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ABNORMAL SUBGROUPS IN CLASSICAL GROUPS THAT CORRESPOND TO CLOSED ROOT SETS

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Abnormal subgroups in classical Chevalley groups, containing a split maximal torus, are classified. The classification problem is reduced to a combinatorial problem in terms of root systems, which is solved with the use of weight graphs. Bibliography: 5 titles.

1. Introduction

First let G be an arbitrary group and $B \leqslant G$. Henceforth, for an arbitrary subset $X \subseteq G$, $\langle X \rangle$ denotes the subgroup generated by X. A subgroup $B \leq G$ is said to be *abnormal* if, for any $g \in G$,

$$g \in \langle B, gBg^{-1} \rangle.$$

We mention several obvious examples of abnormal subgroups.

- Any maximal subgroup in G is either normal or abnormal.
- The normalizer of a Sylow subgroup in a finite group is abnormal (Frattini argument).
- A famous Tits theorem asserts that the Borel subgroup $B(\Phi, K)$ in a Chevalley group $G(\Phi, K)$ is abnormal in it.

It is clear that any overgroup P, $B \leqslant P \leqslant G$, of an abnormal subgroup $B \leqslant G$ is itself abnormal. For this reason, just minimal abnormal subgroups are of most interest. Note that in the last example the group $B(\Phi, K)$ is not only abnormal in $G(\Phi, K)$, but it is also minimal among abnormal subgroups (for example, see [1]). It is known that the overgroups of $B(\Phi, K)$, i.e., parabolic subgroups, are crucial in studying algebraic groups.

We state, in order of decreasing difficulty, several problems on the classification (up to conjugation) of minimal abnormal subgroups (for example, see [1]).

Problem 1. *Classify minimal abnormal subgroups in finite simple groups and in groups of Lie type.*

As is seen even from the first example, this problem is extremely difficult, because such a classification would provide a new insight into the classification of maximal subgroups in finite groups of Lie type. In the last decades, the program of classification of maximal subgroups in finite groups of Lie type has been the focus of attention of many foremost specialists on the theory of finite groups, including Aschbacher, Seitz, Liebeck, and many others. From papers on the description of maximal subgroups, it is seen that, as the first step to the solution of any problem on finite simple groups, one should solve a similar problem on simple algebraic groups over an algebraically closed field. This naturally leads us to the following problem.

Problem 2. *Classify minimal abnormal subgroups in simple algebraic groups over an algebraically closed field.*

However this problem still remains extremely difficult. From the role which is played by parabolic subgroups in studying algebraic groups, it is seen that even the constuction of new interesting classes of minimal abnormal subgroups would be of great importance. In the present paper, we make one of the very first steps in this direction. Namely, we consider the following problem.

Problem 3. *Classify the minimal abnormal subgroups in Chevalley groups that contain a maximal split torus.*

From the description of overgroups of maximal tori, it follows that, in this case, the problem can be stated in purely combinatorial terms. Namely, in papers by Seitz (for a finite field) and by Borevitch, Vavilov, Dybkova, and O. King (for an infinite field), it was shown that if the field K is not too small, then for any subgroup H, $T \leqslant H \leqslant G$, there exists a unique closed subset of roots $S \subseteq \Phi$ such that $G(S) \leqslant H \leqslant N(S)$.

Thus, the description of minimal abnormal subgroups containing T is reduced to the following purely combinatorial problem on root systems.

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Problem 4. *Describe closed root sets $S \subseteq \Phi$ such that*

$$w \in W(\langle S \cup w(S)\rangle^r)$$

for any $w \in W$.

In the present paper, this problem is completely solved for classical root systems. We state the main result.

2. Statement of the problem and underlying definitions

Let Φ be an irreducible root system, $S \subseteq \Phi$ be a closed root subset, i.e.,

$$\alpha, \beta \in S,\ \alpha + \beta \in \Phi \Rightarrow \alpha + \beta \in S.$$

We recall several well-known definitions. The *reductive part* of a closed root subset $S \subseteq \Phi$ is the set $S \cap -S$, which is denoted by S_r. The reductive part is a root subsystem in Φ. The *Weyl group* $W(S)$ of a closed root subset S is the Weyl group $W(S_r)$ of the root subsystem S_r. We denote the action of an element of the Weyl group w on a root ϕ by $w(\phi)$ and on a root system S by $w(S)$. The closure of a root system S is denoted by $\langle S\rangle$.

Notation. Denote $S_w = \langle S \cup w(S)\rangle$.

We consider the following problem.

Statement of the problem. *Describe closed root sets $S \subseteq \Phi$ such that*

$$w \in W(S_w) = W(\langle S \cup w(S)\rangle^r)$$

for any $w \in W$.

We shall call such closed subsets S *good* and all the remaining closed subsets *bad*, respectively. The main result obtained in the present paper is as follows.

Theorem 2.1. *Let Φ be a classical root system, i.e., of type A_n, B_n, C_n, or D_n, and let a closed root subset $S \subseteq \Phi$ be given. Then the following conditions are equivalen:*

1. *there is $w \in W(\Phi)$ such that $w \notin W(S_w)$;*
2. *there is $w \in W(\Phi)$ such that $w \notin W(S_w)$, $w^2 = 1$.*

Remark 2.1. Theorem 2.1 does provide a classification of good root subsets in the classical systems, because the action of involutions on S is described very simply in terms of weight graphs. Namely in these terms shall we state and prove the results for each of the four classical root systems, thus proving Theorem 2.1 as well.

We recall the definition of a weight graph.

The *weight graph* of a subset S of a root system Φ is an oriented graph to the vertices of which correspond the weights of the given system, and an arrow is drawn from a weight u to a weight v if $u - v$ is a root and lies in S.

