The aims of the subject

- Find examples of mathematical statements unprovable in strongest axiomatic systems. (Examples from the past:
  - examples springing from the indicator method of J.Paris (e.g. the Paris-Harrington Principle);
  - unprovability of finite versions of Kruskal's theorem (H.Friedman) and the graph minor theorem (Friedman-Robertson-Seymour);
  - new ZFC-unprovable statements by Friedman;
  - results by A.Weiermann, L.Carlucci, myself and other authors.
- Reverse Mathematics (given a mathematical theorem, determine its logical strength).

RT^2_2 is the following statement in the language of second-order arithmetic:

"for every colouring F of 2-subsets of $\mathbb{N}$ into 2 colours, there is an infinite subset whose 2-subsets are all of the same colour".

The strength of RT^2_2 is an open problem.

Density

A set X is 0-dense$(2,2)$ if $|X| > 3$, X is $(n+1)$-dense$(2,2)$ if for every $f\colon [X]^2 \to 2$ there is an n-dense$(2,2)$ set H such that $f|_{[H]^2} = \mathit{const}$ and $|H| > \min H$.

Consider a theory $T = I\Delta_0 + \bigcup_{n \in \omega} \forall a \, \exists b \, [a, b]$ is n-dense$(2,2)$.

Theorem (joint with A.Weiermann)
The set of Π_2-consequences of $\mathrm{WKL}_0 + \mathrm{RT}^2_2$ coincides with the set of Π_2-consequences of T.

This result may shed light on the strength of RT^2_2 and on whether RT^2_2 proves totality of the Ackermann function.

METHOD: inside a nonstandard model of T, we build an initial segment satisfying $\mathrm{WKL}_0 + \mathrm{RT}^2_2$ (nonstandard chunks of first-order definable sets play the role of future infinite sets).

Canonical Ramsey Theorem for pairs (CRT^2) is the following statement: "for any function $f\colon [\mathbb{N}]^2 \to \mathbb{N}$, there is an infinite set $X \subseteq \mathbb{N}$ such that one of the following four cases occurs:
(1) X is f-homogeneous;
(2) f is injective on X;
(3) f depends only on the first coordinate on X;
(4) f depends only on the second coordinate on X".

Regressive Ramsey Theorem for pairs (RegRT^2) is the following statement: "for any function $f\colon [\mathbb{N}]^2 \to \mathbb{N}$ such that $f(x,y) \le x$, there is an infinite set X such that on X, f depends only on the first coordinate".

With an analogous definition of T, we can prove the following theorem
Theorem (joint with A.Weiermann)
The Π_2 consequences of T are the same as of $\mathrm{WKL}_0 + \mathrm{CRT}^2$.

Using the model-theoretic machinery developed by Paris and Kirby **(strong initial segments in models of Peano Arithmetic)**, it is possible to prove:

- the first-order consequences of RegRT^2 coincide with Peano Arithmetic (the recursion-theoretic approach was developed by P.Clote in the 1980s);
- the first-order consequences of CRT^2 coincide with Peano Arithmetic (the recursion-theoretic proof was first obtained by J.Mileti in 2004).

METHOD: showing that a semi-regular initial segment is strong if and only if it satisfies CRT^2, using a definable ultrapower construction inside a countable model of arithmetic.

Friedman's sine-principle (see FOM internet forum):
"for any k there exists N such that whenever $x_1, x_2, \ldots, x_N$ are rational numbers, there exist $p_1 < p_2 < \cdots < p_{k+2}$ such that

$$|\sin(x_{p_1} \cdot x_{p_2} \cdot \ldots \cdot x_{p_k}) - \sin(x_{p_1} \cdot x_{p_3} \cdot \ldots \cdot x_{p_{k+1}})| < 4^{-p_1}$$

$$|\sin(x_{p_2} \cdot x_{p_3} \cdot \ldots \cdot x_{p_{k+1}}) - \sin(x_{p_2} \cdot x_{p_4} \cdot \ldots \cdot x_{p_{k+2}})| < 4^{-p_2}\text{"}$$

is unprovable in PA.

My sharp versions:

Let SP^n be: "for all m there is N such that for any sequence of rational numbers $A = \langle a_1, a_2, \cdots, a_N \rangle$, there is a subset $H \subset A$ of size m such that for any two n-element subsets $a_{i_1} < a_{i_2} < \cdots < a_{i_n}$ and $a_{i_1} < a_{j_2} < \cdots < a_{j_n}$, we have

$$|\sin(a_{i_1} \cdot a_{i_2} \cdot \ldots \cdot a_{i_n}) - \sin(a_{i_1} \cdot a_{j_2} \cdot \ldots \cdot a_{j_n})| < r^{\log^{n-1}}(i_1)\text{''},$$

where $r \approx 0.5960716\ldots$.

For $n > 1$, SP^{n+1} is not provable in $I\Sigma_n$.

There is also an $I\Sigma_1$ version.

METHOD: the proof uses reduction of Kanamori-McAloon Principle and the Rhin-Viola theorem on irrationality measure of π.

A related series of results about dynamical systems will appear shortly.

Braids

Positive braids are ordered as $\omega^{\omega^{\omega}}$, hence giving rise to several $I\Sigma_2$-unprovable statements.

METHODS: the proofs use 2-extendible initial segments and indiscernible elements in a model of $I\Delta_0 + \exp$.

There is an opportunity to use Artin's Braid Theorem to obtain unprovable statements with physical meaning.

Threshold results

The study of exact unprovability results ('threshold results') started with Weiermann's investigations in 2000 and 2001 of the gap between provable and unprovable instances of the Paris-Harrington principle and of Kruskal's theorem.

Example: let PH_f be the statement "for all n, m, c, there is N such that for all colourings $f\colon [N]^n \to c$, there is an f-homogeneous subset H of size at least m such that $|H| > f(\min H)$". Weiermann proved that for $f(x) = \log^k(x)$, PH_f is PA-unprovable but $\mathrm{PH}_{\log^*}$ is provable in $I\Delta_0 + \exp$.

My model-theoretic approach to threshold results can be found on my webpage `http://logic.pdmi.ras.ru/~andrey`.

The current problem is to find the gap between provability and unprovability in the graph minor theorem. Let GMT_f be the statement "for all K there is N such that whenever $G_1, G_2, \ldots, G_N$ are simple graphs such that $|G_i| < K + f(i)$ then there are $i < j$ such that G_i is a minor of G_j".

The current situation is: the statement $\mathrm{GMT}_{\sqrt{2}\cdot\sqrt{\log}}$ is provable in $I\Delta_0 + \exp$, but the statement $\mathrm{GMT}_{c\cdot\log}$ is unprovable in PA.

METHOD:

- provable instances by Polya's theorem and the asymptotic pigeonhole principle;
- unprovable instances: by combining the Friedman-Robertson-Seymour translation of trees into graphs with Weiermann's threshold for Kruskal's theorem.

thank you