

Задание 5.

DM 21. Докажите, что в \mathbb{R}^n не более $2n$ векторов, скалярное произведение любых двух из которых неположительно.

DM 22. Докажите усиления оценки Варшамова-Гилберта. Докажите, что существует линейный код $\Sigma^k \rightarrow \Sigma^n$ с расстоянием d , где $|\Sigma| = q$, если выполняется а) $q^k V_q(d-1, n) \leq q^n$; б) $q^k V_q(d-2, n) \leq q^n$.

DM 23. A set $S \subseteq \{0, 1\}^n$ is a pairwise independent space, if, for every pair $i \neq j \in \{1, \dots, n\}$, it is the case that if you pick a random element of S and project it onto the i th coordinate and j th coordinate you get a pair of independent bits drawn uniformly from $\{0, 1\}$.

1. Let H be the $(2^\ell - 1) \times \ell$ parity check matrix of a binary Hamming code. Show that the collection of vectors $S = \{\vec{x}H^T \mid \vec{x} \in \{0, 1\}^\ell\}$ forms a pairwise independent space. (H^T denotes the transpose of H .)
2. Show that any pairwise independent space on n bits must contain at least $n + 1$ points.

DM 24. An instance of the MAX 3SAT problem ϕ consists of m “clauses” C_1, \dots, C_m on n Boolean variables x_1, \dots, x_n , where a clause is the disjunction of (exactly) 3 distinct literals; and each literal is either a variable x_i or its negation $\neg x_i$. The goal is to find a 0/1 assignment to the n variables that “satisfies” the maximum number of clauses, where a clause is satisfied if at least one of the literals in the clause is set to 1. For any MAX 3SAT instance ϕ with m clauses, prove that there exists an assignment satisfying at least $\frac{7}{8} \cdot m$ clauses.

DM 25. (An Application of Codes): This is a long exercise whose goal is to “derandomize” Problem DM24. Specifically the final outcome we seek is a deterministic algorithm to compute, given a MAX 3SAT instance ϕ with m clauses, an assignment that satisfies at least $\frac{7}{8} \cdot m$ clauses of ϕ . We start with some definitions.

DEFINITION: A probability space on $\{0, 1\}^n$ is a function $P : \{0, 1\}^n \rightarrow [0, 1]$ such that $\sum_{\alpha \in \{0, 1\}^n} P(\alpha) = 1$. The support of a distribution P is the set of α such that $P(\alpha) > 0$. A probability space is said to be 3-wise independent if for every triple $i, j, k \in \{1, \dots, n\}$ of distinct indices, the marginal distribution P_{ijk} of P on the (i, j, k) th coordinates is the uniform distribution.¹

1. Let P be a 3-wise independent distribution. Let ϕ be a MAX 3SAT instance with m clauses. Show that there exists an assignment α in the support of P such that α satisfies $\frac{7}{8} \cdot m$ clauses of ϕ .
2. Given an $m \times n$ matrix H , define an associated probability space P_H , where $P_H(x) = \frac{1}{M}$ if $H \cdot x = \vec{0}$ and $P_H(x) = 0$ otherwise.
 - (a) For what value of M does the above satisfy the definition of a probability space. (Note that M is not allowed to depend on x .)
 - (b) Give a necessary and sufficient condition (using coding theoretic terms) for P_H to be 3-wise independent.
 - (c) Use the above characterization, to give a 3-wise independent probability space of small support.
3. Put the above together to describe an efficient deterministic algorithm that computes an assignment satisfying $\frac{7}{8} \cdot m$ clauses given any instance of MAX 3SAT with m clauses.

DM 9. Пусть G — это алгебраический (n, d, α) -экспандер. Пусть $k \leq \frac{1}{\alpha}$ и n делится на k . Докажите, что если покрасить вершины в k цветов так, чтобы каждый цвет использовался ровно $\frac{n}{k}$ раз, то найдется хотя бы одна вершина, среди соседей которой встречаются все k цветов.

¹More elaborately, for $b_1, b_2, b_3 \in \{0, 1\}$, let $S_{b_1, b_2, b_3} = \{\alpha \in \{0, 1\}^n \mid \alpha_i = b_1, \alpha_j = b_2, \alpha_k = b_3\}$. Now let $P_{ijk}(b_1, b_2, b_3) = \sum_{\alpha \in S_{b_1, b_2, b_3}} P(\alpha)$. This is the marginal distribution of P onto its i, j, k th coordinates. We require this to be uniform, i.e., $P_{ijk}(b_1, b_2, b_3) = \frac{1}{8}$ for every i, j, k, b_1, b_2, b_3 .