On order-types of models of arithmetic

by

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A thesis submitted to the University of Birmingham for the degree of Ph. D. in the Faculty of Science

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13 April, 2000

Synopsis

In this thesis we study a range of questions related to the order structure of models of first-order Peano Arithmetic.

In Chapter 1 we give necessary definitions and describe the current state of the subject in the literature survey.

In Chapter 2 we study first properties of order-types of models of PA, give examples and place first restrictions on what the order-type of a model of PA can be.

In Chapter 3 we study models generated by indiscernibles and prove that the model generated by a set of indiscernibles ordered as a dense linear order $(C, <)$ is order-embeddable into $C^{<\mathbb{Q}}$.

In Chapter 4 we study a wide variety of questions associated with interpretability in a model of PA. In section 4.1 we prove that there is only one dense linear order interpreted in $\Omega \models \text{PA}$, namely $Q(\Omega)$. In section 4.2 we express the order-type of all inner models in terms of the $(<, \cdot)$-structure of the outer model. In section 4.3 we introduce and study the notion of self-similarity. Section 4.4 briefly studies connections between models of PA and models of ZFC and their mutual interpretability.

Chapter 5 introduces a class of canonical orders of models of PA and makes the first attempt to study it.

In Chapter 6 we prove that every model of cardinality less than λ has 2^λ order-types of elementary end-extensions of cardinality λ, that there are 2^{ω_1} ω_1-like models of PA and that (assuming $\diamondsuit$) there are 2^{ω_1} ω_1-like models of PA whose multiplicative reducts are pairwise non-isomorphic.

Sections 7.1 and 7.3 present our first attempt to solve Friedman's problem in the resplendent case. Section 7.2 connects this theme with the notion of arithmetic saturation. Also, we obtain a consequence of arithmetic saturation for automorphism groups.

The thesis concludes with a list of questions intended to guide and inspire future research.

Acknowledgements

I would like to thank my supervisor, Dr. Richard W. Kaye, for his guidance and patience during my long years of studying in Birmingham.

Also, I thank Professor V.G. Kanovej and Professor H. Kotlarski for mathematical discussions during their visits to Birmingham.

Contents

Introduction

Why study models of arithmetic

There are many reasons to study models of arithmetic coming from different areas of man's interest: Foundations of Mathematics, Philosophy, Methodology of Science, and Mathematics itself. Let me list some of these reasons and some historical landmarks in the development of the subject.

The history of the subject started with the discovery in the 1930s by Gödel and Skolem of nonstandard models. Since then many people studied models of arithmetic having different reasons in mind. For some, it was the usual mathematical curiosity to find out more about these 'beautiful' and 'interesting' structures. For others, they were a tool to prove facts about theories of arithmetic, especially first-order Peano Arithmetic, the theory that has been traditionally the main object of arithmetical research (and which is the main theory considered in this thesis). The discovery by Paris and Harrington in 1977 of a 'mathematical' statement (PH) independent of PA was a landmark. Other independent statements: "Hercules and the Hydra", "for all m the Goodstein sequence for m starting at 2 eventually hits zero" (by Kirby and Paris), and the statement (KM) (due to Kanamori and McAloon) were discovered a few years later (for discussion see [14], pp. 206-222). All these independence results were first proved using models of PA. However, no general method of obtaining such results was developed and research in this direction has stalled. There is an old proof of Gödel's Incompleteness Theorems due to Kreisel that uses models of PA and the Arithmetised Completeness Theorem [34]. Recently H. Kotlarski invented a new proof of Gödel's Theorem, also relying only on models of arithmetic.

Another boost to the subject was given by the discovery of recursive saturation and its interconnections with notions of resplendency, satisfaction classes and automorphism groups, which produced a beautiful and harmonious theory. At the moment "Models of Peano Arithmetic" is an established area of mathematical research profiting from methods developed in other parts of Model Theory and exporting its results and its language to other areas.

Nevertheless, there is a reason which distinguishes study of models of set theories, models of arithmetic, etc, from study of other areas of mathematics. This is the idea of exploring the most basic and fundamental notions of the whole mathematics appealing to man's desire of the 'search for eternal truth'. The intuitive collection $\mathbb{N} = \{|,\ ||,\ |||,\ ||||,\ \ldots\}$ is given to us by Nature (or nature of human mind?) and we, all of us, are challenged to unveil its mysteries. The most striking wonder of $\mathbb{N}$ is inadequacy of classical formal logic (with its "absolute infinity" quantifiers) to describe it as demonstrated by Gödel's Theorems.

At this stage, revolutionary-minded people split from the mass of mathematicians and tried to change the logic and the whole paradigm. Intuitionist and constructivist worlds are beautiful and complex and their set $\mathbb{N}$ is very different from ours. More traditional mathematicians stayed behind in their old (still exciting) world trying to answer new questions about incompleteness and Truth in terms of their old notions and beliefs. This thesis belongs to that old tradition. Throughout the thesis our logic is classical, our infinite sets are "actually infinite" and quantifiers in our formulas are understood in the classical way.

Peano Arithmetic is a strong theory. On one hand, it represents all recursive functions hence giving rise to Incompleteness. On the other hand, it formalises the Arithmetised Completeness Theorem (see later) thus allowing us construct models of consistent theories 'inside' models of PA. (Example: if $M \models \mathrm{PA} + \mathrm{Con}_{\mathrm{ZFC}}$ then there is $V \models \mathrm{ZFC}$ strongly interpreted inside M. The whole of mathematics is expressed by one formula inside a model of arithmetic!)

The study of PA and models of PA should inevitably become connected to the study of its relatives: other 'foundational' theories, such as ZFC (and GNB), NF, KP, Vopěnka's set theory, n^{th} order arithmetic, etc. There are two sorts of connection. Firstly, all foundational theories introduce the notion of a natural number. Different foundational theories may disagree about abstract sets and about cardinalities and about properties of their collections of 'all' subsets of natural numbers but they all agree that natural numbers should satisfy PA. Secondly, PA is one of those foundational theories and a lot of mathematics can be developed in PA or interpreted in models of PA. However, many people who studied PA believe that PA is more reliable than other theories. Even if some foundational theories such as ZFC (which dominates current research) are proved inconsistent (which would not damage much of mathematics though) PA will probably stay intact.

Order-types

Why study order-types? The first reason is that, given a theory T with definable linear order, the question "What are the order-types of models of T?" is one of the first **natural** questions to ask and can be understood by a child! Surely, studying order-types of models is more natural than studying some strange *ad hoc* structural properties.

Secondly, studying order-types enhances our **geometrical** vision of models allowing us to "see" and "touch" them. This geometrical vision is lacking in the study of some of PA's relatives, such as NF.

Thirdly, **linearly ordered sets** are structures of great importance in their own right, not only throughout mathematics but, hopefully (in the remote future), in the rest of science. There are linear orders (which may be used for "modelling" "reality") other than $\mathbb{R}$ and $\mathbb{Q}$!

Finally, there are practical **mathematical** reasons: the order of a model of PA is deeply interconnected with the rest of model's structure. Good illustrations are Pabion's Theorem (see below) and Theorem 55 (page 84).

All countable nonstandard models of PA are order-isomorphic, so throughout most of the thesis our models will be uncountable. In the countable case, the possibility of enumerating the model gave great variety of proofs, results and techniques. Nothing similar exists so far in the uncountable case.

Friedman's problem

Another natural question to ask would be "Do classes of order-types of nonstandard models of different completions of PA coincide?" This question is the 14^{th} Friedman's problem from his "One hundred and two problems in Mathematical Logic" [7].

No-one seems to know what the answer to this question should be. On one hand, we can not think of a good reason for two extensions of PA to have models of different order-types, and the countable case confirms it. On the other hand, there is a theory $\mathrm{Th}(\mathbb{N}) = \{\varphi \mid \mathbb{N} \models \varphi\}$ that looks different from other extensions of PA: only $\mathrm{Th}(\mathbb{N})$ has a model of order-type $(\omega, <)$ (the standard model), every model of $\mathrm{Th}(\mathbb{N})$ has no nonstandard definable points. Can it be a source of a counterexample?

Chapter 7 is devoted to my recent attempts to solve the problem. Assuming resplendency and some coding, the order-type of a model can be expanded to a model of a given completion of PA.

Resplendency

Resplendent models are the most popular kind of models considered in this thesis. A model M is called resplendent if it realizes all Σ^1_1 statements $\Phi(\overline{a})$ consistent with $\mathrm{Th}(M, \overline{a})$. Resplendent models, unlike saturated models, exist in all cardinalities. Very often we shall notice that resplendent models behave similarly to countable models. That is why I believe that solving Friedman's problem in the resplendent case would be a realistic enterprise, though I did not succeed at the first try.

Classification

Non-structure theory was developed by Saharon Shelah in the 1970s. The problem of classifying all models of a given theory led to the development of stability (and unstability) theory. Theory T is called **unstable** if there is a formula $\varphi(\overline{x}, \overline{y})$ and a model $A \models T$ containing tuples $\overline{a}_i$, $i < \omega$ such that for all $i < j < \omega$, $A \models \varphi(\overline{a}_i, \overline{a}_j)$.

Obviously, PA has a formula $x < y$ which defines linear order, hence PA is unstable. Many 'non-structure' results have been proved for unstable theories. The main two are:

Theorem 1 (Shelah)
If T is unstable, $\lambda > |T|$, then there are 2^λ non-isomorphic models of T of cardinality λ.

Theorem 2 (Shelah)
Suppose T_1 has infinite models, $L(T) \subseteq L(T_1)$, $T \subset T_1$, T is unstable, $\lambda > |T_1|$. Then there are 2^λ models of T_1 of cardinality λ whose $L(T)$-reducts are pairwise non-isomorphic.

So, in the case $T_1 =$ PA, $T =$ the theory of linear order, we can conclude that there are 2^λ pairwise non-isomorphic order-types of models of PA in every uncountable cardinality λ. This suggests non-classifiability of order-types of uncountable models of PA.

In section 6.1. we prove related results. One of them is: every model $M \models$ PA, $\mathrm{card}(M) < \lambda$ has 2^λ order non-isomorphic elementary end-extensions of cardinality λ, and our proof is elementary, without requiring the delicate machinery developed by Shelah.

Cells

Let $M \models \mathrm{PA}$. For every $\lambda \leq \mathrm{card}(M)$, M is naturally partitioned into λ-cells:

$$\mathrm{Cell}_\lambda(x) = \{y \mid \text{ there are at most } \lambda \text{ points between } x \text{ and } y\}.$$

Questions rise about the structure of cells and whether we can make every cell 'regular', e.g. order-isomorphic to the saturated discrete order of cardinality λ or whether cells themselves can be ordered as the saturated dense linear order. I call such 'regular' order-types **canonical**.

The idea arose from the desire to have more examples of concrete order-types to work with. You can find the whole story in chapters 2 and 5.

Note that I gave the most general definition of a canonical order but have not proved the existence theorem (it turned out to be a hard problem). Nevertheless, I proved several particular cases of it, enough to make the notion interesting.

We also ask how many models of each canonical order-type exist at all. I spent some time trying to prove that *there are many non-isomorphic models of the saturated order-type* (a false statement) until H.Kotlarski refered me to Pabion's Theorem.

Pabion's Phenomenon

If the order-type of M is λ-saturated then M is λ-saturated.

What happens if we consider a limit of ω saturated elementary end-extensions of the saturated model? Surprisingly, in contrast with Pabion's Theorem, there will exist non-isomorphic elementarily equivalent models of PA of this order-type (see Chapter 5).

Interpretations

In a sense, all model theory is about interpretations. Constructing a model of T is constructing an interpretation of a model of T inside a model of your favorite set theory. Also, most of Set Theory is about constructing models of set theories inside other models of set theories.

But not only models of set theory can act as the domain of our discourse where our mathematics takes place. Any theory formalising the Completeness Theorem is suitable for this purpose. PA is one of those theories.

Arithmetised Completeness Theorem

Let $\Omega \models \mathrm{PA}$. Let T be a set of (Gödel numbers of) sentences in L definable in Ω (possibly with parameters). Let $\Omega \models \mathrm{Con}(T)$, that is $\Omega \models \neg\exists x\, \mathrm{Prov}(x, \ulcorner T \rightarrow 0 = 1 \urcorner)$. Then there is a model $N \models T$ and two formulas $\mathrm{dom}_N(x)$ and $\mathrm{Sat}_N(x, y)$ such that the domain of N is $\{x \in \Omega \mid \mathrm{dom}_N(x)\}$ and for every $\varphi(u_1, \ldots, u_n) \in L$ and $x_1, \ldots, x_n \in \{x \in \Omega \mid \mathrm{dom}_N(x)\}$

$$N \models \varphi(x_1, \ldots, x_n) \quad \Leftrightarrow \quad \Omega \models \mathrm{Sat}_N(\ulcorner \varphi \urcorner, \langle x_1, \ldots, x_n \rangle).$$

This kind of interpretation is called a **strong interpretation** and is studied in Chapter 4.

If $T = \mathrm{PA}$, then we obtain models of PA strongly interpreted inside models of $\mathrm{PA} + \mathrm{Con}(\mathrm{PA})$ (**inner models**). The main theorem of Chapter 4 (Theorem 29) says that there can be only one order-type of inner models inside a given model Ω, namely

$$\Omega + Q(\Omega) \cdot (\Omega^* + \Omega),$$

where $Q(\Omega)$ is the set of fractions of Ω and the order-multiplication $\cdot$ is lexicographic. Later we introduce a new class of models, **self-similar** models. If $\Omega \models \mathrm{PA} + \mathrm{Con}(\mathrm{PA})$ then Ω is self-similar if and only if it is order-isomorphic to its inner models. Many previously studied models (saturated, resplendent, inner) turned out to be self-similar.

If $T = \mathrm{ZFC}$ then we ask which models $\Omega \models \mathrm{PA}$ contain a model $U \models \mathrm{ZFC}$ strongly interpreted in Ω such that $\omega_U \cong \Omega$. This and other questions concerning mutual interpretability of models of PA and models of ZFC are discussed in section 4.4.

Although we do not study interpretations of models of other theories in models of PA, I would like to mention them here. Let G be a group strongly interpreted in Ω. G is called Ω-finite if $\{x \in \Omega \mid \mathrm{dom}_G(x)\}$ is bounded in Ω.

Question 1
What do groups strongly interpreted in Ω look like? What do Ω-finite groups look like? Which properties of $\mathbb{N}$-finite (truely finite) groups do they inherit?

Arithmetic saturation

Since introduction of the notion of recursive saturation and the study of its interconnections with automorphism groups, resplendency and satisfaction relations, there has been demand for other notions of saturation. The only other successful notion introduced was arithmetic saturation. I encountered it by chance while trying to solve Friedman's problem. Understanding the mechanism of arithmetic saturation led further to constructing a nice automorphism of arithmetically saturated models. You can read the whole story in sections 7.1 and 7.2.

About this thesis

This thesis is based on my MPhil Thesis, which occupies chapters 2 and 3. The topics which I decided to exclude completely from the scope of the thesis are Presburger Arithmetic and Borel relations due to lack of material.

The thesis contains 42 Questions. Some questions (4, 25, 35, 40) may turn out to be easy, some (6, 10, 17, 26, 37) point at the major directions of future research, one (33) is a relatively ungrounded conjecture, some are particular cases of others. Altogether, questions have to be read as an undetachable part of the text.

Credits

A lot of ideas in this thesis arose during discussions with my supervisor. Retrospectively, it is usually hard to evaluate the extent of their influence on a particular given proof. Though some ideas were directly suggested and I tried them and they worked. An example is the idea to study interpretations of linear orders in models of PA and internalise the proof of "every countable dense linear order without end-points is isomorphic to $\mathbb{Q}$", which eventually gave rise to Chapter 4.

Originally, Chapter 3 arose from discussions with V.G. Kanovej who suggested to look at how Skolem terms fit together in order to investigate uncountable Borel models of arithmetic. The first versions of section 3.1. and Corollary 19 were written by V.G. Kanovej and presented to me as a gift. However, being not interested in Borel model theory at the time, I instead proved a general theorem about embeddability of $\mathrm{EM}(C, p)$ into $C^{<\mathbb{Q}}$ (Theorem 20) only to discover later that this was done by Christine Charretton and Maurice Pouzet in 1983 (see [4]).

Chapter 1

A survey of the subject

1.1 Notation and useful facts

1. **Model Theory**
   For definitions and basic facts about first-order logic, consistency, complete theories, Robinson's joint consistency test, models, elementary diagrams, compactness, completeness, Löwenheim-Skolem Theorem, elementary chains lemma, arithmetical hierarchy of formulas, Σ_n-elementary chains lemma, ultrafilters, ultraproducts, types, back-and-forth method, universality, homogeneity, λ-saturation, saturation, saturated linear orders, the notion of interpretation of one structure in another, I refer the reader to Chang and Keisler [3] or Hodges [11].

2. **Set Theory**
   For basic set-theoretic knowledge: formalization of ZFC, ordinals, cardinals, regularity, stationary sets, clubs, normal functions, Fodor's Theorem, independence of (CH), (GCH), $\diamondsuit$, $\diamondsuit_\lambda$ see Kunen [20] or Jech [12]. I also assume the standard pseudophilosophical discussion around the Gödel's incompleteness phenomenon in the context of ZFC to be known to the reader. We shall also need the following fact ([12], p.59):

   If λ is regular then λ can be partitioned into λ stationary subsets.

3. **Choice**
   As always in Model Theory, we assume (AC).

4. **Cardinal arithmetic**
   I will use the following facts from cardinal arithmetic: König's Theorem: if $\varkappa$ is infinite and $\mathrm{cf}(\varkappa) \leq \lambda$ then $\varkappa^\lambda > \varkappa$, Bukovský–Hechler Theorem: if $\varkappa$ is singular and there is $\gamma < \varkappa$ such that for all δ such

that $\gamma \leq \delta < \varkappa$, $2^\delta = 2^\gamma$, then $2^\gamma = 2^\varkappa$, and a related fact: if λ is a limit cardinal, $\lim_{i\in\mu} \lambda_i = \lambda$ then $\prod_{i\in\mu} 2^{\lambda_i} = 2^\lambda$. See Jech [12], pp. 42-52.

5. **Arithmetic**
For the discussion of PA, models of PA, standard coding devices, representation of recursive functions and recursive sets, formulas $\mathrm{Con}_{\mathrm{PA}}$ and $\mathrm{Prov}_{\mathrm{PA}}(x)$, Gödel's Theorems, Tarski's undefinability of Truth theorem in models of PA, Standard System, $\mathrm{len}(x), \mathrm{term}(x), \mathrm{form}(x), \mathrm{axiom}(x)$ -formulas in PA and the rest of what we consider 'standard' notation, see [14].

6. **Definability**
Let $M \models$ PA. If $M \models \forall\, \overline{y}\ \exists!\ x\ \ \varphi(\overline{y}, x, \overline{a})$ then we introduce a new function symbol $t(\overline{y}, \overline{a})$ and the axiom $\forall x\ \forall \overline{y}(x = t(\overline{y}, \overline{a}) \leftrightarrow \varphi(\overline{y}, x, \overline{a}))$. The term $t(\overline{y}, \overline{a})$ is called a **Skolem term** or a Skolem function.

   An element $b \in M$ is **definable** in M over A iff $M \models (b = t(\overline{a}))$ for some $\overline{a} \in A$ and some Skolem term $t(\overline{y})$ of M. $\mathrm{Cl}_M(A)$ will denote the set of all elements definable in M over A. We know that $\mathrm{Cl}_M(A)$ is an elementary submodel of M (see [14], pp.91-96).

   A subset A is called **definable** in M if there is a formula $\theta(x, \overline{a})$ with $\overline{a} \in M$ such that $b \in A \Leftrightarrow M \models \theta(b, \overline{a})$. We say that A is M**-finite** if A is definable and bounded in M.

7. **Standard System**
Let $M \models$ PA. Then $\mathbb{N}$ is embedded in M as an initial segment. The **Standard System** of M, $\mathrm{SSy}(M)$ is the collection of all subsets of $\mathbb{N}$ definable in M. Another (equivalent) definition is:

$$\mathrm{SSy}(M) = \{A \subseteq \mathbb{N} \mid \exists x \in M,\ \mathrm{pr}(n) | x \Leftrightarrow n \in A\},$$

   where $\mathrm{pr}(n)$ means "the n^{th} prime".

   A **Scott Set** is a nonempty collection $\mathscr{X}$ of subsets of $\mathbb{N}$ closed under $\cap$, $\cup$, complement and relative recursion and such that if $T \in \mathscr{X}$ codes an infinite tree then there is an infinite path $P \subseteq T$, $P \in \mathscr{X}$.

   If $M \models$ PA then $\mathrm{SSy}(M)$ is a Scott Set. Also, if $\mathscr{X}$ is a Scott Set and $|\mathscr{X}| \leq \omega_1$ then there is $M \models$ PA such that $\mathrm{SSy}\, M = \mathscr{X}$. For detailed discussion, see [14], pp. 172-192 and 261-265.

8. **Overspill**
Let $M \models$ PA and I be a proper cut of M (i.e. I is a proper subset of

M, $x < y \in I \Rightarrow x \in I$ and I is closed under the successor function). Suppose $\overline{a} \in M$ and $\varphi(x, \overline{a})$ is a formula such that $M \models \varphi(b, \overline{a})$ for all $b \in I$. Then there is $c > I$ such that

$$M \models \forall\, x \leq c\, \varphi(x, \overline{a}).$$

9. **Types**
   Let $M \models$ PA. A **type** over M is a set $p(\overline{x}, \overline{a})$ of formulas $\varphi(\overline{x}, \overline{a})$, $\overline{a} \in M$, such that $p(\overline{x}, \overline{a})$ is finitely satisfied in M. If T is a theory, a **type** over T is a set of formulas $\varphi(\overline{x})$ such that $T + \{\varphi(\overline{c}) \mid \varphi \in p\}$ is consistent, where $\overline{c}$ is a tuple of new constants.

10. **Indiscernibility**
    A type $p(x)$ over a complete theory $T \supset$ PA is called **indiscernible** if for any $M \models T$, any $x_1 < x_2 < \cdots < x_n$, $y_1 < y_2 \cdots < y_n$ in M, all satisfying p, and any formula φ we have

    $$M \models \varphi(x_1, x_2, \ldots, x_n) \leftrightarrow \varphi(y_1, y_2, \ldots, y_n).$$

    For the construction of an indiscernible type, see [13], page 82.

11. **Quantifier** $Qx\varphi(x)$
    "$M \models Qx\ \varphi(x)$" is an abbreviation for "$M \models \forall\, x\, \exists\, y > x\ \varphi(y)$", i.e. $Qx\ \varphi(x)$ means "for unboundedly many x, $\varphi(x)$ holds".

12. **Specker-MacDowell-Gaifman Theorem**
    Every model $M \models$ PA has a proper elementary end-extension.

13. **Conservative extensions**
    An extension $N \succ M$ is called **conservative** if for every formula $\varphi(x, \overline{a})$ with $\overline{a} \in N$ there is another formula $\psi(x, \overline{b})$ with $\overline{b} \in M$ such that

    $$M \cap \{x \mid N \models \varphi(x, \overline{a})\} = \{x \in M \mid M \models \psi(x, \overline{b})\}.$$

    In other words, every subset of M that is definable in N is already definable in M. We know that conservative extensions of models of PA are end-extensions. See [14], page 101.

14. **Gaifman's Splitting Theorem**
    If $M \subset N$ are models of PA then $K = \{x \in N \mid \exists y \in M,\ x \leq y\}$ is an initial segment of N such that $M \prec K$. See [14], page 89.

15. **$\varkappa$-like models**
    A model is called **$\varkappa$-like** if it has cardinality $\varkappa$ but all its proper initial segments have cardinality less than $\varkappa$. Existence of $\varkappa$-like models for all $\varkappa$ follows from the Specker-MacDowell-Gaifman Theorem.

16. **Recursive saturation**
A type $p(\overline{x}, \overline{a})$ is called **recursive** if the set $\{\ulcorner \varphi(\overline{x}, \overline{y}) \urcorner \mid \varphi(\overline{x}, \overline{a}) \in p(\overline{x}, \overline{a})\}$ is a recursive subset of $\mathbb{N}$.

A model $M \models \mathrm{PA}$ is called **recursively saturated** if every recursive type $p(\overline{x}, \overline{a})$ over M is realized in M.

A type $p(\overline{x}, \overline{a})$ over M is **coded** if $\{\ulcorner \varphi(\overline{x}, \overline{y}) \urcorner \mid \varphi(\overline{x}, \overline{a}) \in p(\overline{x}, \overline{a})\} \in \mathrm{SSy}(M)$.

A recursively saturated model realizes all its coded types. Also, if M is recursively saturated then $\mathrm{Th}\, M$ is coded in M.

If $M, N \models \mathrm{PA}$ are countable and recursively saturated then $(M \cong N) \Leftrightarrow (\mathrm{Th}\, M = \mathrm{Th}\, N$ and $\mathrm{SSy}\, M = \mathrm{SSy}\, N)$.

17. **Languages**
$L_{\mathrm{PA}} = \{0, +, \cdot, <, S\}$, $L_{\mathrm{ZFC}} = \{\in\}$. The following fact is known as **Craig's trick**. If a set T of formulas in some finite language L is recursively enumerable then it is equivalent to a recursive set of formulas. (For proof see [2], Chapter 15 or [14], p.150.) For this reason a recursively saturated model realises all recursively enumerable types, not only the recursive ones.

18. **Resplendency**
A model $M \models \mathrm{PA}$ is called **resplendent** if for every Σ^1_1-statement $\Phi(\overline{a})$, $\overline{a} \in M$ such that there exists a model $N \models \mathrm{Th}(M, \overline{a}) + \Phi(\overline{a})$, $M \models \Phi(\overline{a})$.

Every resplendent model is recursively saturated. Every countable recursively saturated model is resplendent.

19. **Kleene's Theorem**
Let L be a finite language. Let $\{\varphi_n(\overline{x})\}_{n \in \mathbb{N}}$ be a recursive set of formulas of L (i.e. the set of their Gödel numbers is a recursive subset of $\mathbb{N}$). Then there is a Σ^1_1-formula $\Phi(\overline{x})$ such that in all infinite L-structures M,
$$M \models \forall \overline{x}\, (\Phi(\overline{x}) \leftrightarrow \bigwedge_{i \in \mathbb{N}} \varphi_n(\overline{x})).$$

20. **Wilmers' Theorem**
Let $\mathscr{X}$ be a countable Scott Set, T be a consistent theory, $T \in \mathscr{X}$. Then there is a countable recursively saturated $M \models T$ such that $\mathrm{SSy}(M) = \mathscr{X}$.

21. **Linear orders**

Let $(A,<)$ and $(B,<)$ be linear orders. $A+B$ is a linear order, whose domain is the disjoint union of A and B and the order is defined by:
(i) $a_1 <_{A+B} a_2$ if $a_1, a_2 \in A$ and $a_1 <_A a_2$
(ii) $b_1 <_{A+B} b_2$ if $b_1, b_2 \in B$ and $b_1 <_B b_2$
(iii) $a < b$ for all $a \in A, b \in B$.
AB is a linear order with domain $A \times B$ and the order defined lexicographically (not antilexicographically as in some books!):
$(a_1, b_1) < (a_2, b_2)$ if $(a_1 < a_2$ or $(a_1 = a_2$ and $b_1 < b_2))$.
If $(X,<)$ is a linear order then $(X^n,<)$ is the set X^n with the lexicographic order.
A^* is the ordered set A with the order $<$ reversed.

It is known that there are 2^λ pairwise non-embeddable linear orders in each infinite cardinality λ (see [11]).

22. **Saturated linear orders**

The saturated dense linear order of cardinality λ will be denoted by Q_λ. (This is different from Hausdorff's notation Q_α or "the η_α-set of cardinality λ", where $\lambda = \aleph_\alpha$.) Remember: the subscript λ of Q_λ will always stand for the cardinality of Q_λ. The main theorem about Q_λ is:

$$\lambda \text{ is regular and } \Sigma_{\mu<\lambda} 2^\mu = \lambda \Leftrightarrow \text{ there exists } Q_\lambda.$$

Notice that $Q_\omega = \mathbb{Q}$.
The saturated discrete linear order with first and without last element of cardinality λ is $\mathbb{N} + Q_\lambda \mathbb{Z}$.

23. **Truth**

Throughout the thesis I avoid mentioning philosophy of mathematics, even in section 4.4 which comes close to it. However, the following three issues need special explanation:
1) what do I mean by "true"?
2) what is $\mathrm{Th}\,\mathbb{N}$ then?
3) what are "consistent theories"?
Usually the word "true" will mean "provable in ZFC". I understand the *ad hoc* nature of this convention (other notions of a set will produce different "truths" and there is no mathematical reason for ZFC to be given preference) but I'll stick to it. As in much of current mathematics, I shall not exploit the strength of ZFC anyway.

For the sake of psychological convenience I suggest we may think in the following way. Let us fix a 'big', 'monster' model $\mathfrak{M} \models$ ZFC and 'live in it'. Define $\mathbb{N} = \omega_{\mathfrak{M}}$, $\mathrm{Th}\,\mathbb{N} = \{\varphi \mid \omega_{\mathfrak{M}} \models \text{“}\varphi \in L_{\mathrm{PA}}\text{”} \text{ and } \omega_{\mathfrak{M}} \models \varphi\}$, etc. This is a neutral approach. Those who believe in existence of The Only True Universe can take $\mathfrak{M}$ = The Only True Universe, those who don't believe in it can choose $\mathfrak{M}$ according to their preferences. All classical definitions are easily translated into $\mathfrak{M}$-definitions. For example, let a set T of Gödel numbers of sentences of L be definable in $\mathfrak{M}$. T is called consistent if $\mathfrak{M} \models \neg \mathrm{Pr}_T(\ulcorner 0 = 1 \urcorner)$.

This paragraph was necessary to give meaning to such intuitive objects as $\mathbb{N}$ and $\mathrm{Th}\,\mathbb{N}$. We got rid of foundational difficulties by simply pushing them onto the level of our monster $\mathfrak{M}$.

1.2 Literature survey

Very little has been known in the area of order-types of models of PA so far:

1. In any model of PA, the terms 0, $S0 =: 1$, $SS0 =: 2$, $SSS0 =: 3, \ldots$ comprise an initial segment isomorphic to $\mathbb{N}$. This initial segment is called the **standard cut**. Also, if $c > \mathbb{N}$ then there are the points Sc, SSc, $SSSc, \ldots$ following it and the points Pc, PPc, $\ldots$ preceding it (if $P^{(n)}c = 0$ for some $n \in \mathbb{N}$ then $c = n \in \mathbb{N}$), hence there is the whole $\mathbb{Z}$-block of points around c. Hence the order-type of a nonstandard model is equal to $\mathbb{N} + A\,\mathbb{Z}$ for some linear order A. Sometimes we denote the $\mathbb{Z}$-block of a point c by $[c]$.

2. A is a dense linear order without end-points.

   Proof
   Given two points a and b from different $\mathbb{Z}$-blocks we observe that the integer part of $\frac{a+b}{2}$ is between a and b and belongs to a block other than $[a]$ or $[b]$. A has no end-points because for any $a \in M$, $[\frac{a}{2}]$ and $[2a]$ are blocks different from $[a]$. ■

   In particular the order-type of any countable model is $\mathbb{N} + \mathbb{Q}\,\mathbb{Z}$ because $\mathbb{Q}$ is the only countable dense linear order without end-points.

3. A cannot be the set $\mathbb{R}$ of all reals.
   (Smorynski [33] attributes this result to Klaus Potthoff.)

   Proof
   Suppose the order-type of M is $\mathbb{N}+\mathbb{R}\,\mathbb{Z}$. Consider an arbitrary definable M-sequence $\{a_i\}_{i\in M}$ such that for all $i, j \in \mathbb{N}$ such that $i < j$,

$$[a_i] < [a_j]$$

(e.g. the sequence $\{ia\}_{i\in M}$ where $a \in M \smallsetminus \mathbb{N}$). Let $\sup\{[a_i] \mid i \in \mathbb{N}\} = [b]$. Since for all $n \in \mathbb{N}$,

$$a_n < a_{n+1} \quad \text{and} \quad a_n < b,$$

by overspill, there is $c \in M \smallsetminus \mathbb{N}$ such that for all $n \in \mathbb{N}$,

$$a_n < a_c < b.$$

Now, all real numbers from $\{[a_j]\}_{\mathbb{N}<j<c}$ lie between $\{[a_n]\}_{n\in\mathbb{N}}$ and $[b]$ contradicting the definition of $[b]$. ■

Later in the thesis (section 2.4), the above fact will follow from a more general theorem.

4. **Pabion's Theorem**
Let $M \models$ PA, $\varkappa$ be a cardinal. If $(M,<)$ is $\varkappa$-saturated then M is $\varkappa$-saturated.

Proof for $\varkappa = \omega_1$ (due to D. Richard and J.F. Pabion)
First let us prove that a model of PA is ω_1-saturated if and only if all of its ω-sequences are coded. Suppose $A \models$ PA codes all its ω-sequences. Let $p(x) = \{\varphi_n(x)\}_{n\in\omega}$ be an arbitrary type (possibly with $\aleph_0$ parameters). Let x_n satisfy $\varphi_1(x) \wedge \ldots \wedge \varphi_n(x)$. Let $a \in A$ be a code of $\{x_n\}_{n\in\omega}$. Let

$$\psi_k(u) = \forall y(k < y \leq u \to \varphi_k((a)_y)).$$

Informally $\psi_k(u)$ means $\varphi_k((a)_{k+1}) \wedge \ldots \wedge \varphi_k((a)_u)$ (informally, because it may be an infinite conjunction). For all $u \in \mathbb{N}$, $A \models \psi_k(u)$, hence, by overspill, there is $c_k > \mathbb{N}$ such that $A \models \psi_k(c_k)$. Let b be a code of the sequence $\{c_k\}_{k\in\omega}$. Consider the definable set $B = \{x \mid \forall y \leq x\ x < (b)_y\}$. Obviously $\mathbb{N} \subset B$. Hence there is a point $c \in B \smallsetminus \mathbb{N}$. For every standard n, $n < c < (c)_n$, hence $(a)_c$ satisfies the type $p(x)$. Hence a model which codes all ω-sequences is ω_1-saturated.

Suppose $A \models$ PA is of ω_1-saturated order-type. Let us show that then every ω-sequence will be coded. Let $a_0, a_1, \ldots, a_n, \ldots$ be an arbitrary sequence. Translate it into a strictly increasing sequence by putting $b_0 = a_0, b_{n+1} = a_{n+1} + b_n + 1$. By ω_1-saturation, there is some $c > b_n$ for all n. Put $c_n = c - b_n$, so that $\{c_n\}_{n\in\omega}$ is strictly decreasing. Consider the following conditions:

$$2^{c_0} \ < \ x \ < \ 2^{c_0+1}$$

$$\vdots$$
$$2^{c_0} + \cdots + 2^{c_n} \;<\; x \;<\; 2^{c_0} + \cdots + 2^{c_{n-1}} + 2^{c_n+1}$$
$$\vdots$$

By ω_1-saturation, there is a point $u \in A$ satisfying all of the above conditions. Now, c_k is the k^{th} exponent in the binary expansion of u. It remains to solve a triangular system to get a_k, the process which can be formally expressed in PA. Hence, $\{a_k\}_{k\in\omega}$ is coded in A. ■

The proof for higher cardinalities is more complicated and can be found in [21].

5. **$(\varkappa, \varkappa)$-cuts**
It has been proved by Shelah that every $M \models$ PA has a cut $I \subset M$ such that for some cardinal $\varkappa$, $\mathrm{cf}(I) = \varkappa$ and $\mathrm{cf}(M \setminus I) = \varkappa$. See [29].

6. **Some results of J. Schmerl**
A number of possibly relevant results has been obtained by J. Schmerl. Let me quote two of them.

Fact 1 ([25])
If λ is regular, $N \models$ PA is λ-saturated and $\lambda < |N|$ then there is a rather classless λ-saturated $M \succ N$ such that $|M| = |N|$.

Fact 2 ([24])
If M is a countable recursively saturated model of PA then for every cardinal $\varkappa$ there is a resplendent N of cardinality $\varkappa$ with $N \equiv_{\infty\omega} M$ and N generated by indiscernibles.

7. Models of singular strong limit cardinalities have long sequences.
The following application of an Erdös-Rado theorem is due to Hodges [10].

Fact *Let T be a completion of* PA*, $M \models T$, $\varkappa$ be a singular strong limit cardinal,* $\mathrm{card}\, M = \varkappa$*. Then M contains an increasing sequence of order-type $\varkappa$.*

Proof
Let $\varkappa = \sum_{\alpha<\mu} \lambda_\alpha$, $\mathrm{cf}(\varkappa) = \mu$, $\{\lambda_\alpha\}_{\alpha<\mu}$ be increasing, each λ_α be regular. By a Theorem due to Erdös and Rado, for each $\alpha < \mu$, there is $h_\alpha\colon \lambda_\alpha \to M$ which is either order-preserving or order-reversing. Using subtraction, we can assume that each h_α is order-preserving. Let $B = \bigcup_{\alpha<\mu} \mathrm{Im}(h_\alpha)$.