Notation. In the present paper, the weights are denoted by integers; accordingly, the arrow from the vertex i to the vertex j of the weight graph is denoted by (i, j) (the notation is taken from Malyshev's paper [2], although in that paper the author used another representation of root systems in the form of graphs). If there is an arrow (i, j) in a graph S, we shall write $(i, j) \in S$; similarly for edges. In the sequel, we omit the word *weight* and say merely the *graph* of a root subset.

Definition 2.1. *A graph is said to be good if it corresponds to a good root subset in Φ; a bad graph is defined similarly. The graph corresponding to a root set is denoted by the same letter as the root set itself.*

Definition 2.2. *The arrows (i, j) and (j, i) are said to be reverse to each other.*

Definition 2.3. *The reductive part of a graph is the graph that is obtained from the given graph by removing all the arrows the reverse arrows to which are lacking (this corresponds to constructing the reductive part of a root set).*

Definition 2.4. *A connected component of a graph is a connected component (in the usual sense of graph theory) of its reductive part. To a connected component of the weight graph of a root subset corresponds a root subsystem lying in this subset.*

Notation. The complement to a set of vertices A will be denoted by $\overline{A}$.

3. The main part

3.1. The case $\Phi = A_n$. The set of weights for the system A_n is the orthonormal basis

$$\{e_i, \quad 1 \leqslant i \leqslant n+1\} \subset \mathbb{R}^{n+1},$$

and we denote the corresponding vertices of the graph by the numbers from 1 to $n+1$. The root set of the system A_n is the set of vectors

$$\{e_i - e_j,\ 1 \leqslant i \neq j \leqslant n+1\}.$$

Definition 3.1. *A graph S is said to be transitively closed if it satisfies the following transitivity property: if the graph contains a path by arrows from the vertex i to the vertex j, then it contains an arrow from i to j, i.e., $(i, j) \in S$.*

Remark 3.1. To the *closure* of a root subset in A_n corresponds the *transitive closure* of the graph. Forestalling events, we note that in the case of C_n, this is also valid, but in the cases of B_n and D_n, to the closure of a root subset corresponds the restriction of the transitive closure of the graph to the root sets of B_n and D_n, respectively.

The Weyl group of the system A_n is the group S_{n+1} of permulations on $n+1$ elements, which acts by all possible permutations on the vertices of the graph. For any root system, the union of (weight) graphs is the superposition of them one on the other, i.e., the weight graph the set of edges of which is the union of the sets of edges of the initial graphs. In the language of graphs, $S_w = \langle S \cup w(S)\rangle$ is constructed in the following way: the take the union of graphs $S \cup w(S)$ and close transitively the graph obtained.

Remark 3.2. In the case of A_n, the Weyl group of a root subsystem S is the direct sum of permutation groups acting on the connected components of the graph S.

Theorem 3.1. *Let a closed root subset $S \subseteq A_n$ be given. The following statements are equivalent:*

1. *there is $w \in W(A_n)$ such that $w \notin W(S_w)$.*
2. *There is $w \in W(A_n)$ such that $w \notin W(S_w)$, $w^2 = 1$.*
3. *The set of all vertices of the weight graph S is divided into four sets A, B, C, and D satisfying the following two conditions:*
   (a) *the sets B and C contain an equal number of vertices, there are no arrows between them, and the sets B and C are nonempty.*
   (b) *There are no arrows from $\overline{A}$ to A and from D to $\overline{D}$; the sets A and D may be empty.*

Remark 3.3. The implication $(1 \Rightarrow 3)$ means that a bad graph and its adjacency matrix have the structure given in Fig. 1.

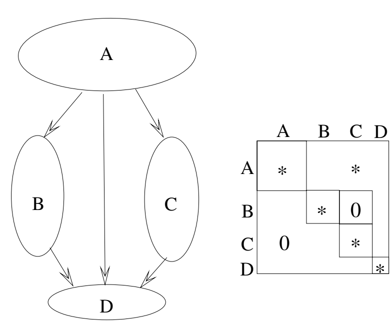

Fig. 1. A bad graph and its adjacency matrix.

Proof of Theorem 3.1. The implication $(2 \Rightarrow 1)$ is obvious. We prove the implication $(3 \Rightarrow 2)$. Let the graph S contain collections of connected components A, B, C, D satisfying conditions (a) and (b). We construct a (nontrivial) involution $w \in W(\Phi)$ in the following way. We set a one-to-one correspondence between the vertices from B and the vertices from C (condition (a) implies that this can be done). Let w carry each vertex from B to the corresponding vertex from C, each vertex from C to the corresponding vertex from B, and let it leave fixed all the vertices in $A \cup D$. It is clear that $w^2 = 1$. It remains to prove that $w \notin W(S_w)$, where $S_w = \langle S \cup w(S)\rangle$.

Lemma 3.1. *In the graph S_w there are no arrows between B and C.*

Proof. We note that the graph $w(S)$ has the same structure as the graph S, i.e., the graph $w(S)$ satisfies conditions (a) and (b) (with the same A, B, C, and D). This can easily be seen in Fig. 1 with a representation of the graph. Indeed, in the graph $w(S)$, the subgraphs of the graph S on the sets B and C interchange. The arrows inside $A \cup D$ do not change. An arrow of the graph S from A to B is converted in the graph $w(S)$ into an arrow from A to C, and conversely; similarly for arrows from B or C to D. Therefore, the graph $S \cup w(S)$ has also the same structure, as well as its transitive closure S_w. Consequently, in S_w there are no arrows between B and C, as required.

