Resplendent linear orders

Andrey Bovykin*
and
Richard Kaye†

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Abstract

This paper introduces 'treatable' theories, a large class of first-order theories each having a unique countable recursively saturated model.Important examples of such theories are DIS and DLO. It turns out that resplendency or parameter-free resplendency for such theories is implied by a $\Sigma^1_1(\Pi_1 \operatorname{Th} \mathbb{N})$-statement. Other results concerning resplendency and Σ^1_1-definability are obtained.

1 Introduction

A model is resplendent if it is expandable to a model of any sentence Φ (in an expanded language) with parameters from the model consistent with the elementary diagram of the model. It is not clear in general what are "all sentences in an expanded language consistent with the elementary diagram of our model" and what they might say. In this paper we are trying to characterize the notion of resplendency for certain theories in a simple intuitive way. This paper should be of interest to those interested in linear orders, resplendency, ω-caregoricity and the power of arithmetic.

Resplendency

Definition 1.1. Let $\mathcal{L}$ be a first-order language. A Σ^1_1-*formula over* $\mathcal{L}$ is an expression of the form $\exists X_1, \ldots, X_n\ \phi(X_1, \ldots, X_n, \bar{x})$ where $X_1, \ldots, X_n$ are new relation or function symbols and $\phi(X_1, \ldots, X_n, \bar{x})$ is a formula of the expanded language $\mathcal{L} \cup \{X_1, \ldots, X_n\}$. If $\phi(X_1, \ldots, X_n)$ has no free first-order variables $\bar{x}$, then $\exists X_1, \ldots, X_n\ \phi(X_1, \ldots, X_n)$ is a Σ^1_1 *sentence*. An $\mathcal{L}$-structure M *satisfies* the Σ^1_1 sentence $\Phi = \exists X_1, \ldots, X_n\ \phi(X_1, \ldots, X_n)$ if there is an expansion $(M, X_1, \ldots, X_n)$ of M to $\mathcal{L} \cup \{X_1, \ldots, X_n\}$ such that

$$(M, X_1, \ldots, X_n) \vDash \phi(X_1, \ldots, X_n).$$

If T is a first-order $\mathcal{L}$-theory then we say that $T + \Phi$ is *consistent* if it has a model, i.e., if the $\mathcal{L} \cup \{X_1, \ldots, X_n\}$-theory $T + \phi(X_1, \ldots, X_n)$ has a model.

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†School of Mathematics and Statistics, University of Birmingham, Edgbaston, Birmingham, B15 2TT, U.K. `http://www.mat.bham.ac.uk/R.W.Kaye/`

Definition 1.2. An $\mathcal{L}$-structure M is *resplendent* if for all tuples $\bar{a} \in M$ of finite length and for all Σ^1_1 sentences $\Psi(\bar{a})$ over $\mathcal{L} \cup \{\bar{a}\}$

$$\text{if } \mathrm{Th}(M, \bar{a}) + \Psi(\bar{a}) \text{ is consistent then } (M, \bar{a}) \vDash \Psi(\bar{a}).$$

Thus resplendent models satisfy as many Σ^1_1-formulas as possible.

Definition 1.3. An $\mathcal{L}$-structure M is *chronically resplendent* if for all tuples $\bar{a} \in M$ of finite length and for all Σ^1_1 sentences $\exists \bar{X}\ \Psi(\bar{X}, \bar{a})$ over $\mathcal{L} \cup \{\bar{a}\}$, if $\mathrm{Th}(M, \bar{a}) + \exists \bar{X}\ \Psi(\bar{X}, \bar{a})$ is consistent then $(M, \bar{a})$ is expandable to a resplendent model of $\Psi(\bar{X}, \bar{a})$.

Definition 1.4. M is called *parameter-free resplendent* if for every parameter-free Σ^1_1-sentence Θ consistent with $\mathrm{Th}(M)$, we have $M \vDash \Theta$.

In the countable case chronical resplendency, resplendency and parameter-free resplendency coincide with recursive saturation. However, the relationship between these notions in the uncountable case is more complicated. Resplendency, parameter-free resplendency and recursive saturation are pairwise different notions. The question whether resplendency=chronical resplendency is open.