If for each $c \in M$, $|\{b \in B \mid b < c\}| < \varkappa$ then $\mathrm{cf}(M) = \mu$. As $\mu < \varkappa$, there is $\beta < \mu$ such that for all $\delta > \beta$, $\lambda_\delta > \mu$, hence for all $\delta > \beta$, $\mathrm{Im}(h_\delta)$ is bounded in M by some point c_δ. Define the increasing sequence $(a_\alpha)_{\alpha\in\varkappa}$ as follows. For all $\alpha < \lambda_{\beta+1}$, put $a_\alpha = h_{\beta+1}(\alpha)$. Let δ be such that $\beta + 1 < \delta < \mu$ and suppose for all $\alpha < \sup_{\gamma<\delta} \lambda_\gamma$, a_α has been defined. As $\delta < \mu$, $\{a_\alpha\}_{\alpha<\sup_{\gamma<\delta}\lambda_\gamma}$ is bounded in M (otherwise cofinality of M would be less than δ), say by point b. For $\alpha \in [\sup_{\gamma<\delta} \lambda_\gamma, \lambda_\delta)$, put $a_\alpha = b + h_\delta(\alpha)$. Obviously, $\{a_\alpha\}_{\alpha<\lambda_\delta}$ is bounded by $b + c_\delta$.

If for some $c \in M$, $|\{b \in B \mid b < c\}| = \varkappa$ then suppose without loss of generality that $B < c$. Let $f\colon \mu \to M$ be order preserving. Then for $\alpha \in [\lambda_\delta, \lambda_{\delta+1})$, put $a_\alpha = c \cdot f(\delta) + h_{\delta+1}(\alpha)$. Obviously, $\{a_\alpha\}_{\alpha\in\varkappa}$ is an increasing $\varkappa$-sequence in M. ■

The condition that $\varkappa$ is a singular strong limit cardinal is important. In Chapter 3 we shall construct a family of models of cardinality 2^ω not even having increasing ω_1-sequences.

8. **Models generated by indiscernibles**

In paper [4] C.Charretton and M.Pouzet give an outline of a construction of the embedding of a model generated by indiscernibles ordered as a left-right symmetric linear order C into $C^{<\mathbb{Q}}$. You can read details of the construction and the proof in my Chapter 3. They also outline a possible proof of the fact that if T is a countable first-order theory with the strict order property then, for every uncountable linear order C, there is a model $M(C) \models T$ such that C is embeddable into C' if and only if $M(C)$ is elementarily embeddable into $M(C')$. I was not at all convinced by the details and suggest questions 7 and 8 as possible ways of proving this fact.

Chapter 2

Density, cells, cofinalities, weight

In this chapter we give first examples of order-types of models of PA and introduce some important notions: density, cell decomposition, cofinality, lower cofinality, weight. Some theorems proved in this chapter will play important role later on in the thesis.

2.1 Density

First let us introduce a class of models M having as many points between any two points as possible, namely, $\operatorname{card} M$.

Definition 1 *Let $M \models \mathrm{PA}$, $\varkappa$ be a cardinal. M is called **$\varkappa$-dense** if $\operatorname{card} M = \varkappa$ and for all $a, b \in M$ such that $a < b$, a and b belong to different $\mathbb{Z}$-blocks,*

$$\operatorname{card}\{x \in M \mid a < x < b\} = \varkappa.$$

Simple Fact 1 *$M \models \mathrm{PA}$ is $\varkappa$-dense if and only if every nonstandard initial segment of M (including M itself) has cardinality $\varkappa$.*

Proof
If M is $\varkappa$-dense and I is a nonstandard initial segment then, for any nonstandard point $a \in I$, $\operatorname{card}[0, a] = \varkappa$. Now,

$$\varkappa = \operatorname{card} M \geq \operatorname{card} I \geq \operatorname{card}[0, a] = \varkappa,$$

hence $\operatorname{card} I = \varkappa$.

If all nonstandard initial segments of M have cardinality $\varkappa$ then, given $a, b \in M$, $a < b$, a,b from different $\mathbb{Z}$-blocks, consider the nonstandard initial segment $I = \{x \mid x < b - a\}$. As $\operatorname{card} I = \varkappa$ then $[a, b] = \{a + x \mid x \in I\}$ has cardinality $\varkappa$ too. ■

Proposition 2 *For every infinite $\varkappa$ there is a $\varkappa$-dense model of* PA.

Proof
An elementary chain argument. Let $M_0 \models \text{PA}$, $\operatorname{card} M_0 \leq \varkappa$. Suppose $i \leq \varkappa$ and for all $j < i$, M_j has already been defined so that $j_1 < j_2 < i$ implies $M_{j_1} \prec M_{j_2}$. By the elementary chains lemma, $\bigcup_{j<i} M_j \succ M_k$ for all $k < i$. Denote $\bigcup_{j<i} M_j$ by K_i. The collection of sentences

$$\{\varphi(\overline{a}) \mid K_i \models \varphi(\overline{a}),\ \overline{a} \in K_i\} \cup \{n < b_i < a \mid n \in \mathbb{N},\ a \in K_i \smallsetminus \mathbb{N}\}$$

is finitely satisfied in K_i, hence, by compactness theorem, there is $N \succ K_i$, which satisfies it. Let us identify the symbol b_i with its interpretation in N. Put $M_i = \operatorname{Cl}_N(K_i \cup \{b_i\})$. We observe that $\operatorname{card} M_i = \operatorname{card} K_i$ and $M_i \succ K_i$. Define $M = M_\varkappa$. The sequence $b_0 > b_1 > \cdots > b_i > \cdots$ $(i \in \varkappa)$ is unbounded below in $M \smallsetminus \mathbb{N}$, hence M is $\varkappa$-dense. ■

Actually, we have proved that any model of cardinality less or equal than $\varkappa$ has a $\varkappa$-dense elementary extension.

In the sequel we shall continue to use standard model-theoretic constructions and results (such as the elementary chains lemma, compactness theorem, etc., and identification of a constant symbol naming an element with the element itself) but without explicit mention.

In the rest of this section we shall study some examples of $\varkappa$-dense models.

First example

Lemma 3 *Let $K \models$ PA and $X \in \operatorname{SSy}(K)$. Then for every $b \in K \smallsetminus \mathbb{N}$, there is $c < b$ which codes X.*

Proof
Let a be a code of X in K. Let

$$\varphi(x, a, b) = \exists y < b\ \forall z < x\ (\operatorname{pr}(z)|a \leftrightarrow \operatorname{pr}(z)|y),$$

where $\operatorname{pr}(x)$ is the xth prime number. For every $n \in \mathbb{N}$, $M \models \varphi(n, a, b)$, hence, by overspill, there is $c \in M$ such that $\mathbb{N} < c < b$ and for every $n \in \mathbb{N}$,

$$M \models \operatorname{pr}(n)|a \leftrightarrow pr(n)|c,$$

hence c codes X. ■

For any non-principal ultrafilter D on ω and any countable model M, denote $\prod_D M$ by M^ω. (Later we shall see that, modulo (CH), $\prod_D M$ does not depend on the ultrafilter D.)

Proposition 4 *M^ω is $2^{\aleph_0}$-dense.*

Proof
Let us prove prove that $\mathrm{SSy}(M^\omega) = \mathscr{P}(\mathbb{N})$. Let $X = \{i_1, i_2, \ldots\}$ be an arbitrary subset of $\mathbb{N}$. Define $\mathbf{a}_X = [(a_1, a_2, a_3, \ldots)_{j\in\omega}]$, where $a_j = \prod_{k\leq j} \mathrm{pr}(i_k)$. Observe that for every $n \in \mathbb{N}$,

$$\{i \in \omega \mid \mathrm{pr}(n) | a_i\} \in D \;\Leftrightarrow\; n \in X,$$

so $\mathbf{a}_X$ codes X. Thus, $\mathrm{SSy}(M^\omega) = \mathscr{P}(\mathbb{N})$. By Lemma 3, every nonstandard initial segment contains a code of X, hence, by Fact 1, M^ω is $2^{\aleph_0}$-dense. ■

A $\varkappa$-dense submodel of a μ-dense model

Proposition 5 *If M is μ-dense then for every $\varkappa < \mu$ there is a $\varkappa$-dense elementary submodel of M.*

The following proof was suggested by my supervisor.
Proof
Let θ be a surjective mapping

$$\varkappa \xrightarrow{\theta} \{(i_1, i_2, ..., i_n, j) \mid n \in \mathbb{N},\; i_1, i_2, ...i_n \in \varkappa,\; j \in \mathbb{N}\}.$$

Partition $\varkappa$ into a disjoint union of $\varkappa$ sets of cardinality $\varkappa$:

$$\varkappa = \coprod_{i\in\varkappa} S_i.$$

For every $i \in \varkappa$, let $\alpha_i \colon S_i \to \varkappa$ be a bijection between S_i and $\varkappa$. Now we are going to introduce $\varkappa$ points of M, whose Skolem closure will be $\varkappa$-dense. Let $a_0 \in M$ be arbitrary. Suppose $\eta < \varkappa$ and for all $i < \eta$, a_i are already defined. As $\{S_i \mid i \in \varkappa\}$ is a partition of $\varkappa$, $\eta \in S_\nu$ for some ν. Let $\theta(\alpha_\nu(\eta)) = (i_1, i_2, \ldots, i_n, j)$. If $i_1, i_2, \ldots, i_n < \eta$ choose $a_\eta \in M \setminus \{a_i \mid i < \eta\}$ such that $a_\eta < t_j(a_{i_1}, a_{i_2}, ...a_{i_n})$, where $t_j(x_1, x_2, \ldots, x_n)$ is the jth element in the enumeration of all nonstandard Skolem terms of $\mathrm{Th}(M)$. Otherwise let a_η be arbitrary. Let $K = \mathrm{Cl}_M\{a_i \mid i < \varkappa\}$. Obviously, $\mathrm{card}\, K = \varkappa$. Consider any point $t_j(a_{i_1}, a_{i_2}, ...a_{i_n}) \in K$. Let $I = \bigcup_{\nu\in\varkappa} \alpha_\nu^{-1}\theta^{-1}(i_1, \ldots, i_n, j)$. I is cofinal in $\varkappa$ because $\mathrm{card}\, I = \varkappa$. Introduce $J = \{a_\eta \mid \eta \in I,\; \eta > i_1, i_2, \ldots, i_n\}$. We observe that $\mathrm{card}(J) = \varkappa$. For every $a_\eta \in J$, $a_\eta < t_j(a_{i_1}, a_{i_2}, ...a_{i_n})$ by definition of a_η. Thus K is $\varkappa$-dense. ■

Ultrapower construction

Proposition 6 *For every cardinal λ such that $\lambda^+ = 2^\lambda$ and any countable model M there is an ultrafilter D on λ such that $\prod_D M$ is 2^λ-dense.*

Proof
It suffices to prove that $\prod_D \mathbb{N}$ is 2^λ-dense, because for any nonstandard $\mathbf{a} \in \prod_D M$ there is a nonstandard $\mathbf{b} \in \prod_D \mathbb{N} \hookrightarrow \prod_D M$ such that $\mathbf{b} < \mathbf{a}$. Assuming the appropriate ultrafilter is already found (we shall specify its properties during the proof) we need to prove that

$$\left| \left\{ x \in \left(\prod_D \mathbb{N} \right) \smallsetminus \mathbb{N} \mid x < \mathbf{a} \right\} \right| = 2^\lambda$$

$$\text{where} \quad \mathbf{a} = [(n_1 n_2 n_3 ... n_j ...)_{j \in \lambda}].$$

Suppose there are λ points $\mathbf{a_i}$ less than $\mathbf{a}$, enumerated as

$$\{\mathbf{a_i} = [(a_{i1} a_{i2} a_{i3} ... a_{ij} ...)_{j \in \lambda}] \mid i \in \lambda\}.$$

We are going to construct a new point $\mathbf{b} < \mathbf{a}$ which is different from any of $\mathbf{a_i}$. Let D have a subset F such that:

1. $|F| = \lambda$;

2. for all $i \in \lambda$, $(n_i = m)$ implies that i belongs to not more than $(m-1)$ elements of F.

Let $F = \{e_i \mid i \in \lambda\}$. Let $j < \lambda$ and $n_j = m$. List all $e_{i_1}, e_{i_2}, e_{i_3}, ..., e_{i_{m-1}} \in F$ such that $j \in e_{i_k}$ for all $k = 1, 2, ..., m-1$. Let b_j be a natural number such that $b_j < m$ and $b_j \neq a_{i_1 j}, b_j \neq a_{i_2 j}, ..., b_j \neq a_{i_{m-1} j}$. It is possible to choose such b_j because not more than $(m-1)$ points from $\{1, 2, ..., m\}$ are occupied by $a_{i_k j}$. So $b_j \neq a_{ij}$ for all $j \in e_i$. Since $e_i \in F \subset D$, $\mathbf{b} = [(b_j)_\lambda] \neq \mathbf{a_i}$ for all $i \in \lambda$. Thus, there are more than λ points below $\mathbf{a}$. Assuming $\lambda^+ = 2^\lambda$ there are exactly 2^λ of them. So $\prod_D M$ is 2^λ-dense. ■

Now we only need to find an ultrafilter with the required properties.

Definition 2 *An ultrafilter D on λ is called **λ-excellent** if it is countably incomplete (i.e. there is a sequence $C_1 \supseteq C_2 \supseteq C_3 \supseteq \cdots$ s.t. $\bigcap_{i \in \omega} C_i = \varnothing$) and for every $\{C_i\}_{i \in \omega} \subset D$ such that $C_1 \supseteq C_2 \supseteq C_3 \supseteq \cdots$ and $\bigcap_{i \in \omega} C_i = \varnothing$ there is $F \subset D$ such that:*

1. $|F| = \lambda$;

2. Every element of $C_n \setminus C_{n+1}$ belongs to not more than $(n-1)$ elements of F.

In the proof of Proposition 6, for our arbitrary nonstandard point $\mathbf{a}$ we can introduce $C_m = \{i \in \lambda \mid n_i \geq m\}$. Observe that $\bigcap_{m\in\mathbb{N}} C_m = \varnothing$ and for all $m \in \mathbb{N}$, $C_m \in D$, because $\mathbf{a}$ is nonstandard. Thus, the only property we require from our ultrafilter is for it to be λ-excellent.

Observation 7 *Let $(M, <)$ be a countable linearly ordered set and D an ω_1-complete ultrafilter (i.e. for any countable $\{C_i\}_{i\in\omega} \subset D$, $\bigcap_{i\in\omega} C_i \in D$) on a set I. Then $\prod_D M = M$.*

Proof
If there is a point $\mathbf{a} = [(a_1 a_2 a_3 ...)] \in \prod_D M$ which is not equal to any of the old points then enumerate $X = \{x \in M \mid x < \mathbf{a}\}$ as $\{x_i\}_{i\in\omega}$ and $Y = \{y \in M \mid y > \mathbf{a}\}$ as $\{y_i\}_{i\in\omega}$. Introduce $C_n = \{i \in I \mid x_n < a_i < y_n\}$. As $X < \mathbf{a} < Y$, $C_n \in D$. However, as $\mathbf{a} \notin M$, $\bigcap_{n\in\mathbb{N}} C_n = \varnothing \notin D$, contradiction. ■

In view of this, all ultrafilters we consider in the future will be not only non-principal but countably incomplete as well.

Definition 3
*An ultrafilter on I is called **λ-regular** if it has a subset F such that:*

1. *$|F| = \lambda$;*
2. *Every point of I belongs to not more than finitely many elements of F.*

Definition 4 *An ultrafilter on I is called **regular** if it is* card I*-regular.*

Thus λ-excellent ultrafilters are a kind of regular ultrafilters.

Lemma 8 $(\lambda > \omega)$ & $(\lambda^+ = 2^\lambda) \Rightarrow$ *there exists a λ-excellent ultrafilter.*

This λ-excellent ultrafilter was constructed in my MPhil Thesis. It is a long chain construction which I decided to omit here.

We do not need a λ-excellent ultrafilter for $\lambda = \omega$, because, as we saw in Proposition 4 , $\prod_D M$ is $2^{\aleph_0}$-dense for any ultrafilter D on ω.

Also, there exist λ-good ultrafilters providing us, modulo (GCH), with saturated ultraproducts. If M is the saturated model of cardinality λ then M is λ-dense as we are going to see in a moment.

Question 2 *Is there an ultrafilter D on λ such that there exists a countable $M \models$ PA with $\prod_D M$ 2^λ-dense but not saturated?*

Saturation

Fact (Keisler)[3]
If D is a countably incomplete ultrafilter then $\prod_D M$ is ω_1-saturated.

Corollary (CH)
Let M and N be two countable models. The following are equivalent:

1. *$M \equiv N$;*
2. *$\prod_D M \cong \prod_E N$ for any non-principal ultrafilters D and E on ω;*
3. *there are ultrafilters D and E such that $\prod_D M \cong \prod_E N$.*

Proof of the Corollary
(1) $\Rightarrow$ (2) because both $\prod_D M$ and $\prod_E N$ are saturated and have the same cardinality and, hence, isomorphic.
(2) $\Rightarrow$ (3) obvious.
(3) $\Rightarrow$ (1) because every model is elementarily equivalent to its ultrapower. ■

We deduce the following theorem [3].

Theorem (CH) *M^ω is saturated and, for every $N \equiv M$ and any two ultrafilters D and E on ω, $\prod_D M \cong \prod_E N$.*

As M^ω is saturated, its order-type is the saturated discrete linear order with first and no last point, $\mathbb{N} + Q_{\omega_1}\mathbb{Z}$.

Proposition 9
A saturated model of cardinality $\varkappa$ is $\varkappa$-dense.

Proof
If there was a nonstandard point a such that $\text{card}\{x \mid x < a\} < \varkappa$ then the type $p(x) = \{x < a\} \cup \{x \neq a_i \mid a_i < a\}$ with fewer then $\varkappa$ parameters would not be realized. This contradicts saturation. ■

Resplendency

Proposition 10 *If $M \models \text{PA}$ is resplendent then M is $\text{card}\, M$-dense.*

Proof
Let $a \in M \smallsetminus \mathbb{N}$. The statement "there are $\text{card}\, M$ points below a" is expressed by the following Σ^1_1 formula: $\exists f\, \forall xy\, (x \neq y \rightarrow f(x) \neq f(y)\ \&\ f(x) < a)$, which is realised in $\text{Cl}(a)$ (because $\{x \in \text{Cl}(a) \mid x < a\}$ is countable, hence

there is a bijection between $\{x \in \mathrm{Cl}(a) \mid x < a\}$ and $\mathrm{Cl}(a)$), hence is realised in M.

Moreover, the Σ_1^1 formula

$$\exists f\ \forall xy\ (x < y\ \rightarrow\ f(x) < f(y) < a\ \&\ \forall z \exists w (z < f(x) \rightarrow f(w) = z))$$

is realised in $\mathrm{Cl}(a)$ because $\{x \in \mathrm{Cl}(a) \mid x < [a]\}$ has order-type $\mathbb{N} + \mathbb{Q}\mathbb{Z}$, i.e. is order-isomorphic to $\mathrm{Cl}(a)$. Hence, for every $a \in M \setminus \mathbb{N}$, there is an initial segment $I < a$ such that $(I, <) \cong (M, <)$. ■

In section 2.3, Lemma 17, we shall actually classify all order-types of initial segments of a resplendent model.

2.2 Cells

In this section we shall divide a model into pieces that have the same density. These pieces will be called **cells**. The notion of a cell extends the notion of a $\mathbb{Z}$-block into larger cardinalities.

Simple Fact 11
Let M be a linear order and suppose there is $a \in M$ such that $\mathrm{card}[0, a] = \lambda$. *Let $I_\lambda = \{a \mid \mathrm{card}[0, a] \leq \lambda\}$. Then* $\mathrm{card}\, I_\lambda = \lambda$ *or* $\mathrm{card}\, I_\lambda = \lambda^+$.

Proof
If $\mathrm{card}\, I_\lambda = \beta \geq \lambda^+$, let $A = \{a_i\}_{i<\lambda^+}$ be the first λ^+ elements in the enumeration of I_λ. A cannot be bounded in I_λ, because if $A < a$ for some $a \in I_\lambda$ then $\mathrm{card}[0, a] \geq \lambda^+$. Hence, $\{a_i\}_{i<\lambda^+}$ is cofinal in I_λ and, thus, I_λ is a union of λ^+ sets $[0, a_i]$, $i \in \lambda^+$ of cardinality λ. Hence, $\beta = \lambda^+ \cdot \lambda = \lambda^+$. ■

Definition 5 *Let $M \models$ PA.*
*For every λ such that there is $a \in M$, $\mathrm{card}[0, a] = \lambda$ and for every $x \in M$, define the **λ-cell** of x as $\mathrm{Cell}_\lambda(x) = x \pm I_\lambda = \{y \mid |x - y| \in I_\lambda\}$. Let $\{\lambda_1, \lambda_2, \ldots, \lambda_i \ldots\}_{i<\mu}$ be the enumeration in increasing order of all λ such that there exists a, $\mathrm{card}[0, a] = \lambda$. For every $i < \mu$, if $\mathrm{card}\, I_{\lambda_i} = \lambda_i$ put $\varepsilon_i = 0$, otherwise (i.e if $\mathrm{card}\, I_{\lambda_i} = \lambda_i^+$) put $\varepsilon_i = 1$. The matrix $\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_i & \cdots \\ \varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_i & \cdots \end{pmatrix}_{i<\mu}$ is called the **sort** of M.*

Properties of cells

Theorem 12

1. For every $i < \mu$, $x \in M$,
   $\mathrm{Cell}_{\lambda_i}(x) = \{y \mid$ there are at most λ_i points between x and $y\}$.

2. For every $i < \mu$ and $x \in M$, $\mathrm{Cell}_{\lambda_i}(x)$ is convex.

3. For every $i < \mu$, $x, y \in M$, either x and y are in the same λ_i-cell or $\mathrm{Cell}_{\lambda_i}(x) \cap \mathrm{Cell}_{\lambda_i}(y) = \varnothing$.
   Thus, for every $i < \mu$, M is partitioned into λ_i-cells.

4. A $\varkappa$-dense model consists of just one cell.

5. For every $i < \mu$, I_{λ_i} is closed under addition and multiplication.

   Proof
   If $\mathrm{card}[0, a] \leq \lambda_i$ and $\mathrm{card}[0, b] \leq \lambda_i$ then $\mathrm{card}[0, a + b] = \mathrm{card}[0, a] + \mathrm{card}\{a + x \mid x < b\} \leq \lambda_i + \lambda_i = \lambda_i$ and $\mathrm{card}[0, a \cdot b] = \mathrm{card}[0, a] + \mathrm{card}[a, 2a] + \mathrm{card}[2a, 3a] + \cdots + \mathrm{card}[(b-1)a, ba] \leq \lambda_i \cdot \lambda_i = \lambda_i$. ■

6. For $i < \mu$, let $(M/I_{\lambda_i}, <)$ be the set of all λ_i-cells in their natural order. Then $(M/I_{\lambda_i}, <)$ is a dense linear order.

   Proof
   Let $a, b \in M$, $\mathrm{Cell}_{\lambda_i}(a) \neq \mathrm{Cell}_{\lambda_i}(b)$. Consider $c = [\frac{a+b}{2}] \in M$. If there was no cell between $\mathrm{Cell}_{\lambda_i}(a)$ and $\mathrm{Cell}_{\lambda_i}(b)$ then c would belong either to $\mathrm{Cell}_{\lambda_i}(a)$ or to $\mathrm{Cell}_{\lambda_i}(b)$. Let $c \in \mathrm{Cell}_{\lambda_i}(a)$. Then $\mathrm{card}[a, c] = \lambda_i$. Then $\mathrm{card}[c, b] = \mathrm{card}[a, c] = \lambda_i$, hence, by property (1), $b \in \mathrm{Cell}_{\lambda_i}(a)$ and $\mathrm{Cell}_{\lambda_i}(a) = \mathrm{Cell}_{\lambda_i}(b)$. Contradiction. ■

7. It follows from the proof above that for every $i < j < \mu$, $I_{\lambda_j}/I_{\lambda_i}$ is a dense linear order too.

8. Denote $(I_{\lambda_1} \smallsetminus \mathbb{N})/\mathbb{Z}$ by B_1, $(I_{\lambda_j} \smallsetminus \bigcup_{i<j} I_{\lambda_i})/\bigcup_{i<j} I_{\lambda_i}$ by B_j. Then
   $I_{\lambda_1} = \mathbb{N} + B_1\mathbb{Z}$,
   $I_{\lambda_2} = \mathbb{N} + B_1\mathbb{Z} + B_2B_1\mathbb{Z}$,
   $I_{\lambda_3} = \mathbb{N} + B_1\mathbb{Z} + B_2B_1\mathbb{Z} + B_3B_2B_1\mathbb{Z}$, etc
   $(M, <) = \mathbb{N} + B_1\mathbb{Z} + B_2B_1\mathbb{Z} + \cdots + (\underbrace{\cdots B_3B_2B_1}_{\mu^*}\mathbb{Z})$.

   So, the problem of specifying the order-type of M is reduced to specifying linear orders B_j for all $j < \mu$. Chapter 5 deals with the situation when the orders B_j are combinations of ordinals and saturated orders Q_λ.

9. (J. Paris and G. Mills [22])
   If $\varepsilon_j = 1$ then $I_{\lambda_j} \models \mathrm{PA}$.

10. (J. Paris and G. Mills [22])
    If $\lambda_1 = \omega$, $\lambda_2 > 2^\omega$ then I_ω is closed under exponentiation and this is the fastest function under which I_ω needs to be closed. More precisely:

if $M \models \mathrm{PA}$ is countable and $I \subseteq M$ in a nonstandard initial segment of M closed under exponentiation then for every infinite $\varkappa > \omega$ there is $N \succ M$ such that $\lambda_1^N = \omega$, $I_\omega^N = I$, $\lambda_2^N = \varkappa$.

11. (GCH) If D is a λ-excellent ultrafilter and M is a model of sort

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_i & \cdots \\ \varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_i & \cdots \end{pmatrix}_{i<\mu},$$

where $\lambda_i \leq 2^\lambda$ for all $i < \mu$, but M is not $(2^\lambda)^+$-like then $\prod_D M$ is 2^λ-dense.

Proof
We observe that card $\prod_D M = 2^\lambda$ because, since D is a regular ultrafilter, card $\prod_D M = (2^\lambda)^\lambda$, (see [3]), and, by König's lemma, $(2^\lambda)^\lambda = 2^\lambda$ because $\mathrm{cf}(2^\lambda) > \lambda$. Actually, the picture is this:

$$\begin{pmatrix} \omega^\lambda & \lambda_1^\lambda & \ldots & (2^\lambda)^\lambda \\ 0 & 0 & \ldots & 0 \end{pmatrix}$$

but the cells $I_{\omega^\lambda}, I_{\lambda_i^\lambda}$ for $\lambda_i \leq 2^\lambda$ will form one single cell because they all have cardinality 2^λ. The sets $\prod_D I_{\lambda_i}$ for $\lambda_i \leq 2^\lambda$ are 2^λ-dense because they have cardinality 2^λ and each of their initial segments contains an initial segment of $\prod_D \omega$, which is 2^λ-dense by Proposition 6. ■

The main theorem about cells

Theorem 13 *For every matrix* $\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_i & \cdots \\ \varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_i & \cdots \end{pmatrix}_{i<\mu}$ *with* $\lambda_1 < \lambda_2 < \cdots < \lambda_i < \cdots$ *and* $\varepsilon_i \in \{0,1\}$ *for all* $i < \mu$, *there is a model of this sort.*

Proof
Let M_{λ_1} be a λ_1-dense model. If ε_1=1 then let N_{λ_1} be any λ_1^+-like end-extension of M_{λ_1}, otherwise put $N_{\lambda_1} = M_{\lambda_1}$. Suppose $j < \mu$ and we have already defined

$$N_{\lambda_1} \prec N_{\lambda_2} \prec \cdots \prec N_{\lambda_i} \prec \cdots \ (i < j).$$

Put $M_0 = \bigcup_{i<j} N_{\lambda_i}$. M_0 has the sort $\begin{pmatrix} \lambda_1 & \cdots & \lambda_i & \cdots \\ \varepsilon_1 & \cdots & \varepsilon_i & \cdots \end{pmatrix}$, $i < j$. Let M_1 be any proper conservative elementary extension of M_0. We are going to construct a sequence $\{c_i\}_{i\in\lambda_j}$ ordered as λ_j^* such that $M_0 < c_i < M_1 \setminus M_0$ for all i, and there are no points between M_0 and $\{c_i\}_{i\in\lambda_j}$.

Lemma (#)
Let $M \models$ PA, $\varphi(x)$ be a formula such that $M \models Qx\varphi(x)$ and $\theta(x, \overline{y})$ an arbitrary formula. Then there is a third formula $\psi(x)$ such that:

1. $M \models Qx\psi(x)$;

2. $M \models \forall x\ (\psi(x) \to \varphi(x))$;

3. *for all* $\overline{a} \in M$,
   either: $M \models \exists y\ \forall x > y\ (\psi(x) \to \theta(x, \overline{a}))$
   or: $M \models \exists y\ \forall x > y\ (\psi(x) \to \neg\theta(x, \overline{a}))$.

For proof see [14], p. 97.

Let us enumerate all formulas of L_{PA} as $\theta_0(x, \overline{y}_0),\ \theta_1(x, \overline{y}_1),\ \theta_2(x, \overline{y}_2), \ldots$ We construct a second sequence of formulas $\varphi_0(x), \varphi_1(x), \varphi_2(x), \ldots$ such that for all $i \in \omega$, $M_0 \models Qx\varphi_i(x)$. Let $\varphi_0(x)$ be "$x = x$". If $\varphi_i(x)$ is given satisfying $M_0 \models Qx\varphi_i(x)$ we may apply Lemma (#) to obtain φ_{i+1} such that $M_0 \models Qx\varphi_{i+1}(x),\ M_0 \models \forall x\ (\varphi_{i+1}(x) \to \varphi_i(x))$ and for all $\overline{a} \in M_0$,
either: $M_0 \models \exists y\ \forall x > y\ (\varphi_{i+1}(x) \to \theta_i(x, \overline{a}))$
or: $M_0 \models \exists y\ \forall x > y\ (\varphi_{i+1}(x) \to \neg\theta_i(x, \overline{a}))$.
This process is repeated until $\varphi_i(x)$ is defined for all $i \in \mathbb{N}$.
This collection $\{\varphi_i(x)\}_{i\in\omega}$ is a particular kind of **definable type**, as constructed first by Gaifman [8]. We shall refer to this type in the future as a **Gaifman type** for M_0. (Also, it is possible to make this type indiscernible over sufficiently small parameters, more precisely: if $K \models$ PA and $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are increasing sequences in K, with each a_i, b_j realizing $p(x)$ and $c \in K$ with $\text{Cl}_K(c) < a_i, b_j$ for each $i, j \leq n$, then for each L_{PA}-formula $\theta(\overline{x}, \overline{y})$,

$$K \models \forall \overline{y} < c\ (\theta(\overline{a}, \overline{y}) \leftrightarrow \theta(\overline{b}, \overline{y}).$$

We are going to use this fact later in Theorem 50.)
Let us consider the type

$$p_1(x) = \{x > a \mid a \in M_0\} \cup \{x < b \mid b \in M_1 \setminus M_0\} \cup \{\varphi_i(x)\}_{i\in\mathbb{N}}.$$

Any finite subset of this set of formulas is satisfied in M_1 by a sufficiently large element of M_0 satisfying $\varphi_i(x)$ for large enough i. Let the constant c_1 realise this type in some $N \succ M_1$. Define $M_2 = \text{Cl}_N(M_1 \cup \{c_1\})$.

Now we are going to show that M_2 is a conservative extension of M_0. Consider an arbitrary formula $\theta(u, \overline{b})$, where $\overline{b} = (b_1, \ldots, b_n)$ and b_j is defined

by the formula $\eta_j(v, \overline{a}, \overline{e}, c_1)$, $j = 1, 2 \dots n$, $\overline{a} \in M_0$, $\overline{e} \in M_1 \setminus M_0$.
As M_1 is a conservative extension of M_0 we can find for the formula

$$\delta(x, y, \overline{a}, \overline{e}) = \exists \overline{v} \left(\bigwedge_{j=1}^{n} \eta_j(v_j, \overline{a}, \overline{e}, x) \ \& \ \theta(y, \overline{v}) \right)$$

another formula $\xi(x, y, \overline{g})$ with a parameter $\overline{g} \in M_0$ such that for all $x, y \in M_0$

$$M_0 \models \xi(x, y, \overline{g}) \ \Leftrightarrow \ M_1 \models \delta(x, y, \overline{a}, \overline{e}).$$

The formula $\xi(x, y, y_1, \dots, y_k)$ is $\theta_i(x, y, y_1, \dots, y_k)$ for some $i \in \mathbb{N}$.
We claim that for all $d \in M_0$,

$$M_2 \models \theta(d, b_1, b_2, \dots, b_n) \Leftrightarrow M_0 \models \exists w \ \forall x > w \ (\varphi_{i+1}(x) \rightarrow \xi(x, d, \overline{g})).$$

($\Leftarrow$) If $w_0 \in M_0$ and $M_0 \models \forall x > w_0 \ (\ \varphi_{i+1}(x) \rightarrow \xi(x, d, \overline{g}) \ )$
then, by our choice of ξ, for all $x \in M_0$, $x > w_0$,

$$M_1 \models \varphi_{i+1}(x) \rightarrow \delta(x, d, \overline{a}, \overline{e}).$$

Then, by overspill, there is $a_0 \in M_1 \setminus M_0$ such that

$$M_1 \models \forall \ x \ (\ w_0 < x < a_0 \ \& \ \varphi_{i+1}(x) \rightarrow \delta(x, d, \overline{a}, \overline{e})).$$

Then, as $M_2 \succ M_1$,

$$M_2 \models \forall \ x \ (\ w_0 < x < a_0 \ \& \ \varphi_{i+1}(x) \rightarrow \delta(x, d, \overline{a}, \overline{e})).$$

Then, putting $x = c_1$ we get $M_2 \models \delta(c_1, d, \overline{a}, \overline{e})$,
which means that $M_2 \models \theta(d, \overline{b})$.
($\Rightarrow$) Let $M_2 \models \theta(d, \overline{b})$. By the construction of $\varphi_{i+1}(x)$,
either: $M_0 \models \exists w \ \forall x > w \ (\varphi_{i+1}(x) \rightarrow \xi(x, d, \overline{g}))$
or: $M_0 \models \exists w \ \forall x > w \ (\varphi_{i+1}(x) \rightarrow \neg\xi(x, d, \overline{g}))$
If $M_0 \models \forall \ x > w_0 \ (\varphi_{i+1}(x) \rightarrow \neg\xi(x, d, \overline{g}))$ then,
by our choice of ξ, for all $x \in M_0$, $x > w_0$,

$$M_1 \models \varphi_{i+1}(x) \rightarrow \neg\delta(x, d, \overline{a}, \overline{e}).$$

Again, by overspill,

$$M_1 \models \forall x \ (\ w_0 < x < a_0 \ \& \ \varphi_{i+1}(x) \rightarrow \neg\delta(x, d, \overline{a}, \overline{e})).$$

Then, as $M_2 \succ M_1$,

$$M_2 \models \forall x \ (\ w_0 < x < a_0 \ \& \ \varphi_{i+1}(x) \rightarrow \neg\delta(x, d, \overline{a}, \overline{x})),$$

which is not true, because

$$M_2 \models\ w_0 < c_1 \ \&\ c_1 < a_0 \ \&\ \varphi_{i+1}(c_1) \ \&\ \delta(c_1, d, \overline{a}, \overline{e}).$$

Hence, $M_0 \models \exists\, w\, \forall\, x > w\ (\ \varphi_{i+1}(x) \to \xi(x, d, \overline{g}))$.
So, the new constant did not spoil the conservativeness of our extension M_1.
Again consider the set of formulas

$$p_2(x) = \{x > a \mid a \in M_0\} \cup \{x < b \mid b \in M_2 \setminus M_0\} \cup \{\varphi_i(x)\}_{i \in \mathbb{N}}.$$

It is finitely satisfied in M_2, so we can realise it by c_2 and set
$M_3 = \mathrm{Cl}(M_2 \cup \{c_2\})$. M_3 is still a conservative extension of M_0.
Analogously we can define M_i for all $i < \lambda_j$.
Let $M_{\lambda_j} = \bigcup_{i<\lambda_j} M_i$. M_{λ_j} has the sort $\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_j \\ \varepsilon_1 & \varepsilon_2 & \cdots & 0 \end{pmatrix}$ because $\bigcup_{i<j} N_{\lambda_i}$ is an initial segment of M_{λ_j} and, given a point $a > \bigcup_{i<j} N_{\lambda_i}$ we can find $k < \lambda_j$ such that $a \in M_k$ and observe that there are λ_j distinct points $\{c_i\}_{i>k}$ below a.
If ε_j=1 we take as N_{λ_j} any λ_j^+-like end-extension of M_{λ_j}, otherwise put $N_{\lambda_j} = M_{\lambda_j}$. Having constructed N_{λ_j} for all $j < \mu$, we observe that $\bigcup_{j<\mu} N_{\lambda_j}$ has the required sort. ■

Actually, we have proved even more:

Corollary 14 *For any sort-matrix, every complete theory $T \supset \mathrm{PA}$ and any $M \models T$ such that $\mathrm{card}\, M \leq \lambda_1$, there is $N \succ M$ of this sort such that for every $\eta < \mu$, $I_{\lambda_\eta} \models T$.*

Another corollary of the proof of Theorem 13 will be required later.