It remains to note that w interchanges the vertices of B with the vertices of C. By Lemma 3.1, these vertices lie in different connected components of the graph S_w, and thus, in view of Remark 3.2, $w \notin W(S_w)$, as required. Now we proceed to proving the implication $1 \Rightarrow 3$. Let S be a bad graph, i.e, there exist $w \in W(A_n)$ such that $w \notin W(S_w)$.

Definition 3.2. *The connected components of the graph S_w are called boxes.*

We represent the graph S_w on the plane (see Fig. 2). Points depict the vertices of the graph, arrows correspond to the arrows of the graph, rectangles represent boxes, and dotted lines depict the actions of the element w. From elementary graph theory, it is well known that any oriented graph can be arranged on the plane in such a way that all arrows are directed from top to bottem (nonstrictly), and with each horizontal arrow the graph contains a reverse arrow to it. We depict our graph precisely in this way. Thus, the different boxes are disposed on different horizontals, all the vertices inside a box are placed on one horizontal, and all arrows between vertices from different boxes are directed strictly from top to bottom.

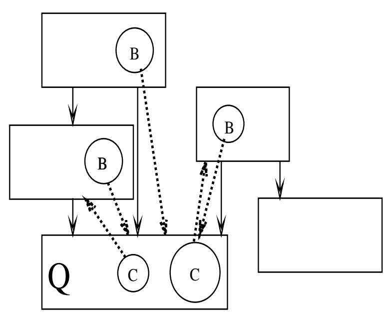

FIG. 2. Representation of the graph S_w on the plane; the sets B and C.

Definition 3.3. *A box is said to be special if there is a vertex in it that leaves this box under the action of w.*

Remark 3.4. In the case of the system A_n, the condition $w \notin W(S_w)$ is equivalent to the fact that, in the graph S_w, there is at least one special box (this is clear from Remark 3.2 on the structure of the Weyl group of a root subsystem in A_n).

In the sequel we consider the representation of the graph S_w on the plane constructed above. Consider the lowest special box, and denote it by Q (by Remark 3.4, its existence follows from the fact that S is a bad graph). We define

$$B = \{b \notin Q \mid w(b) \in Q\}, \ C = \{c \in Q \mid w(c) \notin Q\},$$
$$A = \{a \notin B \cup C \mid \exists x \in B \cup C : (a, x) \in S\}, \quad D = \overline{A \cup B \cup C}.$$

It is easy to see that A, B, C, and D are a partition of the entire set of vertices of the graph. We also note that all the vertices from B lie in special boxes, which are disposed above the box Q. We prove two auxiliary statements.

Lemma 3.2. *Condition* (a) *from part* 3 *of Theorem* 3.1 *holds, i.e., in the graph S, there are no arrows between B and C, they have an equal number of vertices, and they are nonempty.*

Proof. We note at once that C is nonempty, because the box Q is special. The number of vertices in B is equal to the number of vertices arriving at Q from the outside under the action of the permutation w, and the number of vertices in C is equal to the number of vertices leaving Q under the action of w; therefore these numbers are equal. In particular, since C is nonempty, B is also nonempty. Now we prove that in S there are no arrows between B and C. There is no arrow from C to B, because otherwise it would be directed from bottom to top, but our graph is represented in such a way that all arrows are directed from top to bottom. Now we prove that no arrow from B to C exists. Assume the contrary. Let $(b, c) \in S$, where $b \in B$ and $c \in C$. Then $(w(b), w(c)) \in S_w$, i.e., $S_w \supseteq w(S)$. But $w(b)$ lies in Q, but $w(c)$ belongs to some other special box (it is special, because, under the action of w, something enters it, and thus something leaves it under the action of w). Therefore, $w(c)$ lies above $w(b)$ and the arrow $(w(b), w(c))$ in the graph S_w is directed from bottom to top, a contradiction, which completes the proof of the lemma.

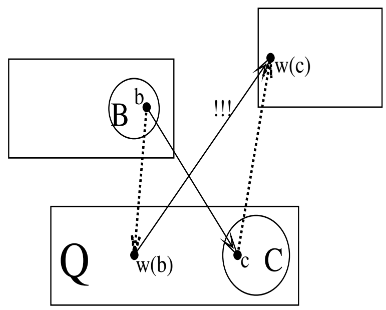

FIG. 3. Illustration to the proof of Lemma 3.2 – impossibility of existence of arrows between B and C.

Lemma 3.3. *In the graph S, there is no vertex $x \notin B \cup C$, out of which an arrow would go to $B \cup C$ and, simultaneously, into which an arrow from $B \cup C$ would come.*

Proof. Assume the contrary. Suppose such a vertex x exists, i.e., in $B \cup C$ there are two vertices y and z such that (y, x), $(x, z) \in S$. If one of the vertices y and z lies in B and the other belongs to C, then there exists a path between B and C (going through x) and, by transitivity, in S there is an arrow between B and C, which is impossible by Lemma 3.2. Thus, either both vertices y and z lie in B, or they belong to C. Consider these two cases.

1. $y, z \in B$. We note that $(w(y), w(x))$, $(w(x), w(z)) \in S_w$. But $w(y)$, $w(z) \in Q$ and thus $w(x) \in Q$; otherwise there is an arrow directed from bottom to top: $(w(y), w(x))$ or $(w(x), w(z))$. Since $w(x) \in Q$, we have $x \in B \cup Q$. By assumption of the lemma, $x \notin B$, and thus $x \in Q$. But then the arrow (x, z) is directed from bottom to top, a contradiction.

