UPPER BOUNDS ON BETTI NUMBERS OF TROPICAL PREVARIETIES

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Abstract. We prove upper bounds on the sum of Betti numbers of tropical prevarieties in dense and sparse settings. In the dense setting the bound is in terms of the volume of Minkowski sum of Newton polytopes of defining tropical polynomials, or, alternatively, via the maximal degree of these polynomials. In sparse setting, the bound involves the number of the monomials.

1. Introduction

In this paper we are concerned with upper bounds on Betti numbers of tropical prevarieties. Basic definitions and statements regarding tropical algebra can be found in [12, 14].

Each *tropical polynomial* f in n variables can be represented as $\min\{L_1, \dots, L_m\}$, where $L_1, \dots, L_m$ are linear functions on $\mathbb{R}^n$ (called *tropical monomials*) with non-negative integer coefficients at variables. For a monomial L_j its *degree* is the sum of these integer coefficients, the maximum among the degrees of the monomials is called the *(tropical) degree* of f. With any tropical polynomial f we associate a concave piece-wise linear function

$$L(\mathbf{x}) := \min_{1 \le j \le m} \{L_j(\mathbf{x})\}$$

defined on $\mathbb{R}^n$. A *tropical hypersurface* $V := V(f) \subset \mathbb{R}^n$ is the set of all points in $\mathbb{R}^n$ at which $L(\mathbf{x})$ is not smooth. Any point $\mathbf{x} \in V$ is called a *zero* of f. A *tropical prevariety* is an intersection of a finite number of tropical hypersurfaces, in other words, the set of all common tropical zeroes of a finite system of multivariate tropical polynomials.

Let $V := V(f_1, \dots, f_k) \subset \mathbb{R}^n$ be the tropical prevariety defined by a system $f_1, \dots, f_k$ of tropical polynomials in n variables of degrees not exceeding d. In case $k = n$, the Tropical Bezout Theorem [14] states that the number of all *stable* tropical zeroes (counted with multiplicities) of the system $f_1, \dots, f_k$ equals the product of the degrees of its polynomials. For arbitrary k, the upper bound

$$\binom{k+7n}{3n} d^{3n}$$

on the number of connected components of V was obtained in [5]. In case $k \le n$, the number of maximal faces of transversal intersections of tropical hypersurfaces defined by polynomials f_i, $1 \le i \le k$, was expressed in [1, 15] in terms of mixed Minkowski volumes of Newton polytopes of polynomials f_i. In [2], for $k = n$, this number was bounded from above using the new concept of a *discrete mixed volume* for sparse tropical polynomials.

The structure of this paper is as follows.

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In Section 1 we prove the upper bound

$$2^{2(n+1)} n!\ \mathrm{Vol}_n(P_1 + \cdots + P_k)$$

on the sum of Betti numbers of a tropical prevariety V via the n-dimensional volume of Minkowski sum of Newton polytopes $P_1, \dots, P_k$ of tropical polynomials $f_1, \dots, f_k$. In terms of the $\max_i \deg f_i = d$ this implies the bound

$$2^{2(n+1)} (kd)^n. \tag{1.1}$$

Note that in [5] a naive upper bound

$$\left(\binom{k+7n}{3n} d^{3n} \right)^n$$

was mentioned. Also in [5], an example is constructed of a prevariety defined by kn polynomials of degrees at most d, containing $(kd/4)^n$ zero-dimensional connected components. Comparing this lower bound with (1.1), we see a gap within a factor n^n. An interesting challenge is to close this gap.

In Section 2 we assume that each tropical polynomial f_i, $1 \le i \le k$ is m-sparse, i.e., consists of at most m monomials. In this setting we prove the upper bound

$$n 2^{2n+1} \binom{k\binom{m}{2}}{n}$$

on the sum of Betti numbers of V. We give an example of a prevariety, with $k = n$, for which this bound is close to sharp up to the factor m^n.

Results in Section 1 can be related to the classical upper bounds on Betti numbers of real algebraic and semi-algebraic sets obtained by Petrovskii, Oleinik, Milnor and Thom, and developed further by various authors. In particular, Milnor [13] proved that if a semi-algebraic set $X \subset \mathbb{R}^n$ is defined by a system of k non-strict polynomial inequalities of degrees less than d, then $\mathrm{b}(X) \le (ckd)^n$ for an absolute constant $c > 0$ (compare with (1.1)). Note that in the case when X is defined by an arbitrary Boolean combination of inequalities, there is a bound $\mathrm{b}_i(X) \le (c\nu kd)^n$ for $\nu = \min\{i+1, n-i, k\}$ and an absolute constant $c > 0$ [7].

