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Minimal solutions of tropical linear differential systems

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To the memory of Joos Heintz whose interests were broad.

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Abstract We introduce and study minimal (with respect to inclusion) solutions of finite systems of tropical linear differential equations. We describe the set of all minimal solutions for a single equation. It is shown that any tropical linear differential equation in a single unknown has either a solution, or a solution at infinity. For a generic system of n tropical linear differential equations in the same number of unknowns, upper and lower bounds on the number of minimal solutions are established. The upper bound involves inversions of a family of permutations, which generalize inversions of a single permutation. For $n = 1, 2$, we show that the bounds are sharp.

Keywords tropical linear differential equations · minimal solutions · inversion of a family of permutations · idempotent algebra

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Statements and Declarations

The authors declare that they have no conflict of interest, and that there are no financial or non-financial interests that are directly or indirectly related to the work submitted for publication.

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1 Introduction

We recall [7] that a tropical ordinary linear differential equation (abbr. TLDE) of order $k \in \mathbb{Z}_{>0}$ is an expression of the form

$$P := \min_{0 \leq i \leq k} \{a_i + u^{(i)}\} \tag{1}$$

where $a_i \in \mathbb{Z}$. For a set $S \subset \mathbb{Z}_{\geq 0}$ define a valuation

$$\mathrm{val}_S(i) := \min\{s - i |\ i \leq s \in S\},$$

provided that such min exists, and $\mathrm{val}_S(i) = \infty$ otherwise. We say that S is a *tropical solution* of (1) if the minimum in $\min_{0 \leq i \leq k}\{a_i + \mathrm{val}_S(i)\}$ is attained at least twice, or it is ∞.

If a classical ordinary linear differential equation (abbr. LDE)

$$\sum_{0 \leq i \leq k} \mathcal{A}_i \frac{d^i w}{dt^i} = 0 \tag{2}$$

with Laurent series coefficients $\mathcal{A}_i = t^{a_i} \sum_{j \geq 0} \mathcal{A}_{ij} t^j,\ \mathcal{A}_{i0} \neq 0$ has a power series w as its solution, then the support $S = \mathrm{supp}(w) \subset \mathbb{Z}_{\geq 0}$ is a tropical solution of (1). In the sequel, when talking about tropical solutions of (1), we omit the adjective "tropical" for brevity.

We call a solution S of (1) *minimal* when S is minimal among solutions with respect to inclusion. We say that the equation (1) is *holonomic* if it has just a finite number of minimal solutions.

Note that a system of (partial) LDE is called holonomic if its space of solutions has a finite dimension. In particular, an ordinary LDE is always holonomic, which is not necessary the case for its tropical counterpart, as we show in Example 3.

Similarly, a TLDE in $n \geq 1$ unknowns is an expression

$$P = \bigoplus_{\substack{1 \leq j \leq n \\ 0 \leq i \leq k_j}} a_{i,j} \odot u_j^{(i)} = \min_{\substack{1 \leq j \leq n \\ 0 \leq i \leq k_j}} \{a_{i,j} + u_j^{(i)}\}, \text{ where } a_{i,j} \in \mathbb{Z}. \tag{3}$$

A tuple $(S_1, \ldots, S_n) \in \mathcal{P}(\mathbb{Z}_{\geq 0})^n$ is a solution of (3) if and only if the value $\min_{\substack{1 \leq j \leq n \\ 0 \leq i \leq k_j}} \{a_{i,j} + \mathrm{val}_{S_j}(i)\}$ is attained at least twice. We call a solution $(S_1, \ldots, S_n)$ of (3) *minimal* when S_j is minimal among solutions with respect to inclusion for all $j \in [n]$.

If a classical ordinary linear differential equation

$$\sum_{\substack{1 \leq j \leq n \\ 0 \leq i \leq k_j}} \mathcal{A}_{i,j} \frac{d^i w_j}{dt^i} = 0$$

with coefficients $t^{a_{ij}}\sum_{k\geq 0}\mathcal{A}_{ijk}t^k=\mathcal{A}_{i,j}\in K[\![t]\!]$ (where $\mathcal{A}_{ij0}\neq 0$ and K a field of characteristic zero) has a power series $w=(w_1,\ldots,w_n)$ as its solution, then the support $(\mathrm{supp}(w_1),\ldots,\mathrm{supp}(w_n))$ is a solution of (3).

Consider now the ring of formal power series in $s\geq 1$ variables $K[\![t_1,\ldots,t_s]\!]$. Let $\mathrm{Sol}(\Sigma)\subset K[\![t_1,\ldots,t_s]\!]^n$ be the set of classical formal power series solutions of an homogeneous linear system Σ of LDE in $n\geq 1$ unknowns $u_1,\ldots,u_n$, and with coefficients in $K[\![t_1,\ldots,t_s]\!]$.

In [2], the authors show that under certain circumstances (Σ has to be of differential type zero, and the cardinality of K has to be large enough) then the *tropicalization* $\mathrm{trop}(\mathrm{Sol}(\Sigma))\subset\mathcal{P}(\mathbb{Z}_{\geq 0}^s)^n$ of $\mathrm{Sol}(\Sigma)$ (i.e. its whole family of supports) is a $\mathbb{B}$-semimodule that has the structure of an (infinite) matroid.

The result from [2] can be presented as the next step in developing the theory of tropical differential algebraic geometry, right after the so-called fundamental theorems of tropical differential algebraic geometry (both the ordinary [3] and the partial version [5]; see also [9] for the case of non-trivial valuation), which offer a connection between this *tropicalization* $\mathrm{trop}(\mathrm{Sol}(\Sigma))$ and the set of solutions of certain systems of TLDEs of the form (3).

Also in [2], the authors say that from a combinatorial perspective, it makes sense to study the structure of $\mathrm{Sol}(\Sigma)\subset\mathcal{P}(\mathbb{Z}_{\geq 0}^s)^n$ associated to tropical linear systems Σ consisting of TLDEs of the form (3).

In this paper we study this problem for the case $s=1$. If Σ is such a system, then the whole set of solutions $\mathrm{Sol}(\Sigma)$ is contained in the upwards closed set generated by its set of minimal elements $\mu(\mathrm{Sol}(\Sigma))$. We show that if $n>1$, then the set $\mu(\mathrm{Sol}(P))$ of tropical solutions of a single TLDE contains in a natural way certain points of a matroid, not infinite, but valuated in this case (Corollary 2).

1.1 Results

We fix $\mathbf{k}=(k_1,\ldots,k_n)\in\mathbb{Z}_{>0}^n$ and we consider $P=P_1(u_1)\oplus\cdots\oplus P_n(u_n)$ a TLDE as in (3), such that the differential order of $P_j(u_j)$ is $k_j\in\mathbb{Z}_{>0}$ for $j\in[n]$.

The first tool in our analysis is the following decomposition of the set $\mu(\mathrm{Sol}(P))$ of non-zero minimal (with respect to inclusion) solutions of P introduced in Definition 7

$$\mu(\mathrm{Sol}(P))=F(P)\sqcup\mathcal{C}_{\mathbb{Z}_{\geq 0}}(P),\tag{4}$$

where $F(P)$ is a finite set, and $\mathcal{C}_{\mathbb{Z}_{\geq 0}}(P)$ is the set of minimal solutions of P having order at least $\mathbf{k}$.

The second tool in our analysis is a family of tropical linear (homogeneous) polynomials $\{A_{\alpha,j},\ j\in[n],\ 0\leq\alpha\}$ introduced in Definition 6:

$$A_{\alpha,j}=\bigoplus_{i=0}^{k_j}val_{t_j^{\alpha}}(i)\odot x_{i,j}\in\mathbb{T}[x_{i,j}\ :\ 1\leq j\leq n,\ 0\leq i\leq k_j,\alpha].\tag{5}$$

Our main results for the first part are complete descriptions of $\mu(\mathrm{Sol}(P))$. If $n > 1$, the expression $P(u_1, \ldots, u_n) = P_1(u_1) \oplus \cdots \oplus P_n(u_n)$ tells us that $\bigcup_j \mu(\mathrm{Sol}(P_j)) \subset \mu(\mathrm{Sol}(P))$, so we see that the case of $n > 1$ depends on the case $n = 1$.

Theorem (Theorem 1 in the text). Consider $P = P(u)$ as in (1). Denote by $V(A_\alpha) \subset \mathbb{Z}^{k+1}$ the tropical hyperplane defined by the tropical polynomial $A_\alpha(x_i) := \min_{0 \leq i \leq k, \alpha}\{x_i + \alpha - i\}$, $0 \leq \alpha$, cf. (5). Then

i) The set $\mathcal{C}_{\mathbb{Z}_{\geq 0}}(P)$ satisfies

$$\mathcal{C}_{\mathbb{Z}_{\geq 0}}(P) = \begin{cases} \emptyset, & \text{if } P \notin V(A_k), \\ \mathbb{Z}_{\geq 0} \odot \{k\} = \{\{k+i\} \,:\, i \in \mathbb{Z}_{\geq 0}\}, & \text{otherwise.} \end{cases}$$

ii) The set $F(P)$ is finite, and it consists of polynomials $S = t^p + t^q$, with $0 \leq p \leq q$ satisfying
   (a) $0 \leq p \leq q < k$, or
   (b) $0 \leq p < k \leq q$ for $q = q(p)$ unique (only when $P \notin V(A_k)$).

Theorem (Theorem 6 in the text). Consider $P = \bigoplus_{j=1}^n P_j(u_j)$ for $n > 1$. The set $\mathcal{C}_{\mathbb{Z}_{\geq 0}}(P)$ satisfies

$$\mathcal{C}_{\mathbb{Z}_{\geq 0}}(P) = \bigcup_{P \in V(A_{k_j, j})} \mathbb{Z}_{\geq 0} \odot t_j^{k_j} \cup \bigcup_{A_{k_b, b}(P) \leq A_{k_a, a}(P)} \mathbb{Z}_{\geq 0} \odot (t_a^{k_a} + t_b^{k_b + A_{k_a,a}(P) - A_{k_b,b}(P)}).$$

The set $F(P)$ is finite. Furthermore, if $S \in F(P)$, then either

1. $S \in \bigcup_j F(P_j)$, or
2. $S = t_i^p + t_j^q$ satisfies
   (a) $(p, q) < (k_i, k_j)$,
   (b) if $p < k_i$ and $q = q(p) \geq k_j$.

In particular, $\mathcal{C}_{\mathbb{Z}_{\geq 0}}(P)$ is contained in a valuated matroid $\mathcal{C}_{\mathbb{R}}(P) \subseteq \mathbb{T}^n$.

We also use the decomposition (4) and the polynomials (5) to give a definition of regularity for the case $n = 1$ in Definition 8, and for $n > 1$ in Definition 12.

Afterwards we consider finite systems Σ of $n \geq 1$ TLDEs in n unknowns of the form

$$P_l = \bigoplus_{1 \leq j \leq n,\ 0 \leq i \leq k_j} a_{ijl} \odot u_j^{(i)} = \min_{1 \leq j \leq n,\ 0 \leq i \leq k_j}\{a_{ijl} + u_j^{(i)}\},\ 1 \leq l \leq n. \tag{6}$$

We say that Σ is holonomic if it has a finite number of minimal solutions. We deduce (Proposition 4) that the system $\Sigma = \{P\}$ is holonomic if and only if $P \notin V(A_k)$.

Our main results for the second part include upper and lower bounds on the number of minimal solutions for a generic system (6). The upper bound involves inversions of a family of permutations which generalize inversions of a single permutation. We show that the bounds are sharp for $n = 1, 2$. For $n = 1$ the maximal number of minimal solutions of a generic regular TLDE (1) equals k (Theorem 2), and for $n = 2$, we give the following result.

Corollary (Corollary 4 in the text). The maximal number of minimal solutions of a generic regular system (6) for $n = 2$ equals $\frac{(k_1+k_2)(k_1+k_2+1)}{2}$.

For generic regular systems of $n > 2$ TLDEs of the type (6) we provide lower and upper bounds on the number of minimal solutions (unlike the case $n = 2$ there is a gap between obtained lower and upper bounds, and it would be interesting to diminish this gap).

1.2 Roadmap

Section 2 is about preliminaries. After that the paper can be divided in two parts as follows. The first part consists of section 3, and it describes the structure of the set $\mu(\mathrm{Sol}(P))$ as well as the definition of regularity for TLDEs.

The second part consists of sections 4-6. In section 4 we start our study of systems of TLDEs (6) by introducing our definition of generic regular system (cf. Definition 13); after that, we study carefully the bounds for the case $n = 2$.

The remaining sections are about providing bounds for generic regular systems of $n > 2$ TLDE of the type (6): in section 5 we establish a lower bound on the number of minimal solutions (Theorem 9), and in section 6 we prove an upper bound (Theorem 11). The proof relates the upper bound with a new concept of inversions of a family of permutations (see Definition 14, Theorem 10) generalizing the customary inversions of a permutation.

1.3 Conventions

If E is a non-empty set, we denote by $|E|$ its cardinality and by $\mathcal{P}(E)$ its power set. If $E = \{1, \ldots, n\}$ is finite, we denote it by $[n]$. We also use the following notation

$$\binom{[n]}{i} = \begin{cases} \{X \subset [n] \;:\; |X| = i\}, & \text{if } 0 \leq i \leq n, \\ \emptyset, & \text{otherwise.} \end{cases}$$

If $A \subset B$ are sets, we denote by $B \setminus A$ its relative complement. We denote by $\mathrm{Sym}(n)$ the symmetric group, and we view a permutation from $\mathrm{Sym}(n)$ as a bijection of the set $[n]$ with itself.

2 Preliminaries

2.1 $\mathbb{B}$-semimodules and $\mathbb{B}$-algebras

In this section we will introduce the main algebraic objects of this work, which are (semi)modules and algebras over the Boolean semifield $\mathbb{B} = (\{0 < \infty\}, \odot =$

$+, \oplus = \min$). This language provides an unified approach that mixes both order-theory and commutative algebra.

Definition 1 A $\mathbb{B}$**-semimodule** is a triple $M = (M, +, 0)$ consisting of a commutative, idempotent semigroup $(M, +)$ with identity element 0. A $\mathbb{B}$-subsemimodule of M is a subset $N \subseteq M$ which is itself a $\mathbb{B}$-semimodule. A morphism of $\mathbb{B}$-semimodules is a map $f : M_1 \to M_2$ that satisfies $f(0) = 0$ and $f(a + b) = f(a) + f(b)$.

Any $\mathbb{B}$-semimodule M bears a canonical max order : $a \leq_M b$ if and only if $a + b = b$; note that $0 \in M$ is always the unique bottom under this order. Sometimes we use the opposite order $a \leq_M b$ if and only if $a + b = a$, and in this case we will denote the identity element by ∞.

Thus we obtain a poset $(M, \leq)$ (we will drop the M from $\leq_M$). In particular, if $f : M_1 \to M_2$ is a morphism of $\mathbb{B}$-semimodules, then f preserves (respectively reverses) these orders if they agree (respectively if they disagree).