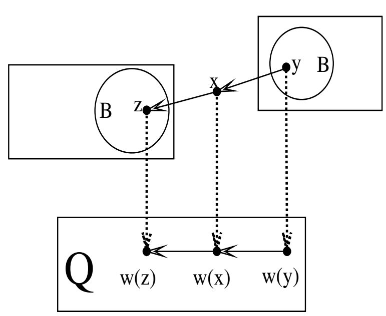

FIG. 4. Illustration to the proof of Lemma 3.3, part 1.

2. $y, z \in C$. In particular, y and z lie in the box Q. Since (y, x), $(x, z) \in S$, we conclude that x also lies in Q (otherwise one of the arrows (y, x) or (x, z) is directed from bottom to top). The vertex z lies in C, and therefore $w(z)$ lies above Q by the construction of C. But the existence of the arrow (x, z) in the graph S implies the existence of the arrow $(w(x), w(z))$ in the graph S_w, and thus $w(x)$ lies above $w(z)$, which is certainly above Q. But then x leaves Q under the action of w, i.e., $x \in C$, which contradicts the assumption $x \notin B \cup C$ of the lemma. The contradiction completes the proof of the lemma.

Thus, the sets A, B, C, and D constructed above satisfy, by Lemma 3.2, condition (a). It remains to verify that condition (b) is fulfilled. Indeed, by Lemma 3.3, no arrow can go to A from $B \cup C$. By the construction of the set A, no arrow exists from D to $B \cup C$. No arrows exist from D to A; otherwise, since the graph S is transitively closed, there is an arrow from D to $B \cup C$. Thus, both conditions (a) and (b) are satisfied, and the implication $1 \Rightarrow 3$ is proved, which completes the proof of Theorem 3.1.

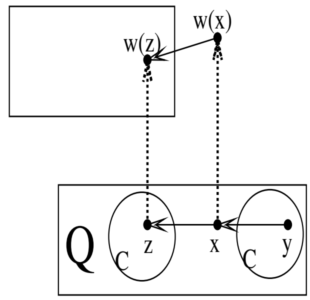

FIG. 5. Illustration to the proof of Lemma 3.3, part 2.

3.2. The case $\Phi = C_n$. The set of weights for the system C_n is $\{\pm e_i,\ 1 \leqslant i \leqslant n+1\}$, where the e_i form an orthonormal basis for $\mathbb{R}^{n+1}$.

Notation. The corresponding vertices of the weight graph will be denoted by the integers from 1 to $n+1$ (these vertices will be depicted on the left and called left or positive) and from -1 to $-n-1$ (they will be shown on the right and called right or negative). We place the left vertices vertically one below another in increasing order, and the right vertices are situated at equal distances from the left ones on the right in decreasing order. Thus, for any i, the vertex $-i$ lies on the same horizontal as the vertex i.

Definition 3.4. *Vertices that lie on one and the same horizontal, i.e., having the numbers i and $-i$, will be called dual (with respect to each other). The set composed of vertices dual to the vertices of a given set X will be called dual to X and denoted by $-X$.*

Remark 3.5. The root set of the system C_n is the set of vectors

$$\{\pm e_i \pm e_j,\ \pm 2e_i,\ 1 \leq i < j \leq n+1\} \subset \mathbb{R}^{n+1};$$

moreover, the pluses and minuses at e_i and e_j are independent. To the root $\pm e_i \pm e_j$ corresponds the pair of arrows $(\pm i,\ \mp j)$ and $(\pm j,\ \mp i)$, and to the root $2e_i$ corresponds the arrow $(i, -i)$.

Similarly to A_n, in the case of C_n to each closed root subset corresponds a transitively closed graph S possessing the property

$$(i,j) \in S \Leftrightarrow (-j,-i) \in S.$$

Definition 3.5. *It is easy to see that either in a connected component of a (transitively closed) graph all the vertices occur together with their dual vertices (such components will be called complete) or, conversely, the component contains none of the dual pairs (we call them incomplete). For any incomplete component, there is a dual one that consists of vertices dual to the vertices of this component.*

Remark 3.6. For the Weyl group $W(c_n)$, we have $W(C_n) = S_{n+1} \ltimes \mathbb{Z}/2\mathbb{Z}^{n+1}$, and S_{n+1} acts by all possible permutations on the set of positive vertices (and by the same permutations on the set of negative vertices), but $\mathbb{Z}_2^{n+1}$ changes the signs of vertices in all possible ways (i.e., interchanges the dual vertices). It is clear that for any $w \in W(C_n)$ if $w(i) = j$, then $w(-i) = -j$. The Weyl group of a root subsystem S is the direct sum of the Weyl groups of root subsystems corresponding to the connected components of the weight graph S; moreover, for an incomplete connected component the corresponding term of the direct sum is the permutation group S_k on this component (and on the dual to it, respectively) and, for a complete connected component, this is $S_k \ltimes \mathbb{Z}/2\mathbb{Z}^k$ and it acts on this component as the Weyl group of the system C_{k-1}.

In terms of weight graphs, the statement of the theorem does not virtually differ from the case of the system A_n.

Theorem 3.2. *Let a closed root subset $S \subseteq C_n$ be given. The following assertions are equivalent:*

1. *There is $w \in W(C_n)$ such that $w \notin W(S_w)$.*
2. *There is $w \in W(C_n)$ such that $w \notin W(S_w)$, $w^2 = 1$.*

3. *The set of all vertices of the weight graph S is divided into four sets A, B, C, and D, satisfying the following three conditions:*
   (a) *the sets B and C contain the equal number of vertices, there are no arrows between them, and the sets B and C are nonempty.*
   (b) *There are no arrows from $\overline{A}$ to A and from D to $\overline{D}$; the sets A and D may be empty.*
   (c) *One of the following three situations occurs:*
   - $B = -C$
   - $-B \subseteq A$, $-C \subseteq A$
   - $B = -B$, $C = -C$.

