Orthogonal tropical linear prevarieties

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Abstract. We study the operation $A^{\perp}$ of tropical orthogonalization, applied to a subset A of a vector space $(\mathbb{R} \cup \{\infty\})^n$, and iterations of this operation. Main results include a criterion and an algorithm, deciding whether a tropical linear *prevariety* is a tropical linear *variety*, formulated in terms of a duality between $A^{\perp}$ and $A^{\perp\perp}$. We give an example of a countable family of tropical hyperplanes such that their intersection is not a tropical prevariety.

Keywords: tropical linear prevarieties, tropical linear varieties, orthogonalization.

Introduction

We study some aspects of tropical linear prevarieties and varieties. General concepts of tropical algebra can be found in [16], [17]. Specific questions of tropical linear algebra were considered in [5], [6], [18], [19].

We introduce the operation $A^{\perp}$ of tropical orthogonalization, applied to a subset A of a vector space $(\mathbb{R} \cup \{\infty\})^n$. The special interest to us presents an interplay between the tropical linear prevariety $A^{\perp}$ and its orthogonalization $A^{\perp\perp}$. Our main results include a criterion and a deciding algorithm for a tropical linear prevariety to be a tropical linear *variety*, formulated in terms of a duality between $A^{\perp}$ and $A^{\perp\perp}$. In this note we present our results without proofs which will appear elsewhere.

In Section 1 we list basic definitions, including the concept of a *tropical hull* of a subset in $(\mathbb{R} \cup \{\infty\})^n$. We recall a fundamental theorem, proved in [2], [8], stating that any tropical linear prevariety is the tropical hull of a finite set of vectors. We give an example, showing that the restriction of a tropical linear prevariety to $\mathbb{R}^n$ may not be representable in this way. We describe a simple algorithm, with polynomial complexity, testing the membership of a vector in a tropical hull.

In Section 2 we study some properties of double orthogonalization. In particular, we state that $A^{\perp\perp}$ is the minimal tropical linear prevariety containing a finite set A, and that dimensions of tropical hulls of A and of $A^{\perp\perp}$ coincide.

In Section 3 we preset a theorem stating that, given two mutually complementary and orthogonal linear subspaces P and Q of the vector space $(\mathbb{C}((t^{1/\infty})))^n$ over Puiseux series, there exists a finite $A \subset (\mathbb{R} \cup \{\infty\})^n$ such that tropicalizations of P and Q coincide with $A^\perp$ and $A^{\perp\perp}$ respectively.

Section 4 contains a criterion and a deciding algorithm for a tropical linear prevariety to be a tropical linear variety. The algorithm has a doubly exponential complexity. We also give a brief description of algorithms which for a given tropical linear variety $A^\perp$ produces a linear subspace P whose tropicalization coincides with $A^\perp$.

Finally, in Section 5 we give an example of a countable family of tropical hyperplanes in $(\mathbb{R} \cup \{\infty\})^6$ such that their intersection is not a tropical prevariety. This strengthens examples in [7] (example of T. Theobald) and [12] about countable intersections of non-linear tropical hypersurfaces.

1 Tropical hull

We use the notation $\mathbb{R}_\infty$ for $\mathbb{R} \cup \{\infty\}$. We assume that for all $a \in \mathbb{R}$ the rules $a < \infty$, $a + \infty = \infty$, $\infty + \infty = \infty$, and, for positive a, $a \cdot \infty = \infty$ hold. The element ∞ is a "tropical zero", being the neutral element with respect to taking minimum.

Definition 1. *A* tropical hyperplane *in $\mathbb{R}^n_\infty$ is the set of all points $(x_1, \dots, x_n) \in \mathbb{R}^n_\infty$ at which a set $\{x_1 + a_1, \dots, x_n + a_n\}$, where $a_i \in \mathbb{R}_\infty,\ 1 \le i \le n$, has at least two minimal elements. A* tropical linear prevariety *in $\mathbb{R}^n_\infty$ is the intersection of a finite number of tropical hyperplanes.*

Remark 1. The point $(\infty, \dots, \infty)$ belongs to every tropical linear prevariety. A tropical hyperplane according to Definition 1 corresponds to the notion of a codimension one linear subspace in classical linear algebra. It can be identified with a special case, when $a_{n+1} = \infty$, of a more general notion of a tropical hyperplane, defined as a set of all points $(x_1, \dots, x_n) \in \mathbb{R}^n_\infty$ at which a set $\{x_1 + a_1, \dots, x_n + a_n, a_{n+1}\}$, where $a_i \in \mathbb{R}_\infty,\ 1 \le i \le n+1$, has at least two minimal elements.

