COMPLEXITY OF SOLUTION OF LINEAR SYSTEMS IN RINGS OF DIFFERENTIAL OPERATORS

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Suppose given a $k_1 \times k_2$ system of linear equations over the Weyl algebra $\mathcal{H}_a = \mathbb{F}[x_1, \ldots, x_n, x_1, \ldots, x_n]$ or over the algebra of differential operators $\mathcal{X}_a = \mathbb{F}[x_1, \ldots, x_n][x_1, \ldots, x_n]$, where the degree of each coefficient of the system is less than $d$. It is proved that if the system is solvable over $\mathcal{H}_a$ or $\mathcal{X}_a$, respectively, then it has a solution of degree at most $(k_1 d)^{O(n)}$.

Introduction

Let $\mathcal{A}_a = \mathcal{A}_a(\mathbb{F}) = \mathbb{F}[x_1, \ldots, x_n, d_1, \ldots, d_n]$ be the Weyl algebra over the field $\mathbb{F}$ ([6]); as is known, it is defined by the relations

$$X_i X_j - X_j X_i = d_j, D_j X_i - D_i X_j = d_i X_i - d_j X_j, i \neq j$$

(1)

One can also interpret the Weyl algebra as the algebra obtained from the polynomial algebra $\mathbb{F}[x_1, \ldots, x_n]$ by adjoining the differentiation operators $D_1, \ldots, D_n$ with respect to the variables $x_1, \ldots, x_n$, respectively. We denote by $\mathcal{H}_a = \mathcal{H}_a(\mathbb{F}) = \mathbb{F}[x_1, \ldots, x_n][D_1, \ldots, D_n] \Rightarrow \mathcal{A}_a(\mathbb{F})$ the algebra of differential operators obtained from the field $\mathbb{F}(x_1, \ldots, x_n)$ of rational functions by adjoining the operators $D_1, \ldots, D_n$ ([6]).

An element $a \in \mathcal{A}_a$ can be represented uniquely in the form $a = \sum_{i,j} a_{i,j} D_i x_j x_i t^i$, where $a_{i,j} \in \mathbb{F}$, the multi-indices $I = (i_1, \ldots, i_s), J = (j_1, \ldots, j_s)$; any element $b \in \mathcal{H}_a$ can be represented uniquely in the form $b = a d^c$, where $a \in \mathcal{A}_a, 0 \neq c \in \mathbb{F}[x_1, \ldots, x_n]$ and the degree $\deg(c)$ is the least possible. We define the degree $\deg(D_i x_j x_k \ldots x_{i_1} i^i) = i_{n+1} + i_{n+2} + \ldots + i_s$, according to the Bernsstein filtration ([6]) and the degree $\deg(a) = \max \{\deg(D_i x_j x_k \ldots x_{i_1} i^i)\}$. Finally, $\deg(b) = \max \{\deg(a), \deg(c)\}$.

The goal of the present paper is to estimate the complexity of the solution of the system of linear equations

$$\sum_{i \leq k \leq m} u_{x, t} V_t = w_k$$

(2)

over the ring $\mathcal{A}_a$ (i.e., the coefficients $u_{x, t} \in \mathcal{A}_a$, and also the unknowns $V_t \in \mathcal{A}_a$) or, respectively, over the ring $\mathcal{H}_a$. Below in the formulation of the theorem and corollary $\mathcal{R}$ denotes either the ring $\mathcal{A}_a$ or the ring $\mathcal{H}_a$.

**THEOREM.** Let the system (2) be solvable in the ring $\mathcal{R}$ and $\deg(u_{x, t})_k \deg(w_k) < d, i \leq k < m, l \leq s$. Then one can find a solution $V_1, \ldots, V_s$ of (2) for which $\deg(V_t) < (\mathbb{P}d)^{O(n)}, i \leq l \leq s$.

**COROLLARY.** Let the field $\mathbb{F}$ be defined effectively, for example, as a finitely generated extension of the field $\mathbb{Q}$ (cf. 2, 3, 5, 7), and the bitwise size of any coefficient of the polynomials $u_{x, t}, w_k$ be at most $M$. Then one can verify the solvability of (2) over the ring $\mathcal{R}$ and construct a solution (if the system is solvable) in polynomial time in $M, (\mathbb{P}d)^{O(n)}, s$.

One derives the corollary from the theorem by representing (2) as a system of linear equations over the field $\mathbb{F}$ in the case $\mathcal{R} = \mathcal{A}_n$ or over the field $\Gamma \{X_1, \ldots, X_n\}$ in the case $\mathcal{R} = \mathcal{K}_n$, respectively, whose unknowns are the coefficients of the elements $V_1, \ldots, V_6$ for monomials in $D_{n_1}, D_1 X_{n_2}, D_1 \ldots$ or for monomials in $D_{n_1}, D_1 \ldots$, respectively.

We note that in the case of a system (2) over the ring of polynomials $\Gamma \{X_1, \ldots, X_n\}$ the estimate of the theorem is well-known [11]; however it is impossible to extend its proof from [11] directly to our case since the rings $\mathcal{A}_n$ and $\mathcal{K}_n$ are noncommutative. However the general approach of [11] is used below in proving the theorem.

