Introduction

This research is devoted to the study of **Gödel incompleteness** phenomenon: finding mathematical statements unprovable in strong axiomatic theories, studying logical strength of mathematical statements. We are going to obtain independence results and study logical strength in the following areas of mathematics:

- finite Ramsey theory, infinite Ramsey theory;
- well-quasi-order theory;
- analytic combinatorics;
- braid theory (and, possibly, knot theory and related topology) ;
- number theory;
- the theory of chaotic dynamical systems;
- Banach spaces.

Task 1 Braids

Find the logical strength of statements about braids, such as "for every infinite sequence of positive braids, there is an infinite increasing subsequence".

Prove $I\Sigma_2$-unprovability of some miniaturisations, e.g., "for every K there is N such that for any sequence $B_1, B_2, \ldots, B_N$ of positive braids such that $|B_i| < K + i$, there are $i < j \leq N$ such that $B_i \prec B_j$".

How to prove it:

1. 2-extendible cuts in models of arithmetic;
2. build a set of indiscernibles from a certain nonstandard braid;
3. show that a braid principle implies termination of the battle with hydras of height 3 (use translation of braids into trees);
4. direct approach via ordinals (using well-orderedness of positive braids as ω^{ω^ω}).

Longer-term goals:

1. catalogue a few other unprovable statements about braids;
2. develop a similar approach to knot theory;
3. translate braid principles into topology of manifolds and geometry (or even mechanics: e.g., recall that braid groups are fundamental groups of certain configuration spaces).

Tasks 2 and 3: Kruskal Theorem and Graph Minor Theorem

For every primitive recursive real number a, introduce the statement GM_a as: "for all K there is N such that for any sequence of simple graphs $G_1, G_2, \ldots, G_N$ such that $|G_i| < K + a \cdot \sqrt{\log i}$, there are $i < j$ such that G_i is (isomorphic to) a minor of G_j".

We have proved that if $a \leq \sqrt{2}$ then GM_a is $I\Sigma_1$-provable and will prove that if $a > \sqrt{2}$ then GM_a is PA-unprovable.

For general pseudographs (loops an multiple edges allowed) there is a difficulty: to determine the first term of the asymptotic of the number of pseudographs of size n (pseudograph size defined as $|V(G)| + |E(G)|$) is an open problem in graph theory.

Weiermann's Program:

- find exact thresholds between provability and unprovability;
- find new independence results by exceeding natural bounds.

Task 2 Model-theoretic approach

Here, our original goal is to develop model theory to re-prove unprovability of Kruskal Theorem and Graph Minor Theorem.

Task 4 Zeta-function

Recall Friedman's sine principle in dimension n (unprovable in $I\Sigma_{n-1}$): "for all m, there is N such that for any sequence $A = \{a_1, a_2, \ldots, a_N\}$ of rational numbers, there is $H \subseteq A$ of size m such that for any two n-element subsets $a_{i_1} < a_{i_2} < \cdots < a_{i_n}$ and $a_{i_1} < a_{k_2} < \cdots a_{k_n}$ in H, we have

$$|\sin(a_{i_1} \cdot a_{i_2} \cdots a_{i_n}) - \sin(a_{i_1} \cdot a_{k_2} \cdots a_{k_n})| < \frac{1}{i_1}\text{''}.$$

Prove a similar theorem with Riemann zeta-function in place of the sine by plugging in existing theorems on the average behaviour of zeros in the critical strip.

Task 5 Dynamical systems

Generalize the sine-principle to produce unprovability results about dynamical systems.

- use logistic map with large parameter;
- find a generalization;
- manifacture a dynamical system with diffeomorphisms of manifolds.

Tasks 6 and 8 Reverse mathematics

Candidates for large logical strength:

- Pudlak-Rödl theorem: for every equivalence relation defined on a uniform barrier B, there is an infinite $M \subseteq \mathbb{N}$ such that $E|_{B|_M}$ is canonical.
- Gowers theorem: an infinite-dimensional normed space over $\mathbb{Q}$ either contains an infinite unconditional basic sequence or a closed infinite-dimensional hereditarily indecomposable subspace.
- Are there manifestations of the Ramsey-Dvoretzki-Milman phenomenon that necessarily require the strength of the Infinite Ramsey Theorem?

Task 6 Density approach

This is one of the ways to find exact strength of a statement in the language of second-order arithmetic:

for an infinitary statement S, introduce a sequence of first-order statements $\{\text{dense}(n) \mid n \in \mathbb{N}\}$ and prove that $\mathrm{WKL}_0 + S$ has the same Π^2_0-consequences as $I\Delta_0 + \cup_{n\in\omega}\text{dense}(n)$.

Task 7 Indiscernibles

Use the method of indiscernibles to produce new PA-unprovable statements:

- experiment more with ramseyan statements;
- play with modern ramseyan theorems;
- use square-free numbers to code subsets;
- use sieve theory;
- adopt some combinatorial number theory.

Longer-term plans:

- Using **automated deduction** (Isabelle/HOL prover), find reformulations of known unprovable statements.
- Try to prove unprovability of **Hypothesis H**: "for any finite collection of irreducible polynomials $F_1(x), F_2(x), \ldots, F_n(x)$ with integer coefficients and such that $\prod_{i \leq n} F_i$ has no fixed prime divisor, there exist infinitely-many integers m such that for all $i \leq n$, $F_i(m)$ are prime".
- Find statements that are arithmetical versions of modern **large cardinal axioms**.
- Find a PA-unprovable statement in terms of existence of a winning strategy in a certain **game of Noughts and Crosses**.
- Learning to build **models of strong theories**.