Example 1 The power set $\mathcal{P}(\mathbb{Z}_{\geq 0})$ of $\mathbb{Z}_{\geq 0}$ equipped with union as binary operation is a $\mathbb{B}$-semimodule (the identity is $\emptyset$). Note that for $S, T \in \mathcal{P}(\mathbb{Z}_{\geq 0})$, we have $S \leq T$ if and only if $S \subseteq T$.

For $n \geq 1$ fixed, we will denote by $\mathcal{P}(\mathbb{Z}_{\geq 0})^n$ the $\mathbb{B}$-semimodule which is the n-fold product of $\mathcal{P}(\mathbb{Z}_{\geq 0})$. The identity is $(\emptyset, \dots, \emptyset)$, and given $S, T \in \mathcal{P}(\mathbb{Z}_{\geq 0})^n$ with $S = (S_1, \dots, S_n)$ and $T = (T_1, \dots, T_n)$, then $S \leq T$ if and only if $S_i \subseteq T_i$ for $i \in [n]$.

Definition 2 Let M be a $\mathbb{B}$-subsemimodule of $\mathcal{P}(\mathbb{Z}_{\geq 0})^n$. We denote by $\mu(M) = \min_{\leq}\{M \setminus \{(\emptyset, \dots, \emptyset)\}\}$ the set of non-zero minimal elements of $(M, \leq)$.

Given $\emptyset \neq X \subset M$, we denote by $\langle X \rangle^{\uparrow} = \{n \in M : x \leq n \text{ for some } x \in X\}$ the upward closed set generated by X.

A $\mathbb{B}$-algebra is equivalent to an (additively) idempotent semiring. The canonical order of a $\mathbb{B}$-algebra is the same as its order as a $\mathbb{B}$-semimodule.

Example 2 The $\mathbb{B}$-semimodule $\mathcal{P}(\mathbb{Z}_{\geq 0})$ becomes a $\mathbb{B}$-algebra when we endow it with the product $ST = \{i + j : i \in S, j \in T\}$ (this denotes the Minkowski sum of subsets of $\mathbb{Z}_{\geq 0}$). The multiplicative identity is $\{0\}$.

For $n \geq 1$ fixed, we will denote by $\mathcal{P}(\mathbb{Z}_{\geq 0})^n$ the $\mathbb{B}$-algebra which is the n-fold product of the $\mathbb{B}$-algebra $\mathcal{P}(\mathbb{Z}_{\geq 0})$.

We denote by $\mathbb{T}$ the $\mathbb{B}$-algebra $(\mathbb{R} \cup \{\infty\}, \odot = +, \oplus = \min)$ known as the tropical semifield. Note that $\mathbb{T}$ bears the min order: $a \leq b$ if and only if $a \oplus b = a$. We also denote by $\mathbb{T}^*$ the tropical torus $\mathbb{T} \setminus \{\infty\} = (\mathbb{R}, 0, \odot = +)$.

Recall that an expression $a = a_1 \oplus \cdots \oplus a_n = \min\{a_1, \dots, a_n\}$ in $\mathbb{T}$ vanishes (tropically), if either

i) $a = \infty$, or
ii) $a \neq \infty$, and $a = a_i = a_j$ for some $i, j \in [s]$ with $i \neq j$,

Condition ii) above is equivalent to say that $n \geq 2$, and the minimum in $a = min\{a_1, \dots, a_n\}$ is achieved at least twice.

If we use set-theoretic notation $a = \min\{a_1, \dots, a_n\}$ instead of $a = a_1 \oplus \cdots \oplus a_n$, we assume that the factors may not necessarily be distinct

If $f \in \mathbb{T}[x_1, \dots, x_n]$ is a tropical polynomial, say $f = \bigoplus_\alpha a_\alpha \odot x^{\odot \alpha}$ it induces an evaluation map $ev_f : \mathbb{T}^n \to \mathbb{T}$ sending $p = (p_1, \dots, p_n) \in \mathbb{T}^n$ to $ev_f(p) = \bigoplus_\alpha a_\alpha \odot p^{\odot \alpha}$. We denote by $V(f) \subset \mathbb{T}^n$ its bend locus, equivalently, the set of points $(p_1, \dots, p_n) \in \mathbb{T}^n$ such that $ev_f(p)$ vanishes in $\mathbb{T}$.

We denote by $\odot : \mathbb{T}^* \times \mathbb{T}^n \to \mathbb{T}^n$ the action of the tropical torus $\mathbb{T}^*$ on $\mathbb{T}^n$ given by $\lambda \odot p = (p_1 + \lambda, \dots, p_n + \lambda)$. Note that if a polynomial f is homogeneous, then $\mathbb{T}^*$ acts naturally on $V(f)$: if $p \in V(f)$ and $\lambda \in \mathbb{T}^*$, then $\lambda \odot p = (p_1 + \lambda, \dots, p_n + \lambda) \in V(f)$.

Definition 3 We say that $a = a_1 \oplus \cdots \oplus a_n = \min\{a_1, \dots, a_n\}$ **vanishes weakly** if $a_i = a_j \neq \infty$ for some $i \neq j$.

Remark 1 Note that if $a = a_1 \oplus \cdots \oplus a_n$ vanishes and $a \neq \infty$, then it vanishes weakly, but the converse is not true (tropical vanishing is weak vanishing plus the condition $a = a_i = a_j$). An equivalent way to say that $a = a_1 \oplus \cdots \oplus a_n$ does not vanish weakly is if all the factors $a_i \neq \infty$ are distinct, so the locus of all $(a_1, \dots, a_n) \in (\mathbb{T}^*)^n$ such that $a = a_1 \oplus \cdots \oplus a_n$ weakly vanishes is the tropical hypersurface $V(\bigodot_{1 \leq i < j \leq n} x_i \oplus x_j)$.

2.2 Differential algebra over $\mathbb{B}$

Fix $n \geq 1$. A TLDE in n unknowns $\{u_1, \dots, u_n\}$ (differential variables) and of differential order $\mathbf{k} = (k_1, \dots, k_n) \in (\mathbb{Z}_{>0})^n$ is a formal expression P as in (3).

Convention 1 *We will take the vector of coefficients of P to lie in $\prod_{j=1}^n \mathbb{Z}^{k_j+1}$.*

This induces a differential evaluation map

$$\begin{aligned} \mathrm{dev}_P : \mathcal{P}(\mathbb{Z}_{\geq 0})^n &\to \mathcal{P}(\mathbb{Z}_{\geq 0}) \\ (S_1, \dots, S_n) &\mapsto P(u_j^{(i)} \mapsto d^i(S_j)), \end{aligned}$$

where $d : \mathcal{P}(\mathbb{Z}_{\geq 0}) \to \mathcal{P}(\mathbb{Z}_{\geq 0})$ is the operator $d(S) = \{i-1 : i \in S, i-1 \geq 0\}$. Note that dev_P is a homomorphism of $\mathbb{B}$-semimodules.

We denote by $\mathrm{ord} : \mathcal{P}(\mathbb{Z}_{\geq 0}) \to \mathbb{T}$ the map defined by $\mathrm{ord}(\emptyset) = \infty$ and $\mathrm{ord}(S) = \min(S)$ otherwise. Note that ord is an (order-reversing) homomorphism of $\mathbb{B}$-semimodules, and its image is contained in $\mathbb{Z} \cup \{\infty\}$. We define $\mathrm{trop}_P : \mathcal{P}(\mathbb{Z}_{\geq 0})^n \to \mathbb{T}$ by $\mathrm{trop}_P(S) = \mathrm{ord} \circ \mathrm{dev}_P(S)$, which is an (order-reversing) homomorphism of $\mathbb{B}$-semimodules.

If we denote $\mathrm{ord}(d^i(S_j)) = \mathrm{val}_{S_j}(i)$, then

$$\mathrm{trop}_P(S_1, \dots, S_n) := \bigoplus_{\substack{1 \leq j \leq n \\ 0 \leq i \leq k_j}} a_{i,j} \odot \mathrm{val}_{S_j}(i) = \min_{\substack{1 \leq j \leq n \\ 0 \leq i \leq k_j}} \{a_{i,j} + \mathrm{val}_{S_j}(i)\}. \tag{7}$$

A tuple $S = (S_1, \ldots, S_n) \in \mathcal{P}(\mathbb{Z}_{\geq 0})^n$ is a solution of (3) if and only if the value $\mathrm{trop}_P(S) = \mathrm{trop}_P(S_1, \ldots, S_n)$ from (7) vanishes tropically (see Section 2.1). We denote by $\mathrm{Sol}(P) \subset \mathcal{P}(\mathbb{Z}_{\geq 0})^n$ the set of solutions of P.

Now fix $m \geq 1$, and consider a system $\Sigma = \{P_1, \ldots, P_m\}$ of TLDEs in $n \geq 1$ unknowns. We denote by $\mathrm{Sol}(\Sigma) = \bigcap_l \mathrm{Sol}(P_l)$ the set of solutions of Σ.

Proposition 1 *The set* $\mathrm{Sol}(\Sigma)$ *is a* $\mathbb{B}$*-subsemimodule of* $\mathcal{P}(\mathbb{Z}_{\geq 0})^n$*.*

Proof We have that if P is linear, then $(\emptyset, \ldots, \emptyset) \in \mathrm{Sol}(P)$, since $\mathrm{trop}_P(\emptyset, \ldots, \emptyset) = \infty$. If $S, T \in \mathrm{Sol}(P)$, then $\mathrm{trop}_P(S \cup T) = \mathrm{trop}_P(S) \oplus \mathrm{trop}_P(T)$ since trop_P is a homomorphism. The result follows from the fact that any intersection of $\mathbb{B}$-subsemimodules of $\mathcal{P}(\mathbb{Z}_{\geq 0})^n$ is again a $\mathbb{B}$-subsemimodule of it.

Proposition 2 *The* $\mathbb{B}$*-semimodule* $\mathrm{Sol}(\Sigma)$ *satisfies*

$$\mathrm{Sol}(\Sigma) \setminus \{(\emptyset, \ldots, \emptyset)\} = \bigcup_{S \in \mu(\mathrm{Sol}(\Sigma))} \mathrm{Sol}(\Sigma) \cap \langle S \rangle^{\uparrow}.$$

Proof It is enough to show the inclusion $\mathrm{Sol}(\Sigma) \setminus \{(\emptyset, \ldots, \emptyset)\} \subset \langle \mu(\mathrm{Sol}(\Sigma)) \rangle^{\uparrow} = \cup_{S \in \mu(\mathrm{Sol}(\Sigma))} \langle S \rangle^{\uparrow}$.

If $\{(\emptyset, \ldots, \emptyset)\} \neq S \in \mathrm{Sol}(\Sigma)$, then it follows from Lemma 1 that there exists $S' \in \mu(\mathrm{Sol}(\Sigma))$ such that $S' \leq S$.

Definition 4 We call a system Σ **holonomic** if it has a finite number of minimal solutions.

In the coming sections we will compute $\mu(\mathrm{Sol}(\Sigma))$ for different cases of systems Σ of $m \geq 1$ polynomials in $n \geq 1$ variables, and we will provide bounds for those which are holonomic.

We will now introduce some functions on the space of parameters of the TLDEs P as in (3) which will be used in the following. We shall start with the case $n = 1$.

Definition 5 For $k \in \mathbb{Z}_{>0}$ fixed, we consider the following family of tropical linear polynomials $\{A_j \,:\, j = 0, \ldots, k\}$, defined by

$$A_j := \bigoplus_{i=0}^{k} \mathrm{val}_{\{j\}}(i) \odot x_i \in \mathbb{T}[x_0, \ldots, x_k].$$

The space of parameters of TLDEs $P(u) = \bigoplus_{i=0}^{k} a_i \odot u^{(i)}$ of order k can be identified with $\mathbb{Z}^{k+1}$ sitting inside $\mathbb{T}^{k+1}$. If we denote by $c(P) = (a_i)_{0 \leq i \leq k} \in \mathbb{Z}^{k_j+1}$ the vector of coefficients of P, then we can evaluate the polynomial A_j at the vector $c(P)$, and we have

$$A_j(c(P)) = \bigoplus_{i=0}^{k} \mathrm{val}_{\{j\}}(i) \odot a_i = \min_{0 \leq i \leq min\{j,k\}} \{a_i + (j - i)\} = \mathrm{trop}_P(\{j\}). \quad (8)$$

We now discuss the case $n > 1$: we assume that P is as in (3) and has fixed differential order $\mathbf{k} = (k_1, \ldots, k_n) \in (\mathbb{Z}_{>0})^n$. We shall use (8) to define the family $\{A_{\alpha,j} \,:\, j \in [n],\, 0 \leq \alpha \leq k_j\}$.

Convention 2 *Representing elements* $S = (S_1, \dots, S_n) \in \mathcal{P}(\mathbb{Z}_{\geq 0})^n$ *for* $n > 1$ *sometimes becomes cumbersome. We propose using multiplicative notation* $S = S_1(t_1) + \cdots + S_n(t_n)$*, where* $S_j(t_j) = \sum_{\alpha \in S_j} t_j^\alpha$ *for* $j \in [n]$*, whenever* $S_j \neq \emptyset$*. If* $S = (\emptyset, \dots, \emptyset)$*, we represent it as the series* $S = 0$*.*

For $j \in [n]$ and $\alpha \in \mathbb{Z}_{\geq 0}$ we consider $S_{\alpha,j} = (S_1, \dots, S_n) \in \mathcal{P}(\mathbb{Z}_{\geq 0})^n$ defined by $S_l = \emptyset$ if $l \neq j$, and $S_j = \{\alpha\}$. So $A_{\alpha,j}$ must satisfy $A_{\alpha,j}(c(P)) := \mathrm{trop}_P(S_{\alpha,j})$, where $c(P) \in \prod_{j=1}^n \mathbb{Z}^{k_j+1}$ is the vector of coefficients of P, which is a point of the space of parameters of these TLDEs. With our convention we can write $S_{\alpha,j} = t_j^\alpha$ instead.

Definition 6 We consider the family of tropical linear polynomials $\{A_{\alpha,j} : j \in [n],\ 0 \leq \alpha \leq k_j\}$ defined by

$$A_{\alpha,j} = \bigoplus_{i=0}^{k_j} \mathrm{val}_{S_{\alpha,j}}(i) \odot x_{i,j} \in \mathbb{T}[x_{i,j} \ : \ 1 \leq j \leq n,\ 0 \leq i \leq k_j].$$

Convention 3 *From now on we will denote* $A_{\alpha,j}(c(P)) = A_{\alpha,j}(P) = \mathrm{trop}_P(t_j^\alpha) = \mathrm{trop}_P(S_{\alpha,j})$*. Note that if we write* $P = P_1(u_1) \oplus P_2(u_2) \oplus \cdots \oplus P_n(u_n)$*, then* $A_{\alpha,j}(P) = \mathrm{trop}_P(t_j^\alpha) = \mathrm{trop}_{P_j}(t_j^\alpha) = A_{\alpha,j}(P_j)$ *for all* $j \in [n]$*.*

3 Minimal solutions of a single TLDE

In this section we will focus on studying the set $\mu(\mathrm{Sol}(P))$ of minimal non-zero solutions of a single P, where P is a TLDE of the form (3) in $n \geq 1$ unknowns $\{u_1, \dots, u_n\}$, and of differential order $\mathbf{k} = (k_1, \dots, k_n) \in (\mathbb{Z}_{>0})^n$.