Fact 1.5. (Kleene) If $\{\varphi_i(\overline{x})\}_{i\in\omega}$ is a recursive set of $\mathcal{L}$-formulas, where $\mathcal{L}$ is a recursive first-order language, then there is a Σ^1_1-sentence $\Phi(\overline{x})$ such that in every infinite $\mathcal{L}$-structure A, we have $A \vDash \forall \overline{x}(\Phi(\overline{x}) \leftrightarrow \bigwedge_{i\in\omega} \varphi_i(\overline{x}))$.

Σ^1_1 definability

The idea of a Σ^1_1 formula forms one level in the hierarchy of classes of second-order formulas. The other levels are defined by saying that a Π^1_n formula is the negation of a Σ^1_n formula, a Σ^1_{n+1} formula is one of the form

$$\exists X_1, \ldots, X_n\ \Phi(X_1, \ldots, X_n)$$

where Φ is Π^1_n, and a Δ^1_n formula is a formula Θ which is equivalent to some formulas $\Phi \in \Pi^1_n$ and $\Psi \in \Sigma^1_n$ in all models of the appropriate signature.

Definition 1.6. A formula Φ over a signature $\mathcal{L}$ is $\Sigma^1_1(\bigwedge)$ if it is of the form

$$\exists X_1, \ldots, X_n \bigwedge_{i\in\mathbb{N}} \phi_i(X_1, \ldots, X_n)$$

where each $\phi_i(X_1, \ldots, X_n)$ is in the extended signature $\mathcal{L} \cup \{X_1, \ldots, X_n\}$.

More specifically, given $C \subseteq \mathbb{N}$, we say that a formula Φ over a signature $\mathcal{L}$ is $\Sigma^1_1(C)$ if it is of the form

$$\exists X_1, \ldots, X_n \bigwedge_{i\in\mathbb{N}} \phi_i(X_1, \ldots, X_n)$$

where each $\phi_i(X_1, \ldots, X_n)$ is in the extended signature $\mathcal{L} \cup \{X_1, \ldots, X_n\}$ and the set of (Gödel-numbers of) the formulas $\phi_i(X_1, \ldots, X_n)$ as i ranges over $\mathbb{N}$ is recursive in C.

So Kleene's theorem says that Σ^1_1 and $\Sigma^1_1(C)$ are the same for recursive sets C and infinite models M, but for nonrecursive C, the class of $\Sigma^1_1(C)$ sentences may have additional expressive power. $\Sigma^1_1(\bigwedge)$ is the union of all the $\Sigma^1_1(C)$.

Definition 1.7. Let P be a property of $\mathcal{L}$-structures and Γ a class of $\mathcal{L}$-sentences. We say that P is Γ if there is a Γ-sentence Φ such that M has property P if and only if $M \models \Phi$.

Resplendency is a Δ^1_2 notion. Recursive saturation is Σ^1_1 (Smith [4]). On the other hand, resplendency is not Σ^1_1, as shown by the following result due to Lachlan, which appears in [4]. (In Proposition 5.1 below, we shall give a full proof of somewhat more than that resplendency isn't Σ^1_1.)

Fact 1.8 (Lachlan). Let $\mathcal{L}$ be a finite or recursive language extending the language of PA, and suppose Ψ is a Σ^1_1 sentence over $\mathcal{L}$ consistent with PA. Then there is a model M of PA of cardinality ω_1 satisfying Ψ with a countable nonstandard initial segment I.

A model M as in the conclusion of this result cannot be resplendent because the statement $\Phi(a)$ saying " there is a bijection between $[0, a]$ and M" is consistent with $\mathrm{Th}(M, a)$ if $[0, a]$ is infinite and hence is realized in resplendent models.

Models of Arithmetic

Definition 1.9. Let $M \models \mathrm{PA}$, T be a theory in the language $\mathcal{L}$. We say that $A \models T$ is strongly interpreted in M there are formulas $\mathrm{Dom}(x)$ and $\mathrm{Sat}(x, y)$ such that the domain of A is $\{x \in M \mid M \models \mathrm{Dom}(x)\}$ and for all $\varphi \in \mathcal{L}$, all $\overline{a} \in A$, $M \models \mathrm{Sat}(\ulcorner\varphi\urcorner, \langle\overline{a}\rangle) \Leftrightarrow A \models \varphi(\overline{a})$.

