski '04]

Edge-Update			
Vertex-Update			
Query			

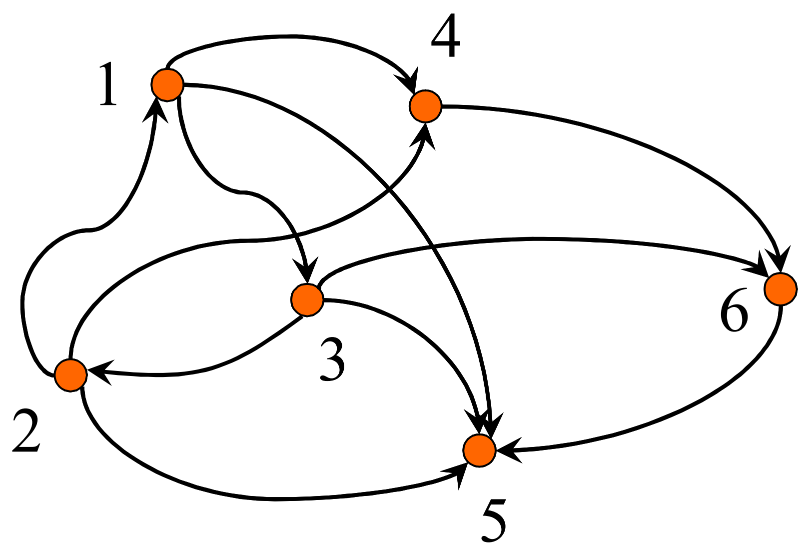
(improving [Demetrescu-Italiano '00], [Roditty '03])

Inserting/Deleting and edge



May change $\Omega(n^2)$ entries of the transitive closure matrix

Symbolic Adjacency matrix



$$\begin{pmatrix} 1 & 0 & x_{13} & x_{14} & x_{15} & 0 \\ x_{21} & 1 & 0 & x_{24} & x_{25} & 0 \\ 0 & x_{32} & 1 & 0 & x_{35} & x_{36} \\ 0 & 0 & 0 & 1 & 0 & x_{46} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & x_{56} & 1 \end{pmatrix}$$

$$\det(A) \not\equiv 0$$

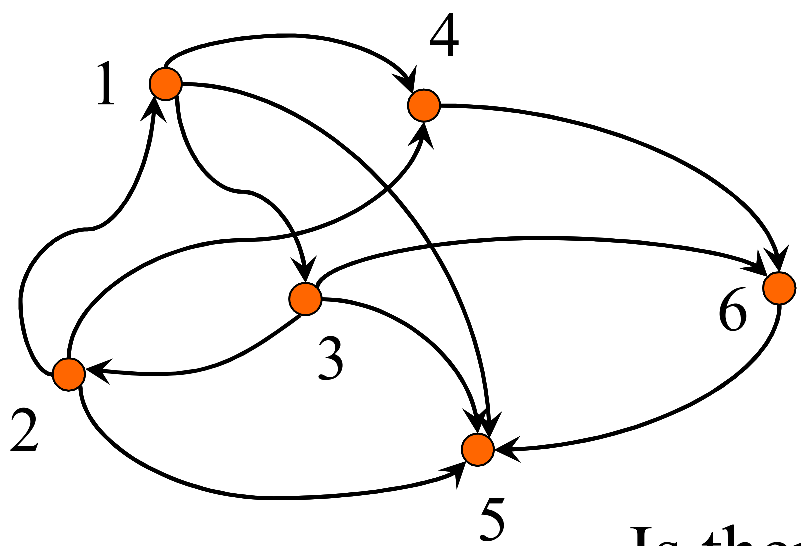
Reachability via adjoint
[Sankowski '04]

Let A be the symbolic adjacency matrix of G. (With 1s on the diagonal.)