Definition 2. *Vectors $\mathbf{v} = (v_1, \dots, v_n),\ \mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}^n_\infty$ are called* tropically orthogonal *if among numbers $v_i + a_i,\ 1 \le i \le n$ there are at least two minimal. Note that $(\infty, \dots, \infty)$ is orthogonal to every vector $\mathbf{a} \in \mathbb{R}^n_\infty$. For a set of vectors $A = \{\mathbf{a}_1, \dots, \mathbf{a}_k\} \subset \mathbb{R}^n_\infty$ denote by $A^\perp$ the set of all vectors in $\mathbb{R}^n_\infty$ orthogonal to each $\mathbf{a}_i,\ 1 \le i \le k$.*

It is clear that $\{\mathbf{a}\}^\perp$ is a tropical hyperplane for a vector $\mathbf{a} \in \mathbb{R}^n_\infty$, while $A^\perp$ is a tropical linear prevariety when $A \subset \mathbb{R}^n_\infty$ is finite. Conversely, every tropical linear prevariety in $\mathbb{R}^n_\infty$ coincides with $A^\perp$ for a suitable finite set of vectors $A \subset \mathbb{R}^n_\infty$.

Definition 3. *For a finite set of vectors* $A = \{\mathbf{a}_1, \dots, \mathbf{a}_k\} \subset \mathbb{R}^n_\infty$ *define its* tropical hull $\mathrm{Trophull}(A)$ *as the set of all vectors in* $\mathbb{R}^n_\infty$ *of the kind*

$$\min_{1 \le i \le k} \{t_i \mathbf{1}_n + \mathbf{a}_i\},$$

where $t_1, \dots, t_k$ *are arbitrary elements in* $\mathbb{R}_\infty$, $\min_{1 \le i \le k}$ *denotes the component-wise minimum of a set of vectors, and* $\mathbf{1}_n$ *is the unit vector in* $\mathbb{R}^n$. *For an arbitrary subset* $X \subset \mathbb{R}^n_\infty$ *define* $\mathrm{Trophull}(X)$ *as the union of sets* $\mathrm{Trophull}(A)$ *over all finite subsets* $A \subset X$.

Note that $\mathrm{Trophull}(A)$ always contains the point $(\infty, \dots, \infty) \in \mathbb{R}^n_\infty$, because all t_i, $1 \le i \le k$ can be chosen to be ∞.

Lemma 1 ([5], [6]). *If* B *finite and* $B \subset A^\perp$, *then* $\mathrm{Trophull}(B) \subset A^\perp$.

Definition 4. *For every partition* $\{i_1, \dots, i_p\} \cup \{i_{p+1}, \dots, i_n\}$ *of* $\{1, \dots, n\}$ *a* chart *is an open convex polyhedron*

$$C_{i_1,\dots,i_p} := \{x_{i_1} = \dots = x_{i_p} = \infty\} \cap \{x_{i_{p+1}} < \infty\} \cap \dots \cap \{x_{i_n} < \infty\} \subset \mathbb{R}^n_\infty.$$

Clearly, $\mathbb{R}^n_\infty$ is the union of all 2^n pair-wise disjoint charts.

One can extend the standard concepts of a convex polyhedron and a finite polyhedral complex to the case of the subsets of the space $\mathbb{R}^n_\infty$ (see [8]). Restriction of a convex polyhedron $P \subset \mathbb{R}^n_\infty$ to a chart $C_{i_1,\dots,i_p}$ coincides with a usual convex polyhedron in $\mathbb{R}^{n-p}$ translated by a vector in $\{0, \infty\}^{n-p}$ with ∞ in positions $i_1, \dots, i_p$. Hence, P is a finite union of translated usual convex polyhedra, and we define the dimension $\dim(P)$ as the maximum of the dimensions of restrictions of P to all charts. The dimension of a finite polyhedral complex is defined as the maximum of dimensions of its convex polyhedra.

The following theorem directly follows from [8, Theorem 1] (part (2) of the theorem was proved earlier in [2, Proposition 2]).

Let $A = \{\mathbf{a}_1, \dots, \mathbf{a}_k\} \subset \mathbb{R}^n_\infty$ be a set of vectors.

Theorem 1 ([8], [2]).

1. *The set* $\mathrm{Trophull}(A)$ *is a union of all convex polyhedra of a polyhedral complex in* $\mathbb{R}^n_\infty$.
2. *For any tropical linear prevariety* $A^\perp \subset \mathbb{R}^n_\infty$ *there exists a finite set of vectors* $\{\mathbf{b}_1, \dots, \mathbf{b}_N\} \subset \mathbb{R}^n_\infty$ *such that* $A^\perp = \mathrm{Trophull}(\{\mathbf{b}_1, \dots, \mathbf{b}_N\})$.