We now introduce basic concepts and general properties of the elements of $\mu(\mathrm{Sol}(P))$.

Remark 2 By Convention 1, our TLDE P has at least two nonzero monomials $a_{0,j} \odot u_j^{(0)}$ and $a_{1,j} \odot u_j^{(1)}$, $a_{0,j}, a_{1,j} \in \mathbb{Z}$ for $j \in [n]$ and $n \geq 1$. Also, we have $a_{0,j} \neq \infty$ for all $j \in [n]$, and these two facts imply that $trop_P(S) \neq \infty$ if and only if $S \neq (\emptyset, \dots, \emptyset)$. In what follows we will focus only on nonzero solutions.

Lemma 1 *Any minimal solution* S *of* P *has at most two monomials.*

Proof Note that if $S \in \mathrm{Sol}(P)$, then there exists $i, j \in [n]$ (not necessarily different), $0 \leq b \leq k_i$, $0 \leq c \leq k_j$, and $t_i^p + t_j^q \leq S$ with $b \leq p$ and $c \leq q$ (again, p, q not necessarily distinct) such that

$$\infty \neq \mathrm{trop}_P(S) = a_{i,b} + p - b = a_{j,c} + q - c \leq a_{i,j} + \mathrm{val}_{S_j}(i),$$

thus $t_i^p + t_j^q$ is also a solution of (3), and $t_i^p + t_j^q = S$ by minimality.

It follows from Lemma 1 that any $S \in \mu(\mathrm{Sol}(P))$ can be expressed as $S = t_i^p + t_j^q$ with $i, j \in [n]$ (not necessarily distinct) and $p, q \in \mathbb{Z}_{\geq 0}$ (not necessarily distinct). From now on we will express the minimal solutions in this form.

Proposition 3 *Let* $S \in \mu(\mathrm{Sol}(P))$.

i) if $S = t_j^p + t_j^q$ *where* $p < q$, *then* $p < k_j$;

ii) If $S_1 = t_i^p + t_j^{q_1}$ *and* $S_2 = t_i^p + t_j^{q_2}$ *are minimal solutions of (3) satisfying* $p < k_i$ *and* $k_j \leq q_1, q_2$, *then* $q_1 = q_2 = q(p)$, *which is then uniquely determined.*

Proof For $i)$ suppose on the contrary that $k_j \leq p$. Then for all $0 \leq i \leq k_j$ it holds

$$a_{i,j} + \mathrm{val}_S(i) = a_{i,j} + p - i \tag{9}$$

Hence t_j^p is a solution of (3) which contradicts the minimality of $S = t_j^p + t_j^q$. For point $ii)$, suppose that $q_1 < q_2$. Then q_1 must satisfy

$$\mathrm{trop}_P(S_1) = a_{i,l} + p - l = a_{j,j_1} + q_1 - j_1$$

for some $0 \leq l \leq p$, and

$$\begin{cases} 0 \leq j_1 \leq k_j \leq q_1, & \text{if } i \neq j, \\ p \leq j_1 \leq k_j \leq q_1, & \text{if } i = j. \end{cases}$$

Likewise, q_2 must satisfy

$$\mathrm{trop}_P(S_2) = a_{i,l'} + p - l' = a_{j,j_2} + q_2 - j_2$$

for some $0 \leq l' \leq p$, and $0 \leq j_2 \leq k_j \leq q_2$ if $i \neq j$, or $p \leq j_2 \leq k_j \leq q_2$ if $i = j$. We claim that $\mathrm{trop}_P(S_1) = \mathrm{trop}_P(S_2)$, otherwise it wouldn't be the global minimum. But due to (9) we have

$$a_r + q_2 - r > a_r + q_1 - r \geq M_1 = M_2,$$

for every $0 \leq r \leq k_j$ if $i \neq j$, or $p \leq r \leq k_j$ if $i = j$, which contradicts that $S_2 = t_i^p + t_j^{q_2}$ is a minimal solution. Here M_1, M_2 represent the common minimal value attained at the two distinct monomials of P. Thus $S_1 = S_2 = S = t_i^p + t_j^q$, and q must satisfy

$$\mathrm{trop}_P(S) = a_{i,l} + p - l = a_{j,j_1} + q - j_1,$$

so it is uniquely determined by p.

We will now introduce the decomposition $\mu(\mathrm{Sol}(P)) = \mathcal{C}_{\mathbb{Z}_{\geq 0}}(P) \sqcup F(P)$ from (4) which will be very important in the following.

Definition 7 We denote by $\mathcal{C}_{\mathbb{Z}_{\geq 0}}(P)$ the set of minimal solutions of P having t-adic order at least $\mathbf{k}$; this is $S = (S_1, \ldots, S_n) \in \mu(\mathrm{Sol}(P))$ such that $\mathrm{ord}(S_i) \geq k_i$ for all $i \in [n]$. We denote by $F(P) = \mu(\mathrm{Sol}(P)) \setminus \mathcal{C}_{\mathbb{Z}_{\geq 0}}(P)$.

The set $\mathcal{C}_{\mathbb{Z}_{\geq 0}}(P)$ will be described in Theorem 1 for the case $n = 1$, and in Theorem 6 for the case $n > 1$.

Remark 3 If $P = \bigoplus_j P_j(u_j)$ for $n > 1$, then $\bigcup_{j=1}^n \mu(\mathrm{Sol}(P_j)) \subset \mu(\mathrm{Sol}(P))$. Indeed, if $S = S_j(t_j)$ for some $j \in [n]$ (see Convention 2), then $\mathrm{trop}_P(S) = \mathrm{trop}_{P_j}(S)$, so $S = S_j(t_j) \in \mathrm{Sol}(P)$ if and only if $S \in \mathrm{Sol}(P_j)$. Minimality also can be inferred from this fact. So in order to understand the case $n > 1$, it is necessary to understand first the case $n = 1$, which we will discuss in the next section.

3.1 The case $n = 1$

In this section we will be exploring the case in which P is a TLDE as in (1) of fixed differential order $k \in \mathbb{Z}_{>0}$. Recall from Definition 5 that $A_k = \bigoplus_{i=0}^{k} \mathrm{val}_{\{k\}}(i) \odot x_i = \min_{0 \leq i \leq k}\{x_i + (k-i)\}$.

Proposition 4 *Let P be a TLDE. The following statements are equivalent:*

i) P is non-holonomic,
ii) $S = \{s\}$ is a solution for some $s \geq k$.

In this case $\{q\}$ is also its solution for any $q \geq k$.

Proof First assume that $\{s\}$ is a solution of (1) for some $s \geq k$. Then the minimum in $\min_{0 \leq i \leq k}\{a_i + s - i\}$ is attained at least twice (cf. (9)). Therefore the minimum in $\min_{0 \leq i \leq k}\{a_i + q - i\}$ is also attained at least twice for any $q \geq k$. Hence the equation (1) is non-holonomic.

Conversely, let (1) be non-holonomic. There are at most finite number of minimal solution of (1) of the form $\{p, q\},\ p \neq q$ taking into account Proposition 3. Thus there exists a minimal solution of (1) of the form $\{s\}$ for suitable $s \geq k$.

Remark 4 One can easily algorithmically test whether (1) is non-holonomic. Due to Proposition 4 ii) the latter is equivalent to $\{k\}$ being a solution of (1), in other words that the minimum in $A_k(P) = \min_{0 \leq i \leq k}\{a_i + k - i\}$ is attained at least twice. Then $Sing := V(A_k) \subset \mathbb{Z}^{k+1}$ is the set of all non-holonomic equations (1).

Let P be a TLDE as in (1) of differential order k. We are now ready to state the main result of this Section, which is describing the decomposition $\mu(\mathrm{Sol}(P)) = \mathcal{C}_{\mathbb{Z}_{\geq 0}}(P) \sqcup F(P)$ from Definition 7.

Theorem 1 (Structure of $\mu(\mathrm{Sol}(P))$ for $n = 1$) *Consider P as above.*

i) The set $\mathcal{C}_{\mathbb{Z}_{\geq 0}}(P)$ satisfies

$$\mathcal{C}_{\mathbb{Z}_{\geq 0}}(P) = \begin{cases} \emptyset, & \text{if } P \notin V(A_k), \\ \{t^{k+i} \ :\ i \in \mathbb{Z}_{\geq 0}\}, & \text{otherwise.} \end{cases}$$

In particular, the system $\Sigma = \{P\}$ is holonomic if and only if $P \notin V(A_k)$.

ii) The set $F(P)$ is finite, and it consists of polynomials $S = t^p + t^q$, with $0 \leq p \leq q$ satisfying
  (a) $0 \leq p \leq q < k$, or
  (b) $0 \leq p < k \leq q$ for $q = q(p)$ unique (only when $P \notin V(A_k)$).

Proof The first part comes from Proposition 4, and the second part from Proposition 3.

3.1.1 A notion of generic position for ordinary TLDEs

Theorem 1 implies that a holonomic equation (1) can contain as its minimal solutions sets of the form either $\{j,l\},\ 0 \leq j \leq l < k$ or for each $0 \leq p < k$ at most one set of the form $\{p,q\}$ where $q \geq k$. We call a family of minimal sets of the form $\{j,l\},\ 0 \leq j \leq l < k$ and of minimal sets of the form $\{p,\star\},\ 0 \leq p < k$ (where $\star$ stays for some indeterminate $q \geq k$) a *configuration.* In this case we use the notation $\{p,q\} \in \{p,\star\}$. Thus, there is a finite number of possible configurations.

Each configuration C (including the empty one) provides a set $U_C \subset \mathbb{Z}^{k+1}$ of holonomic equations P of the form (1) satisfying $\mu(\mathrm{Sol}(P)) = C$.

Theorem 2 *The collection $\{U_C\}_C$ where C runs through all the configurations, is a partition of $\mathbb{T}^{k+1} \setminus Sing$ into polyhedral complexes in which every polyhedron is given by integer linear constraints of the form either $x(j)-x(l) = c,\ 0 \leq j,l \leq k,\ c \in \mathbb{Z}$ or $x(j) - x(l) \geq c$. The polyhedral complex U_{C_0} corresponding to the configuration $C_0 := \{\{i,\star\} \mid 0 \leq i < k\}$ has the full dimension $k+1$.*

Proof First we note that $\dim(Sing) \leq k$ because of Remark 4. By the same token if a configuration C contains a minimal solution of the form $\{j\},\ 0 \leq j < k$ then $\dim U_C \leq k$. If C contains a minimal solution of the form $\{j,l\},\ 0 \leq j \neq l < k$ then it holds $\min_{0 \leq i \leq j}\{a_i + j - i\} = \min_{0 \leq i \leq l}\{a_i + l - i\}$ (cf. (9)), and therefore $\dim U_C \leq k$ as well.

Now consider the configuration C_0. Then the following system of min-plus linear equations holds:

$$\min_{0 \leq i \leq j}\{a_i + j - i\} = \min_{j < i \leq k}\{a_i + q_j - i\},\ 0 \leq j < k \tag{10}$$

for appropriate $q_j \geq k$ (cf. (9)). Taking $a_0, \ldots, a_k$ satisfying inequalities $a_0 \geq a_1 \geq \cdots \geq a_k$, we obtain that

$$\min_{0 \leq i \leq j}\{a_i + j - i\} = a_j,\ \min_{j < i \leq k}\{a_i + q_j - i\} = a_k + q_j - k,\ 0 \leq j < k.$$

Hence (10) implies that $q_j = a_j - a_k + k \geq k,\ 0 \leq j < k$.

Since for any configuration C such that C contains a minimal solution of the form $\{j,l\},\ 0 \leq j \leq l < k$ it holds $\dim U_C \leq k$ (see above), we complete the proof.

Example 3 If $k = 1$ an equation $P = a_0 \odot u^{(0)} \oplus a_1 \odot u^{(1)} = \min\{a_1 + u^{(1)},\ a_0 + u^{(0)}\}$ as in (1) has

- no (non-zero) solutions when $a_1 - 1 > a_0$. Below in Section 3.1.2 we will generalize this result for arbitrary k;
- the minimal solutions $\{q\ :\ q \geq 1\}$ when $a_1 - a_0 = 1$, so the equation (1) is non-holonomic;
- a single minimal solution $\{0, a_0 - a_1 + 1\}$ when $a_0 - a_1 \geq 0$.

Thus, in Theorem 2 the polyhedron $U_{C_0} \subset \mathbb{Z}^2$ is given by the constraint $a_0 - a_1 \geq 0$, where $C_0 = \{0, \star\}$.

Example 4 While Theorem 2 shows that a generic TLDE of order k has k minimal solutions, the (holonomic) equation $P = \bigoplus_{i=0}^{k} 0 \odot u^{(i)} = \min_{0 \leq i \leq k}\{u^{(i)}\}$ has $k(k+1)/2$ minimal solutions $\{j, l\},\ 0 \leq j < l \leq k$. Thus, its configuration contains all possible minimal solutions (cf. Proposition 3).

Definition 8 We say that P is **regular** if

1. $P \notin \bigcap_{j=0}^{k} V(A_j)$, and
2. $A_0(P) \oplus \cdots \oplus A_{k-1}(P)$ does not vanish weakly in $\mathbb{T}$ (see Definition 3).

We have an alternative characterization of regularity in terms of the structure of $\mu(\mathrm{Sol}(P))$.

Proposition 5 *We have that P is regular if and only if all of the elements of $\mu(\mathrm{Sol}(P))$ are of the form $\{p, q\}$ with $0 \leq p < k \leq q$.*

Proof The first condition of Definition 8 is equivalent to the fact that P has no monomial solutions (which implies that it is holonomic). Indeed, note that $A_j(P) = \min_{0 \leq i \leq \min\{j,k\}}\{a_i + j - i\}$, thus $\{j\} \in \mu(\mathrm{Sol}(P))$ if and only if $P \in V(A_j)$.

The second condition is equivalent to the fact that P has no minimal binomial solutions $\{p, q\}$ with $0 \leq p < q < k$. Indeed, for $0 \leq j_1 \neq j_2 \leq k-1$ we have $\mathrm{trop}_P(\{j_1, j_2\}) = \mathrm{trop}_P(j_1) \oplus \mathrm{trop}_P(j_2) = A_{j_1}(P) \oplus A_{j_2}(P)$ since trop_P is a homomorphism, so $\{j_1, j_2\}$ is a solution of P if $A_{j_1}(P) = A_{j_2}(P)$, and it is minimal since P has already no (minimal) monomial solutions.

Remark 5 There exists a tropical hypersurface $\Delta(k) \subset \mathbb{T}^{k+1}$ such that $\Delta(k) \cap \mathbb{Z}^{k+1}$ consists of the TLDEs which are not regular. We call this $\Delta(k)$ the *tropical discriminant*. Indeed, the first condition of Definition 8 provides directly a tropical prevariety, and the second condition also provides a tropical prevariety by Remark 1.