There is a directed path from i to j in G iff

$$(\mathrm{adj}(A))_{ij} \neq 0$$

Reachability via adjoint (example)



$$\begin{pmatrix} 1 & 0 & x_{13} & x_{14} & x_{15} & 0 \\ x_{21} & 1 & 0 & x_{24} & x_{25} & 0 \\ 0 & x_{32} & 1 & 0 & x_{35} & x_{36} \\ 0 & 0 & 0 & 1 & 0 & x_{46} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & x_{65} & 1 \end{pmatrix}$$

Is there a path from 1 to 5?

$$\det \begin{pmatrix} 0 & 0 & x_{13} & x_{14} & x_{15} & 0 \\ 0 & 1 & 0 & x_{24} & x_{25} & 0 \\ 0 & x_{32} & 1 & 0 & x_{35} & x_{36} \\ 0 & 0 & 0 & 1 & 0 & x_{46} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_{65} & 1 \end{pmatrix} = \begin{matrix} -x_{15} \\ -x_{13}x_{32}x_{25} \\ +x_{13}x_{35} \\ -x_{13}x_{36}x_{56} \\ -x_{14}x_{46}x_{65} \\ -x_{13}x_{32}x_{24}x_{46}x_{65} \end{matrix}$$

Dynamic transitive closure

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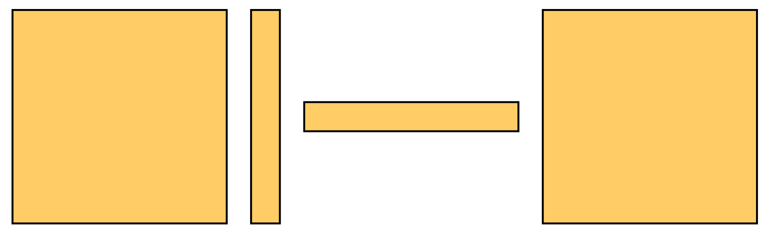
Dynamic matrix inverse

- **Entry-Update**(i,j,x) – Add x to A_{ij}
- **Row-Update**(i,v) – Add v to the i-th row of A
- **Column-Update**(j,u) – Add u to the j-th column of A
- **Query**(i,j) – return $(A^{-1})_{ij}$

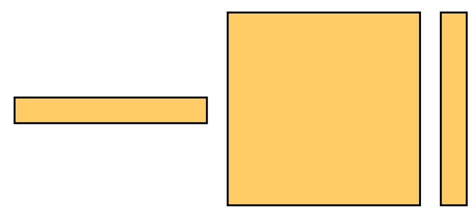
Sherman-Morrison formula

$$(A + uv^T)^{-1} = A^{-1} - \frac{A^{-1}uv^TA^{-1}}{1 + v^TA^{-1}u}$$

$A^{-1}uv^TA^{-1}$:



$v^TA^{-1}u$:



Inverse of a rank one correction
is a rank one correction of the inverse

Inverse updated in $O(n^2)$ time

O(n^2) update / O(1) query algorithm [Sankowski ’04]

Let $p \approx n^3$ be a prime number

Assign random values $a_{ij} 2\ F_p$ to the variables x_{ij}

Maintain A^{-1} over F_p

Edge-Update → Entry-Update

Vertex-Update → Row-Update + **Column-Update**

Perform updates using the Sherman-Morrison formula

Small error probability
(by the Schwartz-Zippel lemma)

Lazy updates

Consider single entry updates

$$A_k = A_{k-1} + a_k u_k v_k$$

$$a_k = \pm a_{i_k,j_k} \quad u_k = e_{i_k} \quad v_k = e_{j_k}^T$$

$$A_k^{-1} = A_{k-1}^{-1} + \alpha_k u'_k v'_k$$

$$\alpha_k = 1 + a_k v_k A_{k-1}^{-1} u_k = 1 + a_k (A_{k-1}^{-1})_{j_k,i_k}$$

$$u'_k = A_{k-1}^{-1} u_k = (A_{k-1}^{-1})_{*,i_k}$$

$$v'_k = v_k A_{k-1}^{-1} = (A_{k-1}^{-1})_{j_k,*}$$

$$A_k^{-1} = A_0^{-1} + \sum_{i=1}^{k} \alpha_i u'_i v'_i$$

Lazy updates (cont.)

$$A_k^{-1} = A_0^{-1} + \sum_{i=1}^{k} \alpha_i u_i' v_i'$$

Do not maintain A_k^{-1} explicitly!