Corollary 1. *Any tropical linear prevariety* $A^\perp \subset \mathbb{R}^n_\infty$ *is a union of all convex polyhedra of a polyhedral complex in* $\mathbb{R}^n_\infty$.

Remark 2. There is an algorithm, with polynomial complexity, which for a given set $A = \{\mathbf{a}_1, \dots, \mathbf{a}_k\} \subset \mathbb{R}^n_\infty$ and a vector $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n_\infty$ tests the inclusion $\mathbf{x} \in \mathrm{Trophull}(A)$. We assume bit complexity, if all input vectors are in $(\mathbb{Z} \cup \{\infty\})^n$, or the complexity of BSS model, if the input vectors are in $\mathbb{R}^n_\infty$. Let, for definiteness, input vectors be in $\mathbb{R}^n_\infty$.

The algorithm attempts to find a vector $(t_1,\dots,t_k)\in\mathbb{R}_\infty^k$ such that

$$\mathbf{x}=\min_{1\le i\le k}\{t_i\mathbf{1}_n+\mathbf{a}_i\},\qquad (1)$$

where minimum is component-wise.

Fix i, $1\le i\le k$. If $\mathbf{a}_i=(\infty,\dots,\infty)$, then choose $t_i\in\mathbb{R}_\infty^k$ arbitrarily. Otherwise, let, for definiteness, elements $a_{i1},\dots,a_{is}$ be all different from ∞ for $1\le s\le n$. Choose $t_i\in\mathbb{R}_\infty$ as $\max_{1\le j\le s}(x_j-a_{ij})$ (note that t_i will turn out to be ∞ if at least one of $x_j=\infty$, where $1\le j\le s$). Chosen t_i is the minimal such that $t_i(1,\dots,1)+\mathbf{a}_i\ge\mathbf{x}$, where the inequality is component-wise.

Repeating the procedure for each i, $1\le i\le k$, we obtain the vector $(t_1,\dots,t_k)$. The vector $\mathbf{x}\in\mathrm{Trophull}(A)$ iff the equality (1) takes place.

Theorem 1, (2) states that any linear tropical prevariety $A^\perp=\{\mathbf{a}_1,\dots,\mathbf{a}_n\}^\perp\subset\mathbb{R}_\infty^n$ coincides with the tropical hull of a finite subset of it's vectors. The following example shows that this fact is not necessarily true for the restriction $A^\perp\cap\mathbb{R}^n$.

Example 1 (cf. [11]). Let $A_0=\{\mathbf{a}_1,\dots,\mathbf{a}_{n-1}\}\subset\mathbb{R}^n$, where

$$\mathbf{a}_i=(\underbrace{1,\dots,1,0}_{i},1,\dots,1,0,0)\text{ for }1\le i\le n-2\text{ and }\mathbf{a}_{n-1}=(1,\dots,1,0,0).$$

It is easy to see that every vector $\mathbf{x}=(x_1,\dots,x_n)\in A_0^\perp\subset\mathbb{R}_\infty^n$ should have minimal elements x_{n-1},x_n, and, conversely, every vector $\mathbf{x}$ with minimal elements x_{n-1},x_n is in $A_0^\perp$. Therefore,

$$A_0^\perp=\{t\mathbf{1}_n+(c_1,\dots,c_{n-2},0,0)|\text{ for all }0\le c_i\in\mathbb{R}_\infty\text{ and }t\in\mathbb{R}_\infty\}.$$

Directly from definitions it follows that $A_0^\perp=\mathrm{Trophull}(\{\mathbf{b}_1,\dots,\mathbf{b}_{n-1}\})$, where

$$\mathbf{b}_i=(\underbrace{\infty,\dots,\infty,0}_{i},\infty,\dots,\infty,0,0)\text{ for }1\le i\le n-2\text{ and }\mathbf{b}_{n-1}=(\infty,\dots,\infty,0,0).$$

On the other hand, for restrictions $A_0^\perp\cap\mathbb{R}^n$, such a representation, as a tropical hull, is generally not true already when $n=3$.

Proposition 1. *For any finite set $\{\mathbf{v}_1,\dots,\mathbf{v}_N\}\subset A_0^\perp\cap\mathbb{R}^3$ we have*

$$A_0^\perp\cap\mathbb{R}^3\not\subset\mathrm{Trophull}(\{\mathbf{v}_1,\dots,\mathbf{v}_N\}).$$

2 Dual tropical linear prevarieties

We extend the operation $X^\perp$, introduced in Definition 2, so that it can be applied to arbitrary (not necessarily finite) subsets $X\subset\mathbb{R}_\infty^n$. Namely, denote by $X^\perp$ the set of all vectors in $\mathbb{R}_\infty^n$ orthogonal to each $\mathbf{a}\in X$. We will use notations $X^{\perp\perp}:=(X^\perp)^\perp$ and $X^{\perp\perp\perp}:=(X^{\perp\perp})^\perp$.