Fact 1.10. (Pabion, Richard) [3]
A model $M \models \mathrm{PA}$ is ω_1-saturated if and only if $(M, <)$ is ω_1-saturated.

2 Treatable theories

Definition 2.1. Theory T in the language $\mathcal{L}_T = \{R_1, \ldots, R_n\}$ is called treatable if

1. T has a unique countable recursively saturated model;
2. there are $\mathcal{L}_{\mathrm{PA}}$-formulas $\mathrm{Dom}(x)$, $\varphi_{R_1}(\overline{x}), \ldots \varphi_{R_n}(\overline{x})$ such that for every $M \models \mathrm{PA}$, the model $\mathrm{Canon}(M, T)$ defined by formulas $\mathrm{Dom}(x)$, $\varphi_{R_1}(\overline{x}), \ldots \varphi_{R_n}(\overline{x})$ satisfies T.
3. for every $M \models \mathrm{PA}$ and any $N, K \models \mathrm{PA}$ strongly interpreted in M, we have $\mathrm{Canon}(N, T) \cong \mathrm{Canon}(K, T)$. Denote this unique model by $\mathrm{Inner}(M, T)$ to emphasize that it depends only on M.

Theorem 2.2. The theories DLO and DIS are treatable.

Proof.
1. DLO. Let $M \models \mathrm{PA}$. We shall prove that every model of DLO interpreted

in M is isomorphic to $Q(M) = \{\langle a, b\rangle \mid a, b \in M \smallsetminus \{0\}\}$ with order defined as $\langle a_1, b_1\rangle < \langle a_2, b_2\rangle$ if and only if $a_1 b_2 < a_2 b_1$.

Let $(Q_1, <)$ and $(Q_2, <)$ be models of DLO interpreted in M, $f_1(n) =$ the nth element of Q_1 and $f_2(n) =$ the nth element of Q_2. Define $i(0) = 0$, $i(x+1) = \min z\ \forall y \leq x\ (f_1(y) <_1 f_1(x+1)) \quad\leftrightarrow\quad (f_2(i(y)) <_2 f_2(z))$. Such element z can always be found because of density: $M \vDash \exists z \max_{w\leq x}\{f_2(i(w)) \mid f_1(w) < f_1(x+1)\} < z < \min_{w\leq x}\{f_2(i(w)) \mid f_1(x+1) < f_1(w)\}$. Now, $f_2 i f_1^{-1}$ is an M-definable isomorphism between $(Q_1, <)$ and $(Q_2, <)$.

2. DIS. It is easy to see that $\mathbb{N} + \mathbb{Q}\mathbb{Z}$ is the only countable recursively saturated model of DIS (every point above $\mathbb{N}$ comes with a $\mathbb{Z}$-block and the blocks are ordered densely because the type $\{x > a+n\} \cup \{x < b-n\}$ for $a < b$ from different $\mathbb{Z}$-blocks is realized).

Let us show that any $N \vDash \mathrm{PA}$ interpreted in $M \vDash \mathrm{PA}$ is order-isomorphic to $M + Q(M)\cdot(M^* + M)$.

Since N is strongly interpreted in M, there is $f\colon M \to N$ definable in M which determines an isomorphism between M and an initial segment of N. Hence, $(N, <) \cong M + A(M^* + M)$ for some linear order A.

For $a, b \in N$, we define $a \sim b \iff M \vDash \exists x(a -_N b = f(x))$ if $a > b$ and $a \sim b \iff M \vDash \exists x(b -_N a = f(x))$ if $b < a$. We interpret A in M by means of the following formulas: $x \in A \leftrightarrow x \in N\ \&\ \forall y < x\ \neg(y \sim x)$, $o(x, y) \leftrightarrow x <_N y$. Let us prove that A is dense. Take $a, b \in N, a < b, a \not\sim b$. If $\left[\frac{b-a}{2}\right]$ belonged to M (i.e. to the image of f) then so would $b - a$ because $f(M)$ is closed under addition. Hence, $a \not\sim a + \left[\frac{b-a}{2}\right] \not\sim b$.

As A is a dense order interpreted in M, $A \cong Q(M)$. Thus, $\mathrm{Canon}(M, T)$ is $(M, <)$ and $\mathrm{Inner}(M, T) = (M + Q(M)(M^* + M), <)$.