The bound in Section 2 can be viewed as a tropical counterpart of the bounds on Betti numbers for fewnomials [11, 8, 3].

2. Betti numbers for dense tropical polynomials

Recall the notation $V := V(f_1, \dots, f_k)$ for a tropical prevariety in $\mathbb{R}^n$ of all common tropical zeroes of tropical polynomials $f_1, \dots, f_k$. Then V is a finite *polyhedral fan* [4] (in Lemmas 3.1, 3.3 below, we provide an explicit representation of V as a fan).

Let $P_i \subset \mathbb{R}^n$, $1 \le i \le k$, be the Newton polytope of f_i. Let $\mathrm{Vol}_n(P_1 + \cdots + P_k)$ denote the n-dimensional volume of the Minkowski sum of polytopes. Without loss of generality we assume that this volume is positive.

Theorem 2.1. *The number of faces of all dimensions of V does not exceed*

$$(2^{n+1} - 1) n!\ \mathrm{Vol}_n(P_1 + \cdots + P_k).$$

Before proving this theorem, let us extract some corollaries regarding Betti numbers.

Semi-algebraic local triviality [10] implies that for any sufficiently large positive $r \in \mathbb{R}$ the intersection $W := V \cap S_r$, where the simplex

$$S_r = \{(x_1, \dots, x_n) \in \mathbb{R}^n |\, x_j \geq -r,\ 1 \leq j \leq n,\ x_1 + \cdots + x_n \leq r\},$$

is homotopy equivalent to V.

Since V is a polyhedral fan, the set W has a natural structure of a finite polyhedral (and hence CW) complex. If $\varphi(V)$ is the number of all faces of V, then the number $\varphi(W)$ of all faces of W does not exceed $(2^{n+1} - 1)\varphi(V)$ because S has $2^{n+1} - 1$ faces. Therefore, Theorem 2.1 implies the bound

$$\varphi(W) \leq (2^{n+1} - 1)^2 n!\ \mathrm{Vol}_n(P_1 + \cdots + P_k).$$

We will use notations $\mathrm{b}_\ell(X) := \mathrm{rank}\ H_\ell(X, \mathbb{R})$, where $H_\ell(X, \mathbb{R})$ is a singular ℓth homology group, and

$$\mathrm{b}(X) := \sum_{0 \leq \ell \leq \dim X} \mathrm{b}_\ell(X).$$

Recalling that $\mathrm{b}(V) = \mathrm{b}(W)$, and applying to the CW complex W the Week Morse Inequality $\mathrm{b}(W) \leq \varphi(W)$ [6, Corollary 3.7], we deduce the following upper bound from Theorem 2.1.

Corollary 2.2. *The sum of Betti numbers of V satisfies the inequality*

$$\mathrm{b}(V) \leq (2^{n+1} - 1)^2 n!\ \mathrm{Vol}_n(P_1 + \cdots + P_k). \tag{2.1}$$

Let $d = \max_{1 \leq i \leq k} \deg f_i$. Then each P_i is contained in the simplex

$$\{(x_1, \dots, x_n) \in \mathbb{R}^n |\, x_j \geq 0,\ 1 \leq j \leq n,\ x_1 + \cdots + x_n \leq d\}.$$

Hence, in terms of d, the inequality (2.1) can be presented in the following form.

Corollary 2.3. *The sum of Betti numbers of V satisfies the inequality*

$$\mathrm{b}(V) \leq (2^{n+1} - 1)^2 (kd)^n.$$

Proof of Theorem 2.1. Let $Q_i \subset \mathbb{R}^{n+1}$, $1 \leq i \leq k$ be the *extended* Newton polytope of f_i [1, 14, 15] . Let Q be the *top* of $Q_1 + \cdots + Q_k$, which is the set of all points $(\mathbf{x}, a) \in Q_1 + \cdots + Q_k$ such that there are no points $(\mathbf{x}, b) \in Q_1 + \cdots + Q_k$ with $b > a$. Note that for the projection map $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ along the last coordinate we have $\pi(Q) = \pi(Q_1 + \cdots + Q_k) = P_1 + \cdots + P_k$.