Theorem 3.2 differs from Theorem 3.1 only by the addition of the third condition in assertion (3), which is due to the presence in the graph of the relationship between dual vertices.

Proof of Theorem 3.2. The implication $(2 \Rightarrow 1)$ is obvious. We prove that $(3 \Rightarrow 2)$. Let A, B, C, and D satisfying conditions (a), (b), and (c) exist. The proof is identical with the proof of $(3 \Rightarrow 2)$ in Theorem 3.1. There are only two minor distinctions. First, it is necessary to prove that there is a $w \in W(C_n)$ that can interchange B and C. This is obvious in each of the situations in condition (c). Second, when the sets B and C interchange, the sets $-B$ and $-C$ also interchange, and we must prove that the structure of the graph does not change under such a permutation. Similarly, in each of the three situations in condition (c) it is obvious that the permutation of $-B$ and $-C$ also does not change the structure of the graph (here, as before, by the structure we mean the fulfillment of conditions (a) and (b)). By construction, $w^2 = 1$.

Now we prove that $(1 \Rightarrow 3)$. First we need an auxiliary statement.

Lemma 3.4. *If $i \in [1,\ n+1]$ is such that $(i, -i), (-i,\ i) \notin S$, then assertion* (3) *from Theorem* 3.2 *holds for the graph S.*

Proof. We take as B the connected component of the vertex i and as C the connected component of the vertex $-i$. Since $(i, -i), (-i,\ i) \notin S$, the components B and C are dual and thus $B = -C$ and condition (c) is fulfilled. The graph S satisfies condition (a) by construction of the sets B and C and by assumption that there are no arrows between i and $-i$ (and thus the arrows are lacking between B and C). We define the sets A and D as in the proof of the implication $(1 \Rightarrow 3)$ in Theorem 3.1. The fact that B and C are connected components of the graph S implies that Lemma 3.3 holds and thus condition (b) is fulfilled in the graph S. Consequently, Lemma 3.4 is proved.

Let the graph S be bad, and let, on the contrary, the set of its vertices cannot be divided into four sets as required. By Lemma 3.4, one may assume that between any pair of dual vertices there is at least one arrow. We define the lowest box Q and the sets A, B, C, and D as in the proof of $(1 \Rightarrow 3)$ in Theorem 3.1:

$$B = \{b \notin Q \mid w(b) \in Q\},\ C = \{c \in Q \mid w(c) \notin Q\},$$
$$A = \{a \notin B \cup C \mid \exists x \in B \cup C : (a, x) \in S\}, \quad D = \overline{A \cup B \cup C}.$$

In this case, the proofs of Lemmas 3.2 and 3.3 are repeated word for word; therefore, conditions (a) and (b) in part (3) of Theorem 3.2 are fulfilled. It remains to prove that condition (c) is satisfied. We prove two lemmas in terms of the sets A, B, C, and D just constructed.

Lemma 3.5. $B \cap -C = \varnothing$.

Proof. On the contrary, let there exist a vertex $x \in B$ such that $-x \in C$. By Lemma 3.4, the vertices x and $-x$ are joined by an arrow in S. But this is impossible by condition (a) just proved, because no arrows exist between B and C in S.

Lemma 3.6. *Either $\{B \cup C\} \cap \{-B \cup -C\} = \varnothing$, or $B = -B$ and $C = -C$.*

Proof. Let the first relation be not valid, i.e., $\{B \cup C\} \cap \{-B \cup -C\} \neq \varnothing$. Then we prove that $B = -B$ and $C = -C$. By Lemma 3.5, $B \cap -C = \varnothing$, and therefore $C \cap -B = \varnothing$. The following two possibilities may occur: either $B \cap -B \neq \varnothing$ or $C \cap -C \neq \varnothing$. The arguments in the second case repeat the arguments in the first one, and we restrict ourselves to the proof of the lemma in the first case. Thus, let $B \cap -B \neq \varnothing$. Consider $x \in B$ such that $-x \in B$. It is necessary to prove that $B = -B$ and $C = -C$. The vertices $w(x)$ and $w(-x)$ lie in Q and

$w(-x) = -w(x)$; therefore, in Q there is a pair of dual vertices and thus Q is a complete connected component of S_w, i.e., $Q = -Q$.

1. We show that $B = -B$. Consider a vertex $y \in B$. We prove that $-y \in B$. We have $x, y \in B$, and therefore $w(x), w(y) \in Q$; since Q is a connected component of the graph S_w, we have $(w(x), w(y))$, $(w(y), w(x)) \in S_w$. Then the pair dual to these arrows belongs to the graph S_w, i.e., $(w(-x), w(-y), (w(y), w(x)) \in S_w$, but $w(-x) \in Q$ and thus $w(-y) \in Q$ (because Q is a connected component of the graph S_w). Therefore, under the action of w, $-y$ goes to Q and either it lies in B (which is required) or it is contained in Q. But $-y$ cannot lie in Q, because $Q = -Q$ and $-y \in Q$ implies that $y \in Q$, a contradiction with the choice of $y \in B$. Thus, for any $y \in B$ we showed that $-y \in B$ and consequently $B = -B$.

2. We prove that $C = -C$. Consider a vertex $z \in C$ and show that $-z \in C$. Since $Q = -Q$ and $C \subseteq Q$, we have $-z \in Q$. Assume the contrary: let $-z \notin C$, i.e., $-z$ does not leave Q under the action of w. In other words, $-w(z) = w(-z) \in Q$. But $Q = -Q$ and $-w(z)$ lies in Q together with $-w(z)$, a contradiction, because $z \in C$. For an arbitrary $z \in C$, we obtain $-z \in C$, and thus $C = -C$.