□

Question 1. Is there an ω-categorical theory which is not treatable?

The main idea behind treatable theories is that $\mathrm{Inner}(M, T)$ behaves similar to the unique countable recursively saturated model.

Theorem 2.3. If T is treatable and $M \vDash \mathrm{PA} + \Pi_1 \operatorname{Th}\mathbb{N}$ then $\mathrm{Inner}(M, T)$ is parameter-free resplendent.

Proof. Suppose $\varphi_{R_1}, \varphi_{R_2}, \ldots, \varphi_{R_n}$ are the $\mathcal{L}_{\mathrm{PA}}$-formulas interpreting a model of T in every model of PA (as in definition of treatability). Let $\Phi(R_1, \ldots R_n, \overline{S})$ be a parameter-free statement in $\{R_1, \ldots, R_n, \overline{S}\}$ consistent with T. Then there is a countable recursively saturated model $(A, R_1, \ldots, R_n) \vDash T + \Phi(R_1, \ldots, R_n, \overline{S})$. Notice that if $K \vDash \mathrm{PA}$ is recursively saturated then $(K, \varphi_{R_1}, \ldots, \varphi_{R_n}) \vDash T$ is recursively saturated. Hence, by treatability, $(A, R_1, \ldots, R_n)$ is expandable to be $(K, \varphi_{R_1}, \ldots, \varphi_{R_n})$ of some model $K \vDash \mathrm{PA}$ (any countable recursively saturated model will do).

Hence the theory Th in the language $\{R_1, \ldots R_n, +, \times, <, \overline{S}, f\}$ which says

$$(K, +, \times, <) \vDash \mathrm{PA} \wedge (A, R_1, \ldots R_n, \overline{S}) \vDash T + \Phi(R_1, \ldots R_n, \overline{S}) \wedge$$

$$\wedge (A, R_1, \ldots, R_n) \cong_f (K, (\varphi_{R_1})_K, \ldots, (\varphi_{R_n})_K)$$

is consistent. Hence $M \vDash \mathrm{Con}(\mathrm{Th})$.

Consider a model

$$(N, +, \times, <) \vDash \mathrm{PA} \text{ such that } (B, R_1, \ldots, R_n, \overline{S}) \vDash T + \Phi(R_1, \ldots, R_n, \overline{S}) \wedge$$

$$\wedge(B, R_1, \ldots, R_n) \cong_f (N, (\varphi_{R_1})_N, \ldots, (\varphi_{R_n})_N)$$

obtained by an application of the Arithmetized Completeness Theorem to Th in M. By treatability of T, $(B, R_1, \ldots, R_n) \cong \text{Inner}(T, M)$, hence $\text{Inner}(T, M)$ is expandable to a model of $\Phi(R_1, \ldots R_n, \overline{S})$. □

3 Categorically treatable theories

Definition 3.1. A theory T in $\mathcal{L}_T = \{R_1, \ldots, R_n\}$ is categorically treatable if

1. T has a unique countable recursively saturated model;
2. for any two models $(A, R_1^A, \ldots, R_n^A) \vDash T$ and $(B, R_1^B, \ldots, R_n^B) \vDash T$ interpreted in M and any $a_1, \ldots, a_m \in A$, $b_1, \ldots, b_m \in B$ of the same $\mathcal{L}_T$-type, $(A, R_1^A, \ldots, R_n^A, a_1, \ldots, a_m) \cong (B, R_1^B, \ldots, R_n^B, b_1, \ldots, b_m)$.

Notice that every categorically treatable theory is treatable and the unique model of T interpreted in M is $\text{Inner}(M, T)$. Choose and fix one interpretation $\varphi_{R_1}, \ldots, \varphi_{R_n}$.

Theorem 3.2. DLO is categorically treatable, DIS is not.