Let F be a face of Q. Its *dual*, $G(F)$, is the set of all supporting hyperplanes for Q (subset of the set of all suporting hyperplanes for $Q_1 + \cdots + Q_k$) such that their intersections with Q coincide with F. Then $G(F)$ can be identified with a face of the dual polytope to $Q_1 + \cdots + Q_k$, and we have $\dim F + \dim G(F) = n$ (see, e.g., [2, 1, 15]). Observe that F is representable as a Minkowski sum $F = F_1 + \cdots + F_k$, where each F_i is a face of Q_i such that any $H \in G(F)$ is a supporting hyperplane for Q and $H \cap Q_i = F_i$. We say that a face F of Q is *tropical* if $\dim F_i \geq 1$ for all $1 \leq i \leq k$. Then V coincides with the union of polytopes $G(F)$ for all tropical faces F (cf. [1, 15]).

Decompose each n-dimensional face of Q into n-dimensional closed simplices without adding new vertices. The number of all subsimplices of these simplices is not less than the total number of faces of V. Since each n-dimensional simplex S in the decomposition has integer vertices, we have $\mathrm{Vol}_n(\pi(S)) \geq 1/n!$. Therefore, the number of all n-dimensional simplices in the decomposition does not exceed $n!\ \mathrm{Vol}_n(P_1 + \cdots + P_k)$. To complete the proof, it remains to notice that the number of all subsimplices of an n-dimensionl simplex is $2^{n+1} - 1$. $\square$

3. Betti numbers for sparse tropical polynomials

In this section we assume that each tropical polynomial f_i, $1 \le i \le k$ is m-sparse, i.e., contains at most m monomials. In other words, each f_i can be represented as $\min\{L_{i,1}, \dots, L_{i,m}\}$, where $L_{i,1}, \dots, L_{i,m}$ are linear functions on $\mathbb{R}^n$. Though, by definition, coefficients in $L_{i,j}$ at variables are non-negative integers, all results in this section hold for arbitrary real coefficients.

Following [9], for any subset B of $D := \{(i,j)|\, 1 \le i \le k,\ 1 \le j \le m\}$, consider the polyhedron U_B consisting of all points $\mathbf{x} \in \mathbb{R}^n$ such that

$$\min_{1 \le j \le m} \{L_{i,j}(\mathbf{x})\} = L_{i,j_0}(\mathbf{x}) \text{ for every } (i, j_0) \in B$$

and

$$\min_{1 \le j \le m} \{L_{i,j}(\mathbf{x})\} < L_{i,j_1}(\mathbf{x}) \text{ for every } (i, j_1) \notin B.$$

Note that each set U_B is open in its linear hull, and $U_B \subset V$ if and only if for each $1 \le i \le k$ there exist $1 \le j_0 < j_1 \le m$ such that $(i, j_0), (i, j_1) \in B$.

Denote by $\mathcal{L}(X)$ the linear hull of a set $X \subset \mathbb{R}^n$.

Lemma 3.1. *The interior of every face F of U_B in $\mathcal{L}(F)$ coincides with U_{B_1} for a suitable subset $B_1 \subset D$ such that $B \subsetneqq B_1$.*

Proof. There exists a subset $B_1 \subset D$ such that $\dim(U_{B_1} \cap F) = \dim(F)$. We prove that B_1 satisfies the requirements of the lemma. Observe that U_{B_1} is contained in $\mathcal{L}(F)$ of F. The condition $\min_j\{L_{i,j}(\mathbf{x})\} = L_{i,j_0}(\mathbf{x})$ for every $(i, j_0) \in B$ and each $\mathbf{x} \in \mathcal{L}(U_B)$ implies that $(i, j_0) \in B_1$. Hence, $B \subset B_1$, and obviously $B \subsetneqq B_1$.

By [5, Theorem 4.4], for any two points in U_{B_1}, their sufficiently small neighbourhoods are homeomorphic (in fact, isomorphic) by a linear translation from one point to another. Therefore, U_{B_1} is contained in the interior of F in $\mathcal{L}(F)$. It remains to show that, conversely, U_{B_1} contains the interior of F.