We mention that in [3] an algorithm of polynomial complexity is constructed for finding the g.c.d. of a family of ordinary linear operators; this in particular implies the theorem for the case $\mathcal{R} = \mathcal{K}_n$. In [10] it is proved that in the polynomial ring $\Gamma \{X_1, \ldots, X_n\}$ even for the special case of a system (2) of one equation $\sum_{1 \leq i \leq 5} u_i V_i = w$ (i.e., the problem of recognizing whether the polynomial $w$ belongs to the ideal with generators $u_1, \ldots, u_5$) the estimate given in the theorem is sharp in order. Thus, the estimate of the theorem is also sharp in order. Indeed let $u_i, w \in \Gamma \{D_{n_1}, \ldots, D_n\} \subset \mathcal{A}_n$ and $\sum_{1 \leq i \leq 5} u_i V_i = w$; then when $\mathcal{R} = \mathcal{A}_n$ we represent $V_i = \sum_{1 \leq i \leq 5} V^{(1)}_{ij} D_{n_1} \ldots D_{n_1} X_{n_2} \ldots X_{n_2} \ldots X_{n_2} \ldots X_{n_2}$ (cf. above), and then $\sum_{1 \leq i \leq 5} u_i V_i = \sum_{1 \leq i \leq 5} u_i V^{(1)}_{ij} D_{n_1} \ldots D_{n_1} X_{n_2} \ldots X_{n_2} \ldots X_{n_2} \ldots X_{n_2}$ is taken over all pairs $i, j$ in the representation of $V_i$, for which $j = 0$; now when $\mathcal{R} = \mathcal{K}_n$ we represent $V_i = \sum_{1 \leq j \leq 5} V^{(1)}_{ij} D_{n_1} \ldots D_{n_1} X_{n_2} \ldots X_{n_2} \ldots X_{n_2} \ldots X_{n_2}$, where $C = C_{ij} X_{n_2} \ldots X_{n_2}$ (cf. above), and $\sum_{1 \leq i \leq 5} u_i V_i = \sum_{1 \leq i \leq 5} u_i V^{(1)}_{ij} D_{n_1} \ldots D_{n_1} X_{n_2} \ldots X_{n_2} \ldots X_{n_2} \ldots X_{n_2}$ is taken over all pairs $i, j = 0$ in the representation of $V_i$, for which $j = 0$. Thus, we have shown that if the equation $\sum_{1 \leq i \leq 5} u_i V_i = w$ has solution in $\mathcal{R}$, then it has a solution $\sum_{1 \leq i \leq 5} V_i \in \Gamma \{D_{n_1}, \ldots, D_n\}$, where $\deg(V_i) < \deg(V_i)$, $1 < i < 5$, which by [10] proves the required sharpness of order in the theorem.

We also mention that the nearly sharp estimate $\deg(\sum_{1 \leq i \leq 5} u_i V_i) = 1$ established in [9] for the special case of recognition of an ideal being the unit ideal in a polynomial ring, i.e., the problem of solvability of the system $\deg(V_i) < d(0(n))$. The author does not know whether an analogous result holds for the rings $\mathcal{A}_n$ and $\mathcal{K}_n$. A number of algorithmic problems in ideal theory in rings of differential operators are also posed in [8].

**Sec. 1. Estimation of the Elements of a Quasi-Inverse Matrix**

**Over the Weyl Algebra**

Let the matrix $A = (a_{i,j})_{1 \leq i, j \leq n+1}$ have elements in the Weyl algebra, $a_{i,j} \in \mathcal{A}_n$, where $\deg(a_{i,j}) < d$.

**Lemma 1.** There exists a vector $0 \neq b = (b_1, \ldots, b_m) \in (\mathcal{A}_n)^m$ such that $\lambda b = 0$, where $\deg(b) < 4n(m-1)d = N$.

**Proof.** We consider the linear space $\mathcal{B} \subset (\mathcal{A}_n)^m$ over $\mathbb{F}$ consisting of all vectors $c = (c_1, \ldots, c_m)$, such that $\deg(c) < N$. Then $\dim(\mathcal{B}) = (N+2n)$. For any vector $c \in \mathcal{B}$ one has $\deg(Ac) \leq N + d$, i.e., $Ac \in \mathcal{T}$, where the space $\mathcal{T}$ consists of all vectors $e = (e_1, \ldots, e_{m-1}) \in (\mathcal{A}_n)^{m-1}$, for which $\deg(e) < N + d$, so $\dim(T) = (N+2^{m-2}n)$. We prove that $\dim(T) > (N+2^{m-2}n)(m-1)$, from which the lemma will follow. Indeed $\frac{N+d+2n}{2n} - \frac{N+2n}{2n} = 1 + \frac{d}{2n}$. It suffices to verify that $\left(\frac{N+d+1}{N+1}\right)^2 < 1 + \frac{d}{2n}$; it is easy to see that $1 + \frac{d}{2n} > 1 + \frac{1}{2n} > 1 + \frac{d}{N+1}$. The lemma is proved.

We call the $m \times m$ matrix $C = (c_{i,j})$ a right (respectively left) quasi-inverse to the matrix $B = (b_{i,j})$ if the matrix $BC$ (resp. $CB$) is diagonal with nonzero elements on the diagonal.
**Lemma 2.** If the $m \times m$ matrix $B$ over the ring $\mathcal{R}$ has a right quasi-inverse over $\mathcal{R}$, then $B$ also has a left quasi-inverse $G$ over $\mathcal{R}$, while $\deg(\mathcal{G}) < 4m(3m-1)d$.

**Proof.** We note first that there does not exist a vector $\mathbf{0} \neq \mathbf{b} \in (\mathcal{R})^m$, for which $\mathbf{b}B = \mathbf{0}$, since $\mathcal{R}$ has no divisors of zero ([6]). We denote by $B^{(i)}$ the matrix obtained from $B$ by removing the $i$-th column. By Lemma 1 one can find a vector $\mathbf{0} \neq \mathbf{q}_i \in (\mathcal{R})^m$, such that $\mathbf{q}_i B^{(i)} = \mathbf{0}$ and $\deg(\mathbf{q}_i) < N$. Then the matrix $G$ whose rows are $\mathbf{q}_{1}, \ldots, \mathbf{q}_m$ is a left quasi-inverse to $B$. The lemma is proved.