Remark 3. Observe that by the definition, for a finite subset $A \subset \mathbb{R}^n_\infty$, the set $A^{\perp\perp}$ is an intersection of an infinite number of tropical hyperplanes in $\mathbb{R}^n_\infty$. As we will show in Section 5 below, not every intersection of even countable number of tropical hyperplanes is a union of cells of a finite polyhedral complex, let alone linear tropical prevariety. However, in the special case of a finite A, the set $A^{\perp\perp}$ is a linear tropical prevariety (Proposition 2).

Lemma 2. *For any subset $X \subset \mathbb{R}^n_\infty$ we have:*

1. $\mathrm{Trophull}(X) \subset X^{\perp\perp}$*;*
2. $X^\perp = X^{\perp\perp\perp}$*.*

Proposition 2. *Let A be a finite set of vectors in $\mathbb{R}^n_\infty$. Then $A^{\perp\perp}$ is the minimal tropical linear prevariety containing A.*

For the proof of the next theorem we recall the following definition.

Let $A = \{\mathbf{a}_1, \dots, \mathbf{a}_k\}$. Since, by Proposition 2 and Corollary 1, both $A^{\perp\perp}$ and $\mathrm{Trophull}(A)$ have the structure of polyhedral complexes in $\mathbb{R}^n_\infty$, dimensions of these sets are defined. By Lemma 2, (1), $\dim(\mathrm{Trophull}(A)) \le \dim(A^{\perp\perp})$.

Theorem 2. $\dim(\mathrm{Trophull}(A)) = \dim(A^{\perp\perp})$.

Remark 4. Lemma 12 in [11] implies that $\mathrm{trk}(A) + \dim(A^\perp) \ge n$ for a finite $A \subset \mathbb{R}^n_\infty$.

Example 2. Consider the set of vectors $A_0 \subset \mathbb{R}^n$ from Example 1. Arguing as in that example, we see that every vector $\mathbf{y} = (y_1, \dots, y_n) \in A_0^{\perp\perp} \subset \mathbb{R}^n_\infty$ should have minimal elements y_{n-1}, y_n, and, conversely, every vector $\mathbf{y}$ with minimal elements y_{n-1}, y_n is in $A_0^{\perp\perp}$. It follows that $A_0^{\perp\perp} = A_0^\perp$. Observe that $\dim(A_0^\perp) = \dim(A_0^{\perp\perp}) = n - 1$.

By Lemma 2, (1), $\mathrm{Trophull}(A_0) \subset A_0^{\perp\perp}$. On the other hand, $\mathrm{Trophull}(A_0) \ne A_0^{\perp\perp}$. Indeed, we have $A_0 \subset A_0^{\perp\perp} = A_0^\perp$, hence, by Lemma 1, $\mathrm{Trophull}(A_0) \subset A_0^\perp$. Then, by Proposition 1, $A_0^{\perp\perp} \cap \mathbb{R}^n = A_0^\perp \cap \mathbb{R}^n \not\subset \mathrm{Trophull}(A_0)$. In particular, $\mathrm{Trophull}(A_0)$ is not a tropical linear prevariety, by Proposition 2.

3 Tropicalization of linear subspaces

Let $\mathbb{F}$ denote the field $\mathbb{C}((t^{1/\infty}))$ of Puiseux series over $\mathbb{C}$. For an element $y \in \mathbb{F}$ different from 0, let $\mathrm{val}(y) \in \mathbb{Q}$ denote the *valuation* of the element y in $\mathbb{F}$, i.e., the power in the lowest term of the Puiseux series y. Separately define $\mathrm{val}(0) = \infty$.

Definition 5 (cf. [16]). *Let $f := a_1x_1 + \cdots + a_nx_n$, where $0 \ne a_i \in \mathbb{F}$ for all $1 \le i \le n$. The* formal tropicalization *of the hyperplane $\{f = 0\} \subset \mathbb{F}^n$ is the tropical hyperplane,* $\mathrm{Tropf}(\{f = 0\}) \subset \mathbb{R}^n$*, defined by the set* $\{y_1 + \mathrm{val}(a_1), \dots, y_n + \mathrm{val}(a_n)\}$ *(see Definition 1).*

By Kapranov's Theorem [16, Theorem 3.1.3], $\text{Tropf}(\{f=0\})$ coincides with the (Euclidean) closure in $\mathbb{R}^n$ of the countable set

$$\{(\text{val}(x_1),\dots,\text{val}(x_n))|\,(x_1,\dots,x_n)\in\{f=0\}\}.$$

The following definition is "dual" to Definition 4.