Corollary 15
Let $M \models \mathrm{PA}$, $p(x)$ be a Gaifman type for M. Then for any cardinal α, there is an end-extension $N \succ M$, $\mathrm{card}\, N = \max\{\mathrm{card}\, M, \alpha\}$, and a sequence $\{c_i\}_{i\in\alpha}$ of elements of $N \setminus M$ ordered as α^* such that for all $i \in \alpha$, c_i satisfies p and for any $a \in N \setminus M$ there is $i \in \alpha$ such that $c_i < a$.

2.3 Cofinalities

Definition 6
Let $(A, <)$ be a linear order. Then $\Upsilon(A) =$ the least cardinal γ such that there is a nondecreasing sequence of length γ unbounded in A. $\reflectbox{$\Upsilon$}$ — $\text{⅄}(A) = \Upsilon(A^*)$.

Definition 7 *Let* $(A, <)$ *be a linear order. Then* $\mathrm{cf}(A)$, ***cofinality*** *of* A, *is the least cardinal* γ *such that there is a set* $B \subseteq A$ *unbounded in* A, $\mathrm{card}\, B = \gamma$. $\mathrm{lcf}(A)$, ***lower cofinality*** *of* A, *is* $\mathrm{cf}(A^*)$.

Theorem 16

1. *For any linear order* A, $\Upsilon(A) = \mathrm{cf}(A)$, $\text{⅄}(A) = \mathrm{lcf}(A)$.
2. *For any linear order* A, $\mathrm{cf}(A)$ *and* $\mathrm{lcf}(A)$ *are regular.*
3. *For any regular* α, β, *there is a model* $M \models \mathrm{PA}$ *such that* $\mathrm{cf}(M) = \alpha$, $\mathrm{lcf}(M \smallsetminus \mathbb{N}) = \beta$.
4. *Let* $\mathrm{card}\, M \le \varkappa$. *Then for any regular* $\alpha, \beta \le \varkappa$, *there is a* $\varkappa$*-dense* $N \succ M$ *with* $\mathrm{cf}(N) = \alpha$, $\mathrm{lcf}(N \smallsetminus \mathbb{N}) = \beta$.
5. *If* M *is* λ*-like then* $\Upsilon(M) = \mathrm{cf}(\lambda)$.
6. *If* $M \models \mathrm{PA}$, $\mathrm{card}\, M = \lambda$ *and* M *is saturated then* λ *is regular and* $\mathrm{cf}(M) = \lambda$.
7. *If* $M \models \mathrm{PA}$ *is resplendent then* $\mathrm{cf}(M) = \mathrm{lcf}(M \smallsetminus \mathbb{N})$.
8. *Let* $M \models \mathrm{PA}$. *For any regular* α *and any cardinal* $\varkappa \ge \alpha$, *there is a resplendent* $N \succ M$, $\mathrm{card}\, N = \varkappa$ *such that* $\mathrm{cf}(N) = \alpha$.

Proof

1. Firstly, $\Upsilon(A) \ge \mathrm{cf}(A)$, because an unbounded sequence is also an unbounded set. Let $B \subseteq A$ be unbounded in A, $\mathrm{card}\, B = \mathrm{cf}(A)$ and construct an increasing sequence $\{b_i\}_{i<\gamma\le \mathrm{cf}\, A}$ unbounded in A. Let $b_0 \in B$ be arbitrary. Suppose we already have an increasing sequence $\{b_i\}_{i<\delta}$. If $\{b_i\}_{i<\delta}$ is unbounded in A, put $\gamma = \delta$, otherwise choose arbitrarily $b_\delta \in B$, $b_\delta > \{b_i\}_{i<\delta}$. As $\mathrm{card}\, B = \mathrm{cf}(A)$, $\gamma \le \mathrm{cf}(A)$. Hence $\Upsilon(A) \le \mathrm{cf}(A)$.
2. If $(a_i)_{i<\alpha}$ is a nondecreasing unbounded sequence in A and $(\beta_j)_{j<\mathrm{cf}(\alpha)}$ is a nondecreasing unbounded sequence in α then $(a_{\beta_j})_{j<\mathrm{cf}(\alpha)}$ is a nondecreasing unbounded $\mathrm{cf}(\alpha)$-sequence in A.
3. The usual chain-argument.
4. The usual chain-argument.

5. Firstly, there is an increasing cf(λ)-sequence, $(a_i)_{i<\mathrm{cf}(\lambda)}$ unbounded in M (the same proof as the proof of (1)). Suppose there is an unbounded sequence $(b_i)_{i<\gamma}$, $\gamma < \mathrm{cf}(\lambda)$. Then $f\colon \gamma \to \mathrm{cf}(\lambda)$ defined as $f(i) = \max j\ a_j < b_i$ is unbounded in cf(λ), which contradicts regularity of cf(λ).

6. If there was a sequence $\{a_i\}_{i<\gamma}$, $\gamma < \lambda$ unbounded in M, then the type $\{x > a_i\}_{i<\gamma}$ would have to be realized by saturation.

7. The statement $\Upsilon(M) = \text{⅄}(M)$ is expressed by the following Σ^1_1 formula: $\exists f\ \exists a \forall xy > a(f(x) < a)\ \&\ (x < y \to f(y) < f(x))\ \&\ \forall b < a\ \exists x > a\ (f(x) < b)$, which is realised in all countable models.

8. A chain-argument, using the fact that any model has a resplendent elementary extension of the same cardinality. ■

Lemma 17 *Let $M \models \mathrm{PA}$ be resplendent, I, J be any two initial segments of M. Then $(I,<) \cong (J,<) \Leftrightarrow \Upsilon(I) = \Upsilon(J)$.*

It follows that $(I,<) \cong (M,<) \Leftrightarrow \Upsilon(I) = \Upsilon(M)$.

Proof
Let a_1 and a_2 belong to different $\mathbb{Z}$-blocks and b_1 and b_2 belong to different $\mathbb{Z}$-blocks. The Σ^1_1-statement

$$\exists\, f : [a_1, a_2] \ \to\ [b_1, b_2], \quad f \text{ is an order-isomorphism}$$

is realized in $\mathrm{Cl}(a_1, a_2, b_1, b_2)$, hence is realized in M, hence $[a_1, a_2]$ and $[b_1, b_2]$ are order-isomorphic. In particular, given an arbitrary $a \in M \setminus \mathbb{N}$, any infinite segment $[b_1, b_2]$ is order-isomorphic to $[0, a]$.

Denote the order-type of $[0, a)$ by $(A,<)$. Let $\Upsilon(I) = \alpha$, $(a_i)_{i\in\alpha}$ be an increasing sequence unbounded in I, $a_0 = 0$. Then $(I,<) = \Sigma_{i\in\alpha}[a_i, a_{i+1}) \cong \Sigma_{i\in\alpha} A \cong \alpha A$. ■

Also, we have obtained the order-type of our resplendent model M in terms of $(A,<)$:

$$(M,<) \ \cong\ \Upsilon(M)\, A.$$

2.4 Weight

Definition 8 *Let I be a linearly ordered set. An **interval** in I is an infinite set of the form (u, v), where $u, v \in I$.*

Definition 9 *The **weight** of I, $w(I)$, is the supremum of cardinals γ such that there is a disjoint collection of intervals of I, which has cardinality γ.*

Examples: $w(\mathbb{Q}) = \aleph_0,\ w(\mathbb{R}) = \aleph_0,\ w(\mathbb{R}^2) = 2^{\aleph_0}$.

Definition 10 *Let I be a set, $(A, <)$ a linear order, $0 \in A$ an arbitrary point, $f : I \to A$. Then* $\operatorname{supp}(f) = \{x \in I \mid f(x) \neq 0\}$.

Definition 11 *If $(I, <), (A, <)$ are linear orders, $0 \in A$, then* $(A, 0)^{<I} = \{f : I \to A \mid \operatorname{supp}(f) \neq \varnothing, \operatorname{supp}(f)$ *is finite*$\}$ *with order defined lexicographically: $f < g$ if for $a = \min\{i \mid f(i) \neq g(i)\}$, $f(a) < g(a)$.*

Elements of I will sometimes be called **coordinates** or **coordinate axes**. The linear orders $(A, 0)^{<I}$ will play important role later, in Chapter 3.
If $(A, <)$ is 2-homogeneous, (i.e. for all $a_1, a_2 \in A$, there is $g \in \operatorname{Aut}(A, <)$ such that $g(a_1) = a_2$) or I is finite then it is easy to see that for any $0_1, 0_2 \in A$,

$$(A, 0_1)^{<I} \ \cong\ (A, 0_2)^{<I}.$$

Also, $(\{0, 1\}, 0)^{<\omega} \ncong (\{0, 1\}, 1)^{<\omega}$ because $(\{0, 1\}, 0)^{<\omega}$ has least element but no last element while $(\{0, 1\}, 1)^{<\omega}$ has last element but no least element. The following question is relevant to the material of the next chapter.

Question 3 *Are there dense linear orders $(I, <)$, $(A, <)$, without end-points and $0_1, 0_2 \in A$ such that*

$$(A, 0_1)^{<I} \ \ncong\ (A, 0_2)^{<I} \ ?$$

Theorem 18 *Let $M \models \mathrm{PA}$.*

1. *If I is a nonstandard initial segment of M then $w(I) = \operatorname{card}(I)$.*
2. *For any $i, j < \mu$ such that $i > j$, $w(I_{\lambda_i}/I_{\lambda_j}) = \operatorname{card}(I_{\lambda_i}/I_{\lambda_j}) = \operatorname{card} I_{\lambda_i}$.*
3. *Let $(X, <)$ be a linear order, $n \in \mathbb{N}$. Let M be λ-dense, $\lambda > w(X)$, I be a nonstandard initial segment of M. Then I is not order-embeddable into $(X^n, <)$.*
4. *With the same conditions as in (3) above, I is not embeddable into $(X^{<\omega^*}, <)$.*

Proof

1. If I is card I-like then, by Theorem 12 (5), I is closed under multiplication. Then, given any $a \in I \setminus \mathbb{N}$, we observe that $\{(ia,(i+1)a)\}_{i\in I}$ is a collection of cardinality card I of disjoint intervals of I.

   If I is not card I-like then let us show that there is $b \in I$ such that $\operatorname{card}[0,b] = \operatorname{card} I$ and $b^2 \in I$. By Theorem 12 (5), $\bigcup_{\mu<\operatorname{card} I} I_\mu =: K$ is closed under multiplication. Let $c \in I \setminus K$. For all $x \in K$, $x^2 \in K$, in particular $x^2 < c$. Hence, by overspill, there is $b \in I \setminus K$ such that $b^2 < c$.

   Now $\{(0,b),(b,2b),\ldots,((b-1)b,b^2)\}$ is a collection of disjoint intervals of cardinality $\operatorname{card}[0,b] = \operatorname{card} I$.

2. If $a \in I_{\lambda_i} \setminus I_{\lambda_j}$ then for any $k \in I_{\lambda_i}$, ka and $(k+1)a$ belong to different λ_j-cells, hence $\{(ka,(k+1)a)\}_{k\in I_{\lambda_i}}$ is a disjoint system of intervals of $I_{\lambda_i}/I_{\lambda_j}$ which has cardinality card I_{λ_i}.

3. For $n = 1$, see part (1) above. Suppose for all nonstandard initial segments I, I is not embeddable into $(X^n,<)$. Let J be a nonstandard initial segment embeddable into $(X^{n+1},<)$ and $f\colon J \to (X^{n+1},<)$ be the order-embedding. For every $c \in X$ introduce $A_c = \{x \in J \mid f(x) = (c,\ldots)\}$. Obviously, each A_c is convex. Let us show that for all $c \in X$, $\operatorname{card} A_c \le w(X)$. Suppose to the contrary that $\operatorname{card} A_c > w(X)$ for some $c \in X$. Let $a \in A_c$. Then one of the sets $A_1 = \{x \in A_c \mid x < a\}$ and $A_2 = \{x \in A_c \mid x > a\}$ has cardinality card A_c.

   $$\text{Let } B = \begin{cases} \{x \mid a - x \in A_1\} & \text{if } \operatorname{card} A_1 = \operatorname{card} A_c \\ \{x \mid a + x \in A_2\} & \text{otherwise.} \end{cases}$$

   We observe that B is an initial segment of cardinality $\operatorname{card} A_c > w(X)$ embeddable into X^n. Contradiction. Hence $\operatorname{card} A_c \le w(X)$ for all $c \in X$.

   As M is λ-dense, J is not card J-like, hence there is $b \in J$ such that $b^2 \in J$ and $\operatorname{card}[0,b] = \operatorname{card} J = \lambda > w(X) \ge \operatorname{card} A_c$. Hence, for all $i,j \in J$, $i \ne j$ implies $ib \in A_{c_i}$, $jb \in A_{c_j}$ for $c_i \ne c_j$ and each interval (c_i,c_j) is infinite. The collection $\{(c_i,c_{i+1})\}_{i\in[0,b]}$ of intervals of X is disjoint and has cardinality $\operatorname{card} J = \lambda > w(X)$. Contradiction.

4. Let $f\colon I \to (X^{<\omega^*},<)$ be an order-embedding. Let $x_1,x_2 \in I$ be such that $\operatorname{card}[x_1,x_2] = \operatorname{card} I$. Let $n \in \omega$ be the first n such that $f(x_1) \ne f(x_2)$. Then $[f(x_1),f(x_2)]$ is embedded into X^n. This contradicts (3). ■

Notice that clauses (3) and (4) of Theorem 18 can be applied not only to models of PA themselves but to all factor-sets $M/\mathbb{Z}$, M/I_λ as well, by the same proof.

An easy modification of the proof of Theorem 18 produces the following corollary.

Corollary 19 *No uncountable model of* PA *is embeddable in any of the following linear orders:* $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{R}^{<\omega^*}$, ω^ω, $\omega_1^*\mathbb{Q}$, $\omega_1^*\mathbb{R}^n$.

Clearly, it would be nice to generalise this result to other ordinals.

Question 4 *Is it true that if* γ *is an ordinal and* $\operatorname{card} M > w(X)$ *then* $(M, <)$ *is not embeddable into* $(X^{<\gamma^*}, <)$*?*

Question 5
Is there an uncountable model of PA *order-embeddable into* $(\mathbb{R}^{<\omega}, <)$*?*

If there is no such model then we can suggest the following conjecture.

Question 6
If $(I, <)$ *is a countable linear order then the following are equivalent:*

1. *For every linear order* $(A, <)$ *there is an uncountable* $M_A \models$ PA *embeddable into* $A^{<I}$.

2. I *contains a copy of* $\mathbb{Q}$.

Chapter 3 gives a partial answer to the (2⇒1)-part of this problem. Theorem 18 proves that there are no such models for I finite or $I = \omega^*$.

Chapter 3

Models generated by indiscernibles

Indiscernibles are one of the major methods of constructing examples of models of PA developed so far. In this chapter we study order-types of models of PA generated by indiscernibles.

Let T be an arbitrary completion of PA, $(C, <)$ a linearly ordered set. By the Ehrenfeucht–Mostowski Theorem (see [3], or [11], section 11.2), there is a model $M \models T$ such that:

1. $(C, <)$ is order-embeddable into M (so, we can identify $(C, <)$ with its image);

2. for all $y \in M$, $y = t(c_1, \ldots c_n)$ for some $c_1, \ldots c_n \in C$ and some Skolem term t;

3. $M \models \varphi(c_{i_1}, \ldots, c_{i_n}) \leftrightarrow \varphi(c_{j_1}, \ldots, c_{j_n})$ for all L_{PA} -formulas $\varphi(x_1, \ldots x_n)$ and all $c_{i_1} < \ldots < c_{i_n}$, $c_{j_1} < \ldots < c_{j_n}$ in C.

The set $p = \{\varphi \mid \text{there are } c_1, \ldots c_n \in C \text{ such that } c_1 < \ldots < c_n \text{ and } M \models \varphi(c_1, \ldots c_n)\}$ is called the **indiscernible type** satisfied by C. C is called an **indiscernible sequence** in M. The model M is called a **model generated by indiscernibles ordered as** C or **Ehrenfeucht-Mostowski model** and is denoted by $\mathrm{EM}(C, p)$. For any $(C, <)$ and any indiscernible type p, $\mathrm{EM}(C, p)$ is uniquely defined.

Theorem 20
If $(C, <)$ is a dense linear order without end-points and there is an order preserving $f : (C, <) \longrightarrow (C^*, <)$ and a point $0 \in C$ such that $f(0) = 0$, then for any indiscernible type p, $\mathrm{EM}(C, p)$ is order-embeddable into $(C, 0)^{<\mathbb{Q}}$.

Notice that if C is not dense or there is no function $f\colon C^* \to C$ fixing a point then we can embed C into some dense linear order D for which there is $f\colon D^* \to D$ which fixes one point. This will induce an embedding of $\mathrm{EM}(C,p)$ into $\mathrm{EM}(D,p)$ and hence, by Theorem 20, into $(D,0)^{<\mathbb{Q}}$. Thus, every model generated by indiscernibles is embeddable into $(D,0)^{<\mathbb{Q}}$ for some D.

The examples of C's I had in mind while proving the Theorem are $\mathbb{R}$, $\mathbb{Q}$ or, more generally, any densely ordered group. This influenced my notation for the rest of the chapter.

Notation. For any $x \in C$ denote $f(x)$ by $-x$. If $x_1, x_2, x_3 \in C$, $x_1 < x_2 < x_3$ are the points we are considering at the moment then $x_2 + \varepsilon$ is any point between x_1 and x_3 which is different from x_2. Sometimes we require $x_2 + \varepsilon \in (x_2, x_3)$. Let us pick any point $a \in C$, $a > 0$ and denote it by 1. Another symbol '1' will appear in the following notation. For every $x \in C$, let $1x = x$, $0x = 0$, $-1x = -x$.

3.1 Analysis of a single term

Consider a Skolem term $t(v_1, \ldots, v_n)$ of T and points $x_1, \ldots, x_n \in C$. When we write $t(x_1, \ldots, x_n)$ we shall always implicitly assume $x_1 < x_2 < \ldots < x_n$. If $x_1, \ldots, x_n$ are ordered differently then reorder them in the right order and consider $t(v_1, \ldots, v_n)$ with its variables reordered (and v_i and v_j such that $x_i = x_j$ substituted by the same new variable) as a new term. This is done for the sake of indiscernibility as we shall see below. Now, let $V_t = \{t(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \in C,\ x_1 < \cdots < x_n\}$. The question for this section will be: "What is the order-type of V_t in M"?

Definition 12
*Let $1 \le i \le n$. We say that t i-**increases** if for any $x_i + \varepsilon \in (x_i, x_{i+1})$,*

$$t(x_1, \ldots, x_i + \varepsilon, \ldots, x_n) > t(x_1, \ldots, x_i, \ldots, x_n).$$

*We define t i-**decreases** or is i-**constant** similarly.*

This definition does not depend on the choice of $x_1, \ldots, x_n \in C$, by indiscernibility. Also, it does not depend on the choice of $x_i + \varepsilon$ because $x_1 < \ldots < x_i < x_i + \varepsilon_1 < x_{i+1} < \ldots < x_n$ and $x_1 < \ldots < x_i < x_i + \varepsilon_2 < x_{i+1} < \ldots < x_n$ are similar distributions, hence, by indiscernibility of C,

$$t(x_1, \ldots, x_i + \varepsilon_1, \ldots x_n) > t(x_1 \ldots, x_i, \ldots, x_n) \longleftrightarrow$$

$$\longleftrightarrow t(x_1 \ldots, x_i + \varepsilon_2, \ldots, x_n) > t(x_1, \ldots, x_i, \ldots, x_n).$$

Definition 13 *Let $1 \leq i \neq j \leq n$. We say that j is* **more valuable in** t *than i and write $i \lhd j$ if, given $x_1, \ldots, x_n \in C$, $x_1 < \ldots < x_n$,*

$$t(x_1, \ldots, x_i, \ldots, x_j + \delta, \ldots, x_n) > t(x_1, \ldots, x_i + \varepsilon, \ldots, x_j, \ldots, x_n) \longleftrightarrow$$

$$\longleftrightarrow t(x_1, \ldots, x_i, \ldots, x_j + \delta, \ldots, x_n) > t(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n)$$

for all $x_j + \delta \in (x_{j-1}, x_{j+1}) \smallsetminus \{x_j\}$, $x_i + \varepsilon \in (x_{i-1}, x_{i+1})$ such that $(x_j, x_j + \delta)$ and $(x_i, x_i + \varepsilon)$ do not overlap.

Again the definition does not depend on the choice of $x_i + \varepsilon$ and $x_j + \delta$.

Lemma
Let $1 \leq i \neq j \leq n$. Then precisely one of the following holds:

1. *$i \lhd j$*
2. *$j \lhd i$*
3. *t is both i-constant and j-constant.*

Proof If t is i-constant, but not j-constant then $i \lhd j$. Without loss of generality assume t is i-increasing and j-increasing. (The proof for the other three cases is the same.) Let $x_i + \varepsilon \in (x_i, x_{i+1})$, $x_j + \delta \in (x_j, x_{j+1})$.

If $t(x_1, \ldots, x_i, \ldots, x_j + \delta, \ldots, x_n) > t(x_1, \ldots, x_i + \varepsilon, \ldots, x_j, \ldots, x_n)$ then $j \rhd i$.

If $t(x_1, \ldots, x_i, \ldots, x_j + \delta, \ldots, x_n) < t(x_1, \ldots, x_i + \varepsilon, \ldots, x_j, \ldots, x_n)$ then $j \lhd i$.

If $t(x_1, \ldots, x_i, \ldots, x_j + \delta, \ldots, x_n) = t(x_1, \ldots, x_i + \varepsilon, \ldots, x_j, \ldots, x_n)$ then, by indiscernibility $t(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) = t(x_1, \ldots, x_i + \varepsilon, \ldots, x_j, \ldots, x_n)$, hence t is i-constant. By the same argument, t is j-constant. ■

So, "$\lhd$" is a linear order on the set $\{i \mid t \text{ is not } i - \text{constant}\}$. Let us order $\{i \mid t \text{ is } i - \text{constant}\}$ arbitrarily to extend "$\lhd$" to a linear order on $\{i \mid 1 \leq i \leq n\}$, and let the permutation τ of $\{1, \ldots, n\}$ be such that $\tau 1 \rhd \tau 2 \rhd \cdots \rhd \tau n$.

In order to illustrate the definition of "j is more valuable than i" let us study two examples. We shall always identify $x_i \in C$ with its value in M. Don't get confused by different $+$'s!

1. If $M \models x_1 + x_2 < x_3$ for some $x_1, x_2, x_3 \in C$ with $x_1 < x_2$, then for $t_1(x_1, x_2) = x_1 + x_2$, $2 \rhd 1$, because the shift $x_2 \to x_2 + \varepsilon$ is equivalent to the shift $x_2 \to x_3$, which increases our term in spite of any shifts of x_1 inside $(-\infty, x_2)$, by indiscernibility. (More formally: by indiscernibility,

   $$x_1 + x_2 < (x_1 - \delta) + (x_2 + \varepsilon) \iff x_1 + x_2 < (x_1 - \delta) + x_3.$$

   The last inequality is true by our assumption $x_1 + x_2 < x_3$.)

2. If $M \models x_3^{x_2} < x_4 \,\&\, x_4^{x_3} > x_4^{x_2} + x_1$ for some $x_1 < x_2 < x_3 < x_4$ in C, then for $t_2(x_1, x_2, x_3) = x_3^{x_2} + x_1$, $3 \triangleright 2 \triangleright 1$. Here, x_3 is the main variable because $(x_3 + \varepsilon)^{x_2} + x_1$ is a point between $x_3 + \varepsilon$ and $(x_3 + \varepsilon, +\infty) = \{y \in C \mid x_3 + \varepsilon < y\}$, while both $x_3^{x_2+\delta} + x_1$ and $x_3^{x_2}+(x_1+\delta)$ are between x_3 and $(x_3, +\infty)$. $2\triangleright 1$ because $x_3^{x_2+\varepsilon} > x_3^{x_2}+x_1$, and, hence, $x_3^{x_2+\varepsilon} + x_1 > x_3^{x_2} + (x_1 + \delta)$.

The conditions $x_1 + x_2 < x_3$, $x_3^{x_2} < x_4$, $x_4^{x_3} > x_4^{x_2} + x_1$ in these two examples are clearly satisfied by $x_1 < x_2 < x_3 < x_4$ when the indiscernible sequence C is cofinal in M. It is unclear how the terms $t_1(x_1, x_2)$ and $t_2(x_1, x_2, x_3)$ behave if C is not cofinal in M.

Denote $\{(x_1, \ldots, x_n) \in C^n \mid x_1 < x_2 < \cdots < x_n\}$ by D_n.
Introduce the function $s : \{1, \ldots, n\} \to \{-1, 0, 1\}$ as follows.

$$s(i) = s_i = \begin{cases} -1 \text{ if } t \text{ is } \tau i - \text{decreasing} \\ \;\;0 \text{ if } t \text{ is } \tau i - \text{constant} \\ \;\;1 \text{ if } t \text{ is } \tau i - \text{increasing.} \end{cases}$$

Let $E_t = \{(z_{\tau 1}, z_{\tau 2}, \ldots, z_{\tau n}) | \; z_{\tau i} = s_i x_{\tau i},\; (x_1, \ldots, x_n) \in D_n\}$.
Define $f\colon V_t \to E_t$ as

$$f(t(x_1, x_2, \ldots, x_n)) = (z_{\tau 1}, z_{\tau 2}, \ldots, z_{\tau n}).$$

Now it is easy to prove (repeatedly applying definitions 12 and 13) that this function is an order-isomorphism between V_t and E_t.

3.2 Separated and interpenetrating terms

Let $t_1(u_1, u_2, \ldots, u_n)$ and $t_2(v_1, v_2, \ldots, v_m)$ be two Skolem terms of T in the language $L_{\mathrm{PA}} \cup C$ with all their free variables shown such that
$\tau 1 \triangleright \tau 2 \triangleright \cdots \triangleright \tau n$ in t_1 and $\sigma 1 \triangleright \sigma 2 \triangleright \cdots \triangleright \sigma m$ in t_2.

Definition 14 *Call t_1 and t_2 **separated** if either*

$$\text{for all } \overline{x} \in D_n,\ \overline{y} \in D_m, \quad M \models t_1(\overline{x}) > t_2(\overline{y})$$

$$\text{or for all } \overline{x} \in D_n,\ \overline{y} \in D_m, \quad M \models t_1(\overline{x}) < t_2(\overline{y})$$

*and **interpenetrating** otherwise.*

Notice that "being interpenetrating" is an equivalence relation on the set of all Skolem terms of T.

Lemma 21 *If t_1 and t_2 are interpenetrating and not constant then*

$$\textit{either } (x_{\tau 1} > y_{\sigma 1} \to t_1(\overline{x}) > t_2(\overline{y})) \;\&\; (x_{\tau 1} < y_{\sigma 1} \to t_1(\overline{x}) < t_2(\overline{y}))$$

$$\textit{or } (x_{\tau 1} > y_{\sigma 1} \to t_1(\overline{x}) < t_2(\overline{y})) \;\&\; (x_{\tau 1} < y_{\sigma 1} \to t_1(\overline{x}) > t_2(\overline{y})).$$

Proof

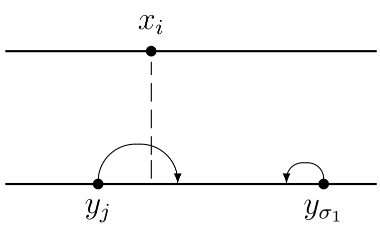
Assume for the moment that t_1 and t_2 are i-increasing for all i.
Suppose there are tuples $(x_1, \ldots, x_n) \in D_n$ and $(y_1, \ldots, y_m) \in D_m$ such that

$$x_1 < y_1 < \ldots < x_{\tau 1} < \ldots < y_{\sigma 1} < \ldots < x_n \qquad (*)$$

and $t_1(x_1, \ldots x_n) > t_2(y_1, \ldots y_m)$ and tuples $(x'_1, \ldots, x'_n) \in D_n$, $(y'_1, \ldots, y'_m) \in D_m$ such that

$$x'_1 < \ldots < y'_{\sigma 1} < \ldots < x'_{\tau 1} < \ldots < x'_n \qquad (**)$$

and still $t_1(x'_1, \ldots, x'_n) > t_2(y'_1, \ldots, y'_m)$.
We are going to show that now t_1 and t_2 will be separated.
Let $x_i < y_{\sigma_1}$ and y_j be the element in the sequence $(*)$ immediately preceding x_i. We are going to swap x_i and y_j preserving the inequality $t_1 > t_2$. Let us shift $y_j \to x_i+\varepsilon$, $y_{\sigma_1} \to y_{\sigma_1}-\varepsilon$, where $x_i < x_i+\varepsilon < x_{i+1}$, $y_{\sigma_1 -1} < y_{\sigma_1}-\varepsilon < y_{\sigma_1}$.



Doing these shifts we are of course using the density of C. As y_{σ_1} is the main variable of t_2, t_2 will decrease after this shift and $t_1 > t_2$ will still be true. By some sequence of such shifts we can obtain any distribution of variables of t_1 and t_2 with $x_{\tau 1} < y_{\sigma 1}$ preserving the inequality $t_1 > t_2$. The same argument for the distribution $(**)$ gives us that $t_1 > t_2$ for the rest of the distributions. The proof would be analogous if for some i our terms were i-decreasing. ■

Thus, if $t_1(u_1, \ldots, u_n)$ and $t_2(v_1, \ldots, v_m)$ are interpenetrating and we want to compare the values

$$t_1(x_1, \ldots, x_n) \;\text{ and }\; t_2(y_1, \ldots, y_m)$$

(where $(x_1, \ldots, x_n) \in D_n$, $(y_1, \ldots, y_m) \in D_m$) then we simply compare their main variables $x_{\tau 1}$ and $y_{\sigma 1}$. But what happens if $x_{\tau 1} = y_{\sigma 1} =: b$? (We shall denote the fact that $x_{\tau 1}$ in t_1 and $y_{\sigma 1}$ in t_2 are equal to the same element $b \in C$ by writing $t_1(x_1, \ldots, x_n \mid x_{\tau 1} = b)$ and $t_2(y_1, \ldots, y_m \mid y_{\sigma 1} = b)$.) If $t_1(\overline{x} \mid x_{\tau 1} = b)$ and $t_2(\overline{y} \mid y_{\sigma 1} = b)$ are separated then the process of comparing has finished. But if $t_1(\overline{x} \mid x_{\tau 1} = b)$ and $t_2(\overline{y} \mid y_{\sigma 1} = b)$ are still interpenetrating then we shall have to compare their second most valuable variables, etc. We continue doing so until either $t_1(\overline{x} \mid x_{\tau 1} = b_1, \ldots, x_{\tau m} = b_m)$ and $t_2(\overline{y} \mid y_{\sigma 1} = b_1, \ldots, y_{\sigma m} = b_m)$ become separated at some stage m or we run out of variables. Lemma 22 (1) and (3) proves the inductive step of this construction.

Lemma 22
Consider $t_1(\overline{x} \mid x_{\tau 1} = b_1, \ldots, x_{\tau m} = b_m)$ *and* $t_2(\overline{y} \mid y_{\sigma 1} = b_1, \ldots, y_{\sigma m} = b_m)$, *where* $b_1, \ldots, b_m \in C$ *and* $b_i < b_j \Leftrightarrow x_{\tau i} < x_{\tau j}$. *Then the following hold.*

1. *If* t_1 *is* $\tau(m+1)$*-increasing and* t_2 *is* $\sigma(m+1)$*-decreasing then* $t_1(\overline{x} \mid x_{\tau i} = b_i, i \leq m)$ *and* $t_2(\overline{y} \mid y_{\sigma i} = b_i, i \leq m)$ *are separated.*

2. *If* $x_{\tau(m+1)}$ *and* $y_{\sigma(m+1)}$ *belong to different intervals* (b_{l_1}, b_{k_1}) *and* (b_{l_2}, b_{k_2}) *then* $t_1(\overline{x} \mid x_{\tau i} = b_i, i \leq m)$ *and* $t_2(\overline{y} \mid y_{\sigma i} = b_i, i \leq m)$ *are separated.*

3. *If* $t_1(\overline{x} \mid x_{\tau i} = b_i, i \leq m)$ *and* $t_2(\overline{y} \mid y_{\sigma i} = b_i, i \leq m)$ *are interpenetrating then*

$$(x_{\tau(m+1)} > y_{\sigma(m+1)} \rightarrow t_1 > t_2) \ \& \ (x_{\tau(m+1)} < y_{\sigma(m+1)} \rightarrow t_1 < t_2)$$

   iff t_1 *and* t_2 *are* $\tau(m+1)$*- and* $\sigma(m+1)$*-increasing, and*

$$(x_{\tau(m+1)} > y_{\sigma(m+1)} \rightarrow t_1 < t_2) \ \& \ (x_{\tau(m+1)} < y_{\sigma(m+1)} \rightarrow t_1 > t_2)$$

   iff they are $\tau(m+1)$*- and* $\sigma(m+1)$*-decreasing.*

Proof
1. Let $t_1 < t_2$ for some distribution of variables $x_{\tau i}$, $y_{\sigma j}$, $(i, j \geq m+1)$ with $x_{\tau(m+1)} < y_{\sigma(m+1)}$. As t_2 is $\sigma(m+1)$-decreasing, the shift $y_{\sigma(m+1)} \rightarrow x_{\tau(m+1)} - \varepsilon$ will only increase t_2, and, hence, preserve the inequality $t_1 < t_2$. Now, let $y_{\sigma j} < x_{\tau i}$, $(i, j \geq m+1)$ be two neighbouring variables belonging to the same interval (b_l, b_k).
If $j \neq m+1$ then the shifts $y_{\sigma j} \rightarrow x_{\tau i} + \varepsilon$, $y_{\sigma(m+1)} \rightarrow y_{\sigma(m+1)} - \varepsilon$,
and if $j = m+1$ then the shifts $x_{\tau_i} \rightarrow y_{\sigma j} - \varepsilon$, $x_{\tau(m+1)} \rightarrow x_{\tau(m+1)} - \varepsilon$
will swap them preserving the inequality $t_1 < t_2$. By some composition of such shifts we can obtain any distribution of $x_{\tau i}, y_{\sigma_j}$ $(i, j \geq m+1)$.

2. Again we can swap any two neighbouring variables $x_{\tau i}$ and $y_{\sigma j}$ which belong to the same (b_l, b_k), preserving the relation between t_1 and t_2.

3. Let t_1 and t_2 be $\tau(m+1)$- and $\sigma(m+1)$-decreasing respectively. (If they were both increasing the proof would be the same.)
Assume $x_{\tau(m+1)} < y_{\sigma(m+1)} \to t_1 < t_2$, and we shall get a contradiction.
As t_1 is $\tau(m+1)$-decreasing, the shift $x_{\tau(m+1)} \to x'_{\tau(m+1)} = y_{\sigma(m+1)} + \varepsilon$ decreases t_1 and, hence, preserves the inequality $t_1 < t_2$.
Thus, we have two distributions of $x_{\tau i}, y_{\sigma j},\ (i, j \geq m+1)$

$$x_1 < \ldots < b_l < \ldots < x_{\tau(m+1)} < \ldots < y_{\sigma(m+1)} < \ldots < b_k < \ldots < y_m$$

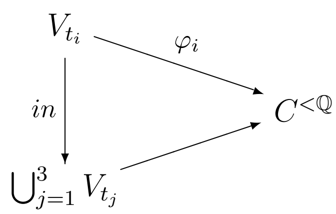
$$\text{and } x_1 < \ldots < b_l < \ldots < y_{\sigma(m+1)} < \ldots < x'_{\tau(m+1)} < \ldots < b_k < \ldots < y_m$$

such that $t_1 < t_2$ for both of them.
Now, again, swapping any two $x_{\tau i}$ and $y_{\sigma j}$, $(i, j \geq m+1)$, which belong two the same (b_{l_1}, b_{k_1}) and preserving $t_1 < t_2$, we get that $t_1 < t_2$ for all distributions, and, hence, $t_1(\overline{x} \mid x_{\tau i} = b_i, i \leq m)$ and $t_2(\overline{y} \mid y_{\sigma i} = b_i, i \leq m)$ are separated. Contradiction. ■

Notice that Lemma 21 is just a particular case of Lemma 22 (3) (when $m = 0$) but I had to include it for better understanding.