For the algebra $\mathcal{H}$, one constructs the skew field of quotients $\mathcal{D}$ (cf. [6]), where any element of $\mathcal{D}$ can be represented in the form $a_1, b_1^{-1}$ and also in the form $b_2^{-1} a_2$ for suitable $a_1, b_1, a_2, b_2 \in \mathcal{D}$ (cf. Lemma 1 for $m = 2$). $\mathcal{D}$ is also the skew field of quotients for the ring $\mathcal{H}$. A matrix over $\mathcal{D}$ (analogously over $\mathcal{H}$) has right and left quasi-inverses (cf. Lemma 2) if and only if it is nondegenerate as a matrix over $\mathcal{D}$, which is equivalent with its Dieudonné determinant being nonzero (cf. [1]). We define the rank $r = \operatorname{rg}(A)$ of the matrix $A$ over $\mathcal{D}$ as the maximal size $r$ of a nondegenerate submatrix. Below in the formulation of the lemma the $m_1 \times m_2$ matrix $A$ of rank $r$ over $\mathcal{D}$ such that its $r \times r$ submatrix $A_{11}$ in the upper left corner is nondegenerate and the elements of $A_{11}$ belong to $\mathcal{H}$.

**Lemma 3.** Let $C_1$ be a left quasi-inverse over $\mathcal{D}$ to $A_{11}$. Then one can find an $(m_1 - r) \times r$ matrix $C_2$ over $\mathcal{D}$, such that

$\begin{pmatrix} C_1 & 0 \\ C_2 & E \end{pmatrix} A = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{pmatrix}$,

$E$ denoting the identity matrix here and below.

**Proof.** The matrix $C_2$ is defined by the condition that in the lower left corner of the product of matrices considered one has the zero matrix (it is obtained by suitable elementary row transformations, by addition of some combinations of the first $r$ rows to the last $(m_1 - r)$ rows). Then in the product obtained the lower right submatrix is also equal to zero in view of the definition of rank, which proves the lemma.

Now we return to the system (2) over the ring $\mathcal{R}$. When $\mathcal{R} = \mathcal{A}$ we apply Lemma 3 to the $m \times 5$ matrix $\left( \begin{array}{c} u_{\kappa,\ell} \\ \ell \end{array} \right)$ and get matrices $C_1, C_2$; making a suitable renumbering of the rows and columns we assume that in the upper left corner of $\left( \begin{array}{c} u_{\kappa,\ell} \\ \ell \end{array} \right)$ there is a nondegenerate submatrix of maximal size $r$. If the vector $(C_2)\left( \begin{array}{c} \omega_1, \ldots, \omega_m \end{array} \right)^T$ is nonzero, the system (2) has no solutions. Now if $(C_2)\left( \begin{array}{c} \omega_1, \ldots, \omega_m \end{array} \right)^T = 0$, then (2) is equivalent to a system of the following form (cf. Lemma 3):

$$a_{\kappa} V_{\kappa} + \sum_{\ell \neq \kappa \leq \ell < \kappa} a_{\kappa,\ell} V_{\ell} = b_{\kappa}, \quad 1 \leq \kappa \leq r.$$  

By virtue of Lemma 2, $\deg(a_{\kappa}), \deg(a_{\kappa,\ell}), \deg(b_{\kappa}) < 4m \times d < 4m \times d$. In the case when the ring $\mathcal{R} = \mathcal{A}$, the elements $u_{\kappa,\ell} = u^{(1)}_{\kappa,\ell} + u^{(2)}_{\kappa,\ell}$, $1 \leq \kappa \leq \ell < \kappa$, where $u^{(1)}_{\kappa,\ell} \in \mathcal{D}$, $\mathcal{D}$, $u^{(2)}_{\kappa,\ell} \in \mathcal{H}[x_\gamma, \ldots, x_n]$, can be reduced to a common denominator $u = \gcd(u^{(1)}_{\kappa,\ell} + u^{(2)}_{\kappa,\ell})$, i.e., $u_{\kappa,\ell} = u^{(1)}_{\kappa,\ell} + u^{(2)}_{\kappa,\ell}$ for all $\ell$, where $u_{\kappa,\ell} \in \mathcal{D}$ for $1 \leq \kappa < \ell$. As above we apply Lemma 3 to the matrix $\left( u^{(2)}_{\kappa,\ell} \right)$ and reduce system (2) to a form analogous to (3). Here since $\deg(u^{(2)}_{\kappa,\ell}) < d m^2$ for $1 \leq \kappa < \ell < \kappa$, one has $\deg(a_{\kappa}), \deg(a_{\kappa,\ell}), \deg(b_{\kappa}) < 4m \times d$ by Lemma 2.

**Remark.** If initially one considers the system $\sum_{\ell \neq \kappa \leq \ell < \kappa} v_{\kappa,\ell} = \omega_{\kappa}, 1 \leq \kappa < m$ instead of (2), then one should introduce a different multiplication of matrices $(a_{\kappa,\ell})(b_{\kappa,\ell}) - \sum_{i \neq \ell} b_{\kappa,\ell} a_{\kappa,\ell}$.  

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Temporarily we fix \( n + 1 \leq \ell \leq 5 \). When \( \mathcal{H} = \mathcal{A}_n \), in view of Lemma 1 one can find \( h_1^{(\ell)}, \ldots, h_{r-\ell}^{(\ell)} \in \mathcal{A}_n \), such that

\[
a_\lambda h_\lambda^{(\ell)} + a_{\lambda, \ell} h_{\lambda, \ell}^{(\ell)} = 0, \quad 1 \leq \lambda \leq r\]

where \( \deg(h_1^{(\ell)}), \ldots, \deg(h_{r-\ell}^{(\ell)}) \leq 16 n^2 m^3 d \). Now we consider the case when \( \mathcal{H} = \mathcal{K}_n \). Let \( a \in \mathcal{K}_n \) be an element of the form \( a = a^{(3)}(a^{(2)})^{-1} \), where \( a^{(2)} \in \mathcal{A}_n \), and \( g \in \mathbb{F}[X_1, \ldots, X_n] \). Using relations of the form