Definition 6. *For every partition* $\{i_1,\dots,i_r\}\cup\{i_{r+1},\dots,i_n\}$ *of* $\{1,\dots,n\}$ *a* chart in $\mathbb{F}^n$ *is an open set*

$$D_{i_1,\dots,i_r}:=\{x_{i_1}=\cdots=x_{i_r}=0\}\cap\{x_{i_{r+1}}\neq 0\}\cap\cdots\cap\{x_{i_n}\neq 0\}\subset\mathbb{F}^n.$$

Let $P\subset\mathbb{F}^n$ be a linear subspace of arbitrary dimension. Clearly,

$$P=\bigcup_{\{i_1,\dots,i_r\}}(P\cap D_{i_1,\dots,i_r}),$$

where the union is taken over all subsets $\{i_1,\dots,i_r\}$ of $\{1,\dots,n\}$.

Definition 7 (cf. [16]). *The* tropicalization $\text{Trop}(P\cap D_{i_1,\dots,i_r})$ *of* $P\cap D_{i_1,\dots,i_r}$ *is the set of all points* $(y_1,\dots,y_n)\in\mathbb{R}^n_\infty$ *such that* $y_{i_1}=\cdots=y_{i_r}=\infty$ *and* $(y_{i_{r+1}},\dots,y_{i_n})$ *belongs to the Euclidean closure in* $\mathbb{R}^{n-r}$ *of the set*

$$\{(\text{val}(x_{i_{r+1}}),\dots,\text{val}(x_{i_n}))|\,(x_{i_{r+1}},\dots,x_{i_n})\in P\cap D_{i_1,\dots,i_r}\}.$$

The tropicalization $\text{Trop}(P)$ *of* P *is defined as* $\bigcup_{\{i_1,\dots,i_r\}}\text{Trop}(P\cap D_{i_1,\dots,i_r})$.

Remark 5. Definition 7 immediately implies that for any two sets $A,B\subset\mathbb{F}^n$ there is the inclusion $\text{Trop}(A\cap B)\subset\text{Trop}(A)\cap\text{Trop}(B)$. The inverse inclusion $\supset$ is not generally true even for linear subspaces.

Let $\dim(P)=d$, and $\mathbf{z}_1,\dots,\mathbf{z}_d\in P$ be a basis in P. Recall that *Plücker coordinates* of P in the Grassmanian $\text{Gr}(d,\mathbb{F}^n)$ are all $(d\times d)$-minors $p_{j_1,\dots,j_d}$ of the matrix with rows $\mathbf{z}_1,\dots,\mathbf{z}_d$, corresponding to columns $1\le j_1<\cdots<j_d\le n$. Any $\mathbf{z}=(z_1,\dots,z_n)\in P$ satisfies *Plücker relation*

$$\sum_{1\le i\le d+1}(-1)^i p_{j_1,\dots,j_{i-1},j_{i+1},\dots,j_{d+1}}z_{j_i}=0, \tag{2}$$

for every subset of columns $1\le j_1<\cdots<j_{d+1}\le n$. Note that relations (2) do not depend on the choice of the basis in P.

Denote the set of points $\mathbf{z}$ satisfying (2) by $P_{j_1,\dots,j_{d+1}}$.

The following statement is a strengthening of [18, Proposition 4.2].

Lemma 3.

$$\text{Trop}(P)=\bigcap_{j_1,\dots,j_{d+1}}\text{Trop}(P_{j_1,\dots,j_{d+1}}). \tag{3}$$

Let $Q\subset\mathbb{F}^n$ be a linear subspace orthogonal to P with $\dim(Q)=n-d$. According to [1], [18], the tropicalizations $\text{Trop}(P)$ and $\text{Trop}(Q)$ are orthogonal, with $\dim(\text{Trop}(P))=d$ and $\dim(\text{Trop}(Q))=n-d$.

Theorem 3. *There is a finite subset* $A\subset\mathbb{R}^n_\infty$ *such that* $\text{Trop}(P)=A^\perp$ *and* $\text{Trop}(Q)=A^{\perp\perp}$.

Corollary 2. *Let* $\text{Trop}(P)=A^\perp$ *for a linear subspace* $P\subset\mathbb{F}^n$ *and* $Q\subset\mathbb{F}^n$ *be the subspace which is complementary orthogonal to* P. *Then* $\text{Trop}(Q)=A^{\perp\perp}$.

4 Criterion and deciding algorithm for being a tropical linear variety

Let $A \subset (\mathbb{Q} \cup \{\infty\})^n$ be a finite set of vectors. Since $A^\perp$ and $A^{\perp\perp}$ are tropical linear prevarieties (see Proposition 2), Theorem 1 implies that

$$A^\perp = \mathrm{Trophull}(\{\mathbf{x}_1, \dots, \mathbf{x}_p\}) \quad \text{and} \quad A^{\perp\perp} = \mathrm{Trophull}(\{\mathbf{y}_1, \dots, \mathbf{y}_q\})$$

for some vectors $\mathbf{x}_1, \dots, \mathbf{x}_p, \mathbf{y}_1, \dots, \mathbf{y}_q \in (\mathbb{Q} \cup \{\infty\})^n$.