3.3 Embedding into $(C, 0)^{<\mathbb{Q}}$

Before describing the general construction, let us study an example. Let $p \supset \{x_1^2 < x_2\}$, $t_1(x) = 0$, $t_2(x) = x$, $t_3(x) = x^2$. We define for all $i \in \{1, 2, 3\}$ embeddings $\varphi_i : V_{t_i} \to C^{<\mathbb{Q}}$ so that the diagram



would commute. The term $t_1(x)$ will be forever separated from all other terms and will be less than them. For all $x \in C$, put $\varphi_1(t_1(x)) =$

$$\begin{pmatrix} \ldots & \underset{100}{1} & \ldots & \underset{101}{0} & \ldots & \underset{102}{1} & \ldots \end{pmatrix}.$$

By this notation we mean the element $f \in C^{<\mathbb{Q}}$,

$$f(q) = \begin{cases} 1 \text{ if } q = 100 \text{ or } q = 102 \\ 0 \text{ if } q = 101 \\ 0 \text{ otherwise.} \end{cases}$$

The term $t_2(x)$ is separated from $t_1(x)$ because for all $x, y \in C$, $M \models t_2(x) > t_1(y)$. We choose a coordinate coming earlier than 100, say $d_1 = 20$, and put $g(20) = 1$ so that $f < g$. Define $\varphi_2(t_2(x)) =$

$$\begin{pmatrix} \dots & 1 & \dots & x & \dots & 1 & \dots \\ & 20 & & 30 & & 31 & \end{pmatrix}.$$

The term $t_2(x_1)$ is 1-increasing, hence, by the analysis of a single term, φ_2 is an order-embedding.

The terms $t_3(x)$ and $t_2(x)$ are interpenetrating, but, once the main variable is fixed (by comparing the main variables, we appeal to Lemma 21), $t_3(x \mid x = b)$ and $t_2(x \mid x = b)$ become separated because $b^2 > b$. Define $\varphi_3(t_3(x)) =$

$$\begin{pmatrix} \dots & 1 & \dots & x & \dots & 1 & \dots \\ & 20 & & 30 & & 30\frac{1}{2} & \end{pmatrix}.$$

Putting the value $h(30\frac{1}{2}) = 1$ makes $t_3(x \mid x = b)$ and $t_2(x \mid x = b)$ separated because $30\frac{1}{2}$ comes earlier than 31.

Now, let us repeat this process carefully for all Skolem terms of T. Let $t_1(x_1, \dots, x_{n_1}), \dots, t_i(x_1, \dots, x_{n_i}), \dots$ be an enumeration of all Skolem terms of T. Let $A_1 = \{d_1 < a_1 < \dots < d_{n_1} < a_{n_1} < d_{n_1+1}\} \subset \mathbb{Q}$. Define $\varphi_1(t_1(x_1, \dots x_{n_1})) =$

$$\begin{pmatrix} 1 & s_1 x_{\tau 1} & 1 & \dots & 1 & s_{n_1} x_{\tau n_1} & 1 \\ d_1 & a_1 & d_2 & \dots & d_{n_1} & a_{n_1} & d_{n_1+1} \end{pmatrix}.$$

Coordinates d_i will be used for separation, a_i for interpenetration. Let $\tau_i \in S_{n_i}$ be the permutation such that $\tau_i 1 \rhd \tau_i 2 \rhd \dots \rhd \tau_i n_i$ in t_i and

$$s_j^i = \begin{cases} 1 \text{ if } t_i \text{ is } \tau_i j - \text{increasing} \\ 0 \text{ if } t_i \text{ is } \tau_i j - \text{constant} \\ -1 \text{ if } t_i \text{ is } \tau_i j - \text{decreasing.} \end{cases}$$

For the sake of better readability we shall write s_j instead of s_j^i and $\tau 1 \rhd \tau 2 \rhd \dots \rhd \tau n_i$ instead of $\tau_i 1 \rhd \tau_i 2 \rhd \dots \rhd \tau_i n_i$ because the index i will always be specified in the expression 't_i' anyway.

Assume that for all $i \leq k$ we have already constructed the sets

$$A_i = \{d_1^i < a_1^i < d_2^i < \dots < d_{n_i}^i < a_{n_i}^i < d_{n_i+1}^i\} \subset \mathbb{Q}$$

and functions

$$\varphi_i : V_{t_i} \longrightarrow C^{A_i}$$

such that the diagram

$$\begin{array}{ccc} V_{t_i} & \stackrel{\varphi_i}{\longrightarrow} & C^{A_i} \\ \Big\downarrow{\scriptstyle in} & & \Big\downarrow{\scriptstyle in} \\ \bigcup_{j\leq k} V_{t_j} & \longrightarrow & C^{\cup_{j\leq k} A_j} \end{array}$$

commutes.

Let us find $A_{k+1} = \{d_1 < a_1 < \cdots < d_{n_{k+1}} < a_{n_{k+1}} < d_{n_{k+1}+1}\}$ such that the order-preserving mapping

$$\varphi_{k+1} : V_{t_{k+1}} \longrightarrow C^{A_{k+1}} \text{ defined as}$$

$\varphi_{k+1}(t_{k+1}(x_1, \ldots, x_{n_{k+1}})) =$

$$\left(\begin{array}{cccccc} 1 & s_1 x_{\tau 1} & 1 & \cdots & s_{n_{k+1}} x_{\tau n_{k+1}} & 1 \\ | & | & | & & | & | \\ d_1 & a_1 & d_2 & \cdots & a_{n_{k+1}} & d_{n_{k+1}+1} \end{array} \right)$$

would make the diagram

$$\begin{array}{ccc} V_{t_{k+1}} & \stackrel{\varphi_{k+1}}{\longrightarrow} & C^{A_{k+1}} \\ \Big\downarrow & & \Big\downarrow \\ \bigcup_{i\leq k+1} V_i & \longrightarrow & C^{\cup_i A_i} \end{array}$$

commutative.
If for all $i \leq k$, t_i and t_{k+1} are separated then choose A_{k+1} as an arbitrary $(2n_{k+1}+1)$-element subset of

$$\{q \in \mathbb{Q} \mid q > A_i, \quad t_i > t_{k+1}\} \cap \{q \in \mathbb{Q} \mid q < A_i, \ t_i < t_{k+1}\},$$

which is an interval in $\mathbb{Q}$. So, bigger terms have earlier coordinates with positive values.

Let t_{k+1} and t_i be interpenetrating. By Lemma 21, first we have to compare their main variables (on the same coordinate axis). Put $d_1 = d_1^i$ and $a_1 = a_1^i$. Suppose we have defined d_j and a_j for all $j < m \leq n_{k+1} + 1$. (Note that m may be equal to $n_{k+1} + 1$, i.e. only the last $d_{n_{k+1}+1}$ is not defined yet.) If for some t_i,

$$t_i(\overline{x} \mid x_{\tau 1} = b_1, \ldots, x_{\tau(m-1)} = b_{m-1}) \text{ and } t_{k+1}(\overline{x} \mid x_{\tau 1} = b_1, \ldots, x_{\tau(m-1)} = b_{m-1})$$

are interpenetrating, then, by Lemma 22 (3), we have to compare their τm-th variables, so we put $d_m = d_m^i$, $a_m = a_m^i$. Otherwise (i.e. if for all i,

$$t_i(\overline{x} \mid x_{\tau 1} = b_1, \ldots, x_{\tau(m-1)} = b_{m-1}) \text{ and } t_{k+1}(\overline{x} \mid x_{\tau 1} = b_1, \ldots, x_{\tau(m-1)} = b_{m-1})$$

are separated) choose $d_m < a_m < \cdots < d_{n_{k+1}} < a_{n_{k+1}} < d_{n_{k+1}+1}$ arbitrarily from the set

$$I = \{q \in \mathbb{Q} \mid q > a_{m-1}\} \bigcap$$

$$\bigcap \{q \in \mathbb{Q} \mid q < d^i_m \ \ t_i(\overline{x} \mid x_{\tau 1} = b_1, \ldots, x_{\tau(m-1)} = b_{m-1}) \ <$$

$$< \ t_{k+1}(\overline{x} \mid x_{\tau 1} = b_1, \ldots, x_{\tau(m-1)} = b_{m-1})\} \bigcap$$

$$\bigcap \{q \in \mathbb{Q} \mid q > d^i_{n_i+1} \ \ t_i(\overline{x} \mid x_{\tau 1} = b_1, \ldots, x_{\tau(m-1)} = b_{m-1}) \ >$$

$$> \ t_{k+1}(\overline{x} \mid x_{\tau 1} = b_1, \ldots, x_{\tau(m-1)} = b_{m-1})\},$$

which is an interval in $\mathbb{Q}$. By construction, this choice of A_{k+1} and φ_{k+1} makes the diagram commutative. ■

Notice how Theorem 20 contrasts with Theorem 18 (3) and (4): if $(X, <)$ is a dense linear order, $|X| > w(X)$ and there is a function $f \colon X \to X^*$ having a fixed point then no $|X|$-dense model M is order-embeddable into $(X^n, <)$ or $(X^{<\omega^*}, <)$ but there is a model (namely $\mathrm{EM}(X)$) which is order-embeddable into $(X^{<\mathbb{Q}}, <)$.

Corollary 23 $\mathrm{EM}(\mathbb{R}, p)$ *is an uncountable model containing no monotonous* ω_1- *or* ω_1^**-sequences.*

Proof Present $\mathbb{R}^{<\mathbb{Q}}$ as a union: $(\mathbb{R}^{<\mathbb{Q}}, <) = \bigcup_{i\in\omega}(A_i, <)$, where $A_i = \mathbb{R}^{n_i}$. Let $f \colon \omega_1 \to \mathbb{R}^{<\mathbb{Q}}$ be an order-embedding. Then $\omega_1 = \bigcup_{i\in\omega} B_i$, where B_i is defined as

$$\alpha \in B_i \ \Leftrightarrow \ f(\alpha) \in A_i.$$

Hence, by regularity, there is $j \in \omega$ such that B_j is unbounded in ω_1, hence order-isomorphic to ω_1. Then $f|_{B_j} \colon B_j \to \mathbb{R}^{n_j}$ is an order-embedding which is impossible. The proof for ω_1^* is the same. ■

Corollary 24
For any dense linear order $(C, <)$ *with no last element,* $\mathrm{EM}(C, p)$ *is order-embeddable into* $(C^* + \{0\} + C)^{<\mathbb{Q}}$.

The next question asks for a generalisation of the Ehrenfeucht–Gaifman lemma (see [16], page 305) for $n \geq 1$.

Question 7
Let p *be an indiscernible type over* T, $(C, <)$ *be a linear order,* $M = \mathrm{EM}(C, p)$. *Prove that if* $t(u_1, \ldots, u_n)$ *is a Skolem term of* T, $x_1, \ldots, x_n \in C$, *then either* $tp(t(x_1, \ldots, x_n)) \neq tp(x_1)$ *or* $t(x_1, \ldots, x_n) = x_i$ *for some* $i = 1, \ldots, n$.

If Question 7 has a positive solution then we can deduce that $\mathrm{EM}(C_1, p)$ is elementarily embeddable into $\mathrm{EM}(C_2, p)$ if and only if C_1 is embeddable into C_2. Furthermore it would follow that there are 2^λ pairwise non-embeddable models of T in any cardinality λ.

Question 8

Let $\lambda > \omega$. Is there a family $\{C_i\}_{i\in 2^\lambda}$ of dense linear orders of cardinality λ with no last element such that $i \neq j$ implies that C_i is not embeddable into $(C_j^* + \{0\} + C_j)^{<\mathbb{Q}}$?

If yes, we can deduce, by Theorem 20, that there are 2^λ pairwise non-embeddable order-types of models of T of cardinality λ.

Chapter 4

Inner models

In this chapter we look at 'inner models' (the models obtained by means of Arithmetised Completeness Theorem) and express their order-type in terms of the $(<, \cdot)$-type of the original model. This leads to a very promising notion of self-similarity.

Interpretation
Let T_1, T_2 be consistent theories in finite languages L_1 and L_2. Let $A \models T_1, B \models T_2$. A is said to be **interpreted** in B if there are formulas (possibly with parameters from B) $\mathrm{dom}(x)$, $\tilde{f}(\overline{x}, y)$ for every functional symbol f of L_1 and $\tilde{R}(\overline{x})$ for every relational symbol R of L_1 such that

$$B \models \forall \overline{x}\ (\mathrm{dom}(\overline{x}) \to \exists! y\ \ \mathrm{dom}(y)\ \&\ \tilde{f}(\overline{x}, y)) \quad \text{and}$$

$$(\{x \in B \mid \mathrm{dom}(x)\}, \tilde{f}, \tilde{R}) \ \cong\ A.$$

Strong interpretation
Let T be a consistent theory in a finite language L. Let $A \models T$, $B \models \mathrm{PA}$. Then A is **strongly interpreted** in B if there are two formulas $\mathrm{dom}(x)$ and $\mathrm{Sat}(x, y)$ (possibly containing parameters from B) such that there is a bijection

$$g\colon A \to \{x \in B \mid \mathrm{dom}(x)\}$$

such that for every $\varphi(\overline{x}) \in L$, $\overline{a} \in A$,

$$A \models \varphi(\overline{a}) \ \Leftrightarrow\ B \models \mathrm{Sat}(\ulcorner \varphi \urcorner, \langle g(\overline{a}) \rangle).$$

Obviously, if A is strongly interpreted in B then A is interpreted in B. If $A \models \mathrm{PA}$ is strongly interpreted in $B \models \mathrm{PA}$ we shall sometimes say that A is an **inner model** in B.

Let $M \models \mathrm{PA}$ be nonstandard, T be a set of statements (in a finite language L) such that $\{\ulcorner \varphi \urcorner \mid \varphi \in T\}$ is coded in M by some point t. Then define $\mathrm{Con}(T)$ to be the following $L_{\mathrm{PA}} \cup \{t\}$-statement:

$$\big(\forall i < \mathrm{len}(t)\ \forall x\ \neg\,\mathrm{Proof}(x, \ulcorner((t)_0 \wedge \ldots \wedge (t)_{\mathrm{len}(t)-1} \rightarrow \exists y\ y \neq y\urcorner)\big)\,.$$

Clearly, this definition depends on the choice of parameter t.
If $M = \mathbb{N}$ and $\{\ulcorner\varphi\urcorner \mid \varphi \in T\}$ is defined in $\mathbb{N}$ by a formula $\theta(x)$ then $\mathrm{Con}(T) = \mathrm{Con}_T =$

$$\forall x(\forall i < \mathrm{len}(x)\ \theta((x)_i) \rightarrow \neg\exists y\,\mathrm{Proof}(y, \ulcorner(x)_0 \wedge \ldots (x)_{\mathrm{len}(x)-1} \rightarrow \exists z z \neq z\urcorner)).$$

$\mathrm{Con}_{\mathrm{PA}}$ is the following L_{PA}-statement:

$$\forall x((\forall i < \mathrm{len}(x)\,\mathrm{axiom}_{\mathrm{PA}}((x)_i)) \rightarrow$$

$$\rightarrow\ \neg\exists y\,\mathrm{Proof}(y, \ulcorner(x)_0 \wedge \ldots (x)_{\mathrm{len}(x)-1} \rightarrow \exists z\ z \neq z\urcorner)).$$

Notice that $\mathbb{N} \models \mathrm{Con}(\mathrm{PA}) \leftrightarrow \mathrm{Con}_{\mathrm{PA}}$. However, if $M \models \mathrm{PA}$ is nonstandard then "$M \models \mathrm{Con}(\mathrm{PA})$" is a weaker condition than "$M \models \mathrm{Con}_{\mathrm{PA}}$" because $\mathrm{Con}_{\mathrm{PA}}$ states the consistency of the collection of all (including all nonstandard) axioms of PA, not only the ones coded by t.

The formula $\mathrm{Pr}_{\mathrm{PC}}(x)$ means "the formula with Gödel number x is provable in the predicate calculus".

Arithmetised Completeness Theorem.
$M \models \mathrm{PA} + \mathrm{Con}(T)$ if and only if there is $N \models T$ which is strongly interpreted in M, i.e. there are formulas

$$\mathrm{dom}_N(x, a) \text{ (domain of } N\text{) and } \mathrm{Sat}_N(x, y, a) \text{ (satisfaction for } N\text{)}$$

such that

$$x \in N \ \Leftrightarrow\ M \models \mathrm{dom}_N(x, a),$$

$$N \models \varphi(\bar{b}) \ \Leftrightarrow\ M \models \mathrm{Sat}_N(\ulcorner\varphi\urcorner, \langle\bar{b}\rangle, a).$$

We shall denote this model by $\mathrm{ACT}(M, T)$. (Warning: even for complete theories T, $\mathrm{ACT}(M, T)$ does not have to be unique!) If $M \models Qx\ \mathrm{dom}(x, a)$ then we can without loss of generality assume $M \models \forall x\ \mathrm{dom}(x, a)$.

Fact
If $M, N \models \mathrm{PA}$, M is nonstandard and N is strongly interpreted in M then N is recursively saturated.

For a more detailed discussion of the Arithmetised Completeness Theorem, see [14], pp. 186-192, 224-246 and [34].

4.1 Interpreted linear orders

Some results about Peano Arithmetic are obtained by "internalising" classical constructions, i.e. simulating their proofs inside a model of PA. E.g. the Arithmetized Completeness Theorem is the internalisation of Completeness Theorem. In this section we also prove some results by internalising classical constructions. Lemma 25 is the internal version of the fact *"every linear order of finite cardinality is isomorphic to* $\{1, 2, \ldots, n\}$ *in natural order for some n."* Theorem 26 is the internalisation of the fact *"every countable dense linear order without end-points is isomorphic to* $\mathbb{Q}$*".* We conclude the section by internalising the classical Fraïssé's construction.

Definition 15 *A linear order A is called* ***boundedly interpreted*** *in M if it is interpreted by the formulas* $\varphi(x, a)$ *(predomain),* $e(x, y, a)$ *(equality),* $o(x, y, a)$ *(order) with* $M \models \exists x \forall y (\varphi(y, a) \to y < x)$*, where* $a \in M$ *is a parameter.*

In the case of PA we can always assume that $e(x, y) \leftrightarrow x = y$, because the order interpreted by $\varphi'(x) \leftrightarrow (\varphi(x) \,\&\, \forall y < x \,\neg e(x, y))$, $e'(x, y) \leftrightarrow x = y$, $o'(x, y) \leftrightarrow o(x, y)$ is isomorphic to A.

Definition 16 *A linear order A is called* M***-finite*** *if there is* $x \in M$*,*
$x = \langle x_1, x_2, x_3 \rangle$*, where* x_1 *codes a bounded subset of M,* x_2 *codes equality,* x_3 *codes order so that* $A \cong x$*.*

From the properties of the coding function we easily observe that the above two definitions are equivalent. If A is M-finite, we shall sometimes write $M \models$ "A is finite" (M believes that A is finite).

Notice that if the formula $\psi(x)$ means "x codes a linear order" then $M \models Qx\psi(x)$.

Lemma 25 *If A is boundedly interpreted in M by the formulas* $\varphi(x, a)$*,* $e(x, y, a)$*,* $o(x, y, a)$ *then* $A \cong [0, b]$ *for some* $b \in \mathrm{Cl}(a)$*.*

Proof
Let $M \models$"A is finite". For every $x \in M$ such that $M \models \varphi(x) \,\&\, \exists z\, o(x, z)$ we define the successor function $s(x)$. Let $f : [0, \max\varphi] \longrightarrow [0, \max\varphi]$ be the following function:
$f(0) = \min z\ o(x, z)$;
$$f(i+1) = \begin{cases} \min z\ o(x, z) \,\&\, o(z, f(i)) \text{ if } \exists z\, o(x, z) \,\&\, o(z, f(i)) \\ f(i) \quad \text{otherwise.} \end{cases}$$
Put $s(x) = f(\max\varphi)$. Let $M \models$"c is the element z of $[0, \max\varphi]$ such that $\forall y \le \max\varphi\ (\varphi(y) \,\&\, \neg e(y, z) \to o(z, y))$".

Let $g(0) = c$, $g(i+1) = s(g(i))$. If $b = \max i \; \exists z \, o(g(i), z)$ we observe that $g : [0, b+1] \to A$ is an order-isomorphism. ■

Definition 17 *Let $(A, +, \cdot, <)$ be an ordered ring, $A^+ = \{x \in A \mid x \geq 0\}$. Then $(Q(A^+), <)$ is the set*

$$\{(x, y) \mid x, y \in A^+, x, y \neq 0\}$$

factored out by the following equivalence relation:

$$(x, y) \sim (z, w) \Leftrightarrow xw = yz$$

with the linear order on it defined as

$$(x, y) < (z, w) \;\; \Leftrightarrow \;\; xw < zy.$$

Actually we do not use addition in this definition. Later we shall be discussing whether there are $(A, +, \cdot, <) \models \mathrm{PA}$, $(B, +, \cdot, <) \models \mathrm{PA}$ such that $(A, +, <) \cong (B, +, <)$ but $(Q(A), <) \; \not\cong \; (Q(B), <)$.
Notice that if A has no last element then $Q(A)$ is a dense linear order.

Theorem 26
If $(A, <)$ is a dense linear order without end-points interpreted in M then

$$(A, <) \;\cong\; (Q(M), <).$$

Proof
Firstly, notice that A is M-infinite, i.e.

$$M \models \forall x \, \exists y \, (\varphi(y) \; \& \; \forall z < x \, \neg e(z, y))$$

because, by Lemma 25, all M-finite linear orders are discrete.

$Q(M)$ is interpreted in M by $\varphi(x) =$ '$\exists x_1 x_2 \; (x = \langle x_1, x_2 \rangle \; \& \; x_1, x_2 \neq 0)$', $e(x, y) =$ '$x_1 y_2 = x_2 y_1$', $o(x, y) =$ '$x_1 y_2 < x_2 y_1$'.

Let Q_1 and Q_2 be two dense linear orders interpreted in M. We are going to prove that they are isomorphic. Moreover, the isomorphism can be taken to be a definable function in M.

Let Q_1 be interpreted by $(\mathrm{dom}_1(x), x <_1 y)$ and Q_2 be interpreted by $(\mathrm{dom}_2(x), x <_2 y)$. Let $f_1(n) =$ the nth element x such that $\mathrm{dom}_1(x)$ and $f_2(n) =$ the nth element x such that $\mathrm{dom}_2(x)$. We are going to find a Skolem term $i(x)$ such that

1. $M \models \forall x \exists y \; i(y) = x$;
2. $M \models \forall zw (f_1(z) <_1 f_1(w)) \longleftrightarrow (f_2(i(z)) <_2 f_2(i(w)))$

i.e. $f_2 i f_1^{-1}$ is a definable order-isomorphism between Q_1 and Q_2. We define $i(x)$ by internalizing the 'forth'-argument in M:
$i(0) = 0$;
$i(x+1) = \min z \; \forall y \leq x \; (f_1(y) <_1 f_1(x+1)) \; \leftrightarrow (f_2(i(y)) <_2 f_2(z))$. Such element z can always be found because of density:

$$M \models \exists z \max_{w \leq x} \{f_2(i(w)) \mid f_1(w) < f_1(x+1)\} < z <$$
$$< \min_{w \leq x} \{f_2(i(w)) \mid f_1(x+1) < f_1(w)\}. \blacksquare$$

Corollary 27
Neither $\mathbb{R}$ nor $\mathbb{Q}$ is interpreted in uncountable models of PA.

Proof
Let $M \models$ PA be uncountable. $\mathbb{Q}$ cannot be interpreted in M because $Q(M)$ is uncountable, hence $\mathbb{Q} \not\cong Q(M)$. $\mathbb{R}$ cannot be interpreted in M because M is order-embeddable into $Q(M)$ but not into $\mathbb{R}$, and hence $\mathbb{R} \not\cong Q(M)$. ■

The next proposition (dealing with the non-dense case) is similar to the following known fact.

Fact (*)(see [14], page 192)
If $M, N \models$ PA and N is strongly interpreted in M then the function $f\colon M \to N$ defined as $f(0) = 0_N$, $f(x+1) = f(x) +_N 1_N$ is a definable isomorphism between M and an initial segment of N.

We shall need Fact (*) in the next section.

Proposition 28 *If $(A, <)$ is a linear order with first point interpreted in M and every element of $(A, <)$ has a successor then*

1. *A is M-infinite, i.e. $M \models Qxy(\varphi(x) \;\&\; \varphi(y) \;\&\; \neg e(x, y))$;*
2. *$(M, <)$ is definably isomorphic to an initial segment of $(A, <)$.*

Proof
2. Define $f\colon M \to A$ as follows:
$f(0) =$ the element w such that $\forall x \, (\varphi(x) \to e(x, w) \vee o(w, x))$, $f(i+1) =$ the element y such that $o(f(i), y) \;\& \forall z (o(f(i), z) \;\& \neg e(y, z) \to o(y, z)))$.
Clearly, f is definable and determines an order-isomorphism between M and an initial segment of A.
1. If A was M-finite then, by Lemma 25, there would exist a definable isomorphism $g\colon A \to [0, b]$ for some $b \in M$. Then $g \circ f$ would be a definable

embedding of M into $[0,b]$ whose existence contradicts the pigeonhole principle. ■

Notice that A does not have to be discrete (example: $A = \text{odds}(M){+}\text{evens}(M)$) but if $a \in A$ has no predecessor then the gap between $\{x \in A \mid x < a\}$ and a has to be big. If $M \models o(b,a)$ then $f\colon M \to A$ defined as:
$f(0) = b$;
$f(i+1) =$ the first w such that

$$\varphi(w)\ \&\ o(f(i),w)\ \&\ \forall y(\varphi(y)\ \&\ o(f(i),y) \to e(w,y) \vee o(w,y))$$

embeds M between b and a.

Fraïssé limit

Another example of a dense linear order interpreted in M is the internal Fraïssé limit. We define an M-finite order with a 'marked' subset as a point $y = \langle y_1, y_2\rangle$, where y_2 codes a subset of the M-finite order $[0, y_1]$. Let $f : M \to M$ be an enumeration of all linear orders with a 'marked' subset.

Let $g(0) = f(0)$. Suppose for $x \in M$, $g(x) = \langle x_1, x_2, x_3\rangle$ be already constructed, where x_2 codes equality in $[0, x_1]$ and x_3 codes order. (Of course we know from Lemma 25 that $g(x)$ is isomorphic to some linearly ordered set of the form $([0,a], <_M)$, but we shall not need it in our construction.) Let $f(x+1) = \langle y_1, y_2\rangle$, where y_2 marks a subset B of $[0, y_1]$.

Let $c = \min\{x_1 + 1, \text{card}(y_2)\}$. We are going to amalgamate $f(x+1)$ with $g(x)$ $(x_1 + 2 - c)$ times by identifying the first c points of the marked subset of $f(x+1)$ with all c-element suborders of $g(x)$ of the form $\{k, k+1, \ldots, k+c-1\}$ and adjusting the order between marked points. At the i^{th} step, $\{(y_2)_0, (y_2)_1, \ldots, (y_2)_{c-1}\}$ will be identified with $\{i-1, i, \ldots, i+c-2\}$.

Let $h(0) = g(x)$. For $h(i)$ already defined, let

$$h(i+1) = \langle x_1 + 1 + (i+1)(y_1+1), eq, order\rangle,$$

where eq defines equality in $[0, x_1 + 1 + (i+1)(y_1+1)]$ and $order$ defines linear order. Let eq code the transitive closure of the following set of pairs:

$$\{(w,w) \mid w \le x_1 + 1 + (i+1)(y_1+1)\} \cup$$

$$\cup \{(w_1,w_2) \mid (w_1,w_2) \in (h(i))_2\}$$

(i.e. all pairs which are already equal in $h(i)$) $\cup$

$$\cup \left\{ (u,v) \mid \exists z \le c-1 \quad \begin{matrix} u = \text{the } (z+i-1)^{th} \text{ point of } g(x) \\ v - x_1 - i(y_1+1) = (y_2)_z \end{matrix} \right\}.$$

Let *order* code the transitive closure of the following set of pairs:

$$\{(u,v) \mid (u,v) \in (h(i))_3\} \cup$$

$$\cup \{(u,v) \mid x_1 + 1 + i(y_1+1) < u < v \leq x_1 + 1 + (i+1)(y_1+1)\}$$

(i.e. all points in our i^{th} copy of $[0, y_1]$ go in their natural order) $\cup$

$$\left\{(u,v) \mid \exists j \leq c-1 \begin{array}{c} (u, (j+i)^{th} \text{ point of } g(x)) \in (h(i))_3 \\ v > (y_2)_j + x_1 + 1 + i(y_1+1) \end{array}\right\}$$

(i.e. points of $h(i)$ smaller than the $(j+i)^{th}$ point of $g(x)$ (which is identified with $(y_2)_{j+1}$) are smaller in $h(i+1)$ than anything greater than the $(y_2)_j$) $\cup$

$$\cup \left\{(u,v) \mid \exists j \leq c-1 \begin{array}{c} x_1 + 1 + i(y_1+1) < u \leq (y_2)_j + x_1 + 1 + i(y_1+1) \\ (v, j+i-1) \in (h(i))_3 \end{array}\right\}$$

(i.e. the points of $f(x+1)$ not greater than $(y_2)_j$ are smaller than anything greater than the point of $h(i)$ identified with $(y_2)_j$) $\cup$

$$\left\{(u,v) \mid \begin{array}{c} (u, (i-1)^{th} \text{ point of } g(x)) \in (h(i))_3, \\ x_1 + 1 + i(y_1+1) < v < (y_2)_0 + x_1 + 1 + i(y_1+1) \end{array}\right\} \cup$$

$$\cup \left\{(u,v) \mid \begin{array}{c} ((i+c-2)^{th} \text{ point of } g(x), u) \in (h(i))_3, \\ (y_2)_{c-1} + x_1 + 1 + i(y_1+1) < v < x_1 + 1 + (i+1)(y_1+1) \end{array}\right\}.$$

Put $g(x+1) = h(x_1 + 2 - c)$.
We define the Fraïssé limit, F, as follows. Predomain $= M$. For $x, y \in M$, let $z=$ the first point w such that $\max\{x,y\} < (g(w))_1$. We define

$$e(x,y) \leftrightarrow (x,y) \in (g(z))_2,$$

$$o(x,y) \leftrightarrow (x,y) \in (g(z))_3.$$

As in the classical Fraïssé construction, we observe that F is dense. Also $M \models$"F is infinite", since otherwise it would be isomorphic to $[0,a]$ for some a by Lemma 25. ■

By Theorem 26, F is definably isomorphic to $Q(M)$ as any dense order interpreted in M is. Also, we can check that F is "M-homogeneous", that is: every definable order-preserving $h\colon [0,a] \to M$ can be extended to a definable automorphism of F. I am not going to write a detailed proof of that here, because it follows the lines of the proof of Theorem 26.

Question 9
Prove internal versions of the uniqueness of the countable atomless boolean algebra and the random graph. What do these structures look like?

4.2 Order-types of inner models

Theorem 29
Let $M \models \mathrm{PA}$ *and* $N \models \mathrm{PA}$ *be an inner model in* M*. Then*

$$(N, <) \cong M + Q(M)(M^* + M).$$

Proof
Since N is strongly interpreted in M, by Fact (*) (page 49) there is $f\colon M \to N$ definable in M which determines an isomorphism between M and an initial segment of N. Hence, $(N, <) \cong M + A(M^* + M)$ for some linear order A.

For $a, b \in N$, we define $a \sim b \;\Leftrightarrow\; M \models \exists x(a -_N b = f(x))$ if $a > b$ and $a \sim b \;\Leftrightarrow\; M \models \exists x(b -_N a = f(x))$ if $b < a$. We interpret A in M by means of the following formulas:

$$\mathrm{dom}_A(x) \longleftrightarrow x \in N \;\&\; \forall y < x \,\neg(y \sim x)$$

$$o(x, y) \longleftrightarrow x <_N y.$$

Let us prove that A is dense. Take $a, b \in N, a < b, a \not\sim b$. If $\left[\frac{b-a}{2}\right]$ belonged to M (i.e. to the image of f) then so would $b - a$ because $f(M)$ is closed under addition. Hence, $a \not\sim a + \left[\frac{b-a}{2}\right] \not\sim b$.

As A is a dense order interpreted in M, by Theorem 26, $A \cong Q(M)$. ■

Is the order-type of an inner model determined by the order-type of the outer model?

Question 10 *Find two models* $A, B \models \mathrm{PA}$ *such that* $A \equiv B$*,* $(A, <) \cong (B, <)$ *but* $(A + Q(A)(A^* + A), <) \not\cong (B + Q(B)(B^* + B), <)$.

Section 6.2 may give insights into how to tackle this problem.

There is nothing special about ZFC from the point of view of Model Theory and we can deal with models of ZFC very much as we deal with models of other theories, such as the theory of groups or PA. Models of ZFC formalise the Completeness Theorem, hence we can talk about models strongly interpreted in models of ZFC and about inner models. Also, models of ZFC can be strongly interpreted in models of $\mathrm{PA} + \mathrm{Con}_{\mathrm{ZFC}}$. Thus, a host of questions rise about order-types (and not only order-types) of models strongly interpreted in another model. Let us list some of them.

Question 11 *If* $M \models \mathrm{PA} + \mathrm{Con(ZFC)}$ *and* $V \models \mathrm{ZFC}$ *is strongly interpreted in* M*, what can we say about* $(\mathrm{Ordinals}(V), <)$ *in terms of* $(M, <)$*?*

Of course, ω_V will be a model of PA strongly interpreted in M, hence, by Theorem 29 has order-type $M + Q(M)(M^* + M)$.

Question 12 *If* $V \models \mathrm{ZFC} + \mathrm{Con}(\mathrm{ZFC})$ *and* $U \models \mathrm{ZFC}$ *is an inner model in* V*, what can we say about* $(\mathrm{Ordinals}(U), <)$ *in terms of* $(\mathrm{Ordinals}(V), <)$*?*

Question 13

1. *How many pairwise non-isomorphic elementarily equivalent models of* ZFC *are strongly interpreted inside* $M \models \mathrm{PA} + \mathrm{Con}(\mathrm{ZFC})$*? (If* M *is countable nonstandard then there is only one (the recursively saturated model whose standard system is* $\mathrm{SSy}(M)$*). What is the order-type of its ordinals?)*
2. *How many non-isomorphic order-types of ordinals of models of* ZFC *interpreted in a model of* $\mathrm{PA} + \mathrm{Con}(\mathrm{ZFC})$ *are there?*
3. *What are those order-types? How about Friedman's problem for models of* ZFC *from the point of view of a model of* $\mathrm{PA} + \mathrm{Con}(\mathrm{ZFC})$*?*

I expect Question 11 to be easier than Question 12 because models of PA are more restrictive: Theorem 29 shows that from the point of view of a model of $\mathrm{PA} + \mathrm{Con}(\mathrm{PA})$, there is only ONE order-type of models of PA, namely $M + Q(M)(M^* + M)$. So, from its point of view, Friedman's problem has a trivial solution: all order-types of models of PA are isomorphic. However, as we shall see in Chapter 6, models of ZFC believe in existence of 2^λ order-types of models of PA in each uncountable cardinality λ.

4.3 Self-similar models

What happens if some model K is strongly interpreted in N and N is an inner model in M? Is $(K, <)$ (which is equal to $N + Q(N)(N^* + N)$) a new order-type? No! K is also strongly interpreted in M, hence, by Theorem 29, $(K, <) \cong M + Q(M)(M^* + M) \cong (N, <)$.
So, N has a nice property: its inner models have order-type $(N, <)$.