Now we turn to the proof of Theorem 3.2; namely, it remains to prove condition (c). Let the situation $B = -B$, $C = -C$ do not hold. Then, by Lemma 3.6, $\{B \cup C\} \cap \{-B \cup -C\} = \varnothing$; in other words, $-B \cup -C \subseteq A \cup D$. We show that $-B \cap D = \varnothing$. Assume the contrary. Let $x \in -B \cap D$. Note that $-x \in B$. The graph S cannot contain the arrow $(x, -x)$ (because there are no arrows from D to B), and thus, by Lemma 3.4, the arrow $(-x, x)$ is contained in S. Then the graph S_w has the arrow $(w(-x), w(x))$; moreover, $w(-x) \in Q$. The vertex $w(x)$ lies in a special box (because $x \in -B$) different from Q (since $B \cap -B = \varnothing$). We have a contradiction, because the arrow $(w(-x), w(x))$ is directed from Q to another apecial box, and thus it goes from bottom to top. Thus, we have proved that $-B \cap D = \varnothing$ and $-B \subseteq A$. It is somewhat easier to prove that $-C \subseteq A$ and condition (c) is fulfilled; namely, we have $-B \subseteq A$, $-C \subseteq A$. Theorem 3.2 is proved.

3.3. The case $\Phi = B_n$. The system of weights for B_n differs from the system of weights for C_n by the zero weight. Recall that the root system B_n consists of the vectors

$$\{\pm e_i \pm e_j,\ \pm e_i,\ 1 \leq i < j \leq n+1\} \subset \mathbb{R}^{n+1},$$

and the pluses and minuses at e_i and e_j are independent. In the graph, to the root $\pm e_i$ corresponds the pair of arrows $(\pm i, 0)$, $(0, \mp i)$. To the closure of a root subset corresponds the intersection of the transitive closure of the graph with the graph B_n, i.e., the closure of the graph on all arrows, except for arrows between dual vertices.

Theorem 3.3. *Let a closed root subset $S \subseteq B_n$ be given. The following assertions are equivalent:*

1. *There is $w \in W(B_n)$ such that $w \notin W(S_w)$.*
2. *There is $w \in W(B_n)$ such that $w \notin W(S_w)$, $w^2 = 1$.*
3. *Either in S there is a vertex $i \neq 0$ that is not joined with 0 by an arrow, or S can be represented in the form $X \sqcup -X \sqcup 0$; moreover, from any vertex of X there is an arrow to 0 and X is divided into four sets A, B, C, and D that satisfy the following two conditions:*
   - (a) *the sets B and C contain equal numbers of vertices, there are no arrows between them, and the sets B and C are nonempty.*
   - (b) *There are no arrows from $\overline{A}$ to A and from D to $\overline{D}$; the sets A and D may be empty.*

Proof. The implication $(2 \Rightarrow 1)$ is obvious. We prove that $(3 \Rightarrow 2)$.

First we consider the case where in the graph S there is a vertex i disjoined with O by an arrow. Consider w interchanging i and $-i$ and leaving fixed all other vertices. It is clear that $w^2 = 1$. We prove that $w \notin W(S_w)$. Assume the contrary. If $w \in W(S_w)$, then in $S \cup w(S)$ there is an arrow between i and 0. Indeed, if this arrow is lacking, then either i and $-i$ lie in different connected components of the graph S_w (and then $w \notin W(S_w)$), or they lie in one and the same component of type D_k, i.e., in a component the root set of which forms a root subsystem of type D_k. But in this case, $w \notin W(S_w)$, because w changes the sign of an odd number of vertices in the corresponding component of type D_k. Therefore, a path between i and 0 must exist in $S \cup w(S)$. We choose the shortest one among the paths joining the set $\{i \cup -i\}$ with 0; without loss of generality, we may assume that this is a path from i to 0 not passing through $-i$. The graph S did not contain this path, but the only arrow on this path that may be lacking in the graph S is the first arrow on the path, say, $(i,\ j)$. But then the initial graph contained the arrow $(-i,\ j)$ and thus a path from $-i$ to 0, a contradiction. Now we assume that in the graph there are no nonzero vertices disjoined with 0. Then, by assertion (3), $S = X \sqcup -X \sqcup 0$, $X = A \sqcup B \sqcup C \sqcup D$, and conditions (a) and (b) are fulfilled. In this case, the proof is similar to the case of A_n, but the structure of the graph is such as given in Fig. 6.

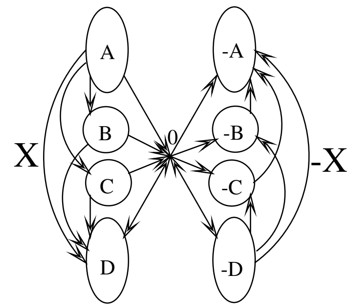

FIG. 6. The structure of the graph S from Theorem 3.3.

In this figure, all arrows from X to 0 are drawn, but the arrows from 0 to X go only to D, for otherwise in S an arrow would be between B and C. We also note that there is no arrow from $-d \in -D$ to $-X \setminus -D$, for otherwise there is the arrow $(0, -d)$, and since the arrow $(d, 0)$ is present, there is an arrow from $d \in D$ to $X \setminus D$ as well, a contradiction. Similarly, there are no arrows from $x \in -X \setminus -D$ to X, for otherwise, by transitivity, $(x,\ 0) \in S$ and thus $(0, -x) \in S$, where $-x \in X \setminus D$, but there are no arrows from 0 to $X \setminus D$. This structure of the graph is not violated under the action of $w \in W(B_n)$ that interchanges B and C (and also $-B$ and $-C$), and similarly to the case of A_n, we derive that w is an involution that does not belong to $W(S_w)$.