Definition 8. *Let* $\mathbf{x} \in (\mathbb{Q} \cup \{\infty\})^n$ *and* $\mathbf{v} \in \mathbb{F}^n$ *such that* $\mathrm{Trop}(\mathbf{v}) = \mathbf{x}$. *Then* $\mathbf{v}$ *is called the* lifting *of* $\mathbf{x}$.

Theorem 4. *The following three statements are equivalent.*

1. *There exist mutually complementary and orthogonal linear subspaces* P, Q *in* $\mathbb{F}^n$ *such that* $A^\perp = \mathrm{Trop}(P)$, *and* $A^{\perp\perp} = \mathrm{Trop}(Q)$ *(in particular,* $A^\perp$, $A^{\perp\perp}$ *are tropical linear* varieties*).*
2. *There exist liftings*

   $$\mathbf{v}_1, \dots \mathbf{v}_p, \mathbf{w}_1, \dots, \mathbf{w}_q \in \mathbb{F}^n \quad \textit{of vectors} \quad \mathbf{x}_1, \dots \mathbf{x}_p, \mathbf{y}_1, \dots, \mathbf{y}_q \in (\mathbb{Q} \cup \{\infty\})^n$$

   respectively, such that $(\mathbf{v}_i, \mathbf{w}_j) = 0$ *for all* $1 \le i \le p$, $1 \le j \le q$.
3. $A^\perp$ *is a tropical linear* variety*.*

Corollary 3. *There is an algorithm which for a given tropical linear* prevariety $A^\perp = \mathrm{Trophull}(\{\mathbf{x}_1, \dots, \mathbf{x}_p\})$ *where* $\mathbf{x}_1, \dots, \mathbf{x}_p \in (\mathbb{Q} \cup \{\infty\})^n$, *decides whether it is a tropical linear* variety*.*

Proof. The input of the algorithm under construction is the set $\{\mathbf{x}_1, \dots, \mathbf{x}_p\} \subset (\mathbb{Q} \cup \{\infty\})^n$. Using the algorithm from the proof of Theorem 1 in [8], construct the set $\{\mathbf{y}_1, \dots, \mathbf{y}_q\} \subset (\mathbb{Q} \cup \{\infty\})^n$ such that $A^{\perp\perp} = \mathrm{Trophull}(\{\mathbf{y}_1, \dots, \mathbf{y}_q\})$.

Consider, over $\mathbb{F}^* \cong \mathbb{F} \setminus \{0\}$, the system of equations

$$\sum_{1 \le \nu \le n} V_{i\nu} W_{j\nu} = 0 \tag{4}$$

for all $1 \le i \le p$, $1 \le j \le q$, $1 \le \nu \le n$ such that $x_{i\nu} \ne \infty$ and $y_{j\nu} \ne \infty$, where $\mathbf{V}_i = (V_{i1}, \dots, V_{in}), \mathbf{W}_j = (W_{j1}, \dots, W_{jn})$ are vectors of variables. Using [1], [13], or [15], the algorithm constructs the tropical basis of the system of equations (4), which is a finite set of polynomials H_ℓ, $1 \le \ell \le N$, where $N \le (p+q)n$, with integer coefficients and variables $V_{i\nu}, W_{j\nu}$ such that $x_{i\nu} \ne \infty$ and $y_{j\nu} \ne \infty$. (See a detailed definition and properties of a tropical basis in [16, Section 2.6].) The algorithm checks whether, for all $1 \le \ell \le N$, vectors $\mathbf{x}_1, \dots \mathbf{x}_p, \mathbf{y}_1, \dots, \mathbf{y}_q$, from which coordinates $x_{i\nu} = \infty$ and $y_{j\nu} = \infty$ are removed, satisfy tropicalizations $\mathrm{Trop}(H_\ell)$. If yes, then, by the definition of tropical basis, there exist liftings $v_{i\nu}, w_{j\nu} \in \mathbb{F}^*$ of all $x_{i\nu} \ne \infty, y_{j\nu} \ne \infty$ respectively, which satisfy the system (4). Let P be the linear hull of vectors $\mathbf{v}_1, \dots \mathbf{v}_p$ such that in every $\mathbf{v}_i$ each coordinate corresponding to $x_{i\nu} \neq \infty$ is the lifting $v_{i\nu}$, while each coordinate corresponding to $x_{i\nu} = \infty$ is 0. Similarly, let Q be the linear hull of vectors $\mathbf{w}_1, \dots \mathbf{w}_q$ such that in every $\mathbf{w}_j$ each coordinate corresponding to $y_{j\nu} \neq \infty$ is the lifting $w_{j\nu}$, while each coordinate corresponding to $y_{j\nu} = \infty$ is 0. Then, by Theorem 4, P and Q are mutually complementary and orthogonal linear subspaces of $\mathbb{F}^n$, while $A^\perp = \mathrm{Trop}(P)$ and $A^{\perp\perp} = \mathrm{Trop}(Q)$. In particular, $A^\perp$ and $A^{\perp\perp}$ are tropical linear varieties. If vectors $\mathbf{x}_1, \dots \mathbf{x}_p, \mathbf{y}_1, \dots, \mathbf{y}_q$ from which coordinates $x_{i\nu} = \infty$ and $y_{j\nu} = \infty$ are removed don't satisfy $\mathrm{Trop}(H_\ell)$ for all $1 \leq \ell \leq N$, then $A^\perp$ is not a tropical linear variety, by Theorem 4.