Definition 18 *A model* $M \models \mathrm{PA}$ *is called self-similar if*

$$(M, <) \cong M + Q(M)(M^* + M).$$

Examples of self-similar models

Proposition 30

1. Every inner model is self-similar.

2. Every countable model is self-similar, because $Q(\mathbb{N}+\mathbb{Q}\mathbb{Z},+,\cdot,<)=\mathbb{Q}$ (because if $\frac{a}{b}<\frac{c}{d}$, then $ad<bc$, then $2ad<2ad+1<2bc$ then

$$\frac{a}{b}=\frac{2ad}{2bd}<\frac{2ad+1}{2bd}<\frac{2bc}{2bd}=\frac{c}{d}$$

   i.e. $Q(\mathbb{N}+\mathbb{Q}\mathbb{Z})$ is dense) hence
   $\mathbb{N}+\mathbb{Q}\mathbb{Z}+Q(\mathbb{N}+\mathbb{Q}\mathbb{Z})(\mathbb{Q}\mathbb{Z})=\mathbb{N}+\mathbb{Q}\mathbb{Z}+\mathbb{Q}\mathbb{Q}\mathbb{Z}=\mathbb{N}+\mathbb{Q}\mathbb{Z}$.

3. No λ-like model is self-similar, because $M+Q(M)(M^*+M)$ always contains an initial segment of cardinality $|M|$.

4. Every saturated model is self-similar, because $Q(\mathbb{N}+Q_\lambda\mathbb{Z})=Q_\lambda$. (If $E<F$ are two subsets of $Q(M)$ of cardinalities $<\lambda$ then consider

$$p(x,y)=\{ay<bx\mid \frac{a}{b}\in E\}\cup\{dx<cy\mid \frac{c}{d}\in F\}.$$

   $p(x,y)$ is a type because, given $\frac{a_1}{b_1},\dots,\frac{a_n}{b_n}<\frac{c_1}{d_1},\dots,\frac{c_m}{d_m}$, by the argument from example 2 above, there are $x,y\in M$ such that

$$\max\{\frac{a_1}{b_1},\dots,\frac{a_n}{b_n}\}<\frac{x}{y}<\min\{\frac{c_1}{d_1},\dots,\frac{c_m}{d_m}\}.$$

   The pair (x,y) realising p separates E and F.) Hence

$$\mathbb{N}+Q_\lambda\mathbb{Z}+Q(\mathbb{N}+Q_\lambda\mathbb{Z})(Q_\lambda\mathbb{Z})=\mathbb{N}+Q_\lambda\mathbb{Z}+(Q_\lambda)Q_\lambda\mathbb{Z}=\mathbb{N}+Q_\lambda\mathbb{Z}.$$

   In particular, every inner model in a saturated model is again saturated by Pabion's Theorem.

Theorem 31 *If M is self-similar and $f\colon M\to M+Q(M)(M^*+M)$ is an order-isomorphism then there is a proper self-similar elementary extension $N\succ M$ such that $|N|=|M|$ and the diagram*

$$\begin{array}{ccc} M & \xrightarrow{f} & M+Q(M)(M^*+M)\\ \downarrow{\scriptstyle in} & & \downarrow{\scriptstyle \widetilde{in}}\\ N & \xrightarrow{\tilde f} & N+Q(N)(N^*+N)\end{array}$$

commutes.

The following short proof was suggested by my supervisor and replaces my original one.

Proof
Let $L = L_{\mathrm{PA}} \cup \{f\}$. If M is self-similar, let us expand it to (M, f), where f is interpreted by an isomorphism between M and $M + Q(M)(M^* + M)$. Any proper elementary extension $(N, \tilde{f}) \succ (M, f)$ of cardinality $|M|$ is as required. ■

Hence any countable model has elementary self-similar extensions of all cardinalities.

Theorem 32 *If M is resplendent then $(M, <) \cong M + Q(M)(M^* + M)$.*

Proof
Every resplendent model is self-similar because the statement

$$\exists\, g\ (\ g \textit{ is an isomorphism between } M \textit{ and } M + Q(M)(M^* + M)\ )$$

is realised in all countable models. Let us show that it is really a Σ^1_1-statement. For that we shall need to interpret $M + Q(M)(M^* + M)$ in model M, which is done easily:

$$\exists D\ (domain)\ \exists f\ \exists S\ (sum)\ \exists E\ (equivalence)\ \exists O\ (order)$$

$$\exists \sim\ (equality\ in\ Q(M)) \exists \triangleleft\ (order\ in\ Q(M))$$

$$\exists g\ (isomorphism\ between\ M\ and\ M + Q(M)(M^* + M))$$

$$[\exists x\ \forall y\ (D(f(y))\ \&\ O(f(y), x))\ \&\ \forall xy\ (x < y \to O(f(x), f(y)))\ \&$$

$$\&\ \forall xyz (f(x) = y\ \&\ O(z, y) \to \exists w\ f(w) = z\ \&$$

(f is an order-isomorphism between M and an initial segment of D)

$\&\ \forall xyz\ (S(x, y) = S(y, x)\ \&\ S(x, f(0)) = x\ \&\ S(x, S(y, z)) = S(S(x, y), z))\&$

$\&\forall x < y\ (E(x, y) \longleftrightarrow \exists z\ S(x, f(z)) = y)$ & *and E is an eqivalence relation* &

(i.e. $E(x, y)$ means $y -_D x \in M$)

$$\&\ \forall xyz (D(x)\ \&\ D(y)\ \&\ D(z) \to \neg O(x, x)\ \&\ O(x, y) \leftrightarrow \neg O(y, x)\ \&$$

$(O(x, y) \& O(y, z) \to O(x, z))$ &

($O(x, y)$ is a linear order on the set D)

$$\&\ \forall x_1\ y_1 \neq 0\ x_2\ y_2 \neq 0\ (\langle x_1, y_1\rangle \sim \langle x_2, y_2\rangle \leftrightarrow x_1 y_2 = x_2 y_1)\ \&$$

$$\& \langle x_1, y_1\rangle \lhd \langle x_2, y_2\rangle \leftrightarrow x_1y_2 < x_2y_1 \;\&$$

(definition of $Q(M)$ with equality $\sim$ and order $\lhd$)

$$\& \; \forall xyzw \; \{D(g(\langle x,y\rangle)) \;\&$$

$$\& \; \langle x,y\rangle \sim \langle z,w\rangle \rightarrow g(\langle x,y\rangle) = g(\langle z,w\rangle) \;\&$$

$$\& \; \langle x,y\rangle \lhd \langle z,w\rangle \rightarrow O(g(\langle x,y\rangle)), g(\langle z,w\rangle)) \; \} \;\&$$

(i.e. $g\colon Q(M) \longrightarrow (D \setminus M)/M$ respects equality and order)

$$\& \; \neg\exists v \; g(\langle x,y\rangle) = f(v) \;\&$$

$$\& \; E(g(\langle x,y\rangle), g(\langle z,w\rangle)) \rightarrow \langle x,y\rangle \sim \langle z,w\rangle \;\&$$

$$\& \; \forall x(D(x) \rightarrow (\exists y(E(x,y) \;\&\; \exists zw \; g(\langle z,w\rangle) = y) \;] \,.\blacksquare$$

So far we know that every inner model is recursively saturated and self-similar and every resplendent model is recursively saturated and self-similar too. Also, if M is not λ-dense then every model of PA strongly interpreted in M will have an initial segment of cardinality $< \lambda$, hence, by Proposition 10, is not resplendent. (This actually produces a family of examples of recursively saturated non-resplendent models.)

Proposition 33 *If $N \models$ PA is strongly interpreted in a resplendent $M \models$ PA then N is resplendent.*

Proof
Suppose $\mathrm{Sat}(x, y, a)$ is the formula defining in M truth for N and $M \models \forall x \; \mathrm{dom}_N(x)$. Suppose $\Phi = \exists \; \overline{R} \; \chi(\overline{R}, \overline{b})$ is consistent with $\mathrm{Th}(N, \overline{b})$, where $\overline{R}$ is a tuple of relation symbols (functional symbols are easily eliminated). Let us transform it into a formula $\Psi = \exists \; \overline{R} \; \hat{\chi}(\overline{R}, \overline{b}, a)$ which in M means "there is $\overline{R}$ such that $\chi(\overline{R}, \overline{b})$ holds in N" and prove that Ψ is consistent with $\mathrm{Th}(M, \overline{b}, a)$.

Define the translation operation $\hat{}$ as follows.
If $\varphi(x_1, \ldots, x_n) = R(x_1, \ldots, x_n)$ then

$$\hat{\varphi}(x_1, \ldots, x_n) = R(x_1, \ldots, x_n);$$

if $\varphi(\overline{x}) = R(t_1(\overline{x}), \ldots, t_n(\overline{x}))$, where $t_1, , \ldots, t_n$ are terms in L_{PA} then

$$\hat{\varphi}(\overline{x}) = \exists \; y_1 \ldots y_n \left(\bigwedge_{i=1}^{n} \mathrm{Sat}(\ulcorner u = v \urcorner, \langle y_i, t_i(\overline{x})\rangle) \;\&\; R(y_1, \ldots, y_n) \right);$$

if $\varphi(\overline{x})$ is an atomic formula in L_{PA} then $\hat{\varphi}(\overline{x}) = \mathrm{Sat}(\ulcorner\varphi(\overline{u})\urcorner, \langle\overline{x}\rangle)$;
if $\varphi = \psi \vee \chi$ then $\hat{\varphi} = \hat{\psi} \vee \hat{\chi}$;
if $\varphi = \neg\psi$ then $\hat{\varphi} = \neg\hat{\psi}$;
if $\varphi = \forall\overline{x}\ \psi(\overline{x})$ then $\hat{\varphi} = \forall\overline{x}\ \hat{\psi}(\overline{x})$.

Let $A \models \mathrm{Th}(M, \overline{b}, a)$ be an arbitrary countable recursively saturated (hence, resplendent) model. Let $B \models \mathrm{Th}(N, \overline{b})$ be the inner model in A defined by $\mathrm{Sat}(x, y, a)$. B is countable and recursively saturated, hence resplendent, hence B realises Φ, hence A realises Ψ, hence, by resplendency, M realises Ψ, hence N realises Φ. Thus, N is resplendent. ■

Let us list some more questions about interconnections of our notions.

Question 14

1. *Is there an uncountable self-similar model which is not recursively saturated?*
2. *Is there a resplendent model which is not an inner model?*
3. *Is there an uncountable self-similar model which is not an inner model?*
4. *Give an example of two elementarily equivalent non-isomorphic inner models in M. (Of course, M will have to be uncountable.)*

4.4 Inside models of ZFC

In this section we briefly investigate different kinds of interpretations of models of PA in models of ZFC, study models of PA as natural numbers of some model of ZFC and discuss models $V \models \mathrm{ZFC}$ strongly interpreted inside $\Omega \models \mathrm{PA}$ such that $\omega_V \cong \Omega$.

Throughout the section we assume $\mathbb{N} \models \mathrm{Con}_{\mathrm{ZFC}}$.
We shall encounter many different omegas: ω_{ZFC}, the Skolem term of the language of ZFC expressing "the first infinite ordinal", ω, the 'outer' 'true' ω, the natural numbers of the environment we are working in, ω_U, the interpretation of ω_{ZFC} in the universe $U \models \mathrm{ZFC}$, and sometimes the symbol ω, which is later interpreted as one of the above omegas.

For convenience let $L_{\mathrm{ZFC}} = \{\in, \varnothing, \omega_{\mathrm{ZFC}}, \times, \sqcup\}$, i.e. already contain the symbols for Skolem terms defining $\varnothing$, ω, direct product and disjoint union of sets.

Definition 19 *Let φ be a closed formula of L_{PA}. Then φ^* will denote the translation of φ into the language $L_{\mathrm{ZFC}} = \{\in, \varnothing, \omega_{\mathrm{ZFC}}, \times, \sqcup\}$ relativized to ω_{ZFC}, namely φ^* is φ with*

$$\forall\, x \quad \textit{substituted by} \quad \forall\, x \in \omega_{\mathrm{ZFC}}$$

$$\exists\, x \quad \textit{substituted by} \quad \exists\, x \in \omega_{\mathrm{ZFC}}$$

$$0 \quad \textit{substituted by} \quad \varnothing$$

$$x < y \quad \textit{substituted by} \quad x \in y$$

$$x + y \quad \textit{substituted by} \quad \mathrm{card}(x \sqcup y),$$

$$x \cdot y \quad \textit{substituted by} \quad \mathrm{card}(x \times y).$$

Definition 20 *Let φ be a closed formula of $L_{\mathrm{ZFC}} = \{\in, \varnothing, \omega_{\mathrm{ZFC}}, \times, \sqcup\}$ all of whose quantifiers are relativised to ω_{ZFC}. We build the formula φ^a in the language L_{PA} as follows. Find all instances of the Skolem terms $\times$ and $\sqcup$ in φ and substitute expressions $\mathrm{card}(s(\overline{x}) \times t(\overline{y}))$ by $s(\overline{x}) \cdot t(\overline{y})$ and $\mathrm{card}(s(\overline{x}) \sqcup t(\overline{y}))$ by $s(\overline{x}) + t(\overline{y})$. After that substitute the rest of the quantifiers $\forall x \in \omega_{\mathrm{ZFC}}$ and $\exists x \in \omega_{\mathrm{ZFC}}$ by $\forall x$ and $\exists x$ and the rest of the symbols '$\in$' by '$<$'.*

For every formula $\varphi \in L_{\mathrm{ZFC}}$ with all quantifiers relativised to ω_{ZFC},
$\mathrm{ZFC} \vdash (\varphi^a)^* \leftrightarrow \varphi$ and for every $\psi \in L_{\mathrm{PA}}$, $\mathrm{PA} \vdash (\psi^*)^a \leftrightarrow \psi$.
(Notice that translation * is preserved under PA-equivalence, i.e.

$$\mathrm{PA} \vdash \varphi \leftrightarrow \psi \quad \text{implies} \quad \mathrm{ZFC} \vdash \varphi^* \leftrightarrow \psi^*$$

but translation ^{a} does not have to be preserved under ZFC-equivalence. However, this is not going to affect our presentation.)

Set-like models and class-like models

There are different kinds of models of PA interpreted inside $U \models \mathrm{ZFC}$. Of course, there is the U-standard model $\omega_U \models \mathrm{Th}(\omega_U)$, where $\mathrm{Th}(\omega_U)$ is 'the true arithmetic from the point of view of U', i.e.

$$\mathrm{Th}(\omega_U) = \{\varphi \in L_{\mathrm{PA}} \mid U \models \varphi^*\}.$$

Also, there are plenty of 'set-like' models of PA inside U, i.e. models $N \models \mathrm{PA}$ strongly interpreted in U such that

$$U \models \exists x\, \forall y\ \mathrm{dom}_N(y) \leftrightarrow y \in x.$$

As Chapter 6 shows, U will believe that there are 2^λ of them in cardinality $\lambda \in U$ such that $U \models \lambda > \omega$. Note that from the point of view of U there will be 2^{ω_U} 'theories' extending $(\mathrm{PA})^U = \{x \in \omega_U \mid \omega_U \models \mathrm{axiom}_{\mathrm{PA}}(x)\}$, while for us (looking at U 'from the outside') the number of theories (in the standard sense) extending PA which are intersections of a definable subset of U with $\mathbb{N}$ will not exceed card(SSy(U)), where

$$\mathrm{SSy}(U) = \{A \subset \mathbb{N} \subseteq \omega_U \mid A \text{ is definable in } U \text{ with parameters }\},$$

so, for example, if U is countable, then only countably-many models of PA will exist in every cardinality. (And a lot of them may turn out to be isomorphic, even if there is no isomorphism inside U.)

Question 15
Let $U \models$ ZFC *be countable,* T *be a completion of* PA *coded in* U*,* $U \models \mathrm{Con}(T)$*. How many non-isomorphic models of* T *are interpreted in* U*?*

By analogy with models of PA interpreted in models of PA, we may ask whether there is a model of PA whose domain is unbounded in U. Let us call such models 'class-like'. (Class-like models are not 'huge', you cannot say 'such model is not a set but a class'. Remember: $U \models$ ZFC is just a model. Think of it as being countable.)

Proposition 34 *Let* $V \models$ ZFC*. There is a class-like model of* PA *in* V*.*

Proof
Let us construct $M \models$ PA by V-transfinite induction. For every ordinal $\alpha \in V$ we define $M_\alpha \models$ PA with domain $\aleph_\alpha$ whose satisfaction relation is $\mathrm{Sat}_\alpha(x, y)$. Let $M_0 = \omega_V$, $\mathrm{Sat}_0(\ulcorner\varphi\urcorner, x) \leftrightarrow x < \omega \;\&\; \varphi^*(x)$. As ZFC $\vdash$ (every model of PA has an elementary end-extension of any larger cardinality), we can define

$$M_{\alpha+1} = \aleph_{\alpha+1}, \quad \mathrm{Sat}_{\alpha+1}(\ulcorner\varphi\urcorner, x) \leftrightarrow (x \in M_\alpha \to \mathrm{Sat}_\alpha(\ulcorner\varphi\urcorner, x)$$

(i.e. $M_{\alpha+1}$ is an elementary extension of M_α) &

$$\& \text{ “}\varphi = \psi \vee \chi\text{” } \to (\mathrm{Sat}_{\alpha+1}(\ulcorner\varphi\urcorner, x) \leftrightarrow \mathrm{Sat}_{\alpha+1}(\ulcorner\psi\urcorner, x) \;\vee\; \mathrm{Sat}_{\alpha+1}(\ulcorner\chi\urcorner, x) \;\&$$

$$\& \text{ “}\varphi = \neg\psi\text{” } \to ((\mathrm{Sat}_{\alpha+1}(\ulcorner\varphi\urcorner, x) \leftrightarrow \neg\,\mathrm{Sat}_{\alpha+1}(\ulcorner\psi\urcorner, x) \;\&$$
$$\& \text{ “}\varphi = \exists y \psi(y, x)\text{” } \to \exists b \;\mathrm{Sat}_{\alpha+1}(\ulcorner\psi\urcorner, \{b\} \cup x))$$

(i.e. $\mathrm{Sat}_{\alpha+1}$ is a satisfaction relation).
If β is a limit ordinal, $M_\beta = \aleph_\beta$, $\mathrm{Sat}_\beta(x,y) \leftrightarrow \exists\alpha < \beta\ \mathrm{Sat}_\alpha(x,y)$. Now define the interpretation of M as follows:

$$\mathrm{dom}(x) \leftrightarrow x = x, \quad \mathrm{Sat}(x,y) \leftrightarrow \exists z\, \mathrm{Sat}_z(x,y). \ \blacksquare$$

In this construction, $\mathrm{Th}(M) = \mathrm{Th}(\omega_V)$. This is by no means necessary. We could start with any nonstandard model of any complete $T \in \mathrm{SSy}(V)$.

Corollary 35 *If* $V \models \mathrm{ZFC}$ *is countable then the linear orders* $\mathbb{Q}$ *and* $\mathbb{N}+\mathbb{QZ}$ *are interpreted in* V*. There are bounded ('set-like') and unbounded ('class-like') interpretations.*

Note that neither $\mathbb{Q}$ nor $\mathbb{N}+\mathbb{QZ}$ can be 'set-like' (boundedly) interpreted in a model of PA because, by Lemma 25, every linear order boundedly interpreted in a model of PA will have to have a last element.

Plausible arithmetics

Definition 21
TA*, the Theory of Arithmetic, is the following collection of statements:*

$$\mathrm{TA} = \{\varphi \in L_{\mathrm{PA}} \mid \mathrm{ZFC} \vdash \varphi^*\}.$$

TA is the most true (and, hence, the most important) of all extensions of PA as it contains all arithmetical statements we shall ever be able to prove from ZFC. TA contains (PH), (KM), $\mathrm{Con}_{\mathrm{PA}}$, $\mathrm{Con}_{\mathrm{PA}\,+\,\mathrm{Con}_{\mathrm{PA}}}$ and all first-order consequences of n^{th} order arithmetic.

Definition 22 *The language of* n^{th} *order arithmetic,* L^n_{PA}*, is*

$$L_{\mathrm{PA}} \cup \{S^{(1)}(u)\} \cup \{S^{(k)}_m(u)\}_{k\le n, m\in\mathbb{N}} \cup \{\in^{(k)}_m\}_{k\le n, m\in\mathbb{N}}$$

with the intended interpretation: $S^{(1)}(x)$ *means "*x *is an element, i.e. a first-order element",* $S^{(2)}_m(X)$ *means "*X *is an* m*-ary relation on first-order elements", i.e. a second-order element,* $S^{(k+1)}_m(Y)$ *means "*Y *is an* m*-ary relation on* k*-order elements". I shall usually omit* $S^{(k)}_m(X)$ *and instead simply attach indices to a letter:* $X^{(k)}(u_1,\dots,u_m)$*.*
The formula $\in^{(k)}_m(X^{(k)}_1,\dots,X^{(k)}_m,Y^{(k+1)})$ *means "the* m*-tuple* $(X^{(k)}_1,\dots,X^{(k)}_m)$ *belongs to* $Y^{(k+1)}$*". In writing, the formula* $\in^{(k)}_m(X^{(k)}_1,\dots,X^{(k)}_m,Y^{(k+1)})$ *will be shortened to* $Y^{(k+1)}(X^{k}_1,\dots,X^{(k)}_m)$*.*

Definition 23 *The theory of n^{th} order arithmetic is the first-order theory in the language L^n_{PA} with the following axioms.*
PA^- *(*PA *without the induction scheme) +*
+ the second-order Induction axiom:

$$\forall X^{(2)}(u)\ ((X(0)\ \&\ \forall x\ (X(x) \to X(x+1)) \to \forall x\ X(x))$$

+ the Comprehension scheme:
for every n^{th} order formula $\varphi(X^{(k)})$, $k < n$, possibly containing parameters of any order, but having the (only) variable $X^{(k)}$ free,

$$\exists Y^{(k+1)} \forall X^{(k)}\ (Y(X)\ \leftrightarrow \varphi(X)).$$

It is known that n^{th} order arithmetic for $n \geq 2$ proves $\mathrm{Con}_{\mathrm{PA}}$ (see [31]).

Observation 36
TA *proves all first-order (L_{PA}) consequences of n^{th} order arithmetic.*

Proof
There is an obvious translation * of n^{th} order formulas into L_{ZFC}:
if φ is first-order, put $\varphi^* = \varphi^*$ in the old sense;
if φ contains higher-order relation symbols then substitute
$\forall X^{(2)}(u_1, \ldots, u_m)$ by $\forall r\ (r \subseteq \omega^m) \to$
and $X^{(2)}(x_1, \ldots, x_m)$ by $(x_1, \ldots, x_m) \in r$;
$\vdots$
$\forall X^{(k+1)}(u_1, \ldots, u_m)$ by $\forall s\ (s \subseteq (\mathscr{P}^{(k)}(\omega))^m) \to$
and $X^{(k+1)}(x_1, \ldots, x_m)$ by $(x_1, \ldots x_m) \in s$.

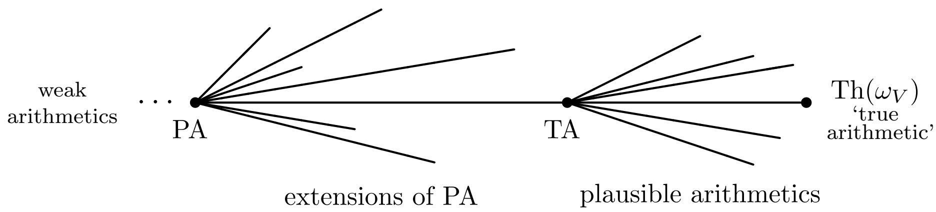
The translations of the axioms of PA^- + Induction follow from ZFC, obviously. The translations of instances of the Comprehension scheme follow from the comprehension scheme of ZFC. Proofs translate and transfer directly in the same way. ■

By Gödel, $\mathrm{Con}_{\mathrm{ZFC}} \notin$ TA. Let us show that TA is recursively aximomatizable. The set of all theorems of ZFC whose quantifiers are relativised to ω_{ZFC} is recursively enumerable and the map $\varphi \ \to\ \varphi^a$ is recursive, hence $\mathrm{TA} = \{\varphi \in L_{\mathrm{PA}} \mid$ there is $\psi \in L_{\mathrm{ZFC}}$ such that $\psi^a = \varphi\}$ is recursively enumerable, hence, by Craig's trick, TA is recursively axiomatizable. Hence, by Gödel's Theorem, TA has 2^ω completions.

Question 16 *(probably a very hard one)*
Find a mathematical statement in the language L_{PA} independent of TA*.*

(By a 'mathematical' statement I mean something different from $\mathrm{Con}_{\mathrm{ZFC}}$ or $\mathrm{Con}_{\mathrm{ZFC}+\mathrm{Con}_{\mathrm{ZFC}}}$. Remember the story of (PH) being the first 'mathematical' statement known to be independent of PA.)

Let us call consistent extensions of TA **plausible arithmetics**. Every 'true arithmetic' $\mathrm{Th}(\omega_V)$ (where $V \models \mathrm{ZFC}$) is plausible, obviously. Now, inside every universe $V \models \mathrm{ZFC}$, the picture looks like this:



The notion of a 'true arithmetic' is relative and depends on the universe. Also, for every plausible arithmetic T, there is a universe $U \models \mathrm{ZFC}$ such that $\omega_U \models T$.

Proof Let T be plausible. Then $\mathrm{ZFC} + \{\varphi^* \mid \varphi \in T\}$ is consistent. Let $V \models \mathrm{ZFC} + \{\varphi^* \mid \varphi \in T\}$. Then $\omega_V \models T$. ■

Now, the question is: is any model of any plausible arithmetic an ω_V in some universe $V \models \mathrm{ZFC}$?

Existence of an ω-standard model

If $U \models \mathrm{ZFC} +$ "there exists a strongly inaccessible cardinal" then obviously $V := \{x \in U \mid x <$ the first strongly inaccessible cardinal$\}$ is a model of ZFC such that $\omega_U \cong \omega_V$. I.e., assuming "there is a strongly inaccessible cardinal" there is a model of ZFC whose natural numbers are $\mathbb{N}$. Is $\mathrm{ZFC} + \mathrm{Con}_{\mathrm{ZFC}}$ enough to construct such a model? I don't know.

Fact (Wilmers)
If $U \models \mathrm{ZFC}$ and ω_U is nonstandard then ω_U is recursively saturated.

Sketch of the proof
Let $p(x) = \{\varphi_i(x,a)\}_{i\in\mathbb{N}}$ be a recursive set of formulas such that for all $n \in \mathbb{N}$, $\omega_U \models \exists x \bigwedge_{i<n} \varphi_i(x,a)$, i.e. $p(x)$ is a type. Then for all $n \in \mathbb{N}$, $U \models \exists x \in \omega_U\ (\omega_U \models \bigwedge_{i<n} \varphi_i(x,a))$ using Tarski's definition of truth in U. Then

$$A = \{n \in \omega_U \mid U \models \exists x \in \omega_U\ (\omega_U \models \bigwedge_{i<n} \varphi_i(x,a))\}$$

is definable in U and contains $\mathbb{N}$, hence contains a nonstandard point, $b \in \omega_U \setminus \mathbb{N}$. Hence $U \models (\omega_U \models \bigwedge_{i<b} \varphi_i(x,a))$, hence $\omega_U \models \bigwedge_{i<b} \varphi_i(x,a)$, that is

$p(x)$ is realised in ω_U. ■

Thus, if $U \models \mathrm{ZFC}$ is strongly interpreted in $V \models \mathrm{ZFC}$ then U and V would usually drastically disagree about ω_U. Model U will think that ω_U is the set of natural numbers $\mathbb{N}$ while model V will think that ω_U is a recursively saturated model of PA.

Question 17 *Prove that if* $\Omega \models \mathrm{TA}$ *is recursively saturated then there is* $V \models \mathrm{ZFC}$ *such that* $\omega_V \cong \Omega$.

Proposition 37
If $\Omega \models \mathrm{TA}$ *is resplendent then there is* $U \models \mathrm{ZFC}$ *such that* $\omega_U \cong \Omega$.

Proof
Consider $\Psi = \exists E\ \exists f\ \exists \mathrm{Sat}\ \ \Phi(E, f, \mathrm{Sat}) =$

$$\exists E(x,y), f(x), \mathrm{Sat}(x,y)$$
$$\text{“}(\Omega, E) \models \mathrm{ZFC}\text{”}\ \&$$
$$\&\ \forall x\ E(f(x), \omega)\ \&\ \forall y\ (E(y,\omega) \to \exists x\ f(x) = y)\ \&$$
$$\&\ \forall xyz\ [x + y = z \to \mathbf{card}(f(x) \sqcup f(y)) = f(z)\ \&$$
$$\&\ x \cdot y = z \to \mathbf{card}(f(x) \times f(y)) = f(z)\ \&\ x < y \to E(f(x), f(y))]$$

is a Σ^1_1-statement (where $\omega, \mathbf{card}, \times$ and $\sqcup$ are expressions in terms of E). Let us show that "$(\Omega, E) \models \mathrm{ZFC}$" is a Σ^1_1-statement. As ZFC is recursively axiomatised, there is a point $z \in \Omega$ coding the set of all axioms of ZFC. We shall identify each formula in the language $\{E\}$ with its Gödel number. The expression "$\varphi = \psi \vee \chi$" will mean "φ is the Gödel number of the disjunction of two formulas with Gödel numbers ψ and χ". Let $\theta_n(x) =$

$$x > n\ \&\ \forall i < x\, \mathrm{Sat}((z)_i, [\,])\ \&$$
$$\&\ \forall \varphi\ \forall a\ (\mathrm{form}(\varphi)\ \&\ \varphi \text{ has } \mathrm{len}(a) \text{ free variables } \longrightarrow$$
$$(\text{“}\varphi = \psi \vee \chi\text{”} \ \to\ (\mathrm{Sat}(\varphi, a) \leftrightarrow \mathrm{Sat}(\psi, a) \vee \mathrm{Sat}(\chi, a)))\ \&$$
$$(\text{“}\varphi = \neg\psi\text{”} \ \to\ (\mathrm{Sat}(\varphi, a) \leftrightarrow \neg\, \mathrm{Sat}(\psi, a)))\ \&$$
$$(\text{“}\varphi = \exists x_1 \psi(x_1, \overline{x})\text{”} \ \to (\mathrm{Sat}(\varphi, a) \leftrightarrow \exists b\ \ \mathrm{Sat}(\psi, \langle b, a\rangle)))).$$

As $\{\theta_n(x)\}_{n\in\mathbb{N}}$ is a recursive set of formulas, by Kleene's Theorem, there is a single Σ^1_1-sentence $\Theta(x)$ such that in any $M \models \mathrm{PA}$, $M \models \forall x\ (\Theta(x) \leftrightarrow \bigwedge_{n\in\mathbb{N}} \theta_n(x))$. Thus, "$(\Omega, E) \models \mathrm{ZFC}$" is expressed by the Σ^1_1-statement $\exists x \Theta(x)$.

Hence, Ψ is a Σ^1_1-statement, which means "there is a model of ZFC, whose natural numbers are Ω". Since $\mathrm{Th}(\Omega)$ is plausible then (as proved above) there is a model $M \models \mathrm{Th}(\Omega)$ such that there is $V \models \mathrm{ZFC}$ of cardinality $\mathrm{card}\, M$ with $\omega_V \cong M$.

Thus Ψ is consistent with $\mathrm{Th}(\Omega)$, hence, by resplendency, is realized. ■

Models of ZFC inside models of PA

Theorem 38 *If* $\Omega \models \mathrm{PA} + \mathrm{Con}(\mathrm{ZFC})$ *is resplendent then there is* $V \models \mathrm{ZFC}$ *strongly interpreted in* Ω *such that* $(\omega_V, <) \cong (\Omega, <)$ *and* $\mathrm{SSy}(\omega_V) = \mathrm{SSy}(\Omega)$*.*

Proof
By Arithmetised Completeness Theorem, there is a formula $\mathrm{Sat}(x, y)$, which defines a satisfaction class for a model $V \models \mathrm{ZFC}$ in Ω. Let ω_V be the point $x \in \Omega$ such that $\Omega \models \mathrm{Sat}(\ulcorner v = \omega \urcorner, x)$. Let

$$\Phi = \ \exists f$$

$$\forall x \, \mathrm{Sat}(\ulcorner v_1 \in v_2 \urcorner, \langle f(x), \omega_V \rangle) \ \&$$
$$\& \ \forall x_1 x_2 \ (x_1 \neq x_2 \to f(x_1) \neq f(x_2)) \ \&$$
$$\& \ \forall y (\mathrm{Sat}(\ulcorner v \in \omega \urcorner, y) \to \exists x \ f(x) = y)$$

(i.e. f is a bijection from Ω to ω_V)

$$\& \ \forall xy \ (x < y \to \mathrm{Sat}(\ulcorner v_1 \in v_2 \urcorner, \langle f(x), f(y) \rangle))$$

(i.e. f is an order-isomorphism).

Φ is consistent because in any countable $M \prec \Omega$,

$$(\omega_{V_M}, <) \ \cong \ (M, <) \ \cong \ \mathbb{N} + \mathbb{Q}\mathbb{Z}.$$

Hence, by resplendency, Φ is realised in Ω, i.e. ω_V is order-isomorphic to Ω. Also, $\mathrm{SSy}(\omega_V) = \mathrm{SSy}(\Omega)$ because $\Omega \prec_{\Delta_0} \omega_V$. ∎

However, in order to obtain so desired $\omega_V \cong \Omega$ we shall have to use the Reflection principles.

Definition 24
Let $M \models \mathrm{PA}$*. Let* $M \models$ *"*$f(a) =$ *the* a^{th} *point* x *such that* $\mathrm{axiom_{PA}}(x)$*". For every* $a \in M$ *define* $\mathrm{Reflection}_a(\mathrm{PA})$ *to be the scheme*

$$\{\varphi \to \mathrm{Con}(\bigwedge_{i<a} f(i) \wedge \varphi) \mid \varphi \in L_{\mathrm{PA}}\}.$$

$\mathrm{Reflection}(\mathrm{PA})$ *is the scheme* $\{\varphi \to \mathrm{Con_{PA}}(\varphi) \mid \varphi \in L_{\mathrm{PA}}\}$.

Obviously, $M \models \mathrm{PA} + \mathrm{Reflection}(\mathrm{PA})$ if and only if $M \models \mathrm{PA} + \mathrm{Reflection}_a(\mathrm{PA})$ for all $a \in M$. It is not surprising that conditions of Theorem 38 even with $M \models \mathrm{TA} + \mathrm{Con}(\mathrm{ZFC})$ do not imply $\omega_V \cong \Omega$ due to the following fact.

Observation 39 *(modification of a result by Kaye and Kotlarski [15])*
Let $M \models \mathrm{PA}$ *be recursively saturated. Then there is* $N \models \mathrm{Th}\, M$ *strongly interpreted in* M *if and only if there is* $a \in M \smallsetminus \mathbb{N}$ *such that* $M \models \mathrm{Reflection}_a(\mathrm{PA})$.

Proof
Suppose there is $N \models \mathrm{Th}\, M$ strongly interpreted in M. For every $n \in \mathbb{N}$, $\varphi \in L_{\mathrm{PA}}$,

$$M \models \varphi \to \mathrm{Con}(\bigwedge_{i<n} f(i) \wedge \varphi)$$

because otherwise (i.e. if for some $n \in \mathbb{N}$, $\varphi \in L_{\mathrm{PA}}$, $M \models \varphi \wedge \neg \mathrm{Con}(\bigwedge_{i<n} f(i) \wedge \varphi))$ there would be no model of $f(0) \wedge \ldots \wedge f(n) \wedge \varphi$ strongly interpreted in M, in particular no model of $\mathrm{Th}\, M$. Hence

$$p(x) = \{x > n\}_{n \in \mathbb{N}} \cup \{\varphi \to \mathrm{Con}(\bigwedge_{i<x} f(i) \wedge \varphi) \mid \varphi \in L_{\mathrm{PA}}\}$$

is a recursive type, hence, by recursive saturation, is realised by, say, $a \in M \smallsetminus \mathbb{N}$. Then $M \models \mathrm{Reflection}_a(\mathrm{PA})$.

Conversely, assume $M \models \mathrm{Reflection}_a(\mathrm{PA})$ for some $a > \mathbb{N}$. Let $e \in M$ code $\mathrm{Th}\, M$. By reflection, for every standard n, $M \models \mathrm{Con}(\bigwedge_{i \le n} (e)_i)$, hence, by overspill, $M \models \mathrm{Con}(\bigwedge_{i \le \alpha} (e)_i)$ for some nonstandard $\alpha < \mathrm{len}(e)$. Now, by Arithmetised Completeness Theorem, there is a model $N \models \bigwedge_{i \le \alpha} (e)_i$ strongly interpreted in M. As $\alpha > \mathbb{N}$, $N \models \mathrm{Th}\, M$. ∎

It is possible to weaken the assumption of recursive saturation (we only need that $\mathrm{Th}\, M$ is coded and $p(x)$ is realised) but in view of Wilmer's Theorem above, we consider only recursively saturated models anyway. So, clearly, $\Omega \models \mathrm{TA} + \mathrm{Reflection}_a(\mathrm{PA})$ for some $a > \mathbb{N}$ is a necessary condition for $\omega_V \cong \Omega$.