Proof. For DLO it follows from the proof of Theorem 2.2. For DIS, notice that the orders M and $M + Q(M)(M^* + M)$ are two nonisomorphic models of DIS interpreted in M. □

Definition 3.3.
If T is categorically treatable and $\varphi_{R_1}, \ldots, \varphi_{R_n}$ is the canonically chosen interpretation of $\text{Inner}(M, T)$ then Φ_T denotes the following parameter-free Σ^1_1-statement:

$$\exists \oplus, \otimes, \ll, f \left(\begin{array}{l} \text{“}M = (A, \oplus, \otimes, \ll) \vDash \text{PA”} \wedge \\ f\colon (A, R_1, \ldots, R_n) \to \text{Inner}(M, T) \text{ is an isomorphism} \end{array} \right).$$

Thus Φ_T is saying “our model of T is isomorphic to $\text{Inner}(M, T)$ for some model $M \vDash \text{PA}$”.

Theorem 3.4. If T is categorically treatable then every parameter-free resplendent model of T is resplendent.

Proof. Let A be parameter-free resplendent, $\bar{a} \in A$, $\Phi = \exists \bar{S}\ \varphi(\bar{S}, \bar{a})$ be consistent with T. Consider $\Psi = \exists \bar{S}\ \exists \bar{x}\ \varphi(\bar{S}, \bar{x})$, which is a parameter-free statement consistent with T. By parameter-free resplendency, there is $\bar{b} \in A$ and relations $\bar{S}'$ on A such that $(A, \bar{S}') \vDash \varphi(\bar{S}', \bar{b})$.

However, parameter-free resplendency ensures that $A \vDash \Phi_{\text{PA}}$, that is $A \cong \text{Inner}(M, T)$ for some $M \vDash \text{PA}$. Hence, by categorical treatability of $\text{Inner}(M, T)$, there is an isomorphism $g\colon A \to A$ such that $g(\bar{b}) = \bar{a}$. Relations $\bar{S}$ defined on A as $\bar{S}(\bar{x}) \Leftrightarrow \bar{S}'(g^{-1}(\bar{x}))$ make $(A, \bar{S})$ into a model of $\varphi(\bar{S}, \bar{a})$. □

Thus, for categorically treatable theories, the following notions are listed in nondecreasing strength:

1. A is recursively saturated;

2. $A \cong (M, \varphi_{R_1}, \ldots, \varphi_{R_n})$ for some $M \models \mathrm{PA}$;

3. A is resplendent = parameter-free resplendent;

4. $A \cong (M, \varphi_{R_1}, \ldots, \varphi_{R_n})$ for some $M \models \mathrm{PA} + \Pi_1 \,\mathrm{Th}\, \mathbb{N}$.

4 Models of DLO

Theorem 4.1. Let $(A, <) \models \mathrm{DLO}$ be ω_1-saturated (i.e., has no (ω, ω)-cuts). Then

1. $(A, <)$ is chronically resplendent;

2. $(A, <) \cong Q(M)$ for some resplendent $M \models \mathrm{PA}$;

3. for every consistent $S \supseteq \mathrm{PA}$, $(A, <) \cong Q(M)$ for some $M \models S$;

4. $(A, <) \cong Q(M)$ for some $M \models \mathrm{PA} + \Pi_1 \,\mathrm{Th}\, \mathbb{N}$;

5. $(A, <)$ is resplendent

are listed in the order of nonincreasing strength, that is (1) $\to$ (2) $\to$ (3) $\to$ (4) $\to$ (5).

Proof. (1) $\to$ (2) because, by chronic resplendency, there is a resplendent model $M \models \Phi_{\mathrm{DLO}}$. (3) $\to$ (4) follows from consistency of $\mathrm{PA} + \Pi_1 \,\mathrm{Th}\, \mathbb{N}$ and (4) $\to$ (5) follows from Theorems 3.2, 2.2 and 3.4.

Suppose $M \models \mathrm{PA}$ is resplendent, $A \cong Q(M)$. Let us first show that $(M, <)$ is ω_1-saturated. Suppose

$$a_1 < a_2 < \cdots < a_n < \cdots \quad \cdots < b_n < \cdots < b_2 < b_1, \qquad n \in \omega.$$

As $Q(M)$ is ω_1-saturated, there are $c, d \in M$ such that

$$\frac{a_1}{1} < \frac{a_2}{1} < \cdots < \frac{a_n}{1} < \cdots < \frac{c}{d} < \cdots < \frac{b_n}{1} < \cdots < \frac{b_1}{1}.$$