The complexity of the algorithm is polynomial in the size of rational coordinates of vectors $\mathbf{x}_i$, $1 \leq i \leq p$, and doubly exponential in n. In this regard, we note that the complexity of the algorithm for computing the tropical basis in [15] is doubly exponential.

Remark 6. There is an algorithm which for a tropical linear *prevariety* $A^\perp = \mathrm{Trophull}(\{\mathbf{x}_1, \dots, \mathbf{x}_p\})$, where $\mathbf{x}_1, \dots, \mathbf{x}_p \in (\mathbb{Q} \cup \{\infty\})^n$, produces bases of linear subspaces P and Q, such that $A^\perp = \mathrm{Trop}(P)$ and $A^{\perp\perp} = \mathrm{Trop}(Q)$, in case these subspaces exist.

More precisely, as in the proof of Corollary 2, construct, using the algorithm from the proof of Theorem 1 in [8], the set $\{\mathbf{y}_1, \dots, \mathbf{y}_q\} \subset (\mathbb{Q} \cup \{\infty\})^n$ such that $A^{\perp\perp} = \mathrm{Trophull}(\{\mathbf{y}_1, \dots, \mathbf{y}_q\})$. Apply the algorithm from [14] to vectors $\mathbf{x}_i, \mathbf{y}_j$, $1 \leq i \leq p$, $1 \leq j \leq q$, from which coordinates $x_{i\nu} = \infty$ and $y_{j\nu} = \infty$ are removed, and to the system (4). The algorithm from [14] will either produce liftings $v_{i\nu}, w_{j\nu} \in \mathbb{F}^*$ of all $x_{i\nu} \neq \infty, y_{j\nu} \neq \infty$ respectively, which satisfy (4), or will indicate that liftings don't exist, i.e., vectors $\mathbf{x}_i, \mathbf{y}_j$ with removed coordinates do not belong to the tropicalization of (4). If liftings exist, then P is the linear hull of vectors $\mathbf{v}_1, \dots \mathbf{v}_p$ such that in every $\mathbf{v}_i$ each coordinate corresponding to $x_{i\nu} \neq \infty$ is the lifting $v_{i\nu}$, while each coordinate corresponding to $x_{i\nu} = \infty$ is 0. Similarly, Q is the linear hull of vectors $\mathbf{w}_1, \dots \mathbf{w}_q$ such that in every $\mathbf{w}_j$ each coordinate corresponding to $y_{j\nu} \neq \infty$ is the lifting $w_{j\nu}$, while each coordinate corresponding to $y_{j\nu} = \infty$ is 0. Herewith, all coordinates of vectors $\mathbf{v}_i, \mathbf{w}_j$ are Puiseux series in $\overline{\mathbb{Q}}((t^{1/\infty}))$ and the algorithm computes their expansions up to a given power of t, representing complex algebraic coefficients as zeroes of irreducible univariate polynomials with integer coefficients.

We sketch briefly an alternative algorithm for producing P and Q. A procedure in [14] reduces the construction of liftings to finding all points in a zero-dimensional algebraic set in $(\overline{\mathbb{Q}}((t^{1/\infty})))^N$, by adding generic linear forms to the tropical basis. All such points (absolutely irreducible components of the zero-dimensional algebraic set) can be computed with singly exponential complexity using algorithms from [3], [9] (these algorithms have much more general scope). Zero-dimensional components are represented as elements of a finite extension of the field $\mathbb{Q}(t)$ via a primitive element. Then, using a procedure from [4] for Puiseux extension of solutions of polynomial equations over $\overline{\mathbb{Q}}((t^{1/\infty}))$, the algorithm checks whether the tropicalization of a solution coincides with $\mathbf{x}_1, \dots \mathbf{x}_p, \mathbf{y}_1, \dots, \mathbf{y}_q$. Because the use of the procedure from [14], this algorithm also has a doubly exponential complexity. The digest of symbolic manipulation technique with Puiseux series can be found in [10].