Now we prove the implication $(1 \Rightarrow 3)$. Let the graph S be bad, i.e., there is $w \in W(B_n)$ such that $w \notin W(S_w)$. If there is a vertex in S disjoined with 0, then assertion (3) is fulfilled. Assume the contrary, i.e., between any vertex and 0 there is at least one arrow, and thus in any pair of dual vertices there is a vertex from which there is an arrow to 0. In each pair of dual vertices, we take the vertex from which there is an arrow to 0 (if there are two such vertices, we take any of them) and place it to X. Then $S = X \sqcup -X \sqcup 0$, and from any vertex of X there is an arrow to 0. For clearness we assume that X is the set of positive vertices (to this end we merely change the signs of negative vertices in X – the graph S does not cease to be bad). We represent w as a composition $\pi \circ w_1$, where $w_1 \in S_{n+1}$ and $\pi \in \mathbb{Z}_2^{n+1}$, i.e. w_1 permutes vertices inside the left-hand and right-hand sides of the graph and π interchanges the pairs of dual vertices. We prove that w_1 is also bad. Assume the contrary, i.e., $w_1 \in W(S_{w_1})$. We note that $S_w \supseteq S_{w_1}$, because between any two dual vertices in S there is at least one arrow and under the action of π the dual vertices belong to one and the same connected component. Denote by $\hat{\pi}$ the set of elementary reflections the composition of which is the element π. Since $S_w \supseteq S_{w_1}$ and the vertices that are interchanged by π in the graph S_w are joined with their duals, we obtain $W(S_w) \supseteq \langle W(S_{w_1}), \hat{\pi} \rangle$. But, in turn, $\langle W(S_{w_1}), \hat{\pi} \rangle \supseteq w$. Therefore, $W(S_w) \supseteq w$, a contradiction with the fact that w is bad. Thus, we have proved that w is bad if and only if w_1 is bad. However, it suffices to consider the action of w_1 only on the positive vertices, and further proof repeats the proof in the case of A_n word for word.

3.4. The case of $\Phi = D_n$. In this case, the set of weights coincides with the set of weights for C_n, and the root set is

$$\pm e_i \pm e_j, \quad 1 \le i < j \le n+1 \subset \mathbb{R}^{n+1},$$

where the pluses and minuses at e_i and e_j are independent. To the closure of a root subset corresponds the intersection of the transitive closure of the graph with the set D_n, i.e., with the set of all arrows, except for arrows between dual vertices, which are lacking in the system D_n. An essential difference of the system D_n from the systems C_n and B_n is the structure of its Weyl group. The Weyl group of D_n is $S_{n+1} \ltimes \mathbb{Z}_2^n$, where S_{n+1}, as before, permutes the positive vertices between themselves and $\mathbb{Z}_2^n$ changes the sign of an even number of positive vertices.

Theorem 3.4. *Let a closed root subset $S \subseteq D_n$ be given. The following statements are equivalent:*

1. *there is $w \in W(D_n)$ such that $w \notin W(S_w)$.*
2. *There is $w \in W(D_n)$ such that $w \notin W(S_w)$, $w^2 = 1$.*
3. *Either there is a pair of nondual vertices i and j such that there are no arrows between $\{i, -i\}$ and $\{j, -j\}$, or the set of all vertices of the graph S is divided into four sets A, B, C, and D satisfying the following three conditions:*

(a) *the sets B and C contain an equal number of vertices, there are no arrows between them, and the sets B and C are nonempty.*

(b) *There are no arrows from $\overline{A}$ to A and from D to $\overline{D}$; the sets A and D may be empty.*

(c) *One of the following two situations occurs:*

- *$B = -C$, and B contains an even number of vertices.*
- *$-B \subseteq A$, $-C \subseteq A$.*

Proof. The implication $(2 \Rightarrow 1)$ is obvious. We prove that $(3 \Rightarrow 2)$.

Let there exist a pair of nondual vertices i and j such that there are no arrows between $\{i, -i\}$ and $\{j, -j\}$. Consider $w \in W(D_n)$ that transposes i with $-i$ and j with $-j$. Next the proof is similar to the proof of $(3 \Rightarrow 2)$ in Theorem 3.3; it is only necessary to replace the 0 weight by the pair of weights $\{j, -j\}$ in the respective place. If such a pair of vertices i and j in the graph is lacking, then the second part of the alternative of assertion (3) holds. In this case the proof is similar to the proof in the case C_n, i.e., to the proof of $(3 \Rightarrow 2)$ in Theorem 3.2. The only distinction is in the proof that we can transpose B and C by an element of the group $W(D_n)$, i.e., by changing the sign of an even number of positive vertices; this is ensured by condition (c).

Now we prove that $(1 \Rightarrow 3)$. Let there exist $w \in W(D_n)$ such that $w \notin W(S_w)$. Depending on the form of w, the following two cases may occur:

1. the permutation w acts only inside the boxes, i.e., for any vertex x, we have $(w(x), x), (x, w(x)) \in S_w$.

In this case, the only reason for which w may not lie in $W(S_w)$ is that there exists a complete box Q in which w changes the sign of an odd number of positive vertices. But in the entire graph, w changes the sign of an even number of positive vertices; consequently, there is at least one complete box R, in which w changes the sign of an odd number of positive vertices. It is sufficient to us that in the graph S_w two complete boxes exist. We fix vertices $q \in Q$ and $r \in R$. Note that between the complete boxes Q and R there are no arrows, because if there is an arrow from Q to R, then there is an arrow from $-R$ to $-Q$, i.e., from R to Q, and thus Q and R form a connected component, a contradiction. But then there are no arrows between $\{q, -q\}$ and $\{r, -r\}$, and the first part of the alternative in statement (3) holds (setting $r = q$, $j = r$).

