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Finding Connected Components of a Semialgebraic Set in Subexponential Time

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Abstract. Let a semialgebraic set be given by a quantifier-free formula of the first-order theory of real closed fields with atomic subformulae of type $(f_i \geq 0)$, $1 \leq i \leq k$ where the polynomials $f_i \in \mathbb{Z}[X_1, \ldots, X_n]$ have degrees $\deg(f_i) < d$ and the absolute value of each (integer) coefficient of f_i is at most 2^M. An algorithm is designed which finds the connected components of the semialgebraic set in time $M^{O(1)}(kd)^{n^{O(1)}}$. The best previously known bound $M^{O(1)}(kd)^{n^{O(n)}}$ for this problem follows from Collins' method of Cylindrical Algebraic Decomposition.

Key words: Semialgebraic set, Connected component, Subexponential-time algorithm, Infinitesimals, Tarski algebra

In this paper we describe a subexponential-time algorithm which finds the connected components of a semialgebraic set, given by a quantifier-free formula of the first-order theory of real closed fields (for a quite wide class of fields, cf. [1,2]). This result generalizes the main theorem from [15] (see also [13]) and is obtained by a modification of the construction from [15]. In [4] (see also [5]) the now well-known method of cylindrical algebraic decomposition was introduced, which allows one to find the connected components within exponential time.

For an ordered field F by $\tilde{F} \supset F$ we denote its unique real closure (see e.g. [8]). In the sequel we consider input polynomials with coefficients in an ordered ring $\mathbb{Z}_m = \mathbb{Z}[\bar{\delta}_1, \ldots, \bar{\delta}_m] \subset \mathbb{Q}_m = \mathbb{Q}(\bar{\delta}_1, \ldots, \bar{\delta}_m)$ where $\bar{\delta}_1, \ldots, \bar{\delta}_m$ are algebraically independent elements over $\mathbb{Q}$ and an order in $\mathbb{Q}_m$ is determined as follows. An element $\delta_1 > 0$ is an infinitesimal w.r.t. $\mathbb{Q}$ (i.e. $0 < \delta_1 < a$ for any rational number $\mathbb{Q} \ni a > 0$), thereupon for each $1 \leq i < m$ the element $\bar{\delta}_{i+1} > 0$ is infinitesimal w.r.t. the field $\mathbb{Q}_i$.

So, let a quantifier-free input formuly Ξ of the first-order theory of real closed fields be given, containing k atomic subformulas of the form $f_i \geq 0$, $1 \leq i \leq k$, where $f_i \in \mathbb{Z}_m[X_1, \ldots, X_n]$.

A rational function $g \in \mathbb{Q}_m(Y_1, \ldots, Y_3)$ is representable in the form $g = g_1/g_2$, where the polynomials $g_1, g_2 \in \mathbb{Z}_m[Y_1, \ldots, Y_3]$ are relatively prime. Denote by $l(g)$

the maximum of bit-sizes of (interger) coefficients of the polynomials g_1, g_2. In the sequel we assume that the following bounds hold:

$$\deg_{X_1,\dots,X_n}(f_i) < d, \qquad \deg_{\delta_1,\dots,\delta_m}(f_i) < d_0, \qquad l(f_i) \le M \qquad 1 \le i \le k \tag{1}$$

for some integers d, d_0, M. Then the bit-size of the formula Ξ does not exceed a value $\mathscr{L} = kMd^n d_0^m$ (cf. [1, 2, 6, 7]).

Note that in the case $m = 0$ (in other words for polynomials with integer coefficients) the algorithms from [4, 5] allow us to find connected components in time polynomial in $M(kd)^{O(n)}$.

The semialgebraic set $\{\Xi\} \subset (\tilde{\mathbb{Q}}_m)^n$ determined by the formula Ξ and consisting of all the points satisfying the formula Ξ can be uniquely represented as the union of its connected components $\{\Xi\} = \bigcup_{1\le i\le t} \{\Xi_i\}$. Each of them in turn is a semialgebraic set determined by a suitable quantifierfree formula Ξ_i of the first-order theory of the field $\tilde{\mathbb{Q}}_m$ (see e.g. [4, 5] for the field $\mathbb{R}$, for an arbitrary real closed field it follows from [3], see also below). Note that the number of connected components $t \le (kd)^{O(n)}$ (cf. [1, 2]).