For contradiction, assume that there exists a point $\mathbf{x}$ in the interior of F such that $\mathbf{x} \in \overline{U_{B_1}} \cap U_{B_2}$ for some subset $B_2 \subset D$ different from B_1. Choose any $(i, j_2) \in B_2 \setminus B_1$, and $(i, j_0) \in B$ such that $\min_j\{L_{i,j}(\mathbf{y})\} = L_{i,j_0}(\mathbf{y})$ for every $\mathbf{y} \in \mathcal{L}(U_B)$. It follows that $\min_j\{L_{i,j}(\mathbf{y})\} = L_{i,j_2}(\mathbf{y})$ for every $\mathbf{y} \in \mathcal{L}(F)$. We get a contradiction with the assumption that $(i, j_2) \notin B_1$, hence U_{B_1} contains the interior of F in $\mathcal{L}(F)$. □

Remark 3.2. Being a polyhedron, the set $\overline{U_B}$ coincides with

$$\{\mathbf{x} \in \mathbb{R}^n|\ \min_j\{L_{i,j}(\mathbf{x})\} = L_{i,j_0}(\mathbf{x}),\ 1 \le i \le k \text{ for every } (i, j_0) \in B\}.$$

Lemma 3.3. *For any subsets $B_1, B_2 \subset D$ there exists a subset $B \subset D$ such that $B \supset (B_1 \cup B_2)$ and $\overline{U_B} = \overline{U_{B_1}} \cap \overline{U_{B_2}}$.*

Proof. For any $(i, j_0) \in B_1 \cup B_2$, $1 \le i \le k$ and any $\mathbf{x} \in \overline{U_{B_1}} \cap \overline{U_{B_2}}$ we have $\min_j\{L_{i,j}(\mathbf{x})\} = L_{i,j_0}(\mathbf{x})$.

Define B as the set of all $(i, j_1) \in D$ such that $\min_j\{L_{i,j}(\mathbf{x})\} = L_{i,j_1}(\mathbf{x})$ for every $\mathbf{x} \in \overline{U_{B_1}} \cap \overline{U_{B_2}}$. Hence, $B \supset (B_1 \cup B_2)$. It remains to prove that $\overline{U_B} = \overline{U_{B_1}} \cap \overline{U_{B_2}}$.

The inclusion $\overline{U_B} \subset (\overline{U_{B_1}} \cap \overline{U_{B_2}})$ follows from Remark 3.2. Conversely, since $\overline{U_{B_1}} \cap \overline{U_{B_2}}$ is a closed convex polyhedron, for every $(i, j_2) \in D \setminus B$ the set of all $\mathbf{x} \in \overline{U_{B_1}} \cap \overline{U_{B_2}}$ such that $\min_j\{L_{i,j}(\mathbf{x})\} < L_{i,j_2}(\mathbf{x})$ contains the interior of $\overline{U_{B_1}} \cap \overline{U_{B_2}}$. Hence, U_B also contains this interior. It follows that $(\overline{U_{B_1}} \cap \overline{U_{B_2}}) \subset \overline{U_B}$. □

Similar to [5], consider an arrangement $\mathcal{A}$ in $\mathbb{R}^n$ consisting of at most $\ell := k\binom{m}{2}$ hyperplanes of the form $L_{i,j_1} = L_{i,j_2}$ for all $1 \le i \le k$, $1 \le j_1 < j_2 \le m$. Without loss of generality, assume that $n \le \ell$.

Observe that for every $B \subset D$ the set U_B is a face of $\mathcal{A}$ whenever $U_B \neq \emptyset$. Then Lemmas 3.1 and 3.3 imply that the number of faces $\varphi(V)$ of V does not exceed the number of faces $\varphi(\mathcal{A})$ of $\mathcal{A}$. According to [16], $\varphi(\mathcal{A}) \le n2^n\binom{\ell}{n}$, thus $\varphi(V) \le n2^n\binom{\ell}{n}$. We proved the following theorem.

Theorem 3.4. *The number of all faces of a tropical prevariety $V \subset \mathbb{R}^n$ defined by k m-sparse tropical polynomials is at most*

$$n2^n\binom{k\binom{m}{2}}{n}.$$

Using the same compactification argument as in the proof of Corollary 2.2, we obtain the following corollary.

Corollary 3.5. *The sum of Betti numbers of V satisfies the inequality*

$$\mathrm{b}(V) \le n(2^{n+1}-1)2^n\binom{k\binom{m}{2}}{n}.$$

In conclusion, we construct an example of a tropical prevariety with $k = n$ which shows that upper bounds in Theorem 3.4 and Corollary 3.5 differ from a lower bound up to a factor m^n.

Let f_i, $1 \le i \le n$ be a tropical polynomial in one variable X_i, of degree m with m tropical zeroes. Then the tropical prevariety defined by the system $f_1, \dots, f_n$ consists of exactly m^n isolated points.

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