If $P = P(u)$, recall that $\mathrm{Sol}(P) = \bigcup_{S \in \mu(\mathrm{Sol}(P))} \mathrm{Sol}(P) \cap \langle S \rangle^{\uparrow}$ from Proposition 2. We will describe explicitly these sets for the case in which P is regular. First we introduce some notation: if P is regular, then $\mu(\mathrm{Sol}(P)) = F(P)$, so any $S \in \mu(\mathrm{Sol}(P))$ is of the form $S_p = t^p + t^q$ with $0 \leq p < k \leq q$ and $q := \deg(S_p)$.

Proposition 6 *Let P be regular with $\mathrm{Sol}(P) \neq \{0\}$. Then the set $\mathrm{Sol}(P) \cap \langle S_p \rangle^{\uparrow}$ is equal to*

$$\{U \supseteq S_p : A_p(P) \leq A_m(P) \text{ for all } m \in U \cap [0, k-1] \text{ and } U \cap [k, \infty) \subseteq \mathbb{Z}_{\geq \deg(S_p)}\}$$

for $S_p \in \mu(\mathrm{Sol}(P))$.

Proof If $0 \neq U \in \mathrm{Sol}(P)$, then there exists $S_p \in \mu(\mathrm{Sol}(P))$ for some $0 \leq p < k$ such that $S_p \subseteq U$. There exists a unique index $v(U)$ such that

$$A_{v(U)}(P) = \bigoplus_{\substack{S_j \in \mu(\mathrm{Sol}(P)) \\ S_j \subset U}} A_j(P)$$

We have $\mathrm{trop}_P(U) = A_{\deg(S_{v(U)})}(P) \oplus A_{\mathrm{ord}(U \cap [k,\infty))}(P)$, and since $A_-(P)$ is increasing for $j \geq k$, we have $\mathrm{ord}(U \cap [k,\infty)) \geq \deg(S_{v(U)})$.

If $m \in U \cap [0, k-1]$, we have $\mathrm{trop}_P(U) = A_{v(U)}(P) \oplus A_m(P)$, so $A_{v(U)}(P) \leq A_m(P)$. If $S_m \subset U$, then again $\mathrm{trop}_P(U) = A_{\deg(S_{v(U)})}(P) \oplus A_{\deg(S_m)}(P)$ implies $A_{\deg(S_{v(U)})}(P) \leq A_{\deg(S_m)}(P)$. Thus $\deg(S_m) \geq \deg(S_{v(U)})$, and we have $U \in \langle S_{v(U)} \rangle^{\uparrow} \cap \mathrm{Sol}(P)$.

3.1.2 A criterion of existence of a nonzero solution of a univariate TLDE

In the last section we saw that if P as in (1) is regular, then $|\mu(\mathrm{Sol}(P))| \leq k$. A natural question to ask is whether or not a regular P has a non-empty set of (minimal) solutions. Here we investigate this question.

For a fixed TLDE P, we use the whole sequence of polynomials $\{A_j : j \in \mathbb{Z}_{\geq 0}\}$ to define the following map.

Definition 9 If $P = \bigoplus_{i=0}^{k} a_i \odot u^{(i)}$ we define a map

$$\begin{aligned} A_-(P) : \mathbb{Z}_{\geq 0} &\to \mathbb{Z} \\ j &\mapsto \mathrm{trop}_P(\{j\}) = A_j(P) \end{aligned}$$

The map $A_-(P)$ is determined by its values on the interval $[0,k] \subset \mathbb{Z}_{\geq 0}$:

$$A_{j+1}(P) \begin{cases} \leq A_j(P) + 1, & 0 \leq j < k, \\ = A_j(P) + 1 = A_k(P) + j - k, & k \leq j. \end{cases}$$

Theorem 3 *i) TLDE (1) has no minimal solutions of the form $\{j_1, j_2\}$, $0 \leq j_1 \neq j_2$ if and only if $a_i - i \geq a_0$, $0 \leq i \leq k$.*
ii) TLDE (1) has no solutions (or equivalently, no minimal solutions) if and only if $a_i - i > a_0$, $1 \leq i \leq k$.

Proof Note that if $A_{j_1}(P) = A_{j_2}(P)$ for some $0 \leq j_1 \neq j_2$ then $\{j_1, j_2\}$ is a solution of (1), since $\mathrm{trop}_P(\{j_1, j_2\}) = A_{j_1}(P) \oplus A_{j_1}(P)$ by linearity.

We shall prove i). First assume that (1) has no solutions of the form $\{j_1, j_2\}$ with $0 \leq j_1 \neq j_2$. We claim that $A_{j+1}(P) = A_j(P) + 1$, $j \geq 0$. Suppose the contrary. Take the minimal integer j_0 such that $A_{j_0+1}(P) \leq A_{j_0}(P)$, clearly $j_0 < k$. For $j \geq j_0 + 1$, it holds that $A_j(P) \leq A_0(P) = a_0$ by hypothesis and the paragraph at the beginning of this proof. Hence $A_k(P) \leq A_0(P)$. Therefore $\{0, k + A_0(P) - A_k(P)\}$ is a solution of (1) contradicting the hypothesis. This proves the claim.

Thus $A_j(P) = a_0 + j$, $j \geq 0$. This implies that $a_i - i \geq a_0$, $0 \leq i \leq k$. Conversely, if the latter inequalities are valid then $A_j(P) = a_0 + j$, $j \geq 0$, and

(1) has no solutions of the form $0 \le j_1 \ne j_2$ since $A_{j_1}(P) \ne A_{j_2}(P)$, which completes the proof of ii).

Now we shall prove ii). Assume that (1) has no solutions. To justify the strict inequalities in i) suppose the contrary and let $i_0 \ge 1$ be the minimal integer such that $a_{i_0} - i_0 = a_0$. Then $\{i_0\}$ is a minimal solution of (1). This proves i) in one direction: if (1) has no solutions then $a_i - i > a_0,\ 1 \le i \le k$. Conversely, assume the latter inequalities. Then $A_j(P) = a_0 + j,\ j \ge 0$. No singleton $\{j\}$ is a solution of (1) since the minimum in the definition of A_j is attained for a single $i = 0$, and that is enough to prove that there are no solutions thanks to i), which proves ii).

3.1.3 Tropicalization of power series solutions of LDE at ∞

Remind that one can treat a tropical solution of (1) as the tropicalization of the starting part of a power series solution $w = \sum_{i\ge 0} b_i t^i$ of a classical LDE (2). One can view this power series as an expansion in a neighborhood of 0. Alternatively, one can consider a power series $w = \sum_{i \ge 0} c_i t^{-i}$ solution of (2) as an expansion in a neighborhood of ∞. This leads to the following definition.

Definition 10 For a subset $\{0\} \ne S \subset \mathbb{Z}_{\le 0}$ introduce a valuation

$$\mathrm{val}_S^{(\infty)}(i) = \begin{cases} \max\{S\}, & \text{if } i = 0,\\ \max\{s - i \mid -1 \ge s \in S\}, & \text{if } 1 \le i \le k. \end{cases}$$

We say that S is a *tropical solution at* ∞ of (1) if the maximum in

$$\max_{0 \le i \le k} \{a_i + \mathrm{val}_S^{(\infty)}(i)\}$$

is attained at least for two values of i.

Lemma 2 *A minimal tropical solution S at ∞ of (1) can be of one of the two following types:*

i) $S = \{-r\}$ for an arbitrary $r \ge 1$, if and only if the maximum in $\max_{0\le i\le k}\{a_i - i\}$ is attained at least twice;
ii) $S = \{0, -r\}$, if and only if $1 \le r := \max_{0 \le i \le k}\{a_i - i\} - a_0$.

Proof Similar to Proposition 3 a minimal solution $S \subset \mathbb{Z}_{\le 0}$ at ∞ consists of at most of two elements. If $0 \notin S$ then $S = \{-r\}$ for an arbitrary $r \ge 1$. In this case the maximum in $\max_{0\le i\le k}\{a_i - i\}$ is attained at least twice. The converse is also true.

Now let $0 \in S$. Then $S = \{0, -r\},\ r \ge 1$ and $a_0 = \max_{0\le i\le k}\{a_i - i\} - r$. Conversely, if $r := \max_{0 \le i \le k}\{a_i - i\} - a_0 \ge 1$ then $\{0, -r\}$ is a minimal solution at ∞ of (1).

Corollary 1 *i) TLDE (1) has either a tropical solution or a tropical solution at ∞.*

ii) If (1) has only minimal tropical solutions being singletons then (1) has a minimal tropical solution at ∞ *of the form* $\{-r\}$ *for any* $r \geq 1$.

Proof i) Let (1) have no tropical solutions. Then $a_i - i > a_0,\ 1 \leq i \leq k$ due to Theorem 3 i). Hence (1) has a (minimal) tropical solution $\{0, a_0 - \max_{0\leq i\leq k}\{a_i - i\}\}$ at ∞ because of Lemma 2 ii).

ii) Due to Theorem 3 it holds $a_i - i \geq a_0,\ 0 \leq i \leq k$, and there exists $1 \leq i \leq k$ such that $a_i - i = a_0$. Therefore (1) has a minimal tropical solution $\{-r\}$ at ∞ for an arbitrary $r \geq 1$ because of Lemma 2 i).

3.2 The case $n > 1$

In this Section we assume that $n > 1$ and P is as in (3) and has fixed differential order $\mathbf{k} = (k_1, \ldots, k_n) \in (\mathbb{Z}_{>0})^n$. As usual, we write $P = \bigoplus_j P_j(u_j)$ with $P_j(u_j)$ of differential order $k_j \in \mathbb{Z}_{>0}$ for $j \in [n]$.

The main result of this Section is a complete description of the parts $\mu(\mathrm{Sol}(P)) = \mathcal{C}_{\mathbb{Z}_{\geq 0}}(P) \sqcup F(P)$ from Definition 7; this is a generalization of Theorem 1.

Definition 11 If $t_i^p + t_j^q = S \in \mu(\mathrm{Sol}(P))$ (see Lemma 1), we define its support $\mathrm{Supp}(S) = \{i, j\} \subset [n]$.

Theorem 4 *For* P *as before, the set* $F(P)$ *is finite. Furthermore, if* $S \in F(P)$*, then either*

1. $\mathrm{Supp}(S) = \{j\}$ *if and only if* $S \in F(P_j)$,
2. $\mathrm{Supp}(S) = \{i, j\}$ *if and only if* $S = t_i^p + t_j^q$ *satisfies one of the following two conditions:*
   (a) $(p, q) < (k_i, k_j)$,
   (b) $p < k_i$ *and* $q = q(p) \geq k_j$ *is unique.*

Proof Let $S \in F(P)$. It is clear that $\mathrm{Supp}(S) = \{j\}$ if and only if $S \in F(P_j)$, and $\bigcup_j F(P_j)$ is finite by Theorem 1. If $\mathrm{Supp}(S) = \{i, j\}$ with $\varphi = t_i^p + t_j^q$, then we have a trichotomy:

1. $(p, q) < (k_i, k_j)$,
2. $p < k_i$ and $q \geq k_j$,
3. $(p, q) \geq (k_i, k_j)$,

and we are ruling out the third possibility (by the definition of the set $F(P)$). So we just need to show that there is a finite number of possibilities (2), but this is just Proposition 3 ii).

We will study the set $\mathcal{C}_{\mathbb{Z}_{\geq 0}}(P)$ for the case $n > 1$ in the next section. Contrary to the case $n = 1$, we will see that it is always non-empty (it is in fact, infinite).

3.2.1 Structure of the set $\mathcal{C}_{\mathbb{Z}_{\geq 0}}(P)$

Let $n > 1$ and $P = P_1(u_1) \oplus P_2(u_2) \oplus \cdots \oplus P_n(u_n)$. The main result of this Section is a complete characterization of $\mathcal{C}_{\mathbb{Z}_{\geq 0}}(P)$.

Consider the order map ord : $\mathcal{P}(\mathbb{Z}_{\geq 0})^n \to \mathbb{T}^n$ sending $S = (S_1, \ldots, S_n)$ to $\text{ord}(S) = (\text{ord}(S_1), \ldots, \text{ord}(S_n))$. This is an (order-reversing) homomorphism of $\mathbb{B}$-semimodules that induces a bijection of sets

$$\mathcal{P}(\mathbb{Z}_{\geq 0})^n \supset \mathcal{C}_{\mathbb{Z}_{\geq 0}}(P) \xrightarrow{\sim} \text{ord}(\mathcal{C}_{\mathbb{Z}_{\geq 0}}(P)) \subset (\mathbb{Z} \cup \{\infty\})^n \qquad (11)$$

Indeed, we know that if $\text{Supp}(S) = \{i\}$, then $S = t_i^p$ is a monomial by Theorem 1, so $\text{ord}(S) = (\infty, \ldots, p, \ldots, \infty)$ for some $p \geq k_i$, and if $\text{Supp}(S) = \{i, j\}$, then $S = t_i^p + t_j^q$, and $\text{ord}(S) = (\infty, \ldots, p, \ldots, q, \ldots, \infty)$ for some $p \geq k_i$ and $q \geq k_j$.

For the rest of this section we will use this identification.

Theorem 5 *Let P be a TLDE with $n > 1$. The set $\mathcal{C}_{\mathbb{Z}_{\geq 0}}(P)$ satisfies*

i) $(\infty, \ldots, \infty) \notin \mathcal{C}_{\mathbb{Z}_{\geq 0}}(P)$,
ii) *if* $S \in \mathcal{C}_{\mathbb{Z}_{\geq 0}}(P)$ *and* $i \in \mathbb{Z}_{\geq 0}$, *then* $i \odot S \in \mathcal{C}_{\mathbb{Z}_{\geq 0}}(P)$,
iii) *if* $S, T \in \mathcal{C}_{\mathbb{Z}_{\geq 0}}(P)$ *with* $\text{Supp}(S) \neq \text{Supp}(T)$, *then* $\text{Supp}(S) \not\subset \text{Supp}(T)$,
iv) *if* $S, T \in \mathcal{C}_{\mathbb{Z}_{\geq 0}}(P)$ *and* $i, j \in [n]$ *with* $s_i = t_i \neq \infty$ *and* $s_j < t_j$, *there exists* $U \in \mathcal{C}_{\mathbb{Z}_{\geq 0}}(P)$ *such that* $u_i = \infty$, $u_j = s_j$ *and* $U \geq \min\{S, T\}$.

Proof Point i) is equivalent to $(\emptyset, \ldots, \emptyset) \notin \mathcal{C}_{\mathbb{Z}_{\geq 0}}(P)$. For point ii), if $S \in \mathcal{C}_{\mathbb{Z}_{\geq 0}}(P)$, then $i \odot S$ for $i \in \mathbb{Z}_{\geq 0}$ represents $t^i \cdot S$, which is once again in $\mathcal{C}_{\mathbb{Z}_{\geq 0}}(P)$. Point iii) follows from minimality.