\[
\mathcal{D}_i (a^{(2)})^{-1} = (a^{(2)})^{p-1} \mathcal{D}_i - (a^{(2)})^{p_1} (a^{(2)})^{-1} \mathcal{D}_i \]

several times we reduce \( a \) to the form \( a = (a^{(3)})^{-1} a^{(2)} \) (cf. the estimates of \( \deg(a) \) obtained above), where \( a^{(3)} \in \mathcal{A}_n \) and \( \deg(a^{(3)}) \leq 16 n^2 m^3 d + 4 n^2 m^3 d + 17 n^2 m^d d^2 \). We reduce all elements \( a_\lambda, a_{\lambda, \ell} \) to the left common denominator \( c \in \mathbb{F}[X_1, \ldots, X_n] \) and get representations \( a_\lambda = c^{-1} a^{(2)}_\lambda, \quad a_{\lambda, \ell} = c^{-1} (a^{(3)}_{\lambda, \ell}) \), where \( \deg(a^{(2)}_\lambda), \deg(a^{(3)}_{\lambda, \ell}) \leq 32 n^2 m^3 d^2 \). Applying Lemma 1 to the matrix

\[
\begin{pmatrix}
a^{(2)}_\lambda & a^{(3)}_{\lambda, \ell} \\
0 & a^{(3)}_{\lambda, \ell}
\end{pmatrix}
\]

we find \( h_1^{(\ell)}, \ldots, h_{r-\ell}^{(\ell)} \), \( h_{\lambda, \ell}^{(\ell)} \in \mathcal{A}_n \), such that (4) holds and \( \deg(h_1^{(\ell)}), \ldots, \deg(h_{r-\ell}^{(\ell)}), \deg(h_{\lambda, \ell}^{(\ell)}) \leq 128 n^2 m^3 d^2 \).

Sec. 2. Solution of Systems of Linear Equations Over the Weyl Algebra

Throughout this section \( \mathcal{H} = \mathcal{A}_n \). The next lemma is an analog of the normalization lemma for the Weyl algebra. Below \( q_1, \ldots, q_{\ell} \in \mathcal{A}_n \) is a finite family of elements.

**LEMMA 4.** There is a nonsingular linear transformation over \( \mathbb{F} \) of the 2n-dimensional space with basis \( X_1, \ldots, X_n, X_1^*, \ldots, X_n^* \),

\[
\begin{array}{ll}
D_1, \ldots, D_n, \text{ under which } & X_i - \Gamma_{i,j} X_j = \sum_j \gamma_{ij}^{(1)} X_j + \sum_k \gamma_{ij}^{(2,1)} D_k J_i + \sum_k \gamma_{ij}^{(2,2)} D_k J_i D_k,
D_i - \Gamma_{i,j} D_j = \sum_j \gamma_{ij}^{(2,1)} X_j + \sum_k \gamma_{ij}^{(2,2)} D_k D_j, & i \neq j,
\end{array}
\]

such that \( \Gamma_{i,j} \) is the generating matrix of \( \mathcal{A}_n \). If \( \gamma_{ij}^{(1)} = 1 \) for \( i, j \neq 1 \) and \( \gamma_{ij}^{(2,1)} = 1 \), then \( \gamma_{ij}^{(2,2)} \) is the identity matrix. In the order listed. Then the relations indicated in the formulation of the lemma are equivalent to \( \Gamma \) belonging to the symplectic group \( \mathcal{S}_n (\mathbb{F}) \), i.e., satisfying \( \Gamma \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix} \). Let

\[ \hat{q}_t = \sum_{i,j} \hat{\epsilon}_{i,j} D_i X_j + \sum_{i,j} \hat{\epsilon}_{i,j} X_j X_i + \sum_{i,j} \hat{\epsilon}_{i,j} J_i J_j, \]

where all terms of highest degree \( \hat{\epsilon}_{i_1 \cdots i_\ell} \) \( \ell \geq \ell \) are gathered in the left summand, i.e., \( \deg(J_i J_j) < \deg(J_i J_j) \). Then after performing the linear transformation the leading coefficient \( \mathcal{L} \mathcal{A}_n (\hat{q}_t) \) is equal to the expression

\[
\sum_{i,j} \hat{\epsilon}_{i,j} \mathcal{L} \mathcal{A}_n (\hat{q}_t) \approx \mathcal{L} \mathcal{A}_n (\hat{q}_t), \]

which one can consider as a homogeneous polynomial in the elements of the last column of \( \Gamma \). Since the symplectic group acts transitively on all vectors \( \mathbb{F}^2 \setminus \{0\} \), one can find a matrix \( \Gamma \in \mathcal{S}_n (\mathbb{F}) \) with arbitrarily preassigned nonzero last column. We choose a column such that all the homogeneous polynomials in its elements indicated are nonzero. Then such a matrix \( \Gamma \) satisfies the requirements of the lemma.