Maintain $\alpha_i, u_i', v_i',\ i = 1, 2, \ldots, k$

Querying $(A_k^{-1})_{r,c}$ – $O(k)$ time

Computing α_k, u_k', v_k' – $O(nk)$ time

Queries and updates get more and more expensive!

Lazy updates (cont.)

$$A_k^{-1} = A_0^{-1} + \sum_{i=1}^{k} \alpha_i u_i' v_i'$$

Query time – $O(k)$

Update time – $O(nk)$

Compute A_k^{-1} explicitly after each K updates

Time required – $O(M(n, K, n))$ time

Amortized update time – $O(nK + M(n, K, n)/K)$

Update time minimized when $K \approx n^{0.575}$

Can be made **worst-case**

Even Lazier updates

$$A_k^{-1} = A_0^{-1} + \sum_{i=1}^{k} \alpha_i u_i' v_i'$$

After ℓ updates in positions

$(r_1, c_1), (r_2, c_2), \ldots, (r_\ell, c_\ell)$

maintain:

$\alpha_i, (u_i')_{c_j}, (v_i')_{r_j}$, for $1 \leq i, j \leq \ell$

Query time – $O(k^2)$

Update time – $O(k^2)$

After K, explicitly update A_k^{-1}

Dynamic transitive closure

- **Edge-Update**(e) – add/remove an edge e
- **Vertex-Update**(v) – add/remove edges touching v.
- **Query**(u,v) – is there are directed path from u to v?

[Sankowski '04]

Edge-Update	n^2	$n^{1.575}$	$n^{1.495}$
Vertex-Update	n^2	–	–
Query	1	$n^{0.575}$	$n^{1.495}$

(improving [Demetrescu-Italiano '00], [Roditty '03])

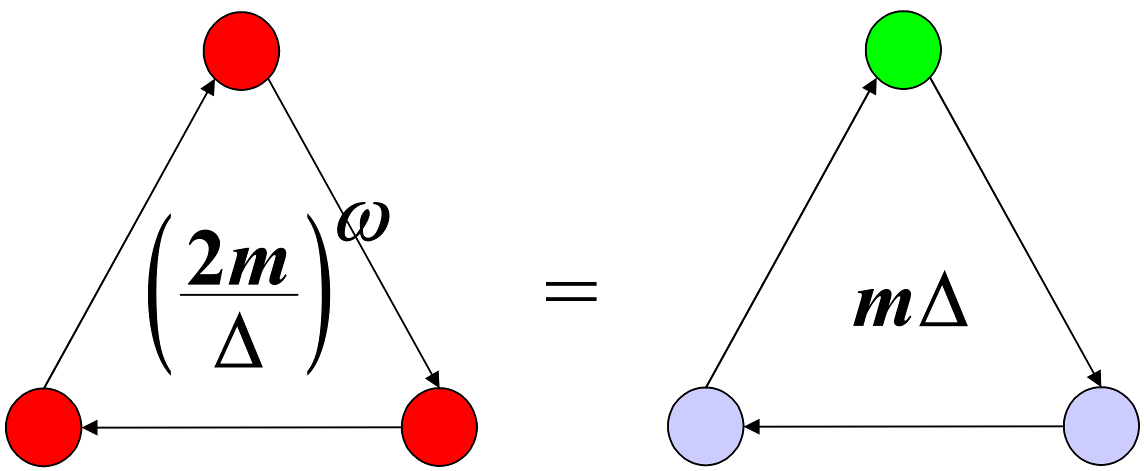
Finding triangles in O($m^{2\omega/(\omega+1)}$) time
[Alon-Yuster-Z (1997)]

Let Δ be a parameter. $\Delta = m^{(\omega-1)/(\omega+1)}$

High degree vertices: vertices of degree $\geq \Delta$.