The proof of iv) is similar to Proposition 3 ii), which uses (9): there exist $S = t_i^p + t_j^q$ with $p, q < \infty$ and $T = t_i^p + t_j^r$ with $r \in [q+1, \infty]$, but this is not possible, since $r = \infty$ contradicts the minimality of S, and if $r \neq \infty$, we have

$$a_{j,s} + q - s = \text{trop}_P(S) = a_{i,l} + p - l = \text{trop}_P(T) = a_{j,t} + r - t$$

but $k_j \leq q < r$, and $a_{j,t} + q - t < a_{j,s} + q - s$, which contradicts the fact that S is a solution. Thus, for any given $S = t_i^p + t_j^q$, such $T = t_i^p + t_j^r$ does not exist.

Remark 6 In Theorem 5, if in point ii) we change "$i \in \mathbb{Z}_{\geq 0}$" by "$i \in \mathbb{R}$", then the resulting object $\mathcal{C}_{\mathbb{R}}(P)$ is a collection of valuated circuits, and the pair $([n], \mathcal{C}_{\mathbb{R}}(P))$ is then called a valuated matroid (or tropical linear space). Thus Theorem 5 says then that $\mathcal{C}_{\mathbb{Z}_{\geq 0}}(P)$ is contained in the valuated matroid $([n], \mathcal{C}_{\mathbb{R}}(P))$.

By point ii), the monoid $(\mathbb{Z}_{\geq 0}, \odot, 0)$ acts on $\mathcal{C}_{\mathbb{Z}_{\geq 0}}(P)$. We will see below that the set $\mathcal{C}_{\mathbb{Z}_{\geq 0}}(P)$ can be recovered from the quotient set $\mathcal{C}_{\mathbb{Z}_{\geq 0}}(P)/\mathbb{Z}_{\geq 0}$.

We denote by $\mathcal{C}(P) \in \mathcal{P}([n])$ the family of supports of elements of $\mathcal{C}_{\mathbb{Z}_{\geq 0}}(P)$, and by $L(P) := \{j \in [n] \,:\, P_j \in V(A_{k_j})\}$.

Convention 4 *Even if* $A_{k_j,j}$ *is a tropical linear polynomial for* $j \in [n]$ *(see Definitions 6, 9), sometimes we will denote the common value* $A_{k_j,j}(P) = A_{k_j,j}(P_j) = \min_{0\le i\le k_j}\{a_{i,j}+k_j-i\}$ *just by* $A_{k_j,j}$ *to simplify the notation.*

Corollary 2 *The pair* $M(P) = ([n], \mathcal{C}(P))$ *is a matroid (of circuits). Moreover, we have* $\mathcal{C}(P) := L(P) \cup \binom{[n]\setminus L(P)}{2}$*, thus* $L(P)$ *is the set of loops of* $M(P)$*.*

Proof By the previous remark, we have that $M(P) = ([n], \mathcal{C}(P))$ is the underlying matroid of the valuated matroid $([n], \mathcal{C}_{\mathbb{R}}(P))$. It is also easy to see that $L(P) \cup \binom{[n]\setminus L(P)}{2}$ satisfies the axioms for circuits (once we prove it).

Let $S \in \mathcal{C}_{\mathbb{Z}_{\geq 0}}(P)$ such that $\mathrm{Supp}(S) = \{j\} \in \mathcal{C}$, then there exists $r \geq k_j$ such that $S = t_j^r \in \mathrm{Sol}(P_j)$, but this is equivalent to $P_j \in V(A_{k_j,j})$ by Proposition 4.

We now consider $\binom{[n]\setminus L(P)}{2}$, which is empty if $|[n]\setminus L(P)| \leq 1$. If $[n]\setminus L(P) = \emptyset$, there is nothing left to do.

If $[n]\setminus L(P) = \{j\}$, then j can not belong to a pair $\{i,j\}$, otherwise we would have $i \in L(P)$, and by this reason there is no minimal solution $S = t_i^p + t_j^q$ with $p \geq k_i$ and $q \geq k_j$.

So we suppose that $|[n] \setminus L(P)| > 1$, and we need to show that if $a, b \in [n] \setminus L(P)$, then there exists $S \in \mathcal{C}_{\mathbb{Z}_{\geq 0}}(P)$ such that $\mathrm{Supp}(S) = \{a,b\}$.

W.L.O.G. we can suppose that $A_{k_a,a} - A_{k_b,b} \geq 0$ (otherwise we choose $A_{k_b,b} - A_{k_a,a} \geq 0$), then $S = t_a^{k_a} + t_b^{k_b + A_{k_a,a} - A_{k_b,b}} \in \mathcal{C}_{\mathbb{Z}_{\geq 0}}(P)$ satisfies $\mathrm{Supp}(S) = \{a,b\}$.

We are now ready to prove the main result of this section.

Theorem 6 (Structure of $\mu(\mathrm{Sol}(P))$ for $n > 1$) *Consider* $P = \bigoplus_{j=1}^n P_j$ *as before, and* $L(P) := \{j \in [n] : P_j \in V(A_{k_j})\}$*. Then*

i) the set $\mathcal{C}_{\mathbb{Z}_{\geq 0}}(P)$ *satisfies*

$$\mathcal{C}_{\mathbb{Z}_{\geq 0}}(P) = \bigcup_{j\in L(P)} \mathbb{Z}_{\geq 0}\odot\{k_j\} \cup \bigcup_{a,b\notin L(P)\,:\,A_{k_b,b}\leq A_{k_a,a}} \mathbb{Z}_{\geq 0}\odot\{(k_a, k_b + A_{k_a,a} - A_{k_b,b})\}.$$

ii) The set $F(P) := \mu(\mathrm{Sol}(P)) \setminus \mathcal{C}_{\mathbb{Z}_{\geq 0}}(P)$ *is finite. Furthermore, if* $S \in F(P)$ *with* $X = \mathrm{Supp}(S)$*, then either*
  (a) $X = \{j\}$ *iff* $X \in F(P_j)$*,*
  (b) $X = \{i,j\}$ *iff* $X = t_i^p + t_j^q$ *satisfies*
    i. $(p,q) < (k_i, k_j)$*,*
    ii. $p < k_i$*, and* $q = q(p) \geq k_j$ *is unique.*

Proof The part ii) is Theorem 4. For part i) it suffices to show the inclusion $\subseteq$. Let $S \in \mathcal{C}_{\mathbb{Z}_{\geq 0}}(P)$. If $\mathrm{Supp}(S) = \{i\} \in L(P)$, then there exists $s \in \mathbb{Z}_{\geq 0}$ such that $S = t^s \cdot t_i^{k_i}$, so $\{S \in \mathcal{C}_{\mathbb{Z}_{\geq 0}}(P) : \mathrm{Supp}(S) = \{i\}\} = \mathbb{Z}_{\geq 0} \cdot \{k_i\}$.

If $\mathrm{Supp}(S) = \{a,b\} \in \binom{[n]\setminus L(M(P))}{2}$, then $S = t_a^{k_a+\alpha} + t_b^{k_b+\beta}$ for some $\alpha, \beta \in \mathbb{Z}_{\geq 0}$. This means

$$\mathrm{trop}_P(S) = A_{P_a}(k_a) + \alpha = A_{P_b}(k_b) + \beta,$$

W.L.O.G. we can suppose that $A_{k_a,a} - A_{k_b,b} = \beta - \alpha \geq 0$ (otherwise we choose $A_{k_b,b} - A_{k_a,a} \geq 0$), so that $A_{k_a,a} = A_{k_b,b} + (\beta - \alpha)$ and $S = t^\alpha \cdot (t_a^{k_a} + t_b^{k_b + A_{k_a,a} - A_{k_b,b}})$. So $\{S \in \mathcal{C}_{\mathbb{Z}_{\geq 0}}(P) \,:\, \mathrm{Supp}(S) = \{a,b\}\} = \mathbb{Z}_{\geq 0} \cdot (t_a^{k_a} + t_b^{k_b + A_{k_a,a} - A_{k_b,b}})$.

The action of the monoid $\mathbb{Z}_{\geq 0}$ on $\mathcal{C}_{\mathbb{Z}_{\geq 0}}(P)$ becomes clear from Theorem 6.

Corollary 3 *The quotient set $\mathcal{C}_{\mathbb{Z}_{\geq 0}}(P)/\mathbb{Z}_{\geq 0}$ is determined by a valuation of circuits, which is the map $\nu : \mathcal{C}(P) \to \mathbb{Z}_{\geq 0}$ given by*

$$\nu(S) = \begin{cases} k_j, & \text{if } j \in L(P), \\ A_{k_a,a} - A_{k_b,b}, & \text{if } \{a,b\} \in \binom{[n]\setminus L(P)}{2}, A_{k_b,b} \leq A_{k_a,a}. \end{cases}$$

And the set $\mathcal{C}_{\mathbb{Z}_{\geq 0}}(P)$ can be recovered from $\mathcal{C}_{\mathbb{Z}_{\geq 0}}(P)/\mathbb{Z}_{\geq 0}$.

The next step is to introduce the concept of regularity for polynomials $P = \bigoplus_{j=1}^n P_j$ for $n > 1$, to explain and simplify Definition 13. To do so, we will use the family $\{A_{\alpha,j} \,:\, j \in [n],\, 0 \leq \alpha \leq k_j\}$ from Definition 6.

Definition 12 We say that $P = \bigoplus_{j=1}^n P_j(u_j)$ is **regular** if

1. (local condition) P_j is regular for all $j \in [n]$ as in Definition 8,
2. (global condition) the expression $\bigoplus_{j=1}^n \bigoplus_{\alpha=0}^{k_j-1} A_{\alpha,j}(P)$ does not vanish weakly in $\mathbb{T}$ (see Definition 3).

Similarly to Proposition 5, we have the following characterization of regularity in terms of controlling the elements of $F(P)$.

Proposition 7 *We have that P is regular if and only if the elements of $F(P)$ from the decomposition* (4) *are of the form*

1. $t_j^p + t_j^q$ *with* $0 \leq p < k_j \leq q$,
2. $t_i^p + t_j^q$ *with* $0 \leq p < k_i$ *and* $k_j \leq q$.

Proof The first condition of Definition 12 controls $\bigcup_j F(P_j)$; the second one prevents the existence of solutions $S = t_i^p + t_j^q$ with $(p,q) < (k_i,k_j)$, which, according to Theorem 4 are the only solutions of $F(P)$ left to be controlled.

If $P = \bigoplus_{j=1}^n P_j(u_j)$ is regular, then $L(P) = \emptyset$ since each P_j is holonomic, and according to Corollary 3, the set $\mathcal{C}_{\mathbb{Z}_{\geq 0}}(P)$ becomes controlled by the values $A_{k_j,j}(P) - A_{i,k_i}(P)$ for $\{i,j\} \in \binom{[n]}{2}$.

Remark 7 There exists a tropical prevariety $\Delta(\mathbf{k}) \subset \prod_{j=1}^n \mathbb{T}^{k_j+1}$ such that $\Delta(\mathbf{k}) \cap \prod_{j=1}^n \mathbb{Z}^{k_j+1}$ consists of the TLDEs which are not regular. We call this $\Delta(\mathbf{k})$ the *tropical discriminant*. Indeed, both conditions of Definition 12 provide tropical prevarieties, the first one by Remark 5, and the second by Remark 1.

4 Systems of tropical ordinary linear differential equations

In this section we define the important concepts of regularity and genericity for holonomic systems of TLDEs. See Definition 13.

Consider a system $\Sigma = \{P_1, \ldots, P_m\}$ of m tropical ordinary linear differential equations in $n \geq 1$ unknowns (cf. (3)), each one of differential order $\mathbf{k} = (k_1, \ldots, k_n) \in (\mathbb{Z}_{>0})^n$:

$$P_l = \bigoplus_{1\leq j\leq n,\ 0\leq i\leq k_j} a_{ijl} \odot u_j^{(i)} = \min_{1\leq j\leq n,\ 0\leq i\leq k_j} \{a_{ijl} + u_j^{(i)}\},\ 1 \leq l \leq m. \tag{12}$$

Recall that the set of solutions of the system (12) is the $\mathbb{B}$-subsemimodule $\mathrm{Sol}(\Sigma) = \bigcap_j \mathrm{Sol}(P_j)$ of $\mathcal{P}(\mathbb{Z}_{\geq 0})$, and if $(S_1, \ldots, S_n) \in \mathrm{Sol}(\Sigma)$ we have that

$$\min_{1\leq j\leq n,\ 0\leq i\leq k_j} \{a_{ijl} + \mathrm{val}_{S_j}(i)\} \tag{13}$$

is attained at least twice for each $1 \leq l \leq m$.

We have that $\mathrm{Sol}(\Sigma)$ is generated by $\mu(\mathrm{Sol}(\Sigma))$, and that Σ is holonomic if $|\mu(\mathrm{Sol}(\Sigma))| < \infty$. Similar to Proposition 3 one can show that if $S = (S_1, \ldots, S_n) \in \mu(\mathrm{Sol}(\Sigma))$, then $S_j \cap \mathbb{Z}_{\geq k_j}$ contains at most one element for $1 \leq j \leq n$.

Proposition 8 *i) When $m < n$ a system (6) is non-holonomic;*
ii) When $m = n$ a generic system (6) is holonomic.

Proof i) Take $S_j := \{q_j\},\ 1 \leq j \leq n$ for unknowns $q_j \in \mathbb{Z}_{\geq k_j}$. Then one can rewrite (13) as a system of m tropical homogeneous linear equations in n unknowns. This system has a solution of the form $(q_1 + b, \ldots, q_n + b)$ for suitable $q_j \in \mathbb{Z},\ 1 \leq j \leq n$ and for an arbitrary $b \in \mathbb{Z}$ (see e.g. [4], [1], [6]).

ii) For each $1 \leq j \leq n$ fix a subset $S_j \cap \{0, \ldots, k_j - 1\}$ and $S_j \cap \mathbb{Z}_{\geq k_j} = \{q_j\}$ for an unknown q_j. One can rewrite (13) as a system of n tropical (non-homogeneous) linear equations in n unknowns. In general, such a system has a unique solution due to the Tropical Bezout Theorem [8].

Remark 8 Now we describe an outline of an algorithm which tests whether a system (6) is holonomic.

First, one fixes $S_j \cap \{0, \ldots, k_j - 1\},\ 1 \leq j \leq n$ and verifies whether a system of tropical linear equations with unknowns $q_j,\ 1 \leq j \leq n$ (as in the proof of Proposition 8 ii)) has a finite number of solutions in $\mathbb{Z}^n_{\geq 0}$. We have the following two situations.

The homogeneous case. If this system is homogeneous one has to check its solvability in $\mathbb{Z}^n$ (say, based on one of the algorithms from [4], [1], [6]). If the system is solvable then the system (6) is non-holonomic (see the proof of Proposition 8 ii)).

The non-homogeneous case. If this system has a non-homogeneous equation of the form

$$\min_{1\leq j\leq n} \{b_j + q_j, b\},\ b \neq \infty \tag{14}$$

we can consider two different cases as candidates for minima, namely

i) $b_{j_1}+q_{j_1} = b_{j_2}+q_{j_2} = \min_{1\leq j\leq n}\{b_j+q_j, b\}$, if we take a pair $1 \leq j_1 < j_2 \leq n$. In this case, it holds $k_{j_1} \leq q_{j_1} \leq b - b_{j_1}$, $k_{j_2} \leq q_{j_2} \leq b - b_{j_2}$, and we have a finite number of possibilities for q_{j_1}, q_{j_2}, and
ii) $b_{j_0} + q_{j_0} = b = \min_{1\leq j\leq n}\{b_j + q_j, b\}$, if we take a singleton $1 \leq j_0 \leq n$. In this case, it holds $q_{j_0} = b - b_{j_0}$.