2. The permutation acts not only inside the boxes, i.e., special boxes exist.

The box Q and the sets B and C are defined in the same way as in the proof of $(1 \Rightarrow 3)$ in Theorem 3.1, and conditions (a) and (b) are fulfilled similarly.

2.1. Let $B \cap -B \neq \varnothing$ or $C \cap -C \neq \varnothing$. Arguing in the same way as in the proof of Lemma 3.6, we derive that $B = -B$ and $C = -C$, and taking $i \in B$ and $j \in C$ arbitrarily, we ensure that the first part of the alternative in statement (3) holds.

2.2. Let $B \cap -B = \varnothing$ and $C \cap -C = \varnothing$. Denote $X = B \cap -C$ (X may be empty). Denote $B_0 = B \setminus X$ and $C_0 = C \setminus -X$.

2.2.1. Suppose $B_0 \neq \varnothing$. We note that for new $B = B_0$ and $C = C_0$, condition (a) is satisfied. We construct A_0 and D_0 in the usual way, i.e.,

$$A_0 = \{a \notin B_0 \cup C_0 \mid \exists x \in B_0 \cup C_0 : (a, x) \in S\}, \ D_0 = \overline{A_0 \cup B_0 \cup C_0}.$$

We prove that $-B_0 \subseteq A_0$. By construction of B_0 and C_0, $-B_0 \cup -C_0 \subseteq A_0 \cup D_0$. Let a vertex $x \in -B_0 \setminus A_0$ exist; consider any vertex $y \in C_0$. If $(x, y) \in S$, then $x \in A_0$. Moreover, $(y, x) \notin S$, because this arrow goes from Q to another special box (containing $-B_0$), i.e., it is directed from bottom to top. Consequently, there are no arrows between x and y. But there are no arrows between $-x$ and y, by condition (a) (because $-x \in B_0$ and $y \in C_0$). Thus, there are no arrows between $\{x, -x\}$ and $\{y, -y\}$, and the first part of statement (3) holds ($i = x$, $j = y$). Thus, we have proved that $B_0 \subseteq A_0$; similarly, $C_0 \subseteq A_0$ and condition (c) is fulfilled. It remains to prove that for A_0, B_0, C_0, and D_0, condition (b) is fulfilled; to this end, it suffices to check that for $B = B_0$ and $C = C_0$, Lemma 3.3 holds. If $x \notin X \cup -X$, the proof of the lemma can be repeated. But $x \notin X \cup -X$ does hold, because there are no arrows from $B_0 \cup C_0$ to $X \cup -X$; this is established similarly to the proof of condition (a). Therefore, condition (a), (b), and (c) of the second part of statement (3) holds.

2.2.2. $B_0 = \varnothing$. In other words, $B = -C$. In this case, $w(\{Q \cup -Q\}) = \{Q \cup -Q\}$. We prove that there are no special boxes different from $-Q$ and Q. Assume the contrary. We take an arbitrary vertex y in a special box different from $Q \cup -Q$; let $x \in C$. If $(x, y) \in S$, then $(w(x), w(y)) \in S_w$, but $w(x) \in Q$ and $w(y)$ lies in another special box, and we obtain an arrow in S_w directed from bottom to top, a contradiction. Therefore, $(x, y) \notin S$; similarly, $(x, -y) \notin S$. If $(y, x) \in S$, then $(-x, -y) \in S$ and $-x \in C_0 \subseteq Q$ and $-y$ lies above Q, again a contradiction. Consequently, $(y, x) \notin S$, and similarly $(y, -x) \notin S$. But then the first assertion of statement (3) holds, where $i = x$ and $j = y$.

It remains to consider the only case where, except for $-Q$ and Q, there are no other special boxes, i.e., on the set $S \setminus \{Q \cup -Q\}$ the element w acts inside the boxes. If on the set $Q \cup -Q$ the element w changes the sign of an odd number of positive vertices, then on the set $S \setminus \{Q \cup -Q\}$ it also changes the sign of an odd number of positive vertices, and for the graph $S \setminus \{Q \cup -Q\}$ the proof from part 1 can be used. Thus, w changes the sign of an even number of positive vertices in $Q \cup -Q$. Suppose B has an odd number of vertices. Consider $C' = w(C)$ and $B' = w(B)$. Since $C = -B$, we have $C' = -B'$. Conside the permutation of vertices π that changes the signs of vertices in B' and C'. It is obvious that it changes the sign of an odd number of positive vertices. Consider the composition $\pi \circ w$. On the one hand, the result of this composition must change the sign of an odd number of positive vertices. On the other hand, $\pi \circ w$ acts inside Q and $-Q$, respectively, and if in Q it changes the sign of k positive vertices, i.e., it takes them from the left to the right, then in Q it must take k vertices from the right to the left, and thus in $-Q$ it must take k vertices from the left to the right; therefore, altogether it changes the sign of $2k$ vertices, i.e., of an even number, a contradiction. Consequently, B has an even number of vertices. In this case, the second part of statement (3) holds, i.e., in condition (c) the first part is valid, and conditions (a) and (b) are proved similarly to the case of C_n.

3.5. Proof of the main result

Proof of Theorem 2.1. Note that in each of the four classical cases we proved the implication $(1 \Rightarrow 2)$, which proves Theorem 2.1.

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