In the present paper the algebraic points $u = (u_1, \dots, u_n) \in (\tilde{\mathbb{Q}}_m)^n$ will be represented in the following way (cf. [1, 2]). A primitive element η (see [6]) of the field $\mathbb{Q}_m(u_1, \dots, u_n)$ is produced such that $\mathbb{Q}_m(u_1, \dots, u_n) = \mathbb{Q}_m[\eta]$. For the element η its minimal polynomial is denoted $\varphi(Z) \in \mathbb{Q}_m[Z]$, moreover $\eta = \sum_{1\le i\le n} \alpha_i u_i$ for appropriate integers $0 \le \alpha_1, \dots, \alpha_n \le \deg_Z(\varphi)$. In addition we obtain an expression $u_i = \sum_{0\le j\le \deg_Z(\varphi)} \beta_i^{(j)} \eta^j$, where $\beta_i^{(j)} \in \mathbb{Q}_m$. Finally, for specifying the root η of φ, a sequence of signs of the derivatives of all orders $\varphi^{(1)}(\eta), \varphi^{(2)}(\eta), \dots, \varphi^{(\deg(\varphi))}(\eta)$ of the polynomial φ in the point η is given. Thom's lemma (see e.g. [9]) implies that the latter condition specific the root η of φ uniquely. We say that a polynomial $g \in \mathbb{Z}_m[X_1, \dots, X_n]$ satisfies a $(D, D_0, \mathscr{M})$-bound if the following inequalities hold: $\deg_{X_1,\dots,X_n}(g) < D$; $\deg_{\delta_1,\dots,\delta_m}(g)$, $\deg_{\delta_1,\dots,\delta_m}(\beta_i^{(i)}) < D_0$; $l(g)$, $l(\beta_i^{(j)}) \le \mathscr{M}$. The point u satisfies a $(D, D_0, \mathscr{M})$-bound if the polynomials φ, $\beta_i^{(j)}$ satisfy this bound.

Then the bit-size of the representation of the point u does not exceed $(\mathscr{M} D D_0^m n)^{O(1)}$ (cf. [1, 2]).

Theorem. *One can design an algorithm, which for any formula of the form Ξ satisfying the bounds* (1), *finds the connected components of the semialgebraic set $\{\Xi\} \subset (\tilde{\mathbb{Q}}_m)^n$ within time $M^{O(1)}(kd)^{n^{O(1)}(m+1)} d_0^{O(n+m)}$ (subexponential in $\mathscr{L}$). The algorithm outputs each connected component by means of a certain quantifier-free formula Ξ_i (see above) with $(kd)^{n^{O(1)}}$ atomic subformulae of the type $g \ge 0$, where a polynomial $g \in \mathbb{Z}_m[X_1, \dots, X_n]$ satisfies a $((kd)^{n^{O(1)}}, d_0(kd)^{n^{O(1)}}, (M + md_0)(kd)^{n^{O(1)}})$-bound.*

For proving the theorem we shall need the following subexponential-time algorithms: (1) the algorithm from [6, 7] for finding irreducible components of an algebraic variety (defined over an algebraically closed field; (2) the algorithm from [1] for solving a system of polynomial inequalities; (3) an algorithm (see [2, 16, 17]) for quantifier elimination in the theory of real closed fields for the formulas with a restricted number of quantifier alternations.

Let us mention that in [15] (see also [13]) an algorithm is describe which counts the number of connected components of a semialgebraic set $\{\Xi\}$ within time

$M^{O(1)}(d_0(kd)^{n^{19}})^{O(n+m)}$. Moreover, this algorithm allows one to recognize for two points from $\{\Xi\}$, whether they are situated in the same connected component. When [15] was already written, the authors learned that a similar result was obtained by J. Heintz, M.-F. Roy, P. Solerno, see [19–22]. The algorithm described in the present paper can be regarded as an "uniformization" of the algorithm from [15]. More precisely, when one of two given points is considered to be "variable", the algorithm expresses the condition that the "variable" point belongs to the same connected component as the second fixed point via a suitable quantifier-free formula.

In the sequel we shall need some geometrical and topological notions and statements about the space $(\tilde{\mathbb{Q}}_m)^n$. To justify them we make use of the following statement called the "transfer principle". Let $K_1 \subset K_2$ be an extension of real closed fields and Π be a certain closed formula of the first-order theory of the field K_1, then its truth values over the fields K_1 and K_2 coincide (see [3]). In particular, in [1] the existence and uniqueness of the decomposition of a semialgebraic set in $(\tilde{\mathbb{Q}}_m)^n$ into a union of its connected components was proved with the aid of the transfer principle.

Let polynomials $f_1, \dots, f_k \in \mathbb{Z}_m[X_1, \dots, X_n]$, satisfying (1) (or (d, d_0, M)-bounds), be given. We call $\{f_1, \dots, f_k\}$-*cell* (cf. [1]) any nonempty semialgebraic set of the kind $\left\{ \underset{i\in I}{\&}(f_i = 0) \& \underset{i_1\in I