We apply Lemma 4 to the family (cf. (4)) of elements \( h_1^{(\ell)}, \ldots, h_{r-\ell}^{(\ell)}, h_{\lambda, \ell}^{(\ell)} \in \mathcal{A}_n \); the linear transformation \( \Gamma \) obtained preserves \( \mathcal{A}_n \), so one can consider \( \mathcal{A}_n \) as the Weyl algebra on \( \Gamma_{i,j} \) and \( \mathcal{D}_i \), and making a change of variables we shall assume that \( 0 \neq \mathcal{D}_n = \mathcal{L} \mathcal{A}_n (h_{n, n}^{(\ell)}) \in \mathcal{F} \), \( 1 \leq \ell \leq 5 \). Then one can divide \( \nu \) (cf. (3)) from the left by \( h_{k, \ell}^{(\ell)} \) with remainder (cf. (4)) with respect to \( \mathcal{D}_n \), in the ring \( \mathcal{A}_n \), i.e., \( \mathcal{D}_n \nu_{(k, \ell)} = \mathcal{D}_n h_{k, \ell}^{(\ell)} \mathcal{D}_n \nu_{(k, \ell)} + \mathcal{D}_n \nu_{(k, \ell)} \in \mathcal{F} \), while \( \deg(\mathcal{D}_n (\mathcal{D}_n \nu_{(k, \ell)})) < \deg(\mathcal{D}_n h_{k, \ell}^{(\ell)}) \). Namely, let \( \nu_{(k, \ell)} = \mathcal{D}_n \nu_{(k, \ell)} = \mathcal{D}_n h_{k, \ell}^{(\ell)} \mathcal{D}_n \nu_{(k, \ell)} + \mathcal{D}_n \nu_{(k, \ell)} \), where \( \nu_{(k, \ell)} h_{k, \ell}^{(\ell)} \in \mathcal{F} \). One can consider the last
ring as a polynomial ring with coefficients from the algebra \( \mathfrak{A}_{n-1} \) in one variable \( X_n \) commuting with it. Let \( V_{j}^{(i)} = V_{j}^{(i)} - X_j \), \( D_{n}^{*} = V_{n}^{*} \), ..., \( p_1 = \text{deg}_2 \mathfrak{A}_{n}, (V_{j}^{(i)}) = p_{1} \), in addition \( V_{j}^{(i)} = V_{j}^{(i)} - X_j \). \( D_{n}^{*} \), \( c_{D_{n}}(V_{j}^{(i)}) = p_{1} \), etc. By induction on \( j \) we prove that \( \text{deg}_2 \mathfrak{A}_{n}, (V_{j}^{(i)}) = p_{j} \). If \( p_{j} = p_{j+1} \), finally, we get \( V_{j}^{(i)} = V_{j}^{(i)} \).

From (3) we subtract (4) multiplied on the right by \( V_{j}^{(i)} \) for each \( n \leq \ell \leq s \), respectively; we get as a result the following system of equations, which has a solution in \( \mathfrak{A}_{n} \) if and only if (3) and thus (2) have a solution:

\[
a_{k} V_{j}^{(i)} + \sum_{\ell = k}^{s} a_{k, \ell} V_{\ell}^{(i)} = b_{k}^{(i)}, \quad 1 \leq k \leq r.
\]

(5)

Since \( \text{deg}_2 \mathfrak{A}_{n}, (a_{k, \ell} V_{\ell}^{(i)}) \leq N_{p} \), where \( N_{p} = (n, m, d) \), one has \( \text{deg}_2 \mathfrak{A}_{n}, c_{D_{n}}(a_{k, \ell} V_{\ell}^{(i)}) \leq N_{p} \) for \( 1 \leq k \leq r \).

We write each \( V_{j}^{(i)} = \sum_{\ell = k}^{s} D_{n}^{i} \), for \( 1 \leq j \leq s \) where \( V_{j}^{(i)} \in \mathfrak{A}_{n-1} \). Then one can replace (5) by an equivalent linear system in the variables \( V_{j}^{(i)} \), \( 1 \leq j \leq s \), \( 0 \leq i \leq N_{p} \), where each equality is replaced by \( (N_{p} + 1) \) equalities, namely:

\[
a_{k} V_{j}^{(i)} + \sum_{\ell = k}^{s} a_{k, \ell} V_{\ell}^{(i)} = \sum_{\ell = k}^{s} D_{n}^{i} \sum_{\ell = k}^{s} a_{k, \ell} V_{\ell}^{(i)} + \sum_{\ell = k}^{s} D_{n}^{i} \sum_{\ell = k}^{s} \sum_{\ell = k}^{s} a_{k, \ell} V_{\ell}^{(i)} \text{ and } b_{k} = \sum_{\ell = k}^{s} D_{n}^{i} b_{k, \ell}^{(i)}, \text{ where } a_{k, \ell}, \sum_{\ell = k}^{s} a_{k, \ell} V_{\ell}^{(i)} = \sum_{\ell = k}^{s} b_{k, \ell}^{(i)}, \text{ for } 0 \leq i \leq N_{p} \). In addition, if \( (a_{k, \ell}, \sum_{\ell = k}^{s} a_{k, \ell} V_{\ell}^{(i)}) \in \mathfrak{A}_{n-1} \) for \( 0 \leq i \leq N_{p} \). Thus we have proved

**LEMMA 5.** The system (2) is equivalent to a linear system

\[
\sum_{\ell = k}^{s} a_{k, \ell} V_{\ell}^{(i)} = b_{k}^{(i)}, \quad 1 \leq k \leq m (N_{p} + 1) = N_{3}
\]

(6)

over the ring \( \mathfrak{A}_{n-1} \), where \( \deg(a_{k, \ell}), \deg(b_{k}^{(i)}), N_{3}, N_{2}, N_{5} = (n, m, d)^{0,0}, N_{3} = s (n, m, d)^{0,0} \). Moreover, if (6) has a solution \( \sum_{\ell = k}^{s} Z_{\ell}^{(i)} = \mathfrak{A}_{n-1} \), such that \( \deg(Z_{\ell}^{(i)}) = N_{2} \) for \( 1 \leq \ell \leq N_{2} \), then (2) has a solution \( V_{1}, \ldots, V_{s} \in \mathfrak{A}_{n} \) for which \( \deg(V_{\ell}) \leq N_{2} + N_{p} \), \( 1 \leq \ell \leq s \).