Then, for each choice of q_{j_1}, q_{j_2} or q_{j_0} respectively, we proceed to the next non-homogeneous equation of the form (14) (substituting our chosen values of q_{j_1}, q_{j_2} or q_{j_0}, respectively).

After exhausting all the equations of the form (14), we take the truncated equation for each considered equation of the form (14) by removing from it all non-assigned unknowns q_j. The algorithm checks whether the truncated tropical system is satisfied. If it is not satisfied, then the algorithm proceeds to another one of its branches by choosing different q_{j_1}, q_{j_2} or q_{j_0} and their values.

If after exhausting all the equations of the form (14),

1. some homogeneous (i.e. with $b = \infty$) linear equations still remain, the algorithm checks the solvability of this homogeneous system (cf. the **homogeneous case** above). If this homogeneous system has a solution then the initial system (6) is non-holonomic (as above). If this homogeneous system has no solution then the algorithm proceeds to another one of its branches.
2. no homogeneous equations remain, and at least one non-assigned unknown q_j remains, then one can take the values of all the latter unknowns as arbitrary large integers. Therefore, in this case the initial system (6) is non-holonomic. Thus, if no branch of the algorithm yields the output that the initial system (6) is non-holonomic, the algorithm concludes that (6) is holonomic.

Unfortunately, the complexity bound of the described algorithm is exponential.

Consider a holonomic system $\Sigma = \{P_1, \dots, P_n\}$ on $n > 1$ unknowns as in (6). We introduce two different notions of general position for Σ, namely regular system, based on the the concept of regular polynomial in $n > 1$ unknowns from Definition 12, and generic system, using also the tropical polynomials $A_{\alpha,j}$ for $j \in [n]$ and $0 \leq \alpha \leq k_j$ from Definition 6.

Definition 13 Consider a system (6).

1. We say that it is **regular** if P_l is regular in the sense of Definition 12 for all $l \in [n]$.
2. We say that it is **generic** if the expressions

$$A_{k_{\sigma(1)},\sigma(1)}(P_1) \odot \cdots \odot A_{k_{\sigma(n)},\sigma(n)}(P_n) = \sum_{1\leq j\leq n} A_{k_{\sigma(j)},\sigma(j)}(P_j) \qquad (15)$$

are pairwise distinct for all $\sigma \in \mathrm{Sym}(n)$, and in addition, for $l_1 \neq l_2$ it holds

$$A_{k_j,j}(P_{l_1}) - A_{\alpha,j_1}(P_{l_1}) \neq A_{k_j,j}(P_{l_2}) - A_{\beta,j_2}(P_{l_2}),\ 0 \leq \alpha < k_{j_1},\ 0 \leq \beta < k_{j_2}. \tag{16}$$

Note that, for fixed $\mathbf{k} = (k_1, \dots, k_n)$, the set of systems (6) being not generic or not regular, lie in a finite union of polyhedra of dimensions less than the full dimension $n(k_1 + \cdots + k_n)$ inside of the space of all systems (6).

Remark 9 Moreover, it follows from Remark 7 that the set of non-regular systems (6) is a tropical prevariety. The same is true for the set of non-generic systems (6), indeed, the first condition of genericity (15) is equivalent to the fact that the *tropical determinant* of the matrix $(A_{k_j,j}(P_l))_{1 \leq l,j \leq n}$ does not vanish weakly (see Definition 3). The second condition of genericity can be reformulated as asking that the tropical determinant tdet $\begin{pmatrix} A_{k_j,j}(P_{l_1}) & A_{\alpha,j_1}(P_{l_1}) \\ A_{k_j,j}(P_{l_2}) & A_{\beta,j_2}(P_{l_2}) \end{pmatrix}$ does not vanish weakly.

Convention 5 *Given* $1 \leq j, l \leq n$ *and* $0 \leq \alpha \leq k_j$*, for the computations that follow we will write* $A_{\alpha,j}(P_l) = A_{\alpha jl}$*. Note that if* $P_l = \bigoplus_{j=1}^n P_{j,l}(u_j)$ *for* $l \in [n]$*, then* $A_{\alpha,j}(P_l) = A_{\alpha,j}(P_{j,l})$ *for every* $j \in [n]$ *and* $0 \leq \alpha \leq k_j$*.*

4.1 Upper bound on the number of minimal solutions for $m = n = 2$

In this section we consider a generic regular system (6) for $n = m = 2$ with unknowns u, v, this is $P_1 = P_{u,1}(u) + P_{v,1}(v)$ and $P_2 = P_{u,2}(u) + P_{v,2}(v)$ each one of differential order $\mathbf{k} = (k_u, k_v)$. Note that

$$A_{i_1 jl} = A_{jl} + i_1 - k_j,\ i_1 \geq k_j,\ 1 \leq j, l \leq n. \tag{17}$$

and also

$$A_{0,1,l}, \dots, A_{k_1-1,1,l}, A_{0,2,l}, \dots, A_{k_2-1,2,l}, \dots, A_{0,n,l}, \dots, A_{k_n-1,n,l}; \tag{18}$$

are pairwise distinct.

For a minimal solution (S_u, S_v) of (6) and $l = 1, 2$ denote by $S_{ul} \subset S_u$, $S_{vl} \subset S_v$ the subsets such that the minimum in (13) is attained at $S_{ul} \cup S_{vl}$. Then $S_u = S_{u1} \cup S_{u2}$, $S_v = S_{v1} \cup S_{v2}$. If i belongs to $S_{u1} \cap S_{u2}$ then we say that i belongs to S_u with the multiplicity 2.

Due to minimality of (S_u, S_v) it holds $|S_u| + |S_v| \leq 4$. Moreover, one can assume that $|S_{ul} \cap \{0, \dots, k_u - 1\}| + |S_{vl} \cap \{0, \dots, k_v - 1\}| \leq 1,\ l = 1, 2$, since otherwise, the system (6) is not regular (cf. Theorem 2). In addition, $|S_u| + |S_v| \geq 3$ since otherwise, the system (6) is again not regular. Now we describe all possible candidates for minimal solutions of a generic regular system (6) (cf. Proposition 3).

Lemma 3 *Any minimal solution of a generic regular system (6) is one of the following 5 types of configurations:*

•*uv)* $S_u \in \{i, \star\},\ S_v \in \{p, \star\},\ 0 \le i < k_u, 0 \le p < k_v$;

•*uu)* $S_u \in \{i, i_0, \star\},\ S_v \in \{\star\},\ 0 \le i \le i_0 < k_u$;

•*vv)* $S_u \in \{\star\},\ S_v \in \{p, p_0, \star\},\ 0 \le p \le p_0 < k_v$;

•*u)* $S_u \in \{i, \star\},\ S_v \in \{\star\}$;

•*v)* $S_u \in \{\star\},\ S_v \in \{p, \star\}$.

For each i, i_0, p, p_0 *(as above) there exist at most one integer* $q_u \ge k_u$ *and at most one integer* $q_v \ge k_v$ *such that* $q_u \in S_u,\ q_v \in S_v$ *for all 5 types, respectively, provided that* (S_u, S_v) *is a minimal solution of (6). Moreover, the subsets* $S_{u1}, S_{u2}, S_{v1}, S_{v2}$ *are uniquely determined by* i, i_0, p, p_0 *for all 5 types, respectively, provided that* (S_u, S_v) *is a minimal solution of (6).*

Proof Denote (cf. Definition 13 ii)) $A := A_{u1} - A_{u2} - A_{v1} + A_{v2}$. Due to genericity of (6), see (15) it holds $A \ne 0$.

•*uv*). Let $S_u = \{i, q_u\},\ S_v = \{p, q_v\},\ 0 \le i < k_u, 0 \le p < k_v$ be a minimal solution of (6). There are 4 cases (taking into account (16)):

$$A_{iu1} = A_{q_u u1},\ A_{pv2} = A_{q_v v2}; \tag{19}$$

$$A_{iu1} = A_{q_v v1},\ A_{pv2} = A_{q_u u2}; \tag{20}$$

$$A_{iu2} = A_{q_u u2},\ A_{pv1} = A_{q_v v1}; \tag{21}$$

$$A_{iu2} = A_{q_v v2},\ A_{pv1} = A_{q_u u1}. \tag{22}$$

We claim that exactly one of (19)-(22) is valid. Suppose the contrary. W.l.o.g. assume that (19) is valid (other three cases can be studied in a similar way). Then (17) implies that

$$A_{iu1} = A_{u1} + q_u - k_u,\ A_{pv2} = A_{v2} + q_v - k_v. \tag{23}$$

This provides the expressions for q_u, q_v. In addition, we have

$$A_{iu1} \le A_{q_v v1},\ A_{pv2} \le A_{q_u u2}; \tag{24}$$

$$A_{iu1} < A_{pv1},\ A_{pv2} < A_{iu2} \tag{25}$$

due to (18).

Observe that (25) contradicts to $A_{iu2} = A_{q_u^{(1)} u2},\ A_{pv1} = A_{q_v^{(1)} v1}$ (see (21)) with any $q_u^{(1)} \ge k_u, q_v^{(1)} \ge k_v$ and also contradicts to $A_{iu2} = A_{q_v^{(2)} v2},\ A_{pv1} = A_{q_u^{(2)} u1}$ (see (22)) with any $q_u^{(2)} \ge k_u,\ q_v^{(2)} \ge k_v$.

Note that (23), (24) imply that

$$A_{u1} - A_{v1} \le (q_v - k_v) - (q_u - k_u) \le A_{u2} - A_{v2}. \tag{26}$$

Equalities $A_{iu1} = A_{q_v^{(3)} v1},\ A_{pv2} = A_{q_u^{(3)} u2}$ (see (20)) for some $q_u^{(3)} \ge k_u,\ q_v^{(3)} \ge k_v$ entail that $A_{iu1} \le A_{q_u^{(3)} u1},\ A_{pv2} \le A_{q_v^{(3)} v2}$ (cf. (24)), therefore

$$A_{v1} - A_{u1} \le (q_u^{(3)} - k_u) - (q_v^{(3)} - k_v) \le A_{v2} - A_{u2}$$

which together with (26) deduces equality $A = 0$. This contradicts to genericity of (6) and proves that (19) entails that (20)-(22) do not hold. The Lemma is justified in case $\bullet uv$).

$\bullet uu$). Let $(S_u = \{i, i_0, q_u\},\ S_v = \{q_v\}),\ 0 \le i \le i_0 < k_u$ be a minimal solution of (6). There are 4 cases (taking into account (16)):

$$A_{iu1} = A_{q_u u1},\ A_{i_0 u2} = A_{q_v v2}; \tag{27}$$

$$A_{iu1} = A_{q_v v1},\ A_{i_0 u2} = A_{q_u u2}; \tag{28}$$

$$A_{i_0 u1} = A_{q_u u1},\ A_{iu2} = A_{q_v v2}; \tag{29}$$

$$A_{i_0 u1} = A_{q_v v1},\ A_{iu2} = A_{q_u u2}. \tag{30}$$

Note that in case $i = i_0$ the equations (27) coincide with (29), and the equations (28) coincide with equations (30).

We claim that exactly one of (27)-(30) is valid (cf. the proof of the case $\bullet uv$) above). Suppose the contrary. W.l.o.g. assume that (27) is valid (other three cases can be studied in a similar way). Then due to (17) it holds

$$A_{iu1} = A_{u1} + q_u - k_u,\ A_{i_0 u2} = A_{v2} + q_v - k_v. \tag{31}$$

This provides the expressions for q_u, q_v. In addition, we have

$$A_{iu1} \le A_{q_v v1},\ A_{i_0 u2} \le A_{q_u u2}; \tag{32}$$

$$A_{iu1} < A_{i_0 u1},\ A_{i_0 u2} < A_{iu2}. \tag{33}$$

due to (18) (in case $i = i_0$ the inequalities (33) are omitted). Observe that (33) contradicts to equalities $A_{i_0 u1} = A_{q_u^{(1)} u1},\ A_{iu2} = A_{q_v^{(1)} v2}$ (see (29)) with $q_u^{(1)} \ge k_u,\ q_v^{(1)} \ge k_v$ and also contradicts to equalities $A_{i_0 u1} = A_{q_v^{(2)} v1},\ A_{iu2} = A_{q_u^{(2)} u2}$ (see (30)) with $q_u^{(2)} \ge k_u,\ q_v^{(2)} \ge k_v$.

Note that (31), (32) imply that

$$A_{u1} - A_{v1} \le (q_v - k_v) - (q_u - k_u) \le A_{u2} - A_{v2}. \tag{34}$$

The equalities $A_{iu1} = A_{q_v^{(3)} v1},\ A_{i_0 u2} = A_{q_u^{(3)} u2}$ (see (28)) with $q_u^{(3)} \ge k_u,\ q_v^{(3)} \ge k_v$ entail that $A_{iu1} \le A_{q_u^{(3)} u1},\ A_{i_0 u2} \le A_{q_v^{(3)} v2}$. Whence together with (31) we obtain that

$$A_{v1} - A_{u1} \le (q_u^{(3)} - k_u) - (q_v^{(3)} - k_v) \le A_{v2} - A_{u2}$$

which leads to the equality $A = 0$ taking into account (34). This contradicts to genericity of (6) and proves that (27) entails that (28)-(30) do not hold. The Lemma is justified in case $\bullet uu$).

The case $\bullet vv$) is studied similarly to $\bullet uu$).

$\bullet u$). Let $(S_u = \{i, q_u\},\ S_v = \{q_v\}),\ 0 \le i < k_u$ be a minimal solution of (6). There are 4 cases:

$$A_{iu1} = A_{q_v v1},\ A_{q_u u2} = A_{q_v v2}; \tag{35}$$
$$A_{iu1} = A_{q_u u1},\ A_{q_u u2} = A_{q_v v2}; \tag{36}$$
$$A_{q_u u1} = A_{q_v v1},\ A_{iu2} = A_{q_v v2}; \tag{37}$$
$$A_{q_u u1} = A_{q_v v1},\ A_{iu2} = A_{q_u u2}. \tag{38}$$

We claim that exactly one of (35)-(38) is valid (cf. the cases $\bullet uv), \bullet uu)$ above). Suppose the contrary. W.l.o.g. assume that (35) is valid (other three cases can be studied in a similar way). Then it holds

$$A_{iu1} = A_{v1} + q_v - k_v,\ A_{v2} + q_u - k_u = A_{v2} + q_v - k_v. \tag{39}$$

This provides the expressions for q_u, q_v. In addition, we have

$$A_{iu1} \leq A_{q_u u1},\ A_{q_u u2} \leq A_{iu2}. \tag{40}$$

Together with (17), (39) this implies that

$$A \geq 0,\ A_{iu2} - A_{iu1} \geq A_{v2} - A_{v1} \tag{41}$$

Observe that the equalities $A_{iu1} = A_{q_u^{(1)} u1},\ A_{q_u^{(1)} u2} = A_{q_v^{(1)} v2}$ (see (36)) with $q_u^{(1)} \geq k_u, q_v^{(1)} \geq k_v$ entail the inequality $A \leq 0$ which contradicts to the first inequality of (41) because of genericity of (6). Similarly, the equalities $A_{q_u^{(2)} u1} = A_{q_v^{(2)} v1},\ A_{iu2} = A_{q_v^{(2)}}$ (see (37)) with $q_u^{(2)} \geq k_u, q_v^{(2)} \geq k_v$ entail the inequality $A \leq 0$ as well, which again leads to a contradiction.