Low degree vertices: vertices of degree $< \Delta$.

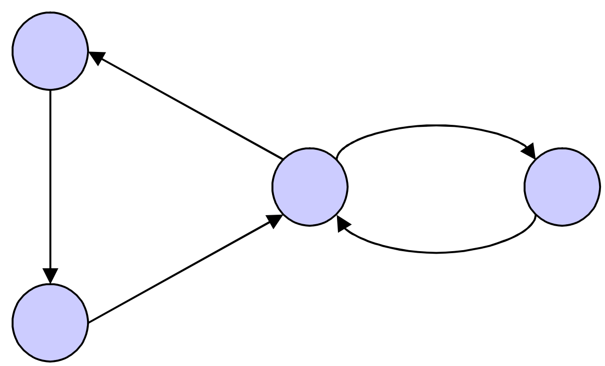
There are at most $2m/\Delta$ high degree vertices



$$\left(\frac{2m}{\Delta}\right)^{\omega} = m\Delta$$

Finding longer simple cycles

A graph G contains a C_k iff $Tr(A^k) \neq 0$?



We want simple cycles!

Color coding [AYZ '95]

Assign each vertex v a random number $c(v)$ from $\{0,1,\ldots,k-1\}$.

Remove all edges (u,v) for which $c(v) \neq c(u)+1 \pmod{k}$.

All cycles of length k in the graph are now simple.

If a graph contains a C_k then with a probability of at least k^{-k} it still contains a C_k after this process.

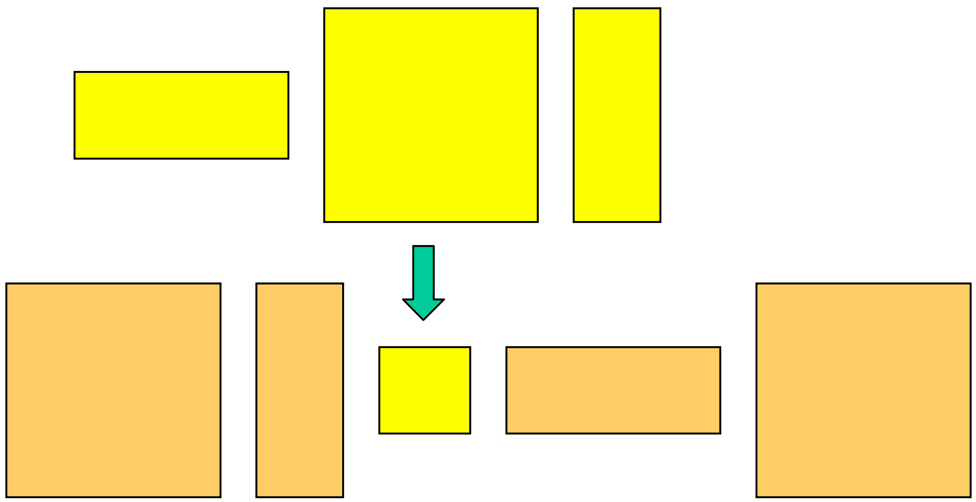
An improved version works with probability $2^{-O(k)}$.

Can be derandomized at a logarithmic cost.

Sherman-Morrison-Woodbury formula

$$(A + UV^T)^{-1} =$$

$$A^{-1} - A^{-1}U\,(I + V^T A^{-1} U)^{-1}\,V^T A^{-1}$$



Inverse of a rank k correction
is a rank k correction of the inverse

Can be computed in $O(M(n,k,n))$ time.