Thus, Lemma 5 lets one eliminate \( D_{n} \) and pass to the consideration of linear equations over the ring \( \mathfrak{A}_{n-1} \). Now we shall similarly eliminate \( X_{n} \). To begin we make the observation that Lemma 1 and hence also Lemma 2 are also valid for the ring \( \mathfrak{A}_{n-1} \). One can verify this by following the proof of Lemma 1 and in addition derive it directly from Lemma 1 in the following way. Let \( A \) be an \((n-1) \times m \) matrix with elements from the ring \( \mathfrak{A}_{n-1} \), so by Lemma 1 one can find a vector \( 0 \neq b = (b_{1, i}, \ldots, b_{m, i}) \in (\mathfrak{A}_{n}, m) \), such that \( Ab = 0 \). We represent each element \( b_{j, i} \), \( 1 \leq j \leq m \), in the form \( b_{j, i} = \sum_{k} b_{j, k} D_{n}^{i} \), where \( b_{j, k} \in \mathfrak{A}_{n-1} \), while \( \deg(b_{j, k}) = \deg(b_{j}) \) (cf. (1)). Let \( i_{0} \) be the smallest index such that \( b_{j, i_{0}} \neq 0 \) for at least one \( j \). Then \( A \) \((b_{1, i_{0}}, \ldots, b_{m, i_{0}}) \) = 0, which proves the analog of Lemma 1 for the ring \( \mathfrak{A}_{n-1} \). Applying the construction described above in Sec. 1 to (6) as to (2), we reduce (6) to trapezoidal form (cf. (3)):

\[
P_{k} Z_{k} + \sum_{\ell = k}^{s} p_{\ell} Z_{\ell} = q_{k}, \quad 1 \leq k \leq \tau_{1}.
\]

(3')

where \( P_{k}, P_{\ell}, Z_{\ell} = \mathfrak{A}_{n-1} \), and \( \tau_{1} \) is the rank of the \( N_{3} \) \times \( N_{2} \) matrix \( (1, \tau_{1}) \). By Lemma 2 and the observation made above, \( \deg(p_{k}), \deg(p_{\ell}), \deg(q_{k}) < (n, m, d)^{0,0} \).

Further, analogously to (4), in view of Lemma 1 and the observation made above, for each \( \tau_{1} \leq \ell \leq N_{2} \) one can find \( q_{1}^{(i)}, \ldots, q_{\ell}^{(i)}, q_{\ell}^{(i)} = \mathfrak{A}_{n-1} \), such that \( \deg(q_{1}^{(i)}), \deg(q_{\ell}^{(i)}) < (n, m, d)^{0,0} \) and

\[
P_{k} q_{1}^{(i)} + \sum_{\ell = k}^{s} p_{k, \ell} q_{\ell}^{(i)} = 0, \quad 1 \leq k \leq \tau_{1}.
\]

(4')

Let \( q_{1}^{(i)}, \ldots, q_{\ell}^{(i)} = \mathfrak{A}_{n-1} \) be a finite family of elements. The next lemma is an analog of Lemma 4.
**Lemma 4'**. There exists a nonsingular linear transformation over \( F \) of \( (2n - 1) \)-dimensional space under which
\[
X_n \rightarrow X_n, \quad X_i \rightarrow \Delta X_i = X_i + S^{(1)}iX_n, \quad D_i \rightarrow \Delta D_i = D_i + S^{(2)}iX_n, \quad 1 \leq i \leq n - 1,
\]
such that \( 0 \neq \ell \in \mathcal{X}_n (\gamma'_K) \subseteq F \) for all \( 1 \leq k \leq t \), where \( \gamma'_K \) is obtained from \( \gamma'_K \) with the help of the linear transformation indicated.

The proof is similar to the proof of Lemma 4. Let
\[
\gamma'_K = \sum_{I, j, 3} \beta^{(3)}_{I, j, 3} D_k^{i_{u-1}} \cdots D_1^{i_1} X_n^{i_{u-n}} X_n^{i_{u-n-1}} \cdots X_n^{i_{u-n-t}} + \sum_{\ell} \gamma_{\ell K},
\]
where all terms of highest degree from \( \gamma'_K \) are gathered into the left summand. Then
\[
\ell \in \mathcal{X}_n (\gamma'_K) = \sum_{I, j, 3} \beta^{(3)}_{I, j, 3} (S_3^{(2)}i)^{i_{u-1}} \cdots (S_3^{(2)}i)^{i_1} (S_3^{(0)}i)^{i_{u-n}} \cdots (S_3^{(0)}i)^{i_1}.
\]
One can find \( S_3^{(1)} \), \( S_3^{(1)} \), \( S_3^{(2)} \), \( S_3^{(2)} \), \( S_3^{(0)} \) \( \in F \) such that the leading coefficients indicated are nonzero for all \( 1 \leq k \leq t \), which proves the lemma.

We apply Lemma 4' to the family (cf. (4')) of elements \( \gamma'(t) \), \( 1 \leq t \leq N_n \); the linear transformation \( \Delta \) constructed obviously preserves (1), so one can again consider \( \mathcal{A}_{u-1} \) as the Weyl algebra on \( \Delta X_n, \ldots, \Delta X_n, \Delta D_1, \ldots, \Delta D_{n-u-1} \), and making a change of variables we shall assume that \( 0 \neq \ell \in \mathcal{X}_n (\gamma'(t)) \subseteq F \), \( u+1 \leq t \leq N_n \).

Similarly to the above, we divide \( Z_\ell \) (cf. (3)) on the left by \( \gamma'(t) \) (cf. (4')) with remainder with respect to \( X_n \) for \( u+1 \leq t \leq N_n \), in the ring \( \mathcal{A}_{n-1} [X_n] \), and thus \( Z_\ell \equiv y'(0) E_\ell + \tilde{E}_\ell \), where \( \deg y'(0) (E_\ell) \leq \deg y'(0) (y'(t)) \leq (u \text{ mod } n) 0(\ell) \) (cf. (4')). Then from (3') we subtract (4') multiplied on the right by \( \tilde{Z}_\ell \) for each \( u+1 \leq t \leq N_n \), respectively. As a result we get the following system of linear equations which has a solution in \( \mathcal{A}_{n-1} [X_n] \), if and only if (3') and thus (6) have a solution:
\[
p_k \sum_{u+1 \leq t \leq N_n} p_k \tilde{Z}_\ell = \gamma'_K, \quad 1 \leq k \leq u.
\]