Finally, suppose that the equalities $A_{q_u^{(3)} u1} = A_{q_v^{(3)} v1},\ A_{iu2} = A_{q_u^{(3)} u2}$ (see (38)) hold with $q_u^{(3)} \geq k_u, q_v^{(3)} \geq k_v$. Hence due to (17) we obtain that $A_{iu2} = A_{u2} + q_u^{(3)} - k_u$. Then the inequality $A_{iu1} \geq A_{q_u^{(3)} u1}$ and (17) imply that $A_{iu1} \geq A_{u1} + q_u^{(3)} - k_u = A_{u1} + A_{iu2} - A_{u2}$ which contradicts to (41). This proves that (35) entails that (36)-(38) do not hold and justifies the case $\bullet u)$.

The case $\bullet v)$ is justified in a similar manner.

In the next lemma we use the notations from Lemma 3.

Lemma 4 *For* $0 \leq i < k_u$ *denote* $A'_i := A_{iu2} - A_{iu1} - A_{v2} + A_{v1}$.

1. Let $(S_u = \{i, q_u\},\ S_v := \{q_v\})$ *be a minimal solution of type* $\bullet u)$ *of a generic regular system (6). Then the following inequalities hold:*

$$A \geq 0, A'_i \geq 0 \textit{ when } S_{u,1} = \{i\}, S_{v,1} = S_{v,2} = \{q_v\}, S_{u,2} = \{q_u\}; \tag{42}$$
$$A \leq 0, A'_i + A \geq 0 \textit{ when } S_{u,1} = \{i, q_u\}, S_{v,1} = \emptyset, S_{u,2} = \{q_u\}, S_{v,2} = \{q_v\}; \tag{43}$$
$$A \leq 0, A'_i \leq 0 \textit{ when } S_{u,1} = \{q_u\}, S_{v,1} = S_{v,2} = \{q_v\}, S_{u,2} = \{i\}; \tag{44}$$
$$A \geq 0, A'_i + A \leq 0 \textit{ when } S_{u,1} = \{q_u\}, S_{v,1} = \{q_v\}, S_{u,2} = \{i, q_u\},\ S_{v,2} = \emptyset. \tag{45}$$

2. Let (S'_u, S'_v) of the type either $\bullet uv)$ or $\bullet uu)$ be a minimal solution of (6) such that $i \in S'_u$. Then it holds

$$A \geq 0, A'_i + A \geq 0 \text{ when } S'_{u,1} = \{i\}, |S'_{v,1}| = 1 \tag{46}$$

and either $|S'_{u,2}| = 1$ in case $\bullet uv)$ or $|S'_{u,2}| = 2$ in case $\bullet uu)$;

$$A \leq 0, A'_i \geq 0 \text{ when } |S'_{u,1}| = 2, S'_{v,1} = \emptyset \tag{47}$$

and either $S'_{u,2} = \emptyset$ in case $\bullet uv)$ or $|S'_{u,2}| = 1$ in case $\bullet uu)$;

$$A \leq 0, A'_i + A \leq 0 \text{ when } S'_{u,2} = \{i\}, |S'_{v,2}| = 1 \tag{48}$$

and either $|S'_{u,1}| = 1$ in case $\bullet uv)$ or $|S'_{u,1}| = 2$ in case $\bullet uu)$;

$$A \geq 0, A'_i \leq 0 \text{ when } |S'_{u,2}| = 2, S'_{v,2} = \emptyset \tag{49}$$

and either $S'_{u,1} = \emptyset$ in case $\bullet uv)$ or $|S'_{u,1}| = 1$ in case $\bullet uu)$.

3. Assume that both (S_u, S_v) as in 1. and (S'_u, S'_v) in 2. are minimal solutions of (6). If $i \in S_{u,1}$ then $S'_{u,1} = S_{u,1}$, $S'_{v,1} = S_{v,1}$, and if $i \in S_{u,2}$ then $S'_{u,2} = S_{u,2}$, $S'_{v,2} = S_{v,2}$.

One can swap the roles of u and of v in 1., 2., 3., simultaneously.

Proof 1. The statement (42) coincides with (41) (see (35)) in the proof of Lemma 3 $\bullet u)$. Other statements (43)-(45) are justified in a similar way (cf. the proof of Lemma 3 $\bullet u)$).

2. The statement (47) in case of the type $\bullet uv)$ follows from (23), (25), (26) (see (19) in the proof of Lemma 3 $\bullet uv)$). The statement (47) in case of the type $\bullet uu)$ follows from (31), (33), (34) (see (27) in the proof of Lemma 3 $\bullet uu)$). Other statements (46), (48), (49) are justified in a similar way (cf. the proof of Lemma 3 $\bullet uv), \bullet uu)$).
3. The first statement of 3. follows from (42), (43), (46), (47), the second statement follows from (44), (45), (48), (49).

Now we establish an upper bound on the number of minimal solutions of a generic regular system (6).

Theorem 7 *A generic regular system (6) has at most*

$$(k_u + k_v)(k_u + k_v + 1)/2$$

minimal solutions.

Proof W.l.o.g. one can assume that $A > 0$ (the case $A < 0$ can be studied in a similar way).

Denote by $B_{u,l}$, $l = 1, 2$ the set of $i \in \{0, \dots, k_u - 1\}$ such that (6) has a minimal solution (S_u, S_v) of the type $\bullet u)$ for which $i \in S_{u,l}$. Then $S_{u,l} = \{i\}$, $|S_{v,l}| = 1$ when $l = 1$, and $|S_{u,l}| = 2$, $S_{v,l} = \emptyset$ when $l = 2$ since $A > 0$ (see Lemma 4 1.). It holds $B_{u,1} \cap B_{u,2} = \emptyset$ due to Lemma 3 $\bullet u)$.

Similarly, we define the sets $B_{v,l} \subset \{0, \dots, k_v - 1\},\ l = 1, 2$. Denote $k_{u,l} := |B_{u,l}|,\ k_{v,l} := |B_{v,l}|, l = 1, 2,\ k := k_{u,1} - k_{v,1} - k_{u,2} + k_{v,2}$.

Due to Lemmas 3, 4 the following upper bounds hold on the number of minimal solutions of the respective types:

$$\bullet uv) : k_{u,1}k_{v,2} + k_{u,2}k_{v,1} + (k_u - k_{u,1} - k_{u,2})(k_v - k_{v,1} - k_{v,2}) +$$

$$(k_{u,1} + k_{u,2})(k_v - k_{v,1} - k_{v,2}) + (k_u - k_{u,1} - k_{u,2})(k_{v,1} + k_{v,2});$$

$$\bullet uu) : k_{u,1}k_{u,2} + (k_u - k_{u,1} - k_{u,2})(k_u - k_{u,1} - k_{u,2} + 1)/2 + (k_{u,1} + k_{u,2})(k_u - k_{u,1} - k_{u,2});$$

$$\bullet vv) : k_{v,1}k_{v,2} + (k_v - k_{v,1} - k_{v,2})(k_v - k_{v,1} - k_{v,2} + 1)/2 + (k_{v,1} + k_{v,2})(k_v - k_{v,1} - k_{v,2});$$

$$\bullet u) : k_{u,1} + k_{u,2};$$

$$\bullet v) : k_{v,1} + k_{v,2}.$$

Summing up all these upper bounds and taking into account the inequality $k_{u,1}k_{v,2} + k_{u,2}k_{v,1} + k_{u,1}k_{u,2} + k_{v,1}k_{v,2} \leq k^2/4$, we obtain the required upper bound

$$\frac{(k_u + k_v)(k_u + k_v + 1)}{2} + \frac{2k - k^2}{4}$$

(taking into account that $(2k - k^2)/4 \leq 1/4$).

4.2 Sharpness of the bound on the number of minimal solutions for $m = n = 2$

The next purpose is to construct a generic regular system (6) which contains all minimal solutions of the types $\bullet uv)$, $\bullet uu)$, $\bullet vv)$ from Lemma 3. This shows that the bound in Theorem 7 is sharp (even without solutions of types $\bullet u), \bullet v)$). First we note some simple growing properties of the maps $A_-(P) : \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ from Definition 9.

Remark 10 If the coefficients of P as in (1) fulfill inequalities $a_{i+1} \geq a_i + 1,\ 0 \leq i < k$ then $A_j(P) = j + a_0,\ j \in \mathbb{Z}_{\geq 0}$; in particular the map $A_-(P)$ is increasing. On the contrary, if $a_{i+1} \leq a_i + 1$ for $0 \leq i < k$, then

$$A_j(P) = a_j,\ 0 \leq j \leq k.$$

Therefore, if $a_{i+1} < a_i$ for $0 \leq i < k$ then $A_-(P)$ is decreasing on the interval $[0, k]$.

Theorem 8 *Assume that the coefficients of a generic system (6) fulfill the following inequalities:*

$$a_{i+1,u,1} \geq a_{iu1} + 1,\ 0 \leq i < k_u; \tag{50}$$

$$a_{i+1,v,1} \leq a_{iv1} - 1,\ 0 \leq i < k_v; \tag{51}$$

$$a_{i+1,u,2} \leq a_{iu2} - 1,\ 0 \leq i < k_u; \tag{52}$$

$$a_{i+1,v,2} \geq a_{iv2} + 1,\ 0 \leq i < k_v; \tag{53}$$

$$a_{0v1} < a_{0u1},\ a_{0u2} < a_{0v2}. \tag{54}$$

Then the system (6) is regular and has minimal solutions of the following configurations:

- $(S_u \in \{i, \star\},\ S_v \in \{p, \star\})$ *of the type* $\bullet uv$), $0 \le i < k_u,\ 0 \le p < k_v$*;*
- $(S_u \in \{i, i_0, \star\},\ S_v \in \{\star\})$ *of the type* $\bullet uu$), $0 \le i \le i_0 < k_u$*;*
- $(S_u \in \{\star\},\ S_v \in \{p, p_0, \star\})$ *of the type* $\bullet vv$), $0 \le p \le p_0 < k_v$*.*

Proof $\bullet uv$) Let $0 \le i < k_u,\ 0 \le p < k_v$. Due to (52) there exists a unique $q_u \ge k_u$ such that $A_{iu2} = A_{q_u u2}$ taking into account Remark 10. By the same argument due to (51) there exists a unique $q_v \ge k_v$ such that $A_{pv1} = A_{q_v v1}$. We claim that $(S_u := \{i, q_u\},\ S_v := \{p, q_v\})$ is a solution of the system (6) described in the Theorem. To this end it suffices to verify the following inequalities:

$$A_{pv1} \le A_{iu1},\ A_{pv1} \le A_{q_u u1},\ A_{iu2} \le A_{pv2},\ A_{iu2} \le A_{q_v v2}. \tag{55}$$

Since $A_{pv1} \le A_{0v1}$ (see (51)) and $A_{0u1} \le A_{iu1}$ (see (50)), the first inequality in (55) follows from (54). Since $A_{q_u u1} \ge A_{0u1}$ (see (50)) we get the second inequality in (55). Since $A_{iu2} \le A_{0u2}$ (see (52)) and $A_{pv2} \ge A_{0v2}$ (see (53)), we get the third inequality in (55) invoking (54). Since $A_{q_v v2} \ge A_{0v2}$ (see (53)) we get the fourth inequality of (55).

$\bullet uu$) Let $0 \le i < i_0 < k_u$. Due to (54) there exists a unique $q_v \ge k_v$ such that $A_{iu1} = A_{q_v v1}$ (see (50), (51)). Due to (52) there exists a unique $q_u \ge k_u$ such that $A_{i_0 u2} = A_{q_u u2}$. We claim that $(S_u := \{i, i_0, q_u\},\ S_v := \{q_v\})$ is a solution of (6).

To this end it suffices to verify the following inequalities:

$$A_{iu1} \le A_{i_0 u1},\ A_{iu1} \le A_{q_u u1},\ A_{i_0 u2} \le A_{iu2},\ A_{i_0 u2} \le A_{q_v v2}. \tag{56}$$

The first and the second inequalities in (56) follow from (50). The third inequality in (56) follows from (52). The fourth inequality in (56) follows from (54) taking into account (52), (53).

$i)_{vv}$ Let $0 \le p < p_0 < k_v$. Due to (54) there exists a unique $q_u \ge k_u$ such that $A_{pv2} = A_{q_u u2}$ (see (52), (53)). Due to (51) there exists a unique $q_v \ge k_v$ such that $A_{p_0 v1} = A_{q_v v1}$. We claim that $(S_u := \{q_u\},\ S_v := \{p, p_0, q_v\})$ is a solution of (6).

To this end it suffices to verify the following inequalities:

$$A_{p_0 v1} \le A_{pv1},\ A_{p_0 v1} \le A_{q_u u1},\ A_{pv2} \le A_{p_0 v2},\ A_{pv2} \le A_{q_v v2}. \tag{57}$$

The first inequality in (57) follows from (51). The second inequality in (57) follows from (54) taking into account (50), (51). The third and the fourth inequalities in (57) follow from (53).

Finally, we have to show that the produced solutions of the types $\bullet uv$), $\bullet uu$), $\bullet vv$) are minimal and they are all the minimal ones. To this end it suffices to verify that the described system (6) has no solutions of types $\bullet u$), $\bullet v$) (see Lemma 3).

Suppose the contrary and let $(S_u = \{i, q_u\},\ S_v = \{q_v\})$ be a solution of (6) of the type $\bullet u$). Taking into account (50)-(53) we have to consider two cases: either $A_{iu1} = A_{q_v v1},\ A_{q_u u2} = A_{q_v v2}$ or $A_{iu2} = A_{q_u u2},\ A_{q_u u1} = A_{q_v v1}$.

In the former case it holds that $A_{iu2} < A_{q_v v2}$ due to (54). In the latter case it holds that $A_{iu1} < A_{q_u u1}$ due to (50). Thus, in both cases we arrive to a contradiction with that $(S_u,\ S_v)$ is a solution of (6).

In a similar way a supposition that there exists a solution of the form $(S_u = \{q_u\},\ S_v = \{p, q_v\})$ of (6) of the type $\bullet v)$ leads to a contradiction as well.

Corollary 4 *The maximal number of minimal solutions of a generic regular system (6) equals* $\frac{(k_u+k_v)(k_u+k_v+1)}{2}$.

Proof The upper bound follows from Theorem 7. On the other hand, the construction from Theorem 8 provides a system with

- $k_u k_v$ minimal solutions of the type $\bullet uv)$ and
- $k_u(k_u+1)/2$ (respectively, $k_v(k_v+1)/2$) minimal solutions of the type $\bullet uu)$ (respectively, $\bullet vv)$).