Question 18 *Prove that if* $\Omega \models \mathrm{TA} + \mathrm{Reflection}(\mathrm{PA})$ *is resplendent then there is* $U \models \mathrm{ZFC}$ *strongly interpreted in* Ω *such that* $\omega_U \cong \Omega$.

Let us discuss a possible solution to this question.

Definition 25 *Let* $\Omega \models \mathrm{PA}$ *be nonstandard,* T *be a set of first-order sentences in any language coded by* $t \in \Omega$. *Then* T^{Ω}_t *is the set of all (standard)* Ω*-consequences of (slightly overspilled)* T, *i.e. statements* $\varphi \in L_T$ *such that* $\Omega \models \mathrm{Pr}_{\mathrm{PC}}(\bigwedge_{i<\mathrm{len}(t)} (t)_i \to \varphi)$.

This definition is very sensitive to the choice of parameter t (different parameters t in general lead to different sets T^{Ω}_t). But once t is fixed we shall omit it from our notation and write T^{Ω}.

Let $M \models \mathrm{PA}$ be nonstandard, $z \in M \setminus \mathbb{N}$ code the set of all theorems of ZFC (again, this is possible because the set of all theorems of ZFC is recursively enumerable) so that $M \models \forall i < \mathrm{len}(z)\, \mathrm{form}_{\mathrm{ZFC}}((z)_i)$ (this is possible by overspill). From the point of view of M there are at least three variants of TA:

1. True $\mathrm{TA} = \{\varphi \in L_{\mathrm{PA}} \mid \mathbb{N} \models \mathrm{Pr}_{\mathrm{ZFC}}(\varphi^*)\} =: \mathrm{TA}^{\mathbb{N}}$.
   $\mathrm{TA}^{\mathbb{N}}$ is recursively enumerable, hence is definable in M. Let

   $$M \models e = \text{ the least element } x \text{ such that}$$

   $$\forall i < \mathrm{len}(z)\ (\ (z)_i \text{ is relativised to } \omega\ \rightarrow\ \exists j < \mathrm{len}(x)\ ((z)_i)^a = (x)_j))\ \&$$

   $$\&\forall j < \mathrm{len}(x) \exists i < \mathrm{len}(z)((z)_i \text{ is relativised to } \omega \text{ and } ((z)_i)^a = (x)_j.$$

   Obviously, e codes $\mathrm{TA}^{\mathbb{N}}$.

2. $\mathrm{TA}^M = \{\varphi \in L_{\mathrm{PA}} \mid M \models \mathrm{Pr}_{\mathrm{PC}}(\bigwedge_{i<\mathrm{len}(e)}(e)_i \rightarrow \varphi)\}$ - the set of all M-consequences of (slightly overspilled) $\mathrm{TA}^{\mathbb{N}}$.

3. $(\mathrm{ZFC}^M)^a = \{\varphi \in L_{\mathrm{PA}} \mid M \models \mathrm{Pr}_{\mathrm{PC}}(\bigwedge_{i<\mathrm{len}(z)}(z)_i \rightarrow \varphi^*)\}$ - the set of all arithmetical sentences which are M-consequences of (slightly overspilled) ZFC.

(There are also other (non-definable) versions, such as $\bigcup_{n\in\omega}\{\varphi \in L_{\mathrm{PA}} \mid M \models \mathrm{Pr}_{\mathrm{PC}}(\bigwedge_{i<n}(e)_i \rightarrow \varphi)\}$ which reflect M-proofs more accurately and may be useful in the future, but I am not going to mention them again.)
Obviously, $\mathrm{TA} \subseteq \mathrm{TA}^M \subseteq (\mathrm{ZFC}^M)^a$. Also, in $\mathbb{N}$ all three notions coincide: $\mathrm{TA} = \mathrm{TA}^{\mathbb{N}} = (\mathrm{ZFC}^{\mathbb{N}})^a$, by the definition of TA.

Observation 40 *If $M \models \mathrm{PA} + \Pi_1 \,\mathrm{Th}\, \mathbb{N}$ then for any $z \in M$ coding* ZFC *and e chosen as above,* $\mathrm{TA} = \mathrm{TA}^M = (\mathrm{ZFC}^M)^a$.

Proof
Let $M \models \exists x \psi(x)$ where $\psi(x) \in \Delta_0$. If $\mathbb{N} \models \neg\exists x\psi(x)$ then $\forall x \neg\psi(x) \in \Pi_1 \,\mathrm{Th}\, \mathbb{N}$ then $M \models \forall x \neg \psi(x)$, contradiction. Hence, $\mathbb{N} \models \Sigma_1 \,\mathrm{Th}\, M$.

Let $\varphi(x)$ be $\mathrm{form}_{\mathrm{PA}}(x)\ \&\ \mathrm{Pr}_{\mathrm{ZFC}}(x^*)$. If $M \models n \in (\mathrm{ZFC}^M)^a$ then $M \models \varphi(n)$ (if n follows from an M-finite fragment of ZFC then it follows from full ZFC) then, as $\mathbb{N} \models \Sigma_1 \,\mathrm{Th}\, M$, $\mathbb{N} \models \varphi(n)$, that is $n \in \mathrm{TA}$. ■

Question 19 *Is it true that in every $M \models$* PA *there is z coding* ZFC *such that* $\mathrm{TA}^M = (\mathrm{ZFC}^M)^a$*?*

Theorem 41 *Let $\Omega \models (\mathrm{ZFC}^{\Omega})^a$ be resplendent. Then there is $U \models$* ZFC *strongly interpreted in Ω such that $\omega_U \cong \Omega$.*

Note that any resplendent model of $\mathrm{TA} + \Pi_1 \operatorname{Th}\mathbb{N}$ satisfies the conditions of Theorem 41.

Proof
Let ZFC be coded by $z \in \Omega \setminus \mathbb{N}$, each $(z)_i$ denoted by ψ_i. Let $\operatorname{Th}\Omega$ be coded by $a \in \Omega \setminus \mathbb{N}$, each $(a)_i$ denoted by φ_i. Let us show that for arbitrary $n \in \mathbb{N}$,

$$\Omega \models \mathrm{Con}(\psi_1 \wedge \ldots \wedge \psi_n \wedge \varphi_1^* \wedge \ldots \wedge \varphi_n^*).$$

Suppose $\Omega \models \mathrm{Pr}_{\mathrm{PC}}(\psi_1 \wedge \ldots \wedge \psi_n \to \neg(\varphi_1^* \wedge \ldots \wedge \varphi_n^*))$. Then, $\neg(\varphi_1^* \wedge \ldots \wedge \varphi_n^*) \in \mathrm{ZFC}^{\Omega}$, hence, as $\Omega \models (\mathrm{ZFC}^{\Omega})^a$, $\neg(\varphi_1 \wedge \ldots \wedge \varphi_n) \in \operatorname{Th}\Omega$, contradiction. Hence, $\Omega \models \mathrm{Con}(\psi_1 \wedge \ldots \wedge \psi_n \wedge \varphi_1^* \wedge \ldots \wedge \varphi_n^*)$.

Now, by overspill, there is $\alpha \in \Omega$ such that

$$\mathbb{N} < \alpha < \min\{\mathrm{len}(a), \mathrm{len}(z)\}$$

and $\Omega \models \mathrm{Con}(\bigwedge_{i<\alpha} \psi_i \wedge \bigwedge_{i<\alpha} \varphi_i^*)$. By the Arithmetised Completeness Theorem, there is $U \models \mathrm{ZFC} + (\operatorname{Th}\Omega)^*$ strongly interpreted in Ω by means of a satisfaction relation $\mathrm{Sat}(x, y)$. (Notice that since we have just proved that $\Omega \models (\mathrm{ZFC}^{\Omega})^a$ implies $\Omega \models \mathrm{Con}(\operatorname{Th}\Omega)$, and hence, by Observation 39 there is $\alpha > \mathbb{N}$ such that $\Omega \models \mathrm{Reflection}_\alpha(\mathrm{PA})$, the condition "$\Omega \models (\mathrm{ZFC}^{\Omega})^a$" should be regarded as one from the family of Reflection principles.)

As in Theorems 37 and 38 above, let

$$\Psi = \ \exists f \, (\ \ \forall x \ \ \mathrm{Sat}(\ulcorner v_1 \in v_2 \urcorner, \langle f(x), \omega_U \rangle) \ \&$$

$$\& \ \forall x_1 x_2 \ (x_1 \neq x_2 \to f(x_1) \neq f(x_2)) \ \&$$

$$\& \ \forall y \ (\mathrm{Sat}(\ulcorner v \in \omega \urcorner, y) \to \exists x \ f(x) = y)$$

(i.e. f is a bijection from Ω to ω_U) &

$$\& \ \forall xyz \ (x + y = z \ \to \ \mathrm{Sat}(\ulcorner u \sqcup v \cong w \urcorner, \langle f(x), f(y), f(z) \rangle)) \ \&$$

$$\& \ \forall xyz \ (x \cdot y = z \ \to \ \mathrm{Sat}(\ulcorner u \times v \cong w \urcorner, \langle f(x), f(y), f(z) \rangle)) \ \ ) \, .$$

Let us prove that Ψ is consistent. Let M be any countable recursively saturated elementary extension of $\mathrm{Cl}_{\Omega}(a, z)$ and let $V \models \mathrm{ZFC} + (\operatorname{Th}\Omega)^*$ be the model defined by $\mathrm{Sat}(x, y)$. By construction, $\operatorname{Th}(\omega_V) = \operatorname{Th}\Omega$. Also, ω_V is strongly interpreted in M, hence $\mathrm{SSy}(\omega_V) = \mathrm{SSy}(M)$ and ω_V is recursively saturated. Hence $\omega_V \cong M$. Thus, Ψ is consistent, hence by resplendency, is realised. Hence, $\omega_U \cong \Omega$. ■

But life would become much easier if Question 19 were settled positively, i.e. if $\mathrm{TA}^M = (\mathrm{ZFC}^M)^a$ in every model M.

Observation 42
If $\Omega \models \mathrm{TA} + \mathrm{Reflection(PA)}$ *is resplendent and* $\mathrm{TA}^\Omega = (\mathrm{ZFC}^\Omega)^a$ *then there is* $U \models \mathrm{ZFC}$ *strongly interpreted in* Ω *such that* $\omega_U \cong \Omega$.

(Thus, the positive solution of Question 19 would provide a characterisation of the theory of such models: there is a model Ω of $T \supset \mathrm{PA}$ strongly interpreting some $U \models \mathrm{ZFC}$ with $\omega_U \cong \Omega$ if and only if $T \supseteq \mathrm{TA} + \mathrm{Reflection(PA)}$.)

Proof
By reflection, $\Omega \models \mathrm{Con}(\mathrm{Th}\,\Omega)$. As $\mathrm{Th}\,\Omega$ is complete and $\Omega \models \mathrm{Con}(\mathrm{Th}\,\Omega)$, $(\mathrm{Th}\,\Omega)^\Omega = \mathrm{Th}\,\Omega$. $\mathrm{TA} \subset \mathrm{Th}\,\Omega$, hence $\mathrm{TA}^\Omega \subset (\mathrm{Th}\,\Omega)^\Omega = \mathrm{Th}\,\Omega$. As $(\mathrm{ZFC}^\Omega)^a = \mathrm{TA}^\Omega$, $\Omega \models (\mathrm{ZFC}^\Omega)^a$ and we find ourselves in the conditions of Theorem 41. ■

Comparison

Can methods developed for models of PA benefit the study of models of ZFC? I hope so. On the superficial level, models of both theories are very similar:

1. their 'ordinals' are linearly ordered, though a model of PA always coincides with its ordinals;
2. every non-empty definable subset has the least element;
3. ω is always an initial segment;
4. Standard System can be defined, and is always a Scott Set;
5. completeness is formalised inside a model.

Can a unified theory of such models be developed?

However, at some point differences occur as well. A conservative extension of a model of PA is always an end-extension. A conservative extension of a model of ZFC is always cofinal (i.e. does not add ordinals above all existing ordinals)[6]. Every model of PA has an elementary end-extension. There are models of ZFC having no elementary end-extensions (i.e. extensions with all new ordinals being greater than the existing ones) [6].

Important notice: most theorems in this section would hold if we replaced ZFC by any other foundational theory. We only assumed consistency of ZFC and the fact that $\mathrm{ZFC} \vdash$ "$\omega \vdash \mathrm{PA}$".

Before closing this section, let us list some open problems which are simply reformulations of established results or established open problems for the the case of ZFC.

Question 20 *What are order-types of ordinals of countable models of* ZFC*? How many of them are there? Can there be some classification result?*

General problem:

Question 21 *What are the order-types of ordinals of models of* ZFC*? How does the order-type of ordinals of a saturated model look like? How many models with the saturated order-type are there?*

Question 22 *Do classes of order-types of ordinals of models of different extensions of* ZFC *coincide?*

Problem 21 asks for a version of Pabion's Theorem, Problem 22 is a reformulation of Friedman's Problem for ZFC.

This area (interconnections between models of PA and ZFC) is very rich and profound. It deals with the very nature of mathematical objects and is waiting to be explored.

Chapter 5

Canonical orders

In this chapter we consider the simplest possible order-types of models of PA: the combinations of Q_λs and ordinals.

This chapter is a relative failure. It is short, and is called a "chapter" because I spent more time and effort on it than on any other two chapters. Also, I believe that the notion introduced here will play an important role in the future. To guarantee existence of Q_λs for regular λs I assume (GCH) throughout the chapter.

We shall use λ-good ultrafilters. For the definition, see [3].

Fact 1 *For every α, there is an α^+-good ultrafilter over α.*

Fact 2 *Ultraproducts over λ-good ultrafilters are λ-saturated.*

Definition

Definition 26 *The class of* ***canonical orders*** *is the smallest collection of linear orders such that*

1. *$\mathbb{N}$ is a canonical order;*
2. *if A is a canonical order then for any regular $\lambda \geq |A|$, any $\mu \leq \lambda^+$, $A + \mu Q_\lambda(A^* + A)$ is a canonical order.*

Let us list a first few canonical orders:

$\mathbb{N}$,

$\mathbb{N} + \mathbb{Q}\mathbb{Z} \quad (A = \mathbb{N},\ \lambda = \omega,\ \mu < \omega \text{ or } \mu = \lambda)$,

$\mathbb{N} + Q_\lambda\mathbb{Z} \quad (A = \mathbb{N},\ \lambda = \lambda,\ \mu < \omega \text{ or } \mu = \lambda)$,

$\mathbb{N} + \omega_1\mathbb{Q}\mathbb{Z} \quad (A = \mathbb{N},\ \lambda = \omega,\ \mu = \omega_1)$,

$\mathbb{N} + \mathbb{Q}\mathbb{Z} + Q_\lambda\mathbb{Q}\mathbb{Z} \quad (A = \mathbb{N} + \omega_1\mathbb{Q}\mathbb{Z},\ \lambda = \lambda,\ \mu < \omega \text{ or } \mu = \lambda)$,

$\mathbb{N} + \omega_1\mathbb{Q}\mathbb{Z} + Q_\lambda(\omega_1^* + \omega_1)\mathbb{Q}\mathbb{Z} \quad (A = \mathbb{N} + \omega_1\mathbb{Q}\mathbb{Z},\ \lambda = \lambda,\ \mu < \omega \text{ or } \mu = \lambda)$,

$\mathbb{N} + \omega Q_\lambda\mathbb{Z} \quad (A = \mathbb{N},\ \lambda = \lambda,\ \mu = \omega)$,

$\mathbb{N} + \omega_1 Q_\lambda \mathbb{Z}$,
...
$\mathbb{N} + \lambda^+ Q_\lambda \mathbb{Z}$.

Note that sometimes the same canonical order can be obtained by different means. If $A = \mathbb{N} + Q_\lambda \mathbb{Z}$ and $B = \mathbb{N}$ then $A + Q_\lambda(A^* + A) \cong B + Q_\lambda(B^* + B)$. Also, $\mathbb{N} + \omega Q_{\aleph_\omega+1}\mathbb{Z} \cong \mathbb{N} + \aleph_\omega Q_{\aleph_\omega+1}\mathbb{Z}$.

The notion of canonical order arises naturally when we start thinking about cells and saturation. I believe that for every canonical A and any $T \supseteq \mathrm{PA}$ there is $M \models T$ such that $(M, <) = A$. Let us prove two particular cases of it.

First example

Proposition 43 *If $T \supseteq \mathrm{PA}$ is complete and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are successor cardinals such that $2^\omega \leq \lambda_1 < \lambda_2 < \ldots < \lambda_n$ then there is $M \models T$ of order-type*

$$\mathbb{N} + Q_{\lambda_1}\mathbb{Z} + Q_{\lambda_2}Q_{\lambda_1}\mathbb{Z} + \cdots + Q_{\lambda_n} \ldots Q_{\lambda_2}Q_{\lambda_1}\mathbb{Z}.$$

Proof
Let M_0 be the model of T of order-type $\mathbb{N} + Q_{\lambda_n}\mathbb{Z}$ (unique by Pabion's Theorem). Suppose $0 \leq i < n$ and M_i is constructed having order-type

$$\mathbb{N} + Q_{\lambda_{n-i}}\mathbb{Z} + Q_{\lambda_{n-i+1}}Q_{\lambda_{n-i}}\mathbb{Z} + \cdots + Q_{\lambda_n} \ldots Q_{\lambda_{n-i+1}}Q_{\lambda_{n-i}}\mathbb{Z}.$$

As λ_{n-i-1} is a successor, there is a λ_{n-i-1}-good ultrafilter D on λ^-_{n-i-1} (where λ^-_{n-i-1} is the predecessor of λ_{n-i-1}). Put $M_{i+1} = \prod_D M_i$.

M_{i+1} is λ_{n-i-1}-saturated because D is λ_{n-i-1}-good. $\prod_D \mathbb{N}$ is an initial segment of M_{i+1} because, as λ_{n-i} is regular, no sequence of λ^-_{n-i-1} points of $M_i \setminus \mathbb{N}$ can reach the bottom of $M_i \setminus \mathbb{N}$. Also, $(\prod_D \mathbb{N}, <) = \mathbb{N} + Q_{\lambda_{n-i-1}}\mathbb{Z}$ because it is saturated. Also,

$$\prod_D (M_i \setminus \mathbb{N}) / \prod_D \mathbb{N} = Q_{\lambda_{n-i}} + \cdots + Q_{\lambda_n} \ldots Q_{\lambda_{n-i}}$$

because if $A, B \subset Q_{\lambda_{n-i+j}} \ldots Q_{\lambda_{n-i}}$, $|A|, |B| < \lambda_{n-i}, A < B$, then, as λ_{n-i} is regular, A and B can be separated by a point of $Q_{\lambda_{n-i+j}} \ldots Q_{\lambda_{n-i}}$. Thus M_{i+1} has order-type

$$\mathbb{N} + Q_{\lambda_{n-i-1}}\mathbb{Z} + Q_{\lambda_{n-i}}Q_{\lambda_{n-i-1}}\mathbb{Z} + \cdots + Q_{\lambda_n} \ldots Q_{\lambda_{n-i}}Q_{\lambda_{n-i-1}}\mathbb{Z}. \quad \blacksquare$$

Second example

Proposition 44 *Let $T \supseteq$ PA. Then for every regular λ and every $\mu \leq \lambda^+$ there is a model of T of order-type $\mathbb{N} + \mu Q_\lambda \mathbb{Z}$.*

Proof
Let us first prove that if M^* is the saturated model of cardinality λ and I is a proper initial segment of M^* then there is a proper initial segment K such that

$$I \subset K \cong M^*.$$

Let N be any elementary end-extension of M^*, $a \in M^* \smallsetminus I$. By universality, there is an elementary embedding $f\colon N \hookrightarrow M^*$ such that $f(a) = a$. Observe that $f(M^*) \prec M^*$ because $M^* \prec N$. Let

$$K = \{x \in M^* \mid \exists y > x,\ y \in f(M^*)\}.$$

By Gaifman's Splitting Theorem, K is an elementary submodel of M^*. Also, K is a proper initial segment of M^* because it is bounded by $f(b)$ where $b \in N \smallsetminus M^*$. $I \subset K$ because $a \in K$. By Pabion's Theorem, as $(K, <) \cong \mathbb{N} + Q_\lambda \mathbb{Z}$, K is saturated, hence isomorphic to M^*.

Let M_0 be the saturated model embedded as an initial segment into M^*. Suppose at stage $\alpha < \lambda^+$ we already have an initial segment $M_\alpha \models T$ of order-type $\mathbb{N} + \alpha Q_\lambda \mathbb{Z}$. If M_α is a proper initial segment let $M_{\alpha+1}$ be a saturated elementary end-extension of M_α that is again a proper initial segment. If $M_\alpha = M^*$, let $\tilde{M}^* \succ M^*$ be a saturated end-extension of M^* and choose $M_{\alpha+1}$ as before.

Now, $M_\mu = \bigcup_{\alpha<\mu} M_\alpha$ is a model of order-type $\mathbb{N} + \mu Q_\lambda \mathbb{Z}$ as required. ■

Pabion's Theorem and the above proof may lead us to the suggestion that the model of order-type $\mathbb{N} + \mu Q_\lambda \mathbb{Z}$ is unique for all $\mu \leq \lambda^+$.

Counterexamples for $\mu = \omega$

Proposition 45 *For any regular λ there are three pairwise non-isomorphic elementarily equivalent models of order-type $\mathbb{N} + \omega \mathbb{Q}_\lambda \mathbb{Z}$.*

Proof
Let T be a completion of PA, $M \models T$ have order-type $\mathbb{N} + Q_\lambda \mathbb{Z}$. Let $[\overline{\mathrm{Cl}\,\varnothing}]$ be the convex closure of $\mathrm{Cl}_M\,\varnothing$ in M. By Gaifman's Theorem, $[\overline{\mathrm{Cl}\,\varnothing}] \prec M$. As $\mathrm{cf}([\overline{\mathrm{Cl}\,\varnothing}]) = \omega$ and $\mathrm{lcf}([\overline{\mathrm{Cl}\,\varnothing}] \smallsetminus \mathbb{N}) = \lambda$, $([\overline{\mathrm{Cl}\,\varnothing}], <) \cong \mathbb{N} + \omega Q_\lambda \mathbb{Z}$. Also, $[\overline{\mathrm{Cl}\,\varnothing}]$ is not isomorphic to M_ω from Proposition 44 because $\mathrm{Cl}\,\varnothing$ is bounded in M_ω and unbounded in $[\overline{\mathrm{Cl}\,\varnothing}]$.

Let $a \in M$, $a > \mathrm{Cl}\,\varnothing$ and $[\overline{\mathrm{Cl}(a)}]$ be the convex closure of $\mathrm{Cl}_M(a)$ in M. Then $[\overline{\mathrm{Cl}(a)}]$ is not isomorphic to M_ω because for every $c \in M_\omega$, $\mathrm{Cl}_{M_\omega}(c)$ is bounded in M_ω. Also, $[\overline{\mathrm{Cl}(a)}]$ is not isomorphic to $[\overline{\mathrm{Cl}\,\varnothing}]$ because $\mathrm{Cl}\,\varnothing$ is bounded in $[\overline{\mathrm{Cl}(a)}]$. ■

I believe that by varying the type of a we can obtain many more pairwise non-isomorphic models of the form $[\overline{\mathrm{Cl}(a)}]$ (and, hence, of order-type $\mathbb{N}+\omega\mathbb{Q}_\lambda\mathbb{Z}$).

Question 23 *How many models of order-type* $\mathbb{N}+\omega Q_\lambda\mathbb{Z}$ *are there?*

Question 24 *Are there non-isomorphic models of order-type* $\mathbb{N}+\omega_2 Q_{\omega_1}\mathbb{Z}$*? What happens in the general case?*

A spectrum-like situation will appear in the general case. For a canonical order A, define $i(A)$ = the number of non-isomorphic models of $T \supseteq \mathrm{PA}$ of order-type A. We already know: $i(\mathbb{N}) = 1$, $i(\mathbb{N}+\mathbb{Q}\mathbb{Z}) = 2^\omega$, $i(\mathbb{N}+Q_\lambda\mathbb{Z}) = 1$, $i(\mathbb{N}+\omega Q_\lambda\mathbb{Z}) \geq 3$. Theorem 50 will show that $i(\mathbb{N}+\omega_1\mathbb{Q}\mathbb{Z}) = 2^{\omega_1}$.

In order to construct examples of non-isomorphic models of order-type $\mathbb{N}+\mu Q_\lambda\mathbb{Z}$ we may need to solve the following problem.

Question 25 *Find four models,* M_1, $\tilde{M}_1$, M_2, $\tilde{M}_2$, *all copies of the saturated model, such that* $\tilde{M}_1$ *is an elementary end-extension of* M_1, $\tilde{M}_2$ *is an elementary end-extension of* M_2, *but there is no isomorphism* $f\colon \tilde{M}_1 \to \tilde{M}_2$ *such that* $f(M_1) = M_2$.

Suggestion: try different lower cofinalities of $\tilde{M}_1 \setminus M_1$ and $\tilde{M}_2 \setminus M_2$.

Future research

Question 26
Prove that there is a model of PA *of every canonical order-type.*

The success in proving Propositions 43 and 44 is due to the fact that we could start with a saturated model and build the model of desired order-type around it. However, the situation is very different if our canonical order-type has a countable initial segment. We can not start with a saturated model because if M is saturated and $N \succ_{\Delta_0} M$ then $\mathrm{SSy}(N) = \mathscr{P}(\omega)$, hence, by Lemma 3 each initial segment of N is uncountable.

The first two obstacles we encounter are order-types $\mathbb{N}+\mathbb{Q}\mathbb{Z}+Q_{\omega_1}\mathbb{Q}\mathbb{Z}$ and $\mathbb{N}+\omega_1\mathbb{Q}\mathbb{Z}+Q_{\omega_1}(\omega_1^*+\omega_1)\mathbb{Q}\mathbb{Z}$.

Question 27
Prove that there is a model of PA *of order-type* $\mathbb{N}+\mathbb{Q}\mathbb{Z}+Q_{\omega_1}\mathbb{Q}\mathbb{Z}$.

Question 28 *Prove that there is a model of* PA *of order-type* $\mathbb{N} + \omega_1\mathbb{Q}\mathbb{Z} + Q_{\omega_1}(\omega_1^* + \omega_1)\mathbb{Q}\mathbb{Z}$.

If we attempt to construct models of these order-types as chains, we may need to solve the following problem.

Question 29
Prove that there is a countable $M \models \mathrm{PA}$ *such that for every countable end-extension* $N \succ M$ *and subsets* $A, B \subset N/M$, $A < B$, *there is* $K \succ N$ *such that there is* $c \in K$ *such that* $A < c < B$ *and* $\{x \in K \mid \exists y \in M,\ y \geq x\} = M$.

Can arithmetic saturation (see Chapter 7) in the case $\mathrm{Th}\, M = \mathrm{Th}\, \mathbb{N}$ help construct by back-and-forth an elementary embedding $f\colon N \to N$ fixing M such that there is a point $x \in N$ such that $f(A) < x < f(B)$? This is a future project.

Chapter 6

Counting

6.1 Order-types of cardinality λ

Theorem 46 *Let λ be an uncountable cardinal. For any consistent theory $T \supseteq \mathrm{PA}$ there are 2^λ order-types of models of T of cardinality λ.*

Lemma 47 *If $M \models \mathrm{PA}$ then for every uncountable regular $\lambda \geq \operatorname{card} M$, there are 2^λ order-types of elementary end-extensions of M of cardinality λ.*

Proof
We know from Corollary 15 that every model M has an end-extension $N_1 \succ M$ such that $\text{⅄}(N_1 \smallsetminus M) = \omega$, $\operatorname{card} N_1 = \operatorname{card} M$ and an end-extension $N_2 \succ M$ such that $\text{⅄}(N_2 \smallsetminus M) = \omega_1$ and $\operatorname{card} N_2 = \max\{\operatorname{card} M, \omega_1\}$. Let $\{A_\alpha \mid \alpha < \lambda\}$ be a partition of λ into λ stationary subsets. For every set $W \subseteq \lambda$, define the model M^W as the limit of the following elementary chain. Let $M^W_0 = M$. Suppose for all $\gamma < \alpha$, M^W_γ are already defined. Denote $\bigcup_{\gamma<\alpha} M^W_\gamma$ by K_α. Let

$$M^W_\alpha = \begin{cases} \text{an end-extension } N \succ K_\alpha \text{ such that } \text{⅄}(N \smallsetminus K_\alpha) = \omega \\ \qquad \text{and } \operatorname{card} N = \operatorname{card} K_\alpha, \text{ if } \alpha \in \bigcup_{\beta \in W} A_\beta \\ \text{an end-extension } N \succ K_\alpha \text{ such that } \text{⅄}(N \smallsetminus K_\alpha) = \omega_1 \\ \qquad \text{and } \operatorname{card} N = \max\{\operatorname{card} K_\alpha, \omega_1\}, \text{ if } \alpha \notin \bigcup_{\beta \in W} A_\beta. \end{cases}$$

Now we are going to prove that if W and U are different subsets of λ then $(M^W, <) \not\cong (M^U, <)$. Suppose $f \colon (M^W, <) \to (M^U, <)$ is an order-isomorphism. Introduce the functions $i, j : \lambda \to \lambda$ as

$$i(\alpha) = \min\{\gamma \in \lambda \mid f(K^W_\alpha) \subseteq K^U_\gamma\}$$

$$j(\beta) = \min\{\delta \in \lambda \mid f^{-1}(K^U_\beta) \subseteq K^W_\delta\}.$$

Let $\mathrm{Fix}(i)$ and $\mathrm{Fix}(j)$ be the sets of all fixed points of i and j respectively. As for every limit ordinal μ, $\bigcup_{\alpha<\mu} K^W_\alpha = K^W_\mu$ and, hence $\bigcup_{\alpha<\mu} f(K^W_\alpha) = f(\bigcup_{\alpha<\mu} K^W_\alpha) = f(K^W_\mu)$, thus $\sup_{\alpha<\mu} i(\alpha) = i(\mu)$, i.e. i is continuous. Also, i is unbounded, hence normal, hence $\mathrm{Fix}(i)$ is a club. By the same argument, $\mathrm{Fix}(j)$ is a club.

Let $\gamma \in \lambda$ be such that $\gamma \in W \Leftrightarrow \gamma \notin U$. As A_γ is stationary, there is a point $\xi \in \lambda$ such that

$$\xi \in \mathrm{Fix}(i) \cap \mathrm{Fix}(j) \cap A_\gamma.$$

As $\xi \in \mathrm{Fix}(i) \cap \mathrm{Fix}(j)$, $f|_{K^W_\xi}\colon (K^W_\xi, <) \cong (K^U_\xi, <)$ is an order-isomorphism. However, by the construction of M^W_ξ and M^U_ξ, neither $(M^W_\xi \setminus K^W_\xi)$ can be mapped onto an initial segment of $(M^U_\xi \setminus K^U_\xi)$ nor $(M^U_\xi \setminus K^U_\xi)$ can be embedded into an initial segment of $(M^W_\xi \setminus K^W_\xi)$. Contradiction. Hence, $(M^W, <) \ncong (M^U, <)$. ■

Theorem 48 *Let $M \models \mathrm{PA}$. Then for every $\lambda > \max\{\mathrm{card}(M), \omega_1\}$, there are 2^λ order-types of λ-like elementary end-extensions of M.*

Proof
Case 1: λ is regular is given by Lemma 47. Notice that at every stage α, $\mathrm{card}\, M^W_\alpha = \mathrm{card}\, K^W_\alpha$, hence $\bigcup_{\alpha<\lambda} M^W_\alpha$ is λ-like.
Case 2: λ is singular, $\mathrm{cf}(\lambda) = \mu$.
If there is $\gamma < \lambda$ such that for all δ such that $\gamma \leq \delta < \lambda$,

$$2^\delta = 2^\gamma$$

then, by Bukovský–Hechler theorem (see page 8), $2^\gamma = 2^\lambda$. Now, given a regular δ such that $\mathrm{card}(M) < \gamma \leq \delta < \lambda$, by Lemma 47, we have $2^\delta = 2^\lambda$ order-types of elementary end-extensions of M of cardinality δ. Extend them to λ-like models. No pair of their order-types will be isomorphic because initial δ-cells have to map onto initial δ-cells.
Now, suppose for every $\gamma < \lambda$ there is δ such that $\gamma < \delta < \lambda$ and $2^\gamma < 2^\delta < 2^\lambda$. Then there is an increasing sequence $\{\lambda_i\}_{i<\mu}$ such that: $\lambda = \sum_{i<\mu} \lambda_i$, $\lambda_0 = \varnothing$, for all i, λ_i is regular and

$$2^{\Sigma_{j<i}\lambda_j} < 2^{\lambda_i} < 2^\lambda.$$

For every $g \in \prod_{i<\mu} 2^{\lambda_i} = \{g\colon \mu \to 2^\lambda \mid g(i) \in 2^{\lambda_i}\}$, we shall define M^g as an elementary chain so that $f \neq g$ implies $(M^f, <) \ncong (M^g, <)$. As $\mathrm{card}(\prod_{i<\mu} 2^{\lambda_i}) = 2^\lambda$, we shall have 2^λ of those models M^g.

Let $M^{\{\varnothing\}} = M$. Suppose for $i \in \mu$ and for all $f \in \prod_{j<i} 2^{\lambda_j}$, M^f are defined. Denote $\lim_{j<i} \lambda_j$ by δ. Notice that $\mathrm{card}(\prod_{j<i} 2^{\lambda_j}) = 2^\delta < 2^{\lambda_i}$. Enumerate all elements of $\prod_{j<i} 2^{\lambda_j}$ as $\{f_k \mid k \in 2^\delta\}$.

Let $\ell < 2^{\lambda_i}$ and suppose that for all $k \in 2^\delta$, all $j < \ell$, $M^{f_k,j}$ has been defined so that

$$(M^{f_k,j}, <) \not\cong (M^{f_n,m}, <) \qquad\qquad \text{if } \ k \neq n \text{ or } j \neq m.$$

Define $\{M^{f_k,\ell}\}_{k\in 2^\delta}$ to be a collection of λ_i-like models such that for every $k \in 2^\delta$, $M^{f_k,\ell}$ is an elementary end-extension of M^{f_k} and

$$(M^{f_k,j}, <) \not\cong (M^{f_n,\ell}, <) \qquad\qquad \text{if } \ k \neq n \text{ or } j \neq \ell.$$

It can be done by Lemma 47 because

$$\left|\left\{M^{f_k,j} \mid j < \ell,\ k \in 2^\delta\right\}\right| = \max\{2^\delta, \ell\} < 2^{\lambda_i}.$$

Hence, we have defined M^g for all $g \in \prod_{j<i} 2^{\lambda_j} \times 2^{\lambda_i} = \prod_{j\le i} 2^{\lambda_j}$.
For every $f \in \prod_{i\in\mu} 2^{\lambda_i}$, define

$$M^f = \bigcup_{i\in\mu} M^{f\restriction i}.$$

Suppose $\gamma < \mu$, $f(\gamma) \neq g(\gamma)$ and $h\colon (M^f, <) \to (M^g, <)$ is an order-isomorphism. Then $\bigcup_{i\le\gamma} M^{f\restriction i} \cong \bigcup_{i\le\gamma} M^{g\restriction i}$. Contradiction. ■

Corollary 49 *Let λ be uncountable.*

1. *If λ is regular then there are 2^λ order-types of λ-dense models.*
2. *Let λ be regular and*
$$\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda \\ \varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_\mu \end{pmatrix}$$
be a sort. If $\varepsilon_\mu = 0$ then there are 2^λ order-types of models of this sort. If $\varepsilon_\mu = 1$ then there are $2^{(\lambda^+)}$ order-types of models of this sort.
3. *Let*
$$\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots \\ \varepsilon_1 & \varepsilon_2 & \cdots \end{pmatrix}$$
be a λ-like sort, $\lambda > \omega_1$. Then there are 2^λ order-types of models of this sort.
4. *Let $T \supseteq \mathrm{PA}$ be consistent. Then there are 2^λ order-types of self-similar models of T of cardinality λ.*

Proof

1. Take M_0 to be λ-dense and apply Lemma 47.

2. Consider any model of the sort

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda \\ \varepsilon_1 & \varepsilon_2 & \cdots & 0 \end{pmatrix}$$

(it exists by Theorem 13). Consider 2^λ order-types of its elementary end-extensions of cardinality λ (use Lemma 47) if $\varepsilon_\mu = 0$ and $2^{(\lambda^+)}$ order-types of its λ^+-like end-extensions if $\varepsilon_\mu = 1$.

