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ON THE EISENBUD-LEVINE FORMULA OVER A PERFECT FIELD

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1. Recently Eisenbud and Levine [1] gave an algebraic formula for the local degree of a smooth mapping. It is natural to use this formula to define the degree of a polynomial mapping over an arbitrary ground field. In [2] Eisenbud suggests the following definition. Suppose k is a field of characteristic 0 and $f: (k^n, 0) \longrightarrow (k^n, 0)$ is a polynomial mapping. Consider $Q_k(f) = k[[x_1, \ldots, x_n]]/(f^1, \ldots, f^n)$, the local algebra of the mapping f at 0, and assume it is finite-dimensional. Let J be the image in $Q_k(f)$ of the Jacobian of f. Choose a linear functional $\varphi: Q_k(f) \longrightarrow k$ such that $\varphi(J) \neq 0$, and put $\langle a, b\rangle_\varphi = \varphi(a, b)$. It is shown in [1] that if $\varphi(J)/\psi(J) \in (k^*)^2$, then the forms $\langle\cdot,\cdot\rangle_\varphi$ and $\langle\cdot,\cdot\rangle_\psi$ are equivalent. Eisenbud [2] suggests defining the degree of f at 0 to be the equivalence class of the form $\langle\cdot,\cdot\rangle_\varphi$ (it is necessary to take a functional φ with $\varphi(J) = \dim_k Q_k(f)$). The degree thus defined carries information about the degree of an extension of f to any field larger than k. In particular, on **R** it carries information not only about the usual degree, which is equal to the signature of the form $\langle\cdot,\cdot\rangle_\varphi$ in view of [1], but also about the multiplicity at 0, which is equal to $\dim_{\mathbf{R}} Q_{\mathbf{R}}(f) = \operatorname{rank}\langle\cdot,\cdot\rangle_\varphi$ and agrees with the degree of the complexification.

In the case of a field k of arbitrary characteristic it is necessary to replace J in this definition by the following element D: take series $a_{ij} \in k[[x_1, \ldots, x_n]]$ such that $\Sigma_j a_{ij} x_j = f^i$, $1 \leqslant i \leqslant n$, put $\bar{D} = \det(a_{ij})$, and let D be the image of $\bar{D}$ in $Q_k(f)$ (the condition $\varphi(J) = \dim_k Q_k(f)$ must be replaced by $\varphi(D) = 1$). If char $k = 0$, then $J = \dim_k Q_k(f) \cdot D$ and therefore the new definition agrees with the old one (see [2]).

We propose a definition of degree reflecting only the behavior of f at k-points. Such a definition dictates a global point of view, which suggests definite invariance requirements (see Theorem 2). In what follows, k is a perfect field and char $k \neq 2$.

Let $WG(k)$ be the Witt-Grothendieck group of symmetric bilinear forms over k. For each proper finite extension $k \subsetneq E$ we define a form $\langle\cdot,\cdot\rangle_E: E \times E \longrightarrow k$ by the rule $\langle a, b\rangle_E = \operatorname{Tr}_k^E(ab)$ and consider the subgroup $TF(k)$ of $WG(k)$ generated by the forms $\langle\cdot,\cdot\rangle_E$. Put $\Delta(k) = WG(k)/TF(k)$. The degree $\deg_0 f$ of the mapping f at 0 is defined to be the image in $\Delta(k)$ of a form $\langle\cdot,\cdot\rangle_\varphi$ with $\varphi(D) = 1$. The definition obviously extends to formal mappings $(k^n, 0) \longrightarrow (k^n, 0)$ (understood to be sets of series $f^1, \ldots, f^n \in k[[x_1, \ldots, x_n]]$ without free terms).

2. THEOREM 1. (a) *If k is algebraically closed or real closed, then $\Delta(k) \simeq \mathbf{Z}$. In the first case an isomorphism is defined by the rank; in the second case, by the signature.*

(b) *If all finite extensions of the field k have degree p^m for a fixed prime p and if k is neither real closed nor algebraically closed, then $\Delta(k) \simeq \mathbf{Z}/p\mathbf{Z}$. An isomorphism is defined by the rank of the form, considered modulo p.*

(c) *In the remaining cases,* $\Delta(k) \simeq 0$.