5 Bounds on the number of minimal solutions of a generic system for $m = n > 1$

First, we construct a generic regular system (6) for arbitrary $m = n$ with *many* minimal solutions, extending Theorem 8.

Theorem 9 *For each* $1 \le j, l \le n, 0 \le i \le k_j$ *take the coefficients* a_{ijl} *of a generic regular system (6) satisfying the following inequalities (cf. Theorem 8):*

$$a_{i+1,l,l} \le a_{ill} - 1,\ 0 \le i < k_l; \tag{58}$$

$$a_{i+1,l+1(mod\ n),l} \ge a_{i,l+1(mod\ n),l} + 1,\ 0 \le i < k_{l+1(mod\ n)}; \tag{59}$$

$$a_{0,l,l} < a_{0,l+1(mod\ n),l}; \tag{60}$$

$$a_{ijl} > a_{k_{l+1(mod\ n)},l+1(mod\ n),l} + k_j,\ j \ne l, j \ne l+1(mod\ n). \tag{61}$$

Then for each family of integers $1 \le p_1 < \cdots < p_s \le n$ *such that* $p_{j+1} \ge p_j + 2(mod\ n),\ 1 \le j \le s,$ *and for any* $0 \le i_{j1} \le i_{j2} < k_{p_j},\ 1 \le j \le s$ *and for any* $0 \le i_r < k_r,\ 1 \le r \le n,\ r \ne p_j,\ r \ne p_j - 1(mod\ n),\ 1 \le j \le n$ *the system (6) has a minimal solution* S *of (6) whose configuration satisfies the following:*

$$S_{u_{p_j}} \in \{i_{j1}, i_{j2}, \star\},\ S_{u_{p_j-1(mod\ n)}} \in \{\star\},\ S_{u_r} \in \{i_r, \star\}. \tag{62}$$

Remark 11 It holds $|S_{u_1}| + \cdots + |S_{u_n}| = 2n$, and the number of minimal solutions of the form (62) equals

$$\prod_{1\le j\le s} k_{p_j}(k_{p_j}+1)/2 \cdot \prod_{1\le r\le n, r\ne p_j, r\ne p_j-1(\mathrm{mod}\ n), 1\le j\le s} k_r.$$

Proof First, let $1 \le r \le n, r \ne p_j, r \ne p_j - 1 (\mathrm{mod}\ n), 1 \le j \le s$. Due to (58) (cf. also Remark 10) there exists (a unique) $q_r \ge k_r$ such that

$$A_{q_r,\ r,\ r} = A_{i_r,\ r,\ r}.$$

Then $S_{u_r} = \{i_r, q_r\} = S_{u_r,r}$.

For $p_j, 1 \le j \le s$ due to (58) there exists (a unique) $q_{p_j} \ge k_{p_j}$ such that

$$A_{q_{p_j},\ p_j,\ p_j} = A_{i_{j2},\ p_j,\ p_j}.$$

Due to (58)-(60) there exists (a unique) $q_{p_j-1(\mathrm{mod}\ n)} \ge k_{p_j-1(\mathrm{mod}\ n)}$ such that

$$A_{q_{p_j-1(\mathrm{mod}\ n)},\ p_j-1(\mathrm{mod}\ n),\ p_j-1(\mathrm{mod}\ n)} = A_{i_{j1},\ p_j,\ p_j-1(\mathrm{mod}\ n)}.$$

Then it holds

$$S_{u_{p_j}} = \{i_{j1},\ i_{j2},\ q_{p_j}\},\ S_{u_{p_j-1(\mathrm{mod}\ n)}} = \{q_{p_j-1(\mathrm{mod}\ n)}\},$$

thus

$$S_{u_{p_j},\ p_j} = \{i_{j2}, q_{p_j}\},\ S_{u_{p_j},\ p_j-1(\mathrm{mod}\ n)} = \{i_{j1}\},\ S_{u_{p_j-1(\mathrm{mod}\ n)},\ p_j-1(\mathrm{mod}\ n)} = \{q_{p_j-1(\mathrm{mod}\ n)}\}.$$

Taking into account (58)-(61) one can verify (62) and that $(S_{u_1}, \dots, S_{u_n})$ is a solution of (6) similar to the proof of Theorem 8, and once again, similar to the proof of Theorem 8 one can verify that the produced solution is minimal.

Similar to Lemma 3 one can prove the following lemma.

Lemma 5 *Let* $(S_{u_1}, \dots, S_{u_n})$ *be a minimal solution of a generic regular system (6). Then for any* $1 \le j \le n$ *there exists a unique* $q_j \ge k_j$ *such that* $q_j \in S_{u_j}$. *It holds* $|S_{u_1}| + \dots + |S_{u_n}| \le 2n$. *The subsets* $S_{u_j,l} \subset S_{u_j},\ 1 \le j, l \le n$ *are uniquely defined by the sets* $S_{u_1} \cap \{0, \dots, k_1 - 1\}, \dots, S_{u_n} \cap \{0, \dots, k_n - 1\}$.

Proof For $1 \le l \le n$ assume that the minimum (which is unique due to regularity of (6)) among $\{A_{ijl} \mid i \in S_{u_j}, 0 \le i < k_j, 1 \le j \le n\}$ is attained at $A_{i_0,j_0,l}$. One can verify (cf. the proof of Lemma 3) that $i_0 \in S_{u_{j_0},l}$. This determines the intersections $S_{u_j,l} \cap \{0, \dots, k_j - 1\},\ 1 \le j, l \le n$.

For each $1 \le l \le n,\ i \in S_{u_{j_0},l},\ 0 \le i < k_{j_0},\ 1 \le j, j_0 \le n$ take (a unique) $q_{jl} \ge k_j$ such that $A_{q_{jl},j,l} = A_{i,j_0,l}$, provided that q_{jl} does exist. For each $1 \le j \le n$ denote by q_j the maximum (which is unique due to genericity of (6), see (16)) among existing $q_{jl},\ 1 \le l \le n$. One can verify that $q_j \in S_{u_j}$ (cf. the proof of Lemma 3). This determines the sets $S_{u_j},\ 1 \le j \le n$.

We claim that the sets $S_{u_j,l},\ 1 \le j, l \le n$ are determined in a unique way. Suppose the contrary. Then there exist two different permutations $p_1, p_2 \in \mathrm{Sym}(n)$ such that

$$A_{i,j_0,l} = A_{p_1(l),l} + q_{p_1(l)} - k_{p_1(l)} = A_{p_2(l),l} + q_{p_2(l)} - k_{p_2(l)},\ 1 \le l \le n$$

(see (17)). Summing up over $1 \le l \le n$ the second and the third parts of the latter equalities, we arrive to a contradiction with genericity of (6), see (15).

Relying on Lemma 5 one can bound from above the number of minimal solutions of a generic regular system (6) by

$$\sum_{d_1+\cdots+d_n\leq n} \binom{k_1+d_1-1}{d_1} \cdots \binom{k_n+d_n-1}{d_n}. \tag{63}$$

Unlike the case $n=2$ (see Corollary 4) the gap between the obtained lower bound on the maximal possible number of minimal solutions (see Theorem 9 and Remark 11) and the upper bound is quite big. It would be interesting to diminish this gap.

6 Inversions of families of permutations and an upper bound on the number of minimal solutions of a generic system

In this section we improve the upper bound (63) on the number of minimal solutions of a generic regular system (6). First we extend the concept of an inversion to families of permutations and prove an upper bound on the number of inversions. We view a permutation from $\mathrm{Sym}(r)$ as a bijection of the set $[r]$ with itself.

Definition 14 For $w \in \mathrm{Sym}(r)$ and an n-tuple $1 \leq i_1, \dots, i_n \leq r$, denote

$$m_w(i_1,\dots,i_n) := w^{-1}(\min\{w(i_1),\dots,w(i_n)\}).$$

We say that $\{i_1,\dots,i_n\}$ is an **inversion** of a family of permutations $w_1,\dots,w_n \in \mathrm{Sym}(r)$ if integers $m_{w_1}(i_1,\dots,i_n),\dots,m_{w_n}(i_1,\dots,i_n)$ are pairwise distinct. Clearly, in this case it holds $\{m_{w_1}(i_1,\dots,i_n),\dots,m_{w_n}(i_1,\dots,i_n)\} = \{i_1,\dots,i_n\}$.

The concept of inversions of a family of permutations generalizes the definition of an inversion of a permutation w which coincides with inversions of a pair of permutations e, w where e denotes the identical permutation.

Theorem 10 *Consider $n \geq 2$. The number of inversions of a family $w_1,\dots,w_n \in \mathrm{Sym}(r)$ is bounded by*

$$\frac{2r^n}{n \cdot n!} + O(r^{n-1})$$

for a fixed n and growing r.

Remark 12 i) For $n=2$ it is well known that the maximal number of inversions equals $r(r-1)/2$.

ii) An obvious bound on the number of inversions is

$$\binom{r}{n} \leq \frac{r^n}{n!} + O(r^{n-1}).$$

Thereby, Theorem 10 improves the obvious bound asymptotically in $n/2$ times.

Lemma 6 *Consider $n \geq 3$. For a family $w_1, \dots, w_n \in \mathrm{Sym}(l)$ there exists $1 \leq j \leq n$ such that the family $\{w_1, \dots, w_n\} \setminus \{w_j\}$ has at most of*

$$\frac{2l^{n-1}}{n!} + O(l^{n-2})$$

inversions.

Proof of Lemma 6. If $1 \leq i_1, \dots, i_{n-1} \leq l$ is an inversion of a family $\{w_1, \dots, w_n\} \setminus \{w_j\}$ then among n integers $m_{w_1}(i_1, \dots, i_{n-1}), \dots, m_{w_n}(i_1, \dots, i_{n-1})$ there are $n-1$ pairwise distinct coinciding with $i_1, \dots, i_{n-1}$. Therefore there exist exactly two integers $1 \leq j \neq j_0 \leq n$ such that $\{i_1, \dots, i_{n-1}\}$ is an inversion of the family $\{w_1, \dots, w_n\} \setminus \{w_j\}$ and of the family $\{w_1, \dots, w_n\} \setminus \{w_{j_0}\}$ as well. In other words, $m_{w_j}(i_1, \dots, i_{n-1}) = m_{w_{j_0}}(i_1, \dots, i_{n-1})$.

This completes the proof of the lemma taking into account that there are

$$\frac{l^{n-1}}{(n-1)!} + O(l^{n-2})$$

$(n-1)$-tuples of the form $1 \leq i_1, \dots, i_{n-1} \leq l$ and that there are exactly n subsets of the size $n-1$ of the set $\{w_1, \dots, w_n\}$.

Proof of Theorem 10. We proceed by induction on r. W.l.o.g. one can assume that $n \geq 3$. Denote $i_j := w_j^{-1}(1),\ 1 \leq j \leq n$. If $i_1, \dots, i_n$ are not pairwise distinct, say $i_1 = i_2$, then there are no inversions which contain i_1. Thus, w.l.o.g. one can assume that $i_1, \dots, i_n$ are pairwise distinct.

There is a unique bijection

$$R := R_{i_1, \dots, i_n} : \{1, \dots, r-n\} \xrightarrow{\sim} \{1, \dots, r\} \setminus \{i_1, \dots, i_n\}$$

which preserves the order. For any permutation $w \in \mathrm{Sym}(r)$ denote

$$\overline{w} := R^{-1} \circ w \circ R_{i_1, \dots, i_n} \in \mathrm{Sym}(r-n)$$

(note that R^{-1} is defined for any triple $\{i'_1, \dots, i'_n\}$ in place of $\{i_1, \dots, i_n\}$). Apply Lemma 6 to the family $\{\overline{w_1}, \dots, \overline{w_n}\}$ and suppose for definiteness that the family $\{\overline{w_1}, \dots, \overline{w_{n-1}}\}$ has at most of

$$\frac{2(r-n)^{n-1}}{n!} + O((r-n)^{n-2}) \tag{64}$$

inversions.

Denote by $I(r)$ the maximal number of inversions in n-families from $\mathrm{Sym}(r)$. The number of inversions $\{i'_1, \dots, i'_n\}$ of the family $w_1, \dots, w_n$ does not exceed

- $I(r-1)$ if $i_n \notin \{i'_1, \dots, i'_n\}$;
- (64) if $i_n \in \{i'_1, \dots, i'_n\}$, $\{i_1, \dots, i_{n-1}\} \cap \{i'_1, \dots, i'_n\} = \emptyset$;
- $O(r^{n-2})$ if $i_n \in \{i'_1, \dots, i'_n\}$, $\{i_1, \dots, i_{n-1}\} \cap \{i'_1, \dots, i'_n\} \neq \emptyset$.

Thus, we get an inductive inequality

$$I(r) \leq I(r-1) + \frac{2(r-n)^{n-1}}{n!} + O(r^{n-2}),$$

which completes the proof of the Theorem.

Theorem 11 *A generic regular system (6) has at most of*

$$\frac{2(k_1 + \cdots + k_n)^n}{n \cdot n!} + O((k_1 + \cdots + k_n)^{n-1})$$

minimal solutions for a fixed n *and growing* $k_1 + \cdots + k_n$.

Proof We define the following permutations $w_1, \ldots, w_n \in \mathrm{Sym}(k_1 + \cdots + k_n)$. The permutation $w_l, 1 \leq l \leq n$ corresponds to the sequence (18) in the increasing order. In other words, $w_l(j)$ equals the place in the increasing order of j-th element of the sequence (18). Note that the elements of the sequence (18) are pairwise distinct because of regularity of the system (6).

Observe that for any minimal solution of a configuration $(S_{u_1}, \ldots, S_{u_n})$ such that

$$\star \in S_{u_1} \cap \cdots \cap S_{u_n}, \sum_{j=1}^{n} |S_{u_j}| = 2n, \{i_1, \ldots, i_n\} = S_{u_1} \cup \cdots \cup S_{u_n} \setminus \{\star\} \quad (65)$$

it holds that $\{i_1, \ldots, i_n\}$ is an inversion of the family of permutations $w_1, \ldots, w_n$ (cf. (25), (33)).

Thus, the number of minimal solutions of a configuration (65) is bounded by

$$\frac{2(k_1 + \cdots + k_n)^n}{n \cdot n!} + O((k_1 + \cdots + k_n)^{n-1})$$

due to Theorem 10 taking into account Lemma 5. The number of minimal solutions of a configuration $(S_{u_1}, \ldots, S_{u_n})$ such that

$$\star \in S_{u_1} \cap \cdots \cap S_{u_n}, \ |S_{u_1}| + \cdots + |S_{u_n}| \leq 2n - 1$$

is less than $O((k_1 + \cdots + k_n)^{n-1})$. This completes the proof of the theorem.

Remark 13 i) The bound from Theorem 11 improves the upper bound (63) asymptotically in $n/2$ times.

ii) In case $n = 3$ and $k := k_1 = k_2 = k_3$ we obtain a lower bound $5k^3/2 - O(k^2)$ (Theorem 9) on the number of minimal solutions of a generic regular system (6), and the upper bound $3k^3 + O(k^2)$ (Theorem 11).

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