3. Follows the lines of the proof of Theorem 48.

4. Consider the new language $L \cup \{a\}$. The theory $S = \mathrm{PA} +$ "a codes T" $+ \mathrm{Con}(T)$ in the language $L \cup \{a\}$ is consistent. Consider 2^λ order non-isomorphic λ-like models of S. By applying the Arithmetised Completeness Theorem to each of them, we obtain 2^λ inner models (hence, self-similar) of T of which no pair can be order-isomorphic because the unions of their initial δ-cells for $\delta < \lambda$ are not order-isomorphic. ■

Question 30 *Extend Corollary 49 (1) and (2) to the singular case.*

Note, that our use of Corollary 15 in this section was essential. Lemma 47, Theorem 48 and Corollary 49 are not true for models of ZFC because there are models of ZFC without elementary end-extensions [6].

Question 31 *Prove that there are* 2^λ *order-types of resplendent models of* PA *of cardinality* λ.

For this you may want to use Shelah's methods of constructing many non-isomorphic structures in a pseudoelementary class and the following theorem:

Theorem (J. Schmerl [24])
Let $T \supseteq \mathrm{PA}$. *Then there is an indiscernible type* Σ *such that whenever* $N \models T$ *is generated by a set* I *of indiscernibles having indiscernible-type* Σ *and having no last element then* N *is resplendent.*

6.2 ω_1-like models

We already know that all ω_1-like models of PA have the same order-type $\mathbb{N} + \omega_1\mathbb{Q}\mathbb{Z}$.

Theorem 50
For any $T \supseteq \mathrm{PA}$ *there are* 2^{ω_1} *models of* T *of order-type* $\mathbb{N} + \omega_1\mathbb{Q}\mathbb{Z}$.

Proof
We can easily obtain 2^ω ω_1-like models by taking 2^ω countable models non-embeddable in each other and considering their end-extensions of order-type $\mathbb{N} + \omega_1\mathbb{Q}\mathbb{Z}$. Let us find 2^{ω_1} of them.

Firstly, by Corollary 15, if $N \models \mathrm{PA}$ and $p(x)$ is a Gaifman type then there is an end-extension $K_1 \succ N$ such that $\{x \in K_1 \mid p(x)\}$ is unbounded below in $K_1 \smallsetminus N$. Let us prove that there is an end-extension $K_2 \succ N$ such that $\{x \in K_2 \mid p(x)\}$ is not unbounded below in $K_2 \smallsetminus N$. Let K_2 be an end-extension obtained from N by means of a Gaifman type indiscernible over N, (see proof of Theorem 13 or [14], pages 97-101), i.e. $K_2 = \mathrm{Cl}(N \cup \{c\})$, where c satisfies such a type $p(x)$. Let $t(c, \overline{a}) \in K_2 \smallsetminus N$, $t(c, \overline{a}) < c$, $\overline{a} \in N$. Let us show that then there is a formula $\varphi(x, \overline{a})$ such that $K_2 \models \varphi(c, \overline{a}) \wedge \neg\varphi(t(c, \overline{a}), \overline{a})$.

Define $t^{(u)}(x, \overline{a})$ as the u^{th} iteration of t, i.e.

$$K_2 \models t^{(0)}(x, \overline{a}) = x \;\wedge\; \forall\, u\; t^{(u+1)}(x, \overline{a}) = t(t^{(u)}(x, \overline{a}), \overline{a}).$$

Put $g(x, \overline{a}) = \min\ u\ (t^{(u+1)}(x, \overline{a}) \geq t^{(u)}(x, \overline{a}))$. Obviously,

$$t(x, \overline{a}) < x \rightarrow g(x, \overline{a}) = g(t(x, \overline{a}), \overline{a}) + 1.$$

Hence, $g(c, \overline{a}) = g(t(c, \overline{a}), \overline{a}) + 1$, which implies that one of them is even and the other is odd. Hence c and $t(c, \overline{a})$ realise different types. Thus, K_2 is an elementary end-extension of N such that $\{x \in K_2 \mid p(x)\}$ is not unbounded below in $K_2 \smallsetminus N$.

Let $\{A_\gamma \mid \gamma < \omega_1\}$ be a partition of ω_1 into ω_1 stationary subsets. Let $W \subseteq \omega_1$, $M_0^W \models T$, $\mathrm{card}\, M_0^W = \omega$, $p(x)$ be a Gaifman type. Suppose for all $i < \alpha$, M_i^W is already defined. Denote $\bigcup_{i<\alpha} M_i^W$ by K_α^W. Define

$$M_\alpha^W = \begin{cases} \text{an end-extension } N \succ K_\alpha^W \text{ such that } N \text{ is countable and} \\ \{x \in N \smallsetminus K_\alpha^W \mid p(x)\} \text{ is unbounded below in } N \smallsetminus K_\alpha^W \\ \text{if } \alpha \in \bigcup_{\gamma \in W} A_\gamma \\ \text{an end-extension } N \succ K_\alpha^W \text{ such that } N \text{ is countable and} \\ \{x \in N \smallsetminus K_\alpha \mid p(x)\} \text{ is not unbounded below in } N \smallsetminus K_\alpha^W \\ \text{if } \alpha \notin \bigcup_{\gamma \in W} A_\gamma. \end{cases}$$

Put $M^W = \bigcup_{\alpha<\omega_1} M_\alpha^W$. Let $W \neq U$, $\gamma \in W \Leftrightarrow \gamma \notin U$. Suppose $f : M^W \longrightarrow M^U$ is an isomorphism. Define $i(\alpha) = \min\{\gamma \in \omega_1 \mid f(K_\alpha^W) \subseteq K_\gamma^U\}$, $j(\beta) = \min\{\delta \in \omega_1 \mid f^{-1}(K_\beta^U) \subseteq K_\delta^W\}$. Again, i and j are normal functions, hence $\mathrm{Fix}(i)$ and $\mathrm{Fix}(j)$ are clubs, so

$$C = \mathrm{Fix}(i) \cap \mathrm{Fix}(j) \cap A_\gamma \neq \varnothing.$$

Let $\beta \in C$. We observe that $f\colon K^W_\beta \cong K^U_\beta$, but f cannot be extended further, because in only one of $M^W_\beta \smallsetminus K^W_\beta$ and $M^U_\beta \smallsetminus K^U_\beta$, $\{x \mid p(x)\}$ is unbounded below. ■

It has been proved by Harnik [9] that all models of PA of order-type $\mathbb{N} + \omega_1\mathbb{Q}\mathbb{Z}$ have the same $(M, <, +)$-isomorphism type. The 2^{ω_1} models in our Theorem 50 all have different $(M, <, +, \cdot)$-isomorphism types (they are non-isomorphic); however can some of them be $(M, \cdot)$-isomorphic?

Question 32 *Prove that there are* 2^{ω_1} *models* $M_i \models \mathrm{PA}$, $i < 2^{\omega_1}$ *of order-type* $\mathbb{N} + \omega_1\mathbb{Q}\mathbb{Z}$ *such that* $(M_i, \cdot) \ncong (M_j, \cdot)$ *for* $i \neq j$.

The following Theorem solves this Question for the universes satisfying $(\diamondsuit)$. The proof generalises Harnik's proof in [9], where he constructs two such models. This is an example of the common situation in model theory where you can get 2^λ models for the price of just two.

Diamonds

Theorem 51 $(\diamondsuit)$ *There are* 2^{ω_1} *models* $M_i \models \mathrm{PA}$, $i < 2^{\omega_1}$ *of order-type* $\mathbb{N} + \omega_1\mathbb{Q}\mathbb{Z}$ *such that* $(M_i, \cdot) \ncong (M_j, \cdot)$ *for* $i \neq j$.

We shall use the following formulation of $(\diamondsuit)$ which is equivalent to the usual one (see [20], page 92 or [12], page 229):

$(\diamondsuit)$: *There is a sequence* $\{f_\alpha \mid \alpha < \omega_1\}$ *of functions* $f_\alpha\colon (1+\alpha)\omega \to (1+\alpha)\omega$ *such that for all* $F\colon (1+\omega_1)\omega \to (1+\omega_1)\omega$ *the set* $\{\alpha \mid F\restriction_{(1+\alpha)\omega} = f_\alpha\}$ *is stationary.*

If $M \models \mathrm{PA}$, let $p(M)$ be the set of all M-primes, N be an end-extension of M. We say that $X \subseteq p(M)$ is coded in N if for some $a \in N$,

$$\{x \in p(M) \mid N \models x|a\} = X.$$

Also, if K is an end-extension of N then the same subsets of $p(M)$ are coded in K as in N. Let $a \in K \smallsetminus N$ code $X \subseteq p(M)$, $c \in N \smallsetminus M$. Define

$$b = \prod_{x<c,\ \mathrm{pr}(x)|a} \mathrm{pr}(x)$$

where $\mathrm{pr}(x)$ means 'the xth prime'. As $b < (\mathrm{pr}(c))!$, $b \in N$. Also b codes X. The following fact from [23] or [32] will be used in the proof:

Proposition (#)
If $M \models \mathrm{PA}$ *is countable then there are continuum many* $X \subset p(M)$ *that are*

codable in some end-extension $N \succ M$.

Proof of Theorem 51

For every $f\colon \omega_1 \to 2$, we construct M^f as a union of elementary chain of countable models $\{M^{f\restriction\alpha}\}_{\alpha<\omega_1}$.

Let $M^{\varnothing}$ be an arbitrary countable model with domain ω. Suppose for all $\beta < \alpha$, $f \in 2^{\omega_1}$, $M^{f\restriction\beta}$ are already constructed having domains $(1+\beta)\omega$. If α is a limit ordinal, let $M^{f\restriction\alpha} = \bigcup_{\beta<\alpha} M^{f\restriction\beta}$. Obviously, $M^{f\restriction\alpha}$ has domain $(1+\alpha)\omega$.

Let $\alpha = \delta + 1$. As $\mathrm{card}\{f\colon \alpha \to 2\} \leq 2^{\omega} =^{\diamondsuit} \omega_1$, we can enumerate all functions $f\colon \alpha \to 2$ as $\{g_i\colon \alpha \to 2 \mid i < \omega_1\}$. Consider $M^{g_0\restriction\delta}$. Let M^{g_0} be any elementary end-extension of $M^{g_0\restriction\delta}$ with domain $(1+\delta+1)\omega$.

Suppose for all $j < i$, M^{g_j} is already constructed. i is a countable ordinal, and each M^{g_j}, $j < i$ codes only countably many subsets of $p(M^{g_j\restriction\delta})$, hence, by Proposition (#), there is $Y \subseteq p(M^{g_i\restriction\delta})$ such that $f_\alpha^{-1}(Y)$ is not coded in any M^{g_j}, $j < i$. Let M^{g_i} be any elementary end-extension of $M^{g_i\restriction\delta}$ which codes Y and has domain $(1+\delta+1)\omega$. Thus define M^{g_i} for all $i < \omega_1$. Now, for $h \in 2^{\omega_1}$, let $M^h = \bigcup_{\alpha<\omega_1} M^{h\restriction\alpha}$.

Let $f, h \in 2^{\omega_1}$, $\gamma < \omega_1$, $f(\gamma) \neq h(\gamma)$. Suppose there is $F\colon (1+\omega_1)\omega \to (1+\omega_1)\omega$, which is an isomorphism between $(M^f, \cdot)$ and $(M^h, \cdot)$. Then $\{\alpha \mid F\restriction_\alpha = f_\alpha\}$ is stationary, hence unbounded. Hence, for some $\alpha > \gamma$, $F\restriction_\alpha\colon (1+\alpha)\omega \to (1+\alpha)\omega$ is an isomorphism between $(M^{f\restriction\alpha}, \cdot)$ and $(M^{h\restriction\alpha}, \cdot)$. Let at stage $(\alpha+1)$ of our construction, $M^{f\restriction\alpha+1} = M^{g_j}$, $M^{h\restriction\alpha+1} = M^{g_i}$ with $j < i$. Let $Y \subseteq p(M^{h\restriction\alpha})$ be the set from the definition of M^{g_i}, coded in $M^{h\restriction\alpha+1}$ by the point a. Then $f_\alpha^{-1}(Y)$ is coded in M^f by $F^{-1}(a)$, hence is already coded in $M^{f\restriction\alpha+1}$, which contradicts our definition of Y. ■

We believe that $(\diamondsuit)$ is not necessary for the conclusion of Theorem 51 to be true and propose a more general Problem.

Question 33 *If $M \models$ PA has cardinality ω_1 and is not saturated then $(M, <)$ has 2^{ω_1} $(M, \cdot)$-non-isomorphic expansions to a model of* PA.

This hard problem is deeply connected to the material of Chapter 7, which also deals with non-isomorphic (and even not elementarily equivalent) expansions of linear orders to models of PA.

Question 34

Try tackling Question 10 in the universes satisfying $\diamondsuit$.

Chapter 7

Around Friedman's Problem

In this chapter we present our first two attempts to solve Friedman's problem in the resplendent case and investigate the connection with the notion of arithmetic saturation we encountered. Also, section 7.2 gives two other applications of arithmetic saturation.

Definition 27 *Let* $A \subseteq \mathbb{N}$ *be arbitrary. Let* $L = L_{\mathrm{PA}} \cup \{A(x)\}$*, where* $A(x)$ *is a new predicate symbol which will later be interpreted as* $x \in A$*. As constructed in [14], chapter 9, there is a* Σ_1*-formula* $\mathrm{Sat}_{\Sigma_1}(x, y)$ *in* L *such that for every* Σ_1*-formula* $\theta(v_1, \ldots, v_n)$ *in* L*,*

$$(\mathbb{N}, A) \models \forall x \ \mathrm{Sat}_{\Sigma_1}(\ulcorner\theta(v_1, \ldots, v_n)\urcorner, \langle (x)_1, \ldots, (x)_n \rangle) \leftrightarrow \theta((x)_1, \ldots, (x)_n).$$

Define $A' = \{n \in \mathbb{N} \mid (\mathbb{N}, A) \models \mathrm{Sat}_{\Sigma_1}((n)_1, (n)_2)\}$*.*

From this definition we can prove that if $\mathrm{SSy}(M)$ is closed under jump and $\theta(x)$ is a formula of L_{PA} then $\{n \in \mathbb{N} \mid \mathbb{N} \models \theta(n)\}$ is coded in M. In particular, $\Pi_n \operatorname{Th} \mathbb{N}$ is coded in M for every n because $\Pi_n \operatorname{Th} \mathbb{N}$ is defined in $\mathbb{N}$ by the formula "x is a Gödel number of a closed Π_n-formula" & $\mathrm{Sat}_{\Pi_n}(x, [\,])$. (Here, $\mathrm{Sat}_{\Pi_n}(x, y)$ is the Π_n-satisfaction predicate in L_{PA} (see [14], pp 126-128).)

Fact (Kirby and Paris)
Let $M \models \mathrm{PA}$ be recursively saturated. Then the following conditions are equivalent.

1. For any $f \in M$ coding a function $f\colon \mathbb{N} \to M$, there is $c \in M \setminus \mathbb{N}$ such that for all $n \in \mathbb{N}$, $f(n) > \mathbb{N} \Leftrightarrow f(n) > c$.

2. $\mathrm{SSy}(M)$ is closed under jump.

A model satisfying the above conditions is called **arithmetically saturated**. Arithmetic saturation is discussed in [16], pp. 253-256. Also, in this chapter we shall need the following variation of the Friedman's embedding theorem.

Fact Let $M, N \models \mathrm{PA}$, M be countable. Then for all $n \in \omega$, the following conditions are equivalent.

1. There is an embedding $h\colon M \to N$ such that $h(M) \prec_{\Sigma_n} N$.
2. $\mathrm{SSy}(M) \subseteq \mathrm{SSy}(N)$ and $N \models \Sigma_{n+1} \,\mathrm{Th}\, M$.

7.1 Resplendency and coding

We already know from Chapter 4 that if $M \models \mathrm{PA}$ is self-similar, $T \supseteq \mathrm{PA}$ is coded in M and $M \models \mathrm{Con}(T)$ then $(\mathrm{ACT}(M, T), <) \cong (M, <)$. In the resplendent case we do not need to assume $M \models \mathrm{Con}(T)$ as the following theorem shows.

Theorem 52 *If $M \models \mathrm{PA}$ is resplendent and $c \in M$ codes a consistent theory $T \supseteq \mathrm{PA}$ then $(M, <)$ can be expanded to a model of T.*

Proof

The following Σ^1_1-statement is realised in any countable elementary submodel of M containing c (since all countable models are order-isomorphic), hence is realized:

$$\exists \oplus, \otimes, \ll, \mathbb{O}, \mathbb{S}, \mathrm{Sat}$$
$$\Big(\forall xy \ (x \ll y \leftrightarrow x < y) \ \bigwedge \ \text{“}(\oplus, \otimes, \ll, \mathbb{O}, \mathbb{S}) \models T\text{”}\Big).$$

Let us write “$(\oplus, \otimes, \ll, \mathbb{O}, \mathbb{S}) \models T$” formally. Introduce new Gödel numbers for the symbols $\oplus, \otimes, \ll, \mathbb{O}, \mathbb{S}$ and extend this numeration to all formulas in the language $\{\oplus, \otimes, \ll, \mathbb{O}, \mathbb{S}\}$ in some standard way. From now on we shall identify each formula φ in the language $\{\oplus, \otimes, \ll, \mathbb{O}, \mathbb{S}\}$ with its Gödel number. Let $\theta_n(x, c) =$

$$(x > n) \ \& \ (\forall i < x \ \ \mathrm{Sat}((c)_i, [\,])$$
$$\& \ \forall \varphi \ \forall a \ (\mathrm{form}(\varphi) \ \& \ \varphi \text{ has } \mathrm{len}(a) \text{ free variables } \longrightarrow$$
$$(\text{“}\varphi = \psi_1 \vee \psi_2\text{”} \ \to \ (\mathrm{Sat}(\varphi, a) \leftrightarrow \mathrm{Sat}(\psi_1, a) \vee \mathrm{Sat}(\psi_2, a))) \ \&$$
$$(\text{“}\varphi = \neg\psi\text{”} \ \to \ (\mathrm{Sat}(\varphi, a) \leftrightarrow \neg \,\mathrm{Sat}(\psi, a))) \ \&$$
$$(\text{“}\varphi = \exists x_1 \psi(x_1, \overline{x})\text{”} \ \to (\mathrm{Sat}(\varphi, a) \leftrightarrow \exists b \ \mathrm{Sat}(\psi, \langle b, a\rangle))))).$$

As $\{\theta_n(x, c)\}_{n \in \mathbb{N}}$ is a recursive set of formulas, by Kleene's Theorem, there is a single Σ^1_1-sentence $\Theta(x, c)$ such that in any $M \models \mathrm{PA}$, $M \models \forall x \ (\Theta(x, c) \leftrightarrow \bigwedge_{n \in \mathbb{N}} \theta_n(x, c))$. Thus, “$(\oplus, \otimes, \ll, \mathbb{O}, \mathbb{S}) \models T$” is expressed by the Σ^1_1-statement $\exists \, x \ \Theta(x, c)$. ■

Corollary 53
If M is resplendent and $\mathrm{SSy}(M) = \mathscr{P}(\mathbb{N})$ *then for any consistent* $T \supseteq \mathrm{PA}$*,* $(M, <)$ *can be expanded to a model of* T*.*

Actually, what this Corollary requires from $\mathrm{SSy}(M)$ is that it contains all completions of PA. The following argument (by my supervisor) shows that this will imply $\mathrm{SSy}(M) = \mathscr{P}(\mathbb{N})$.

Fact(Kaye) *For every* $X \in \mathscr{P}(\mathbb{N})$ *there is a complete* $T^* \supset \mathrm{PA}$ *such that* X *is recursive in* T^**.*

Proof By Rosser's Theorem, given a recursive consistent $T \supseteq \mathrm{PA}$ (as usual we identify each formula of T with its Gödel number, so T is a recursive subset of $\mathbb{N}$), there are canonical sentences L_T and $R_T = \neg L_T$ such that $T + L_T$ and $T + R_T$ are both consistent, and this function $T \to L_T$ is computable. Let $T_0 = \mathrm{PA}$, $T_{n+1} = T_n \cup \{L_{T_n}\}$ if $n \in X$ and $T_{n+1} = T_n \cup \{R_{T_n}\}$ otherwise. Let T^* be any completion of $\bigcup_{n \in \mathbb{N}} T_n$. To determine if $n \in X$ first determine whether $i \in X$ for each $i < n$, thus obtaining T_n. Compute L_{T_n}. Ask if $L_{T_n} \in T^*$. Hence X is recursive in T^*. ■

Corollary 54 *If M is resplendent and ω_1-saturated then $(M, <)$ can be expanded to a model of any consistent extension of* PA*.*

(We also know that, by Pabion's Theorem, this expansion of M will have to be ω_1-saturated because it has an ω_1-saturated order-type. Can we also make it resplendent?)

Theorem 55
If M is resplendent and $\mathrm{cf}(M) > \omega$ *then for all* $n \in \omega$*,* $\Pi_n \operatorname{Th} \mathbb{N}$ *is coded in* M*.*

(Hence, as PA is recursive, by Theorem 52, for every $n \in \mathbb{N}$, $(M, <)$ is expandable to a model of $\mathrm{PA} \cup \Pi_n \operatorname{Th} \mathbb{N}$, i.e. $\operatorname{Th} \mathbb{N}$ can be "approximated" as closely as you want.)

Proof (Resembles the process of fishing.)
For any $n \in \omega$, let us introduce $\Sigma_n \operatorname{Def}$ = the set of all nonstandard definable points of M defined by a Σ_n-formula.

As $\mathrm{cf}(M) > \omega$, there is $a > \mathbb{N}$ such that $\Sigma_1 \operatorname{Def} > a$. Define $A_1 = \{\ulcorner \forall x \varphi(x) \urcorner \mid \varphi \in \Delta_0,\ M \models \forall x < a\ \varphi(x)\}$. Now, $A_1 \subseteq \Pi_1 \operatorname{Th} \mathbb{N}$ because $\mathbb{N} \prec_{\Delta_0} M$. Also, $\Pi_1 \operatorname{Th} \mathbb{N} \subseteq A_1$ because if for some $\varphi \in \Delta_0$ such that $\mathbb{N} \models \forall x \varphi(x)$ there existed $x < a\ \neg\varphi(x)$ then $\min x \neg\varphi(x)$ would be a nonstandard Σ_1-definable point less than a. Hence, $A_1 = \Pi_1 \operatorname{Th} \mathbb{N}$. A_1 is definable, hence coded in M.

Suppose at stage n we already know that $\Pi_n \operatorname{Th} \mathbb{N} \in \mathrm{SSy}(M)$. Let $b \in M$ code $\Pi_n \operatorname{Th} \mathbb{N}$. Consider the statement

$$\Phi_n(b) \;=\; \exists \oplus_n, \otimes_n, \ll_n, \mathbb{O}_n, \mathbb{S}_n$$

$$\forall xy\ (x \ll_n y \longleftrightarrow x < y)\ \bigwedge$$

$$\bigwedge \text{“}(\oplus_n, \otimes_n, \ll_n, \mathbb{O}_n, \mathbb{S}_n) \models \mathrm{PA} \cup \Pi_n \operatorname{Th} \mathbb{N}\text{”} \bigwedge$$

$$\bigwedge \text{“}\,\mathrm{SSy}(M, \oplus_n, \otimes_n, \ll_n, \mathbb{O}_n, \mathbb{S}_n)\ \subseteq \mathrm{SSy}(M)\text{”}.$$

Let us show that the last line is expressible by a Σ^1_1-sentence. Let $\varphi_m(x) = (x > m)\ \&\ \forall\, z\, \exists\, y\, \forall\, i < x\ ((z))_i = (y)_i$, where $((z))_i$ means $(z)_i$ in the language $\{\oplus_n, \otimes_n, \ll_n, \mathbb{O}_n, \mathbb{S}_n\}$. $\{\varphi_m(x)\}$ is a recursive set of formulas, hence, by Kleene's Theorem, there is a Σ^1_1-sentence $\Theta(x)$ such that in any $K \models \mathrm{PA}$, $K \models \forall\, x\, (\bigwedge_{m\in\mathbb{N}} \varphi_m(x) \leftrightarrow \Theta(x))$. Then $\mathrm{SSy}(M, \oplus_n, \otimes_n, \ll_n, \mathbb{O}_n, \mathbb{S}_n) \subseteq \mathrm{SSy}(M)$ is implied by the Σ^1_1-sentence $\exists x \Theta(x)$. Hence, $\Phi_n(b)$ is a Σ^1_1-sentence.

$\Phi_n(b)$ is consistent because, by Wilmers' Theorem, as $(\mathrm{PA} \cup \Pi_n \operatorname{Th} \mathbb{N}) \in \mathrm{SSy}(M)$, there is a countable model

$$N \models \mathrm{PA}\ \cup \Pi_n \operatorname{Th} \mathbb{N}, \quad \mathrm{SSy}(N) = \mathrm{SSy}(\mathrm{Cl}_M(b)).$$

Hence, by resplendency, $\Phi_n(b)$ is already realised in M.

Denote the model $(M, \oplus_n, \otimes_n, \ll_n, \mathbb{O}_n, \mathbb{S}_n)$ by M_n. By construction, $M_n \models \mathrm{PA} \cup \Pi_n \operatorname{Th} \mathbb{N}$, $(M_n, <) \cong (M, <)$, $\mathrm{SSy}(M_n) \subseteq \mathrm{SSy}(M)$.

Let $(\Sigma_n \operatorname{Def})_{M_n} > a > \mathbb{N}$. Consider

$$A_{n+1} = \{\ulcorner \forall x \varphi(x) \urcorner \mid \varphi \in \Sigma_n,\ M_n \models \forall x < a\ \varphi(x)\}.$$

$A_{n+1} \subseteq \Pi_{n+1} \operatorname{Th} \mathbb{N}$ because if $M_n \models \forall x < a\ \varphi(x)$ but $\mathbb{N} \models \exists x\ \neg\varphi(x)$ then for some $k \in \mathbb{N}$, $\mathbb{N} \models \neg\varphi(k)$, which is a Π_n-statement. Hence, as $M_n \models \Pi_n \operatorname{Th} \mathbb{N}$, $M_n \models \neg\varphi(k)$, contradiction.

$\Pi_{n+1} \operatorname{Th} \mathbb{N} \subseteq A_{n+1}$. Let $\mathbb{N} \models \forall x \varphi(x)$, where $\varphi(x) \in \Sigma_n$. If $M_n \models \exists x < a\ \neg\varphi(x)$ then $c =: \min x \neg\varphi(x)$ is a Σ_n-definable point less than a. If $c \in \mathbb{N}$ then $M_n \models \neg\varphi(c)$, which is a Π_n-statement not belonging to $\Pi_n \operatorname{Th} \mathbb{N}$. Contradiction with $M_n \models \Pi_n \operatorname{Th} \mathbb{N}$. Hence $\mathbb{N} < c < a$, which contradicts the assumption that $(\Sigma_n \operatorname{Def}) > a$.

Therefore $\Pi_{n+1} \operatorname{Th} \mathbb{N} = A_{n+1}$, which is coded in M_n. As $\mathrm{SSy}(M_n) \subseteq \mathrm{SSy}(M)$, $\Pi_{n+1} \operatorname{Th} \mathbb{N}$ is also coded in M. ■

At this point my supervisor noticed that I actually proved nothing new, because resplendency and uncountable cofinality imply arithmetic saturation, and any arithmetically saturated model codes all $\Pi_n \operatorname{Th} \mathbb{N}$.

Indeed, resplendency implies recursive saturation and for any $f : \mathbb{N} \longrightarrow M$ there is $a \in M$ such that $\forall n \in \mathbb{N}\ (f(n) > \mathbb{N} \Rightarrow f(n) > a)$ because $\mathrm{cf}(M) > \omega$.

In the next section we shall investigate whether recursive saturation and uncountable lower cofinality give us more information about coding than just arithmetic saturation. The answer will be "No". But first let us study a corollary.

A consistent theory T is called arithmetic if it has an axiomatization S such that $S = \{n \in \mathbb{N} \mid \mathbb{N} \models \theta(n)\}$ for some formula $\theta(x) \in L_{\mathrm{PA}}$. Recursive extensions of PA are examples of arithmetic theories. Also, there are complete arithmetic theories. See [14], Chapter 13.

Proposition 56
For any arithmetic theory $T \supseteq \mathrm{PA}$, if $M \models \mathrm{PA}$ is resplendent and $\mathrm{SSy}(M)$ is closed under jump then there is $N \models T$ such that $(N, <) \cong (M, <)$.

Proof Let $T = \{n \in \mathbb{N} \mid \mathbb{N} \models \theta(n)\}$. As $\mathrm{SSy}(M)$ is closed under jump, T is coded in M. Hence, as M is resplendent, $(M, <)$ is expandable to a model of T, by Theorem 52. ■

7.2 Digression: arithmetic saturation

Lemma 57 *Let $M \models \mathrm{PA}$ be recursively saturated. Then M is arithmetically saturated if and only if for all $a \in M$, $\mathrm{Cl}(a) \setminus \mathbb{N}$ is bounded below in $M \setminus \mathbb{N}$.*

Proof
Suppose, for all $a \in M$, $\mathrm{Cl}(a) \setminus \mathbb{N}$ is bounded below. Let $f \in M$ code a function. For every $n \in \mathbb{N}$, $f(n) \in \mathrm{Cl}(f)$. If $\mathrm{Cl}(f) > b$ then for all $n \in \mathbb{N}$, $(f(n) > \mathbb{N} \Leftrightarrow f(n) > b)$.

Let M be arithmetically saturated, $c > \mathbb{N}$. The type which says: $F \in M$ codes a function $F : [0, c] \longrightarrow M$ with $F(\ulcorner\theta\urcorner) = t_\theta(a)$ (where θ ranges over all formulas of L_{PA} with two variables and t_θ is the Skolem term defined by θ) is recursive, hence realized. But if $\mathrm{Cl}(a) \setminus \mathbb{N}$ is unbounded below then $\{F(\ulcorner\theta\urcorner)\} \cap (M \setminus \mathbb{N})$ is not separated from $\mathbb{N}$, which contradicts arithmetic saturation. ■

Let $E = \{x \in M \mid$ there are no nonstandard definable points below $x\}$.
If $a \in M \setminus \mathrm{Cl}\,\varnothing$, define

$$E_a = \{x \in M \mid \text{ for all } c \in \mathrm{Cl}\,\varnothing, c < x \leftrightarrow c < a\ \}.$$

By Lemma 57, $E \neq \varnothing$ and for any a such that $\mathbb{N} < a < \mathrm{Cl}\,\varnothing \setminus \mathbb{N}$, $E_a = E$. The following lemma establishes some homogeneity properties of E_a which will be important in the rest of this section.

Lemma 58 *Let M be recursively saturated, $a \in M \smallsetminus \mathrm{Cl}\,\varnothing$.*

1. *If $p(x, \bar{b})$ is realized by $c \in E_a$, $c > \mathrm{Cl}(\bar{b}) \cap E_a$ then for all $x \in E_a$ there is $y > x$ such that $p(y, \bar{b})$.*
2. *If $p(x, \bar{b})$ is realized by $c \in E_a$, $c < \mathrm{Cl}(\bar{b}) \cap E_a$ then for all $x \in E_a$ there is $y < x$ such that $p(y, \bar{b})$.*

Proof
1. Let $A_{upper} = \{x \in M \mid \exists\ y \in \mathrm{Cl}\,\varnothing,\ a < y < x\}$. For an arbitrary $e \in E_a$, let us find $d > e$ such that $p(d, \bar{b})$. Let us show that for all $\theta(x, \bar{b}) \in p(x, \bar{b})$, $M \models \theta(y, \bar{b})$ for unboundedly-many $y \in E_a$. Consider the two cases. If $A = \{x \in A_{upper} \mid M \models \theta(x, \bar{b})\}$ is unbounded below then $M \models \theta(y, \bar{b})$ for unboundedly-many $y \in E_a$ by overspill. Otherwise, let $k \in \mathrm{Cl}\,\varnothing$, $a < k < A$. Define $g = \max x < k\ \theta(x, \bar{b})$. We observe that $g \in \mathrm{Cl}(\bar{b})$, while $c \leq g < A_{upper}$, which is a contradiction.

Thus for any $e \in E$, $p(x, \bar{b}) \cup \{x > e\}$ is finitely satisfied. By recursive saturation, $p(x, \bar{b}) \cup \{x > e\}$ is coded, hence realized.
2. Analogous proof, left to the reader. ■

Lemma 59 *Let $M \models \mathrm{PA}$ be a countable arithmetically saturated model, $\mathbb{N} < e < \mathrm{Cl}\,\varnothing \smallsetminus \mathbb{N}$. Then there is an elementary embedding $h : M \to M$ such that for all $x > \mathbb{N}$, $h(x) > e$.*

Proof
A forth-argument. Let us enumerate M as $\{a_1, a_2, \ldots, a_i, \ldots\}_{i<\omega}$ and build inductively a sequence $\{b_1, b_2, \ldots, b_i, \ldots\}_{i<\omega}$ with $tp(b_1, \ldots, b_i) = tp(a_1, \ldots, a_i)$ and $b_i > e \Leftrightarrow b_i > \mathbb{N}$ for all i and define $h(a_i) = b_i$.

Suppose at stage i we already have $tp(a_1, \ldots, a_i) = tp(b_1, \ldots b_i)$, $e < \mathrm{Cl}(b_1, \ldots, b_i) \smallsetminus \mathbb{N}$. By Lemma 57, $\mathrm{Cl}(a_1, \ldots, a_{i+1}) \smallsetminus \mathbb{N}$ is bounded below. Let $c < \mathrm{Cl}(a_1, \ldots, a_{i+1}) \smallsetminus \mathbb{N}$. By Lemma 58 (2),

$$p(x, a_1, \ldots, a_i, c) = \{\theta(a_1, \ldots, a_i, x) \mid M \models \theta(b_1, \ldots, b_i, e)\} \cup \{x < c\}$$

is satisfied, say, by $e^* \in E$.

As $tp(a_1, \ldots, a_i, e^*) = tp(b_1, \ldots, b_i, e)$, by recursive saturation, there is an elementary embedding (actually, an automorphism) $h : M \to M$ such that $h(a_1) = b_1, \ldots, h(a_i) = b_i, h(e^*) = e$. Put $b_{i+1} = h(a_{i+1})$. By construction, $e < \mathrm{Cl}(b_1, \ldots, b_{i+1})$. ■

Hence, M has an elementary extension $N \succ M$, $N \cong M$ and there is $a \in N \smallsetminus M$ such that $\mathbb{N} < a < M$. Since a union of an elementary chain of recursively saturated models is recursively saturated, we can repeat this extension ω_1 times to obtain the following Theorem, which was promised earlier.

Theorem 60
Let $\mathscr{X}$ be a countable Scott Set. Then $\mathscr{X}$ is closed under jump if and only if there is a recursively saturated $M \models \mathrm{PA}$, $\mathrm{cf}(M) > \omega$, $\mathrm{SSy}(M) = \mathscr{X}$.

Later I discovered that this fact is also known to J. Schmerl [26].

The countability assumption cannot be dropped yet because for Scott Sets $\mathscr{X}$ with $\mathrm{card}\,\mathscr{X} > \omega_1$, the existence of a model $M \models \mathrm{PA}$ such that $\mathrm{SSy}(M) = \mathscr{X}$ is still an open problem.

Question 35
Modify the proof of Lemma 59 to include the case $\mathrm{card}\,\mathscr{X} = \omega_1$. (Start with a countable Scott Set $\mathscr{Y}_0 \subset \mathscr{X}$ and construct M as an elementary union of $M_i \models \mathrm{PA}$, $\mathrm{SSy}(M_i) = \mathscr{Y}_i$, $\mathrm{card}\,\mathscr{Y}_i = \omega$, $\mathscr{Y}_i \subseteq \mathscr{Y}_{i+1}$, $\bigcup \mathscr{Y}_i = \mathscr{X}$.)

Automorphisms moving all non-definable points

Now, as we are discussing arithmetic saturation, let us turn for the rest of this section to the rapidly developing area of automorphism groups where arithmetic saturation has profound consequences.

Fact 1 (Kaye, Kossak, Kotlarski)[16]
If $M \models \mathrm{PA}$ is countable and arithmetically saturated then M has an automorphism which moves every nondefinable point.

Fact 2 (Kossak)[19]
If $M \models \mathrm{Th}\,\mathbb{N}$ is countable and arithmetically saturated then there exists $h \in \mathrm{Aut}(M)$ such that for all $x > \mathbb{N}$, $h(x) > x$, i.e. h moves every nonstandard point up.

For some reason this automorphism is becoming known in the literature as 'Kossak's bump'. We are going to prove a theorem which generalises both of the above results. In general, if $\mathrm{Th}(M) \neq \mathrm{Th}\,\mathbb{N}$, there exists no $h \in \mathrm{Aut}(M)$ such that for all $x \notin \mathrm{Cl}\,\varnothing$, $h(x) > x$.

Proof Let $a < b < e$, $e \in \mathrm{Cl}\,\varnothing \setminus \mathbb{N}$, $h(a) = b$. Then $e - a > e - b$, hence $h(e-a) > h(e-b)$, hence $e - b > h(e-b)$. ■

But what we can expect is the following Theorem.