Thus, $a_1 \cdot d < \cdots < a_n \cdot d < \cdots < c < \cdots < b_n \cdot d < \cdots < b_1 \cdot d$. Introduce the initial segments (cuts) $I = \sup_{i \in \omega} a_i$, $J = \inf_{i \in \omega} b_i$. Let $e = \left[\frac{c}{d}\right]$, hence $e \cdot d < c < (e+1) \cdot d$. Notice that $e + 1 > I$ because for all $x \in I$, $c > x \cdot d$, thus $e > I$ because I is a cut. Also, $e < J$ because for all $x > J$, $c < x \cdot d$. Hence $I < e < J$. Thus $(M, <)$ is an ω_1-saturated model of DIS. Now, by Pabion's Theorem [3], M is ω_1-saturated. Hence, $S \in \mathrm{SSy}(M)$. Let $s \in M$ be a code for S. The following Σ^1_1-statement (with parameter s)

$$\exists K \;= (\oplus, \otimes, \ll) \models S \;\wedge (Q(M), <) \cong (Q(K), <)$$

is consistent with $\mathrm{Th}(M, s)$ because all countable nonstandard models of $N \models \mathrm{PA}$ have the same $Q(N)$, hence, by resplendency of M, $(A, <)$ is isomorphic to $Q(K)$ for some $K \models S$. $\square$

Thus, if resplendency = chronic resplendency for dense linear orders then in the class of ω_1-saturated models of DLO, resplendent models are picked by the Σ^1_1-statement Φ_{DLO}, i.e, $(A, <) \models \mathrm{DLO}$ is resplendent $\Leftrightarrow A \models \Phi_{\mathrm{DLO}}$.

5 More on Σ^1_1-definability

Proposition 5.1. Let $\mathcal{L}$ contain the order relation $<$, and let T be an $\mathcal{L}$-theory such that every model of T contains an $<$-initial segment order-isomorphic to $\mathbb{N}$. Then for every $\Sigma^1_1(\bigwedge)$ sentence Ψ over $\mathcal{L}$ consistent with T there is $N \models T$ satisfying Ψ of cardinality ω_1 with $a \in N$ such that $|[0,a]| = \omega$, and hence N is not resplendent.

Proof. Let Ψ be $\exists R_1, \ldots, R_n \bigwedge_i \theta_i(\bar{R})$, and suppose $(M, \bar{R}) \models T + \bigwedge_i \theta_i(\bar{R})$ be a countable recursively saturated structure for the language $\mathcal{L}_{\bar{R}} = \mathcal{L} \cup \{\bar{R}\}$. We identify $\mathbb{N}$ with the corresponding $<$-initial segment of M. Let I, K be new unary predicates and let $\mathcal{L}_{I,\bar{R}} = \mathcal{L} \cup \{I, \bar{R}\}$. By chronic resplendency it is easy to check that there are $I, K \subseteq M$ such that $\mathbb{N} \subseteq I \subseteq K \subsetneq M$, $K \prec_{\mathcal{L}_{I,\bar{R}}} M$, and $I \subseteq_{\mathrm{e}} M$. By Vaught's 2-cardinal theorem there is $N \succ M$ an elementary extension for the language $\mathcal{L}_{I,\bar{R}}$, in which $I^M = I^N$. The result follows since, by recursive saturation of (M, I, K), there is $a \in I \setminus \mathbb{N}$. Resplendency would imply that N is in 1–1 correspondence with a subset of the set of predecessors of a. □

This shows that resplendency for DIS isn't Σ^1_1, in fact cannot be implied by any $\Sigma^1_1(\bigwedge)$ notion.

Theorem 5.2. There are models of DIS of cardinality $\aleph_1$ which are parameter-free resplendent but not resplendent.

Proof. Theorem **??** shows that there is a $\Sigma^1_1(\bigwedge)$ sentence over the signature $\{<\}$ of linearly ordered sets that implies parameter-free resplendency in all models of DIS. The result follows by Proposition 5.1. □

Question 2. Is resplendency a Σ^1_1 notion over the theory DLO?

The contradiction to resplendency in Lachlan's result seems to require a as a parameter. It is not clear, even for theories extending PA, that parameter-free resplendency isn't Σ^1_1.

Question 3. Is 'parameter-free resplendency' a Σ^1_1 notion?

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