5 Infinite intersections of tropical linear prevarieties

By the definition, the intersection of a finite number of tropical hyperplanes is a tropical linear prevariety. In this section we give an example of a *countable* family of tropical hyperplanes in $\mathbb{R}^6_\infty$ such that their intersection is not a finite union of convex polyhedra, in particular, not a tropical prevariety. This strengthens examples in [7] (example of T. Theobald) and [12], in which intersections of a countable families of tropical (non-linear) prevarieties were shown not to be finite unions of convex polyhedra.

Choose a sequence $\{\varepsilon_i\}_{i=1}^n$ of pair-wise distinct real numbers ε_i such that $0 < \varepsilon_i < 1/4$, and consider a tropical hyperplane $L_i \subset \mathbb{R}^6_\infty$ defined by the set

$$\{-i + x_1, -i + x_2, -i/2 - \varepsilon_i + y_1, -i/2 + y_2, z_1, z_2\}$$

(see Definition 1).

Let

$$M := \bigcap_{1 \le i < \infty} L_i \subset \mathbb{R}^6_\infty.$$

Proposition 3. *The set $M \cap \mathbb{R}^6$ is not a finite union of convex polyhedra. In particular, M is not a tropical prevariety.*

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References

1. T. Bogart, A. N. Jensen, D. Speyer, B. Sturmfels, R.R. Thomas, Computing tropical varieties, J. Symbolic Comput. 42, 1–2 (2007) 54–73.
2. P. Butkovic, G.Hegedüs, An elimination method for finding all solutions of the system of linear equations over an extremal algebra, Ekon.-Mat. Obzor 20 (1984) 203–214.
3. A. Chistov, An algorithm of polynomial complexity for factoring polynomials, and determination of the components of a variety in a subexponential time, J.Soviet Math., 34 (1986) 1838–1882.
4. A. Chistov, Polynomial complexity of Newton-Puiseux algorithm, Lect. Notes Comput. Sci., 233 (1986) 247–255.

5. M. Develin, F. Santos, B. Sturmfels, On the rank of a tropical matrix, In: Combinatorial and Computational Geometry, MSRI Publications, v. 52 (2005).
6. M. Develin, B. Sturmfels, Tropical convexity, Doc. Math., 9 (2004) 1–27.
7. A. Dress, W. Wenzel, Algebraic, tropical, and fuzzy geometry, Beitr. Algebra Geom. 52, 2 (2011) 431–461.
8. S. Gaubert, R.D. Katz, Minimal half-spaces and external representation of tropical polyhedra, J. Algebraic Combin. 33, 3, (2011) 325–348.
9. D. Grigoriev, Polynomial factoring over a finite field and solving systems of algebraic equations, J. Soviet Math., 34 (1986) 1762–1803.
10. D. Grigoriev, Polynomial complexity recognizing a tropical linear variety, Lect. Notes Comput. Sci., v. 9301 (2015) 152–157.
11. D. Grigoriev, V. Podolskii, Complexity of tropical and min-plus linear prevarieties, Computational Complexity, 24, 1 (2015) 31–64.
12. D. Grigoriev, N. Vorobjov, Upper bounds on Betti numbers of tropical prevarieties, arXiv:1709.08594.
13. K. Hept, T. Theobald, Tropical bases by regular projections, Proc. Amer. Math. Soc., 137, 7 (2009) 2233–2241.
14. A. N. Jensen, H. Markwig, T. Markwig, An algorithm for lifting points in a tropical variety, Collect. Math., 59, 2 (2008) 129–165.
15. Ya. Kazarnovskii, A. G. Khovanskii, Tropical noetherity and Gröbner bases, St. Petersburg Math. J., 26, 5 (2015) 797–811.
16. D. Maclagan, B. Sturmfels, Introduction to Tropical Geometry, American Math. Society, 2015.
17. J. Richter-Gebert, B. Sturmfels, T. Theobald, First steps in tropical geometry, In: G. Litvinov, V. Maslov (Eds.), Idempotent Mathematics and Mathematical Physics (Proceedings Vienna 2003), Contemporary Mathematics, 377, American Math. Society (2005) 289–317.
18. D. Speyer, Tropical linear spaces, SIAM J. Discrete Math., 22, 4 (2008) 1527–1558.
19. J. Yu, D.S. Yuster, Representing tropical linear spaces by circuits, The 19th International Conference on Formal Power Series and Algebraic Combinatorics, 2007, arXiv:0611579, 2006.