Theorem 61
If $M \models \mathrm{PA}$ is countable and arithmetically saturated then there is $h \in$

Aut(M) *such that for all* $x \in E$*,* $h(x) > x$ *and* h *moves every nondefinable point.*

Since in the case of $\mathrm{Th}(M) = \mathrm{Th}\,\mathbb{N}$, $E = M \setminus \mathbb{N}$, this Theorem generalizes Kossak's result. The proof uses Kossak's method and the following two lemmas.

Lemma (#) (Kaye, Kotlarski)
If M *is arithmetically saturated,* $tp(\overline{a}) = tp(\overline{b})$ *and for any Skolem term* t*,*

$$t(\overline{a}) = t(\overline{b}) \Rightarrow t(\overline{a}) \in \mathrm{Cl}\,\varnothing$$

then for any $c \in M$ *there is* d *such that* $tp(\overline{a}, c) = tp(\overline{b}, d)$ *and for any Skolem term* t*,*

$$t(\overline{a}, c) = t(\overline{b}, d) \Rightarrow t(\overline{a}, c) \in \mathrm{Cl}\,\varnothing.$$

Notice that Fact 1 follows from Lemma (#) by a back-and-forth argument.

Lemma 62 *Let* $M \models \mathrm{PA}$ *be recursively saturated.*

1. *If* $c < \mathrm{Cl}(\overline{a}) \setminus \mathbb{N}$ *then for any* b *there is* b' *such that* $tp(\overline{a}, b) = tp(\overline{a}, b')$*,* $c < \mathrm{Cl}(\overline{a}, b') \setminus \mathbb{N}$*.*
2. *If* $c > \mathrm{Cl}(\overline{a}) \cap E$ *then for any* b *there is* b' *such that* $tp(\overline{a}, b) = tp(\overline{a}, b')$*,* $c > \mathrm{Cl}(\overline{a}, b') \cap E$*.*

Proof
1) By Lemma 58, (1), there is $d < \mathrm{Cl}(\overline{a}, b)$ such that $tp(\overline{a}, c) = tp(\overline{a}, d)$. By recursive saturation, there is $h : M \to M$, $h(\overline{a}) = \overline{a}, h(d) = c$. Denote $h(b)$ by b'. As h is elementary, $tp(\overline{a}, b) = tp(\overline{a}, b')$. As $d < \mathrm{Cl}(\overline{a}, b)$, $c < \mathrm{Cl}(\overline{a}, b')$.
2) Similar proof. ■

Proof
We shall construct a string of points $\{d_i\}_{i\in\mathbb{Z}}$ unbounded above and below in E such that our future automorphism h takes d_i to d_{i+1} which will guarantee that each point of E moves upwards: if $a \in (d_i, d_{i+1})$ then $h(a) \in (hd_i, hd_{i+1}) = (d_{i+1}, d_{i+2})$. Also, it obviously follows that there will be no h-fixed initial segment in E other than $\sup E$ and $\mathbb{N}$.

By Lemma (#) there are $c_0, c_1 \in E$ such that $tp(c_0) = tp(c_1)$ and $(t(c_0) = t(c_1) \Rightarrow t(c_0) \in \mathrm{Cl}\,\varnothing)$, hence, considering the type $\{\varphi(x) \mid M \models \varphi(c_1)\} \cup \{t(x) \neq t(c_1) \mid t(c_1) \notin \mathrm{Cl}\,\varnothing\}$, we deduce, using Lemma 58, that there are $d_0, d_1 \in E$ such that

$$tp(d_0) = tp(d_1),$$

$$t(d_0) = t(d_1) \Rightarrow t(d_0) \in \mathrm{Cl}\,\varnothing,$$
$$\mathrm{Cl}(d_0) \cap E < d_1,$$
$$\mathrm{Cl}(d_1) \smallsetminus \mathbb{N} > d_0.$$

Let $\{s_i\}_{i\in\omega}$ be an enumeration of the whole of $M \smallsetminus \mathrm{Cl}\,\varnothing$. By stage n we shall already have:

$$\overline{a} = a_0, \dots, a_{2n-1},$$
$$\overline{b} = b_0, \dots, b_{2n-1},$$
$$\overline{d} = d_{-n}, d_{-n+1}, \dots, d_n, d_{n+1}$$

satisfying the following conditions:

$$tp(\overline{a}, d_{-n}, \dots, d_n) = tp(\overline{b}, d_{-n+1}, \dots, d_{n+1}),$$
$$d_{-n} < \mathrm{Cl}(\overline{b}, d_{-n+1}, \dots, d_{n+1}) \cap E,$$
$$d_{n+1} > \mathrm{Cl}(\overline{a}, d_{-n}, \dots, d_n) \cap E,$$
$$t(\overline{a}, d_{-n}, \dots, d_n) = t(\overline{b}, d_{-n+1}, \dots, d_{n+1}) \;\Rightarrow\; t(\overline{a}, d_{-n}, \dots, d_n) \in \mathrm{Cl}\,\varnothing.$$

(At stage $n = 0$, $\overline{a}$ and $\overline{b}$ are empty.)

Back

Let $b_{2n} = s_n$. Let $e < \mathrm{Cl}(b_{2n}, \overline{b}, d_{-n}, \dots, d_{n+1})$. By Lemma (#) (applied to the tuples $(\overline{b}, d_{-n+1}, \dots, d_{n+1})$ and $(\overline{a}, d_{-n}, \dots, d_n)$ and the new point d_{-n}), the set of formulas

$$p(x) = \{\varphi(\overline{a}, x, d_{-n}, \dots, d_n) \mid M \models \varphi(\overline{b}, d_{-n}, d_{-n+1}, \dots, d_{n+1})\} \cup$$
$$\cup\, \{t(\overline{a}, x, d_{-n}, \dots, d_n) \neq t(\overline{b}, d_{-n}, d_{-n+1}, \dots, d_{n+1}) \mid$$
$$\mid\; t(\overline{b}, d_{-n}, d_{-n+1}, \dots, d_{n+1}) \notin \mathrm{Cl}\,\varnothing\}$$

is realized, hence, by Lemma 58 (2), is realized by a point less than e, hence, by Lemma 62 (2), is realized by a point $d_{-n-1} < e$ such that

$$\mathrm{Cl}(\overline{a}, d_{-n-1}, \dots, d_n) \cap E < d_{n+1}.$$

$$\text{Let } q(x) = \{\varphi(\overline{a}, d_{-n-1}, \dots, d_n, x) \mid M \models \varphi(\overline{b}, d_{-n}, \dots, d_{n+1}, b_{2n}\} \cup$$
$$\cup\, \{t(\overline{a}, d_{-n-1}, \dots, d_n, x) \neq t(\overline{b}, d_{-n}, \dots, d_{n+1}, b_{2n}) \mid$$
$$\mid t(\overline{b}, d_{-n}, \dots, d_{n+1}, b_{2n}) \notin \mathrm{Cl}\,\varnothing\}.$$

By Lemma (#), $q(x)$ is realized, hence, by Lemma 62 (2), is realized by some point a_{2n} such that

$$\mathrm{Cl}(\overline{a}, d_{-n-1}, \dots, d_n, a_{2n}) \cap E < d_{n+1}.$$

By construction,

$$t(\overline{a}, d_{-n-1}, \ldots, d_n, a_{2n}) = t(\overline{b}, d_{-n}, \ldots, d_{n+1}, b_{2n}) \Rightarrow$$
$$\Rightarrow t(\overline{b}, d_{-n}, \ldots, d_{n+1}, b_{2n}) \in \mathrm{Cl}\,\varnothing,$$

i.e., every nondefinable point of $\mathrm{Cl}(\overline{b}, d_{-n}, \ldots, d_{n+1}, b_{2n})$ moves. Let us show that if $t(\overline{b}, d_{-n}, \ldots, d_{n+1}, b_{2n}) \in E$ then

$$t(\overline{a}, d_{-n-1}, \ldots, d_n, a_{2n}) < t(\overline{b}, d_{-n}, \ldots, d_{n+1}, b_{2n}).$$

If $t(\overline{b}, d_{-n}, \ldots, d_{n+1}, b_{2n}) > d_{n+1}$ then $t(\overline{a}, d_{-n-1}, \ldots, d_n, a_{2n}) \in (d_n, d_{n+1})$ because $\mathrm{Cl}(\overline{a}, d_{-n-1}, \ldots, d_n, a_{2n}) \cap E < d_{n+1}$. If $t(\overline{b}, d_{-n}, \ldots, d_{n+1}, b_{2n}) \in (d_i, d_{i+1})$, $i = -n, \ldots, n$, then $t(\overline{a}, d_{-n-1}, \ldots, d_n, a_{2n}) \in (d_{i-1}, d_i)$.
If $t(\overline{b}, d_{-n}, \ldots, d_{n+1}, b_{2n}) < d_{-n}$ then, by construction of d_{-n-1},
$t(\overline{b}, d_{-n}, \ldots, d_{n+1}, b_{2n}) \in (d_{-n-1}, d_{-n})$, hence $t(\overline{a}, d_{-n-1}, \ldots, d_n, a_{2n}) < d_{-n-1}$.

Forth

Let $a_{2n+1} = s_n$. Using Lemmas (#), 58 (1), 62 (1), we choose d_{n+2} such that

$$tp(\overline{b}, b_{2n}, d_{-n}, \ldots, d_{n+2}) = tp(\overline{a}, a_{2n}, d_{-n-1}, \ldots, d_{n+1}),$$
$$t(\overline{b}, b_{2n}, d_{-n}, \ldots, d_{n+2}) = t(\overline{a}, a_{2n}, d_{-n-1}, \ldots, d_{n+1}) \Rightarrow$$
$$\Rightarrow t(\overline{a}, a_{2n}, d_{-n-1}, \ldots, d_{n+1} \in \mathrm{Cl}\,\varnothing,$$
$$d_{n+2} > \mathrm{Cl}(\overline{a}, a_{2n}, a_{2n+1}, d_{-n-1}, \ldots, d_{n+1}) \cap E,$$
$$d_{-n-1} < \mathrm{Cl}(\overline{b}, b_{2n}, d_{-n}, \ldots, d_{n+2}) \cap E.$$

Now, using Lemmas (#) and 62 (1), we choose b_{2n+1} such that

$$tp(\overline{b}, b_{2n}, b_{2n+1}, d_{-n}, \ldots, d_{n+2}) = tp(\overline{a}, a_{2n}, a_{2n+1}, d_{-n-1}, \ldots, d_{n+1}),$$
$$t(\overline{b}, b_{2n}, b_{2n+1}, d_{-n}, \ldots, d_{n+2}) = t(\overline{a}, a_{2n}, a_{2n+1}, d_{-n-1}, \ldots, d_{n+1}) \Rightarrow$$
$$\Rightarrow t(\overline{a}, a_{2n}, a_{2n+1}, d_{-n-1}, \ldots, d_{n+1}) \in \mathrm{Cl}\,\varnothing,$$
$$\text{and} \quad d_{-n-1} < \mathrm{Cl}(\overline{b}, b_{2n}, b_{2n+1}, d_{-n}, \ldots, d_{n+2}).$$

Having obtained the points a_i, b_i for all $i \in \omega$, we observe that $h\colon M \to M$ defined as $h(a_i) = b_i$ for all $i \in \omega$ is an elementary isomorphism posessing the required properties. ■

I was told that the automorphism constructed above in Theorem 61 may be useful in the study of NF, in particular the problem of consistency of NF. For an up-to-date discussion of related automorphisms and their connection to models of NF and models of PA occuring as natural numbers of models of NF, see [18].

The following problem has been suggested by my supervisor:

Question 36 *Let $M \models \mathrm{Th}\,\mathbb{N}$ be countable and arithmetically saturated. Find $g, h \in \mathrm{Aut}(M)$ such that $\mathrm{Aut}(M)$ is generated by g and h.*

7.3 Initial segments in resplendent models

In this section we present our second attempt to solve Friedman's problem for resplendent models of PA.

Theorem 63
Let T be a completion of PA. *Suppose M is resplendent,* $\mathrm{cf}(M) = \omega$ *and M codes $\Sigma_n T$ for all n. Then $(M, <)$ is expandable to a model of T.*

Proof
Let $b \in M$ be a code of $\mathrm{PA} \cup \Sigma_1 T$ in M. Consider the following statement:

$$\Phi(b) \;=\; \exists \oplus, \otimes, \ll, \mathbb{O}, \mathbb{S}$$

$$\Big(\forall xy\ (x \ll y \longleftrightarrow x < y)\ \bigwedge\ \text{“}(\oplus, \otimes, \ll, \mathbb{O}, \mathbb{S}) \models \mathrm{PA} \cup \Sigma_1 T\text{”}\ \bigwedge$$

$$\bigwedge \mathrm{SSy}(M, \oplus, \otimes, \ll, \mathbb{O}, \mathbb{S}) = \mathrm{SSy}(M)\Big)\,.$$

The last line $\mathrm{SSy}(M, \oplus, \otimes, \ll, \mathbb{O}, \mathbb{S}) = \mathrm{SSy}(M)$ can be written as a Σ^1_1-sentence because if

$$\varphi_n(x) = (x > n)\ \&\ \forall\, v\ \exists\, y \forall\, i < x\ ((v))_i = (y)_i$$

$$\forall\, z\ \exists\, w\ \forall\, i < x\ (z)_i = ((w))_i$$

then $\{\varphi_n(x)\}_{n \in \mathbb{N}}$ is a recursive set of sentences, hence, by Kleene's Theorem, there is a Σ^1_1-sentence $\Psi(x)$ such that in any $A \models \mathrm{PA}$, $A \models \forall x (\bigwedge_{n \in \mathbb{N}} \varphi_n(x) \leftrightarrow \Psi(x))$. Thus, the Σ^1_1-sentence $\exists\, x\ \Psi(x)$ implies $\mathrm{SSy}(M, \oplus, \otimes, \ll, \mathbb{O}, \mathbb{S}) = \mathrm{SSy}(M)$. Hence, $\Phi(b)$ is a Σ^1_1-sentence. Again, it is realised in $\mathrm{Cl}(b)$ by Wilmers' Theorem, hence realised.

Define $K = (M, \oplus, \otimes, \ll, \mathbb{O}, \mathbb{S})$ such that

$$K \models \mathrm{PA} \cup \Sigma_1 T,$$

$$(K, <) \;\cong\; (M, <),$$

$$\mathrm{SSy}(K) \;=\; \mathrm{SSy}(M).$$

Let us construct a countable $N \models T$ with $\mathrm{SSy}(N) \subseteq \mathrm{SSy}(K)$. Let $\mathscr{X} \subseteq \mathrm{SSy}(K)$ be a countable Scott Set containing all $\Sigma_n T$. As $(\mathrm{PA} \cup \Sigma_1 T) \in \mathscr{X}$, there is a model $N_1 \models \mathrm{PA} \cup \Sigma_1 T$, $\mathrm{SSy}(N_1) = \mathscr{X}$. Suppose we already have

$$N_i \models \mathrm{PA} \cup \Sigma_i T,\ \ \mathrm{SSy}(N_i) = \mathscr{X}.$$

As $(\mathrm{PA} \cup \Sigma_{i+1} T) \in \mathscr{X}$, there is a countable

$$N_{i+1} \models \mathrm{PA} \cup \Sigma_{i+1} T, \quad \mathrm{SSy}(N_{i+1}) = \mathscr{X}.$$

By Friedman's Theorem, as $N_i \models \Sigma_i \operatorname{Th} N_i$,

$$N_i \prec_{\Sigma_{i-1}} N_{i+1}.$$

Define $N = \bigcup_{i \in \omega} N_i$. Obviously, $\mathrm{SSy}(N) = \mathscr{X}$. By the Σ_n-elementary chains lemma, for all $n \in \omega$, $N \models \Sigma_n T$, hence $N \models T$.

Now, by the variation of Friedman's Theorem, as $K \models \Sigma_1 T$, there is an embedding $h\colon N \to K$ such that $h(N) \prec_{\Delta_0} K$. Consider

$$N^* = \{x \in K \mid \exists y \in h(N),\ y \geq x\}.$$

By Gaifman's splitting theorem, $N^* \succ h(N)$, hence N^* is an initial segment of K satisfying T. As $h(N)$ is cofinal in N^*, $\Upsilon(N^*) = \omega$, hence, by Lemma 17, $(N^*, <) \cong (M, <)$. ■

Question 37 *If M is resplendent and codes $\Sigma_n T$ for all n, is $(M, <)$ expandable to a model of T?*

The obvious attempt to generalise Theorem 63 to higher cofinalities by proving that if $I \models T$ is an initial segment of $M \not\models T$ then there is an initial segment $J \succ I$ fails, because given $I_0 = I$ we define $I_j = \bigcup_{i<j} I_i$ if j is a limit ordinal, $I_{i+1} =$ an initial segment $J \succ I_i$ if $I_i \neq M$, which gives us an elementary chain of some length γ. Now, $M = \bigcup_{i<\gamma} I_i \models T$, contradiction. (The initial segments satisfying T exist as far up as one wants (by the variation of Friedman's Theorem) but they are not elementary substructures of each other.)

However, in the case $T = \operatorname{Th} \mathbb{N}$ we may hope that arithmetic saturation gives us extra information:

Question 38 *If M is resplendent and $\Upsilon(M) > \omega$, is $(M, <)$ expandable to a model of* $\operatorname{Th} \mathbb{N}$*?*

Conclusion

What is next

I believe that in every subject, self-awareness and understanding of its motivation and higher goals should not be lost and replaced by indulging in accumulating knowledge about "structural properties" and feeding basic mathematical curiosity.

The ultimate question of Foundations of Mathematics is to try to understand the mystery of Arithmetical Truth. Even Set Theory can be assumed to be taking place in $\mathbb{N}$ (in a model of ZFC strongly interpreted in $\mathbb{N}$), so, in this sense, set-theoretic questions are again questions about Arithmetical Truth.

Models of PA are one of the approaches to Arithmetical Truth (other approaches include Constructivism, Proof Theory, Intuitionism), the one which is trying to answer the question "How do structures satisfying some axioms, which we believe should hold for the intuitive set $\mathbb{N}$ look like?" As Proposition 37 shows, some of these structures are so indistinguishable from $\mathbb{N}$ that there are models of ZFC that belive that they *are* $\mathbb{N}$.

Also, models of PA promise to be mathematically rich because they stand on the crossroads of many different areas of mathematics: Set Theory (because uncountable models always require some infinitary combinatorics and because the initial segment ω_U of every universe $U \models$ ZFC is a model of PA), Proof Theory (because of Gödel's Incompleteness, importance of reflection principles, independence results like (PH) and various uses of Gödel numbers of proofs in connection with Standard System (e.g. see [14], p. 177) and with recursive saturation), Recursion Theory (needed in the study of Scott Sets and jump-operator needed in the study of arithmetic saturation, etc.) and, of course, mainstream Model Theory. I propose the following seven directions of future research. (The rest can be found in the List of Questions.)

1. Models of TA as $\omega_{\mathbf{U}}$ for $\mathbf{U} \models \mathrm{ZFC}$

Theorems 37-42 are just the first results with the rest of excitement to follow. To characterise all models of PA which occur as ω_U of some $U \models \mathrm{ZFC}$ (see discussion on pages 60-63 and Question 17) is a question of great importance for foundations of mathematics and I am unaware whether anyone has attempted to answer it before.

2. Friedman's Problem

What we know so far:

1. Every countable nonstandard model has order-type $\mathbb{N} + \mathbb{Q}\mathbb{Z}$.
   Every ω_1-like model has order-type $\mathbb{N} + \omega_1\mathbb{Q}\mathbb{Z}$.

2. The saturated model of cardinality λ has order-type $\mathbb{N} + Q_\lambda\mathbb{Z}$.

3. (GCH) For any $T \supseteq \mathrm{PA}$ and any $\mu < \lambda^+$ there is $N \models T$ of order-type $\mathbb{N} + \mu Q_\lambda \mathbb{Z}$.
   (GCH) For any $T \supseteq \mathrm{PA}$ and any successor $\lambda_1 < \lambda_2 < \cdots < \lambda_n$, there is $K \models T$ of order-type $\mathbb{N} + Q_{\lambda_1}\mathbb{Z} + \cdots + Q_{\lambda_n} Q_{\lambda_{n-1}} \ldots Q_{\lambda_1}\mathbb{Z}$.

4. For any $T \supseteq \mathrm{PA}$ and any dense linear order $(C, <)$, $\mathrm{EM}(C)$ is embeddable into $(C^* + \{0\} + C)^{<\mathbb{Q}}$.

5. If $M \models \mathrm{PA} + \mathrm{Con(PA)}$ is self-similar then for any inner model $N = \mathrm{ACT}(M, \mathrm{PA})$, $(N, <) \cong (M, <)$.

6. If M is resplendent and $T \supseteq \mathrm{PA}$ is coded in M then $(M, <)$ is expandable to a model of T.

7. The order-type of any ω_1-saturated resplendent model can be expanded to a model of any consistent extension of PA.

8. If T is an arithmetic theory, $M \models \mathrm{PA}$ is resplendent and $\mathrm{cf}(M) > \omega$ then there is $N \models T$ such that $(N, <) \cong (M, <)$.

9. If M is resplendent, $\mathrm{cf}(M) = \omega$ and M codes $\Sigma_n T$ for all n, then $(M, <)$ is expandable to a model of T.

10. If M is resplendent, $\mathrm{cf}(M) = \omega$ and $\mathrm{SSy}(M)$ is closed under jump then $(M, <)$ is expandable to a model of $\mathrm{Th}\,\mathbb{N}$.

Still now, I cannot even guess what the answer to the Friedman's problem will be. We may assume different degrees of saturation, resplendency, coding and their combinations and prove that order-types of models satisfying those conditions can be expanded to models of other theories (sometimes "all theories") but again and again we shall keep returning to the same problem: what to do with different Standard Systems?

Question 39 *Let* $T \supset \mathrm{PA}$*,* $M \models T$ *be resplendent. Let* $S \supset \mathrm{PA}$ *be consistent,* $S \notin \mathrm{SSy}(M)$*. Can* $(M, <)$ *be expanded to a model of* S*?*

A solution of this problem gives a solution of Friedman's problem in the resplendent case.

A new approach, different from the two approaches we tried (resplendency & coding and resplendency & initial segments) would be to make a step backwards and first study resplendent linear orders and resplendent dense linear orders instead of resplendent models of PA.

Observation 64

1. *If* $(A, <)$ *is a resplendent discrete linear order with first and no last element then* $(A, <)$ *is expandable to a model of* PA.
2. *If* $M \models \mathrm{PA}$ *is resplendent then* $(M, <)$ *is resplendent.*

Proof
1. Let $\{\varphi_i\}_{i\in\omega}$ be the set of all axioms of PA in the language $\{\oplus, \otimes, \ll, \mathbb{O}, \mathbb{S}\}$. By Kleene's Theorem, $\bigwedge_{i\in\omega} \varphi_i$ is equivalent to a Σ^1_1-statement Ψ. Let $\Phi = \exists \oplus, \otimes, \ll, \mathbb{O}, \mathbb{S}\ (\Psi\ \&\ \forall x, y(x < y \leftrightarrow x \ll y))$. Φ is consistent with the theory of discrete linear order with first and no last element because Φ is realised in the order-type of any model of PA. Hence, by resplendency, Φ is realised in $(A, <)$.
2. Let $\exists \overline{R}\ \overline{f}\ \Phi(\overline{R}, \overline{f}, \overline{a})$ be consistent with $\mathrm{Th}(M, <, \overline{a})$. Then, as $\overline{R}$ and $\overline{f}$ are new symbols not occuring in L_{PA}, $\exists \overline{R}\ \overline{f}\ \Phi(\overline{R}, \overline{f}, \overline{a})$ is consistent with $\mathrm{Th}(M, +, \cdot, <, 0, \overline{a})$ (by Robinson's joint consistency test), hence realised. ■

Question 40

1. *Prove that if* $(A, <)$ *is a resplendent dense linear order then* $\mathbb{N} + A\mathbb{Z}$ *is a resplendent discrete linear order with first and no last element.*
2. *Can every resplendent discrete linear order be expanded to a resplendent model of* PA*?*

3. Canonical orders and their spectrum

The main two problems here are to prove the general case of the existence theorem and to find the number of non-isomorphic models of each canonical order. Sophisticated methods seem to be required in both cases because all obvious approaches failed.

4. Diamonds etc

Diamonds and other combinatorial statements have been knocking at model theorists' door since 1970s. There exists a broad program to investigate the influence of infinitary combinatorics on construction of models and a number of methods have beed developed in this area. Also, there are constructions that use V=L ('morasses'). The plan is to learn existing methods having models of PA in mind. We can start off with an attempt to generalise the proof of Theorem 51.

5. Interpretations

A myriad of questions is rising about interpretations: to develop a theory of how models of strong theories (i.e. the theories described on page 68) interpret each other and what they think about each other, very much in the spirit of sections 4.2 and 4.4, in particular to develop the theory of inner models of PA thus investigating interpretational strength of models of PA, to state and solve problems similar to Questions 1 and 9, giving and studying examples of mathematical structures interpreted in models of PA and internalising classical results in the spirit of section 4.1.

6. Arithmetic Saturation

Arithmetic saturation is a relatively unexplored notion promising exciting results and examples (see Questions 29 and 36). Also, there is still demand for new useful notions of saturation that would give consequences for automorphism groups, satisfaction classes and the Standard System.

Question 41 *Investigate the following notion of saturation: "M is recursively saturated and* $\mathrm{SSy}(M)$ *is closed under ω-jump: $A \to A^{(\omega)}$".*

By the properties of ω-jump, it implies that $\mathrm{SSy}(M)$ is closed under jump, hence M is arithmetically saturated. Also, $\mathrm{Th}\,\mathbb{N}$ will have to be coded in M.

7. Embeddability

A problem wider than just "What are the order-types of models of PA?" would be to introduce different kinds of linear orders and study their mutual embeddability with models of PA as we did in parts of chapters 2 and 3. For example, many questions rise about embeddability of linear orders $\varkappa$ and $\varkappa^*$ into models of PA. Another possibility is to study the following family of linear orders.

Definition 28 *Let* $(A,<)$, $(I,<)$ *be linear orders,* $0 \in A$.
$(A,0)^{\triangleleft I} = \{f\colon I \to A \mid \operatorname{supp} f$ *is well-founded* $\}$, *with order defined lexicographically:* $f < g$ *if for* $a = \min\{\operatorname{supp} f, \operatorname{supp} g\}$, $f(a) < g(a)$.

Question 42
Study mutual embeddability of orders $(A,0)^{\triangleleft I}$ *and models of* PA.

List of questions

Question 1
Let $\Omega \models \mathrm{PA}$. *What do groups strongly interpreted in* Ω *look like? What do* Ω*-finite groups look like? Which properties of* $\mathbb{N}$*-finite (truely finite) groups do they inherit?*

Question 2 *Is there an ultrafilter* D *on* λ *such that there exists a countable* $M \models \mathrm{PA}$ *with* $\prod_D M$ 2^λ*-dense but not saturated?*

Question 3 *Are there linear orders* $(I,<)$, $(A,<)$, *dense, without end-points and* $0_1, 0_2 \in A$ *such that*

$$(A,0_1)^{<I} \ \not\cong\ (A,0_2)^{<I} \ ?$$

Question 4 *Is it true that if* γ *is an ordinal and* $\operatorname{card} M > w(X)$ *then* $(M,<)$ *is not embeddable into* $(X^{<\gamma^*},<)$*?*

Question 5
Is there an uncountable model of PA *order-embeddable into* $(\mathbb{R}^{<\omega},<)$*?*

Question 6
If $(I,<)$ *is a countable linear order then the following are equivalent:*

1. *For every linear order* $(A,<)$ *there is an uncountable* $M_A \models \mathrm{PA}$ *embeddable into* $A^{<I}$.

2. *I contains a copy of* $\mathbb{Q}$.

Question 7
Let p *be an indiscernible type over* T, $(C, <)$ *be a linear order,* $M = \mathrm{EM}(C, p)$. *Prove that if* $t(u_1, \ldots, u_n)$ *is a Skolem term of* T, $x_1, \ldots, x_n \in C$ *then either* $tp(t(x_1, \ldots, x_n)) \neq tp(x_1)$ *or* $t(x_1, \ldots, x_n) = x_i$ *for some* $i = 1, \ldots, n$. *Deduce that* $\mathrm{EM}(C_1, p)$ *is elementarily embeddable into* $\mathrm{EM}(C_2, p)$ *if and only if* C_1 *is embeddable into* C_2. *Deduce that there are* 2^λ *pairwise non-embeddable models of* T *in any cardinality* λ.

Question 8 *Let* $\lambda > \omega$. *Is there a family* $\{C_i\}_{i \in 2^\lambda}$ *of dense linear orders of cardinality* λ *with no last element such that* $i \neq j$ *implies* C_i *is not embeddable into* $(C_j^* + \{0\} + C_j)^{<\mathbb{Q}}$? *If yes, deduce, by Theorem 20 that there are* 2^λ *pairwise non-embeddable order-types of models of* T *of cardinality* λ.

Question 9
Prove internal versions of the uniqueness of the countable atomless boolean algebra, the random graph etc. How do these structures look like?

Question 10 *Find two models* $A, B \models \mathrm{PA}$ *such that* $A \equiv B$,
$(A, <) \cong (B, <)$ *but* $(A + Q(A)(A^* + A), <) \ncong (B + Q(B)(B^* + B), <)$.

Question 11 *If* $M \models \mathrm{PA} + \mathrm{Con(ZFC)}$ *and* $V \models \mathrm{ZFC}$ *is interpreted in* M, *what can we say about* $(\mathrm{Ordinals}(V), <)$ *in terms of* $(M, <)$?

Question 12 *If* $V \models \mathrm{ZFC} + \mathrm{Con(ZFC)}$ *and* $U \models \mathrm{ZFC}$ *is an inner model in* V, *what can we say about* $(\mathrm{Ordinals}(U), <)$ *in terms of* $(\mathrm{Ordinals}(V), <)$?

Question 13

1. *How many pairwise non-isomorphic elementarily equivalent models of* ZFC *are interpreted inside* $M \models \mathrm{PA} + \mathrm{Con(ZFC)}$? *(If* M *is countable and nonstandard then there is only one (the countable recursively saturated model with standard system* $\mathrm{SSy}(M)$*). What is the order-type of its ordinals?)*

2. *How many non-isomorphic order-types of ordinals of models of* ZFC *interpreted in a model of* $\mathrm{PA} + \mathrm{Con(ZFC)}$ *are there?*

3. *What are those order-types? How about Friedman's problem for models of* ZFC *from the point of view of a model of* $\mathrm{PA} + \mathrm{Con(ZFC)}$?

Question 14

1. *Is there an uncountable self-similar model which is not recursively saturated?*

2. *Is there a resplendent model which is not an inner model?*

3. *Is there an uncountable self-similar model which is not an inner model?*

4. *Give an example of two elementarily equivalent non-isomorphic inner models in M. (Of course, M will have to be uncountable.)*

Question 15
Let $U \models \mathrm{ZFC}$ be countable, T be a completion of PA *coded in U, $U \models \mathrm{Con}(T)$. How many non-isomorphic models of T are interpreted in U?*

Question 16 *(probably a very hard one)*
Find a mathematical statement in the language L_{PA} independent of TA.

Question 17
Is it true that if $\Omega \models \mathrm{TA}$ is recursively saturated then there is $V \models \mathrm{ZFC}$ such that $\omega_V = \Omega$.

Question 18 *Prove that if $\Omega \models \mathrm{TA} + \mathrm{Reflection(PA)}$ is resplendent then there is $U \models \mathrm{ZFC}$ strongly interpreted in Ω such that $\omega_U \cong \Omega$.*

Question 19 *Is it true that in every $M \models \mathrm{PA}$ there is z coding* ZFC *such that $\mathrm{TA}^M = (\mathrm{ZFC}^M)^a$?*

Question 20 *What are order-types of ordinals of countable models of* ZFC*? How many of them are there? Can there be some classification result?*

Question 21 *What are the order-types of ordinals of models of* ZFC*? How does the order-type of ordinals of a saturated model look like? How many models with the saturated order-type are there?*

Question 22 *Do classes of order-types of ordinals of models of different extensions of* ZFC *coincide?*

Question 23 *How many models of order-type $\mathbb{N} + \omega Q_\lambda \mathbb{Z}$ are there?*

Question 24 *For every canonical order of the form $\mathbb{N} + \mu Q_\lambda \mathbb{Z}$, $(\mu \leq \lambda^+)$, find the number of pairwise non-isomorphic models of* PA *of this order-type. What happens for arbitrary canonical orders?*

Question 25 *Find four models,* M_1, $\tilde{M}_1$, M_2, $\tilde{M}_2$, *all copies of the saturated model, such that* $\tilde{M}_1$ *is an elementary end-extension of* M_1, $\tilde{M}_2$ *is an elementary end-extension of* M_2, *but there is no isomorphism* $f\colon \tilde{M}_1 \to \tilde{M}_2$ *such that* $f(M_1) = M_2$.

Question 26
Prove that there is a model of PA *of every canonical order-type.*

Question 27
Prove that there is a model of PA *of order-type* $\mathbb{N} + \mathbb{QZ} + Q_{\omega_1}\mathbb{QZ}$.

Question 28 *Prove that there is a model of* PA *of order-type* $\mathbb{N} + \omega_1\mathbb{QZ} + Q_{\omega_1}(\omega_1^* + \omega_1)\mathbb{QZ}$.

Question 29
Prove that there is a countable $M \models \mathrm{PA}$ *such that for every countable end-extension* $N \succ M$ *and subsets* $A, B \subset N/M$, $A < B$, *there is* $K \succ N$ *such that there is* $c \in K$ *such that* $A < c < B$ *and* $\{x \in K \mid \exists y \in M,\ y \geq x\} = M$.

Question 30 *Prove that if* λ *is singular,* $|M| \leq \lambda$, *then there are* 2^λ λ*-dense elementary extensions of* M *of cardinality* λ.

Question 31 *Prove that there are* 2^λ *order-types of resplendent models of* PA *of cardinality* λ.

Question 32 *Prove without* $\diamondsuit$ *that there are* 2^{ω_1} *models* $M_i \models \mathrm{PA}$, $i < 2^{\omega_1}$ *of order-type* $\mathbb{N} + \omega_1\mathbb{QZ}$ *such that* $(M_i, \cdot) \ncong (M_j, \cdot)$ *for* $i \neq j$.

Question 33 *If* $M \models \mathrm{PA}$ *has cardinality* ω_1 *and is not saturated then* $(M, <)$ *has* 2^{ω_1} $(M, \cdot)$*-non-isomorphic expansions to a model of* PA.

Question 34
Try tackling Question 10 in the universes satisfying $\diamondsuit$.

Question 35
Modify the proof of Lemma 59 to include the case $\mathrm{card}\,\mathscr{X} = \omega_1$. *(Start with a countable Scott Set* $\mathscr{Y}_0 \subset \mathscr{X}$ *and construct* M *as an elementary union of* $M_i \models \mathrm{PA}$, $\mathrm{SSy}(M_i) = \mathscr{Y}_i$, $\mathrm{card}\,\mathscr{Y}_i = \omega$, $\mathscr{Y}_i \subseteq \mathscr{Y}_{i+1}$, $\bigcup \mathscr{Y}_i = \mathscr{X}$.*)*

Question 36 *Let* $M \models \mathrm{Th}\,\mathbb{N}$ *be countable and arithmetically saturated. Find* $g, h \in \mathrm{Aut}(M)$ *such that* $\mathrm{Aut}(M)$ *is generated by* g *and* h.

Question 37 *If* M *is resplendent and codes* $\Sigma_n T$ *for all* n*, then* $(M,<)$ *is expandable to a model of* T*.*

Question 38 *If* M *is resplendent and* $\Upsilon(M) > \omega$ *then* $(M,<)$ *is expandable to a model of* $\mathrm{Th}\,\mathbb{N}$*.*

Question 39 *Let* $T \supset \mathrm{PA}$*,* $M \models T$ *be resplendent. Let* $S \supset \mathrm{PA}$ *be consistent,* $S \notin \mathrm{SSy}(M)$*. Can* $(M,<)$ *be expanded to a model of* S*?*

Question 40

1. *Prove that if* $(A,<)$ *is a resplendent dense linear order then* $\mathbb{N} + A\mathbb{Z}$ *is a resplendent discrete linear order with first and no last element.*
2. *Find an example of a resplendent model of* PA *whose order-type is not resplendent.*

Question 41 *Investigate the following notion of saturation: "*M *is recursively saturated and* $\mathrm{SSy}(M)$ *is closed under* ω*-jump:* $A \to A^{(\omega)}$*".*

Question 42
Study mutual embeddability of orders $(A,0)^{\lhd I}$ *and models of* PA*.*

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