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Note that in case (a) our definition of degree is equivalent for $k = \mathbf{R}$ or $k = \mathbf{C}$ to the usual one.

3. In differential topology the concept of local degree makes it possible to define the index of a zero of finite multiplicity of a section of an n-dimensional bundle on an n-dimensional compact manifold. In the oriented case this index lies in **Z**, and in the general case in **Z**/2**Z**. If all zeros of a section are of finite multiplicity, then the sum of their indices is equal to the Euler number of the bundle and does not depend on the section. We will extend this theorem to algebraic manifolds defined over k (the class of manifolds is specified below).

In what follows, all objects (manifolds, bundles, sections) are assumed to be defined over k.

We now suppose that X is a smooth projective n-dimensional manifold, Y is an open subset of X containing all of its k-points (this condition is an analog of compactness in the real case), η is an n-dimensional vector bundle over Y, and v is some section of η. In the case of a real closed field k we assume that η and Y are oriented.(¹) Suppose y is a k-point of the manifold Y and $v(y) = 0$. By means of a trivialization of the bundle η near y the section v induces n germs $v^1, \ldots, v^n \in O_y$ (O_y is the local ring at the point y). Let $\widehat{v^1}, \ldots, \widehat{v^n}$ be the images of these germs in the completion $\widehat{O_y}$ of the ring O_y. The images of the elements $\widehat{v^1}, \ldots, \widehat{v^n}$ under some isomorphism $\widehat{O_y} \xrightarrow{\sim} k[[x_1, \ldots, x_n]]$ induce a formal mapping $f: (k^n, 0) \dashrightarrow (k^n, 0)$. In the case of a real closed field k we will assume that this isomorphism and the trivialization are compatible with the orientations. The zero y of the section v is said to be of finite multiplicity if $\dim_k Q_k(f) < \infty$. If $\dim_k Q_k(f) < \infty$, then $\deg_0 f$ is defined and depends only on v and y (this is true both for a real closed field and the tangent bundle if the isomorphism and trivialization are induced by the same system of local parameters). Put $\operatorname{ind}_y v = \deg_0 f$. We call a section general if all of its zeros at k-points are of finite multiplicity. For a general section v we put $e(v) = \Sigma \operatorname{ind}_y v$, where the sum extends over all k-zeros y of v.

THEOREM 2. *If v and w are two general sections, then $e(v) = e(w)$.*

For the fields considered in part (a) of Theorem 1 this is well known.

Assume that the bundle η has a general section v. We call $e(v)$ the Euler number of η and denote it by $e(\eta)$. If θ is a bundle over Z (Z satisfies the same requirements as Y) and $e(\theta)$ is defined, then $e(\pi_1^*\eta \oplus \pi_2^*\theta)$ is defined and is equal to $e(\eta) \cdot e(\theta)$, where $\pi_1: Y \times Z \longrightarrow Y$ and $\pi_2: Y \times Z \longrightarrow Z$ are projections. If η is a trivial bundle, then $e(\eta) = 0$.

It follows from Theorem 2 that on an irreducible smooth manifold defined over k, the set of k-points is everywhere dense in the Zariski topology if it is nonempty. For $k = \mathbf{R}$ this fact is known.

4. Let us now consider the fact that many affine manifolds defined over a nonclosed field k do not have k-points at infinity, i.e. they contain all k-points of their projective closure. Suppose Y is such a manifold. Then a bundle η over Y can be regarded as a projective

(¹) Orientation of smooth manifolds and vector bundles is introduced by analogy with the classical case. Namely, we call two bases in a vector space over k *equivalent* if the determinant of the matrix of transition from one basis to the other lies in (k^*): an equivalence class of bases is called an *orientation* of the vector space. An orientation of a vector bundle is a set of orientations of its fibers over the k-points of the base that is locally trivial in the obvious sense: an orientation of a smooth manifold is an orientation of its tangent bundle.

module (of rank dim Y) over $k[Y]$. Thus e becomes an invariant of the projective modules over such rings. If k is a field as in part (b) of Theorem 1, then this invariant vanishes on stably free modules.