Just as before we estimate
\[
\deg y'(0) (\tilde{Z}_\ell) \leq \deg y'(0) (p_k \tilde{Z}_\ell) \leq \max_{u+1 \leq t \leq N_n} \{ \deg y'(0) (p_k \tilde{Z}_\ell) \} \leq \deg y'(0) (y'(t)) \leq (u \text{ mod } n) 0(\ell). \]
Similarly we write
\[
\tilde{Z}_{j, i} = \sum_{0 \leq s \leq N_n} x_s \tilde{Z}_{j, i}, \quad \text{where } \tilde{Z}_{j, i} = \mathcal{A}_{n-1} \text{, and we replace each of the equations from (5') by } (N_{u+1}) \text{ linear equations in the variables } \tilde{Z}_{j, i}, \quad 1 \leq j \leq N_n, \quad 0 \leq i \leq N_n \text{ over the ring } \mathcal{A}_{n-1} \text{ (cf. above). Analogously to Lemma 5 one proves the following:}

**Lemma 5'**. The system (6) is equivalent to a system of linear equations
\[
\sum_{j} \tilde{Z}_{j, i} Y_j = h'_i
\]
over the ring \( \mathcal{A}_{n-1} \), where the number of equations and the degrees of all \( \tilde{Z}_{j, i} \), \( h'_i \) are bounded by \( (u \text{ mod } n) 0(\ell) \). In addition if (6) has a solution \( \{ Y_j \in \mathcal{A}_{n-1} \} \), where \( \deg (Y_j) \in N_n \) for some \( N_n \) and all \( j \), then (6) has a solution \( \{ Z_{j, i} \in \mathcal{A}_{n-1} [X_n] \} \), for which \( \deg (Z_{j, i}) \leq N_n + N_n \), where \( N_n \leq (u \text{ mod } n) 0(\ell) \).

To prove the theorem (cf. Introduction), arguing by induction on \( n \) one can assume that the theorem is proved for the ring \( \mathcal{A}_{n-1} \) and the system (6') has a solution \( \{ Y_j \in \mathcal{A}_{n-1} \} \) (if it is solvable over the ring \( \mathcal{A}_{n-1} \)), such that \( \deg (Y_j) \leq (u \text{ mod } n) c^2 (c+1) \), where the constant \( c_1 \) is chosen from the estimates of Lemma 5' and the constant \( c \) from the inductive hypothesis, while by increasing \( c \) one can assume that \( 2 c_1 < 2 < c^2 \). Then it follows from Lemmas 5' and 5 in succession that (6) and (2) have solutions of degrees at most \( (u \text{ mod } n) c^2 \), which proves the inductive hypothesis for the ring \( \mathcal{A}_{n} \) and thus the theorem for the case of the ring \( \mathcal{R} = \mathcal{A}_{n} \).

**Sec. 3. Solution of Systems of Linear Equations Over the Algebra of Differential Operators**

Now we return to consideration of the system (3) over the ring \( \mathcal{R} = \mathcal{K}_n \). For the element \( q_0 = q_0, q_0 \in \mathcal{K}_n \), where
\[
q_0 = \sum_{0 \leq j \leq n} D_n^{j} q_0 = \sum_{0 \leq j \leq n} D_n^{j} \xi_{n} \in \mathcal{A}_n, \quad q_0 \in \mathcal{K}_n, \quad q_0 \neq 0, \quad q_0 \in \mathcal{A}_n, \quad q_0 \neq 0, \quad q_0 \in \mathcal{K}_n.
\]

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the leading coefficient \( \mathcal{C}_{D_n} \left( \frac{q}{l} \right) = \frac{q}{l} \mathcal{C}_{D_n} \left( \frac{q}{l} \right) \subseteq \mathcal{C}_{D_n} \left( l \mathcal{C}_{D_n} \left( \frac{q}{l} \right) \right) \). The next lemma is an analog of Lemma 4. Let \( q^{(1)}, \ldots, q^{(t)} \subseteq \mathcal{C}_{D_n} \) be a family of operators. \( \frac{q^{(1)}}{l} \subseteq \mathcal{C}_{D_n} \)

**Lemma 6.** There is a nonsingular linear transformation over \( \mathcal{F} \) of \( 2n \)-dimensional space under which the vector \( D_{\mathcal{F}} = (\mathbf{1}, \ldots, \mathbf{1})^{T} \), the vector \( (X_1, \ldots, X_n)^T \rightarrow (\mathbf{1}, \ldots, \mathbf{1})^{T} \), in addition for any \( 1 \leq l \leq t \) the leading coefficient \( \mathcal{C}_{D_n} \left( \frac{q^{(l)}}{l} \right) \subseteq \mathcal{C}_{D_n} \) is obtained from \( \frac{q^{(l)}}{l} \) with the help of the linear transformation indicated.

**Proof.** Obviously (1) is preserved under the linear transformation indicated. Let \( g_1 \) denote one of the elements \( g^{(1)} g_2, \ldots, g^{(t)} g_2 \); we use the notation introduced before the lemma and write \( \frac{q}{l} = \sum_{i=0}^{t} D_{\mathcal{F}}^{(i)} \sum_{i=1}^{n} \sum_{i=0}^{t} \frac{q^{(i)}}{l} = \sum_{i=0}^{t} \sum_{i=0}^{t} \frac{q^{(i)}}{l} \) where \( \sum_{i=0}^{t} \frac{q^{(i)}}{l} \) is obtained from \( \frac{q}{l} \) and the elements \( g^{(1)}, \ldots, g^{(t)} \) are different from zero. We conclude that \( \frac{q}{l} \) is obtained from \( \frac{q}{l} \) and the elements \( g^{(1)}, \ldots, g^{(t)} \) are different from zero; we supplement it to a nonsingular matrix and this completes the proof of the lemma.

We apply Lemma 6 to the family of operators \( l_\mu \left( \frac{q}{l} \right), 1 \leq \mu \leq s \); the linear transformation obtained preserves (1) and in addition both the \( n \)-dimensional linear spaces generated by \( X_1, \ldots, X_n \) and \( X_2, \ldots, X_n \), respectively, so one can consider \( \mathcal{F}_{D_n} \) as an algebra of operators on a vector of variables \( \left( \mathbf{1}, \ldots, \mathbf{1}, X_0, \ldots, X_n \right) \) and a vector of differential operators \( \mathcal{F}_{D_n} \) and \( \mathcal{F}_{D_n} \), and making a change of variables we shall assume that \( \mathcal{C}_{D_n} \left( \frac{q}{l} \right) \subseteq \mathcal{C}_{D_n} \) for \( 1 \leq l \leq s \).

Then just as above in Sec. 2 one can divide \( \mathcal{F}_{D_n} \) (cf. (3)) on the left by \( \frac{q}{l} \) with remainder with respect to \( D_{\mathcal{F}} \) for \( 1 \leq l \leq s \) in the algebra \( \mathcal{F}_{D_n} \), i.e., \( \mathcal{F}_{D_n} = \mathcal{F}_{D_n} \mathcal{F}_{D_n} \), where \( \mathcal{F}_{D_n} = \mathcal{F}_{D_n} \mathcal{F}_{D_n} \mathcal{F}_{D_n} \) while \( \deg \mathcal{F}_{D_n} (\frac{q}{l}) \leq \deg \mathcal{F}_{D_n} (\frac{q}{l}) < \deg \mathcal{F}_{D_n} (\frac{q}{l}) = (\deg \mathcal{F}_{D_n}) \). From (3) we subtract (4) multiplied on the right by \( \mathcal{F}_{D_n} \) for each \( 1 \leq l \leq s \), respectively; we get as a result the following system of equations, which has a solution in \( \mathcal{F}_{D_n} \) if and only if (3) and thus (2) have a solution:

\[
a_k \mathcal{F}_{D_n} = \sum_{1 \leq \ell \leq s} a_{\ell,k} \mathcal{F}_{D_n} (\ell), \quad 1 \leq k \leq n
\]

(7)

Similarly to Sec. 2 we establish the estimates \( \deg \mathcal{F}_{D_n} (\mathcal{F}_{D_n}) = \deg \mathcal{F}_{D_n} (\mathcal{F}_{D_n}) \leq m \) and thereby (7) for \( 1 \leq k \leq n \).

We write \( \mathcal{F}_{D_n} = \sum_{1 \leq \ell \leq s} \sum_{1 \leq \ell \leq s} a_{\ell,k} \mathcal{F}_{D_n} \), where \( \mathcal{F}_{D_n} = \mathcal{F}_{D_n} \mathcal{F}_{D_n} \mathcal{F}_{D_n} \) and each of the equations of (7) is replaced by (8) equations. Namely, let \( a \) be one of the elements of the form \( a_{\ell,k} \), \( a_{\ell,k} \), and \( \mathcal{F}_{D_n} \) be the corresponding element \( \mathcal{F}_{D_n} \), \( \mathcal{F}_{D_n} \mathcal{F}_{D_n} \), and \( \mathcal{F}_{D_n} \mathcal{F}_{D_n} \mathcal{F}_{D_n} \), and hence (7) for \( 1 \leq k \leq n \). Analogously to the proof of Lemma 5 in the case of the equations of (7) we equate the coefficients of \( \mathcal{F}_{D_n} \) for \( 1 \leq k \leq n \), and we get the \( n+1 \) equations required. Thus, we have proved

**Lemma 7.** The system (2) is equivalent to a system of linear equations

\[
\sum_{1 \leq \ell \leq s} b_{\ell,k} \mathcal{F}_{D_n} = a_k, \quad 1 \leq k \leq n + 1
\]

(8)

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over the algebra \( \mathcal{H}_{n-1}(F(X_n)) \), where \( \deg(\delta_{X_n}), \deg(\alpha_{X_n}), \mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3 \leq (nmd)^{\delta_{n}} \). In addition, if system (8) has a solution \( Z_1, \ldots, Z_{n-1} \in \mathcal{H}_{n-1}(F(X_n)) \), such that \( \deg(Z_1) \leq \mathcal{N}_1 \) for \( 1 \leq l \leq \mathcal{N}_2 \), then (2) has a solution \( V_1, \ldots, V_{n-1} \in \mathcal{H}_{n}(F) \), for which \( \deg(V_1) \leq \mathcal{N}_1(n-1), \ 1 \leq l \leq s \).

To prove the theorem (cf. Introduction), arguing by induction on \( n \) one can assume that (8) has a solution \( Z_1, \ldots, Z_{n-1} \), \( \mathcal{N}_2, \mathcal{N}_3 \leq \mathcal{H}_{n-1}(F(X_n)) \) (if it is solvable over the algebra \( \mathcal{H}_{n-1}(F(X_n)) \), such that \( \deg(Z_1) \leq (nmd)^{c_2} \), \( 1 \leq l \leq \mathcal{N}_2 \), where the constant \( c_2 \) is chosen from the estimates of Lemma 7 and the constant \( c_0 \) from the inductive hypothesis, while by increasing \( c_0 \), one can assume that \( 2c_2 \leq c_1^{c_2(n-1)} \leq 2c_1^{c_1} \). Then it follows from Lemma 7 that system (2) has a solution of degree at most \( (nmd)^{c_1} \), which proves the inductive hypothesis. We also note that the base of the induction for the field \( \mathcal{H}_{n-1}(F(X_1, \ldots, X_n)) = \mathcal{H}_n(F(X_1, \ldots, X_n)) \) follows from the estimates on determinants over this field. This completes the proof of the theorem for the case of the ring \( \mathcal{H}_n(F) \).

REFERENCES