Let us calculate the invariant e in the case dim $Y = 1$ (we assume that the field k is as in part (b) of Theorem 1). In this case the ring $A = k[Y]$ is Dedekind and the isomorphism classes of projective modules of rank 1 form a group Pic(A) under tensor multiplication, which is also isomorphic to the group of ideal classes of A. The invariant e defines an epimorphism $\mathrm{Pic}(A) \longrightarrow \Delta(k)$. Suppose $0 \neq a \in \mathrm{Pic}(A)$, a is an ideal, and γ is the multiplicity of a (if $a = m_1^{\rho_1} \cdots m_l^{\rho_l}$ is a decomposition into a product of maximal ideals, then $\gamma = \Sigma \rho_i$). Then $e(a) = -\gamma \pmod p$.

In conclusion, we give an explicit example of a projective module M with $e(M) = 1$. Suppose k is a field as in part (b) of Theorem 1 and $\alpha \in k^* \backslash (k^*)^p$. Let

$$A = k[x, y]/(x^p - x - \alpha y^p)$$

and suppose the module M is generated by the rows (x^{p-1}, y) and $(-\alpha x^{p-2} y^{p-1}, 1 - x^{p-1})$. Then M is projective of rank 1 and $e(M) = 1$ (note that the curve $x^p - x - \alpha y^p = 0$ has no k-points at infinity).

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ON PHOTOMETRIC QUANTITIES ON A RIEMANNIAN MANIFOLD

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The purpose of the paper is to give a formal definition of multidimensional analogues of the main photometric quantities for Riemannian manifolds of a wide enough class (called F-manifolds) and to write the basic relations between these quantities. The known systems of theoretical photometry in 3-space [1], [2] use the light vector field $\mathscr{E}$ as the main quantity characterizing the stationary distribution of the light field in free space; the theory of the field $\mathscr{E}$ is known as light field theory. We reserve the term "light field theory" (l.f.t.) for the system of photometric quantities and relations defined below. Our version of l.f.t. considers the light field either in an empty space or in a space filled with an absorbing medium. The main photometric quantities depend upon the time parameter; this allows one to study some questions on dynamics of light flows under the Galilean space-time structure of the space of events. Besides that, in this system one can eliminate an asymmetry characteristic of usual l.f.t. systems, due to the lack of a formal notion of a light field receiver localized in a point; this notion is dual to that of a point source of general (i.e. anisotropic) type. In fact there exist instruments with a selective susceptibility to directions of beams forming a light flow. The simplest of these is the camera obscura. Introducing into l.f.t. an idealized presentation of such a light receiver enables us to eliminate the asymmetry; this can be achieved by passing to the description of a light field in terms of the quantity Π (see below).

Throughout the paper the term "smooth" means C^∞. All smooth manifolds are assumed to be connected, countable at infinity, orientable, and without a boundary if the contrary is not stated. Given a smooth manifold, we denote its tangent bundle by TM, and the tangent space of M at the point $x \in M$ by TM_x. Points of TM are denoted by z with or without indices; tangent vectors in TM_x are denoted by X, Y, Z. If $z \in TM$, $z = (x, X)$, $x \in M$, $X \in TM_x$, then tz, $t \in R$, denotes the point $(x, tX) \in TM$. A smooth Riemannian manifold (R.m.) is a pair (M, g), where M is a smooth manifold and g is a smooth Riemannian metric on M. Given the R.m. (M, g), we denote the sphere bundle of unit tangent spheres by SM, and the fiber of this bundle over the point $x \in M$ by SM_x. We denote by $W_x(M, g)$ the maximal domain of TM_x where the exponential mapping $\exp_x$ of the R.m. (M, g) at x is defined. Let $W(M, g) = \bigcup_{x \in M} W_x(M, g) \subseteq TM$. We denote by π the fiber map of SM onto M.

The support function of a R.m. (M, g) is, by definition, the nonnegative function p on SM with values in the extended reals $R \cup \{\infty\}$ defined by

$$p(z) = \begin{cases} 1/\tilde{p}(z), & \text{if } \tilde{p}(z) \neq 0, \\ \infty, & \text{if } \tilde{p}(z) = 0, \end{cases}$$

where

$$\tilde{p}(z) = \inf_{s \in \Phi(z)} s, \qquad \Phi(z) = \{ s \in R \mid s \geq 0,\ s^{-1} z \in W(M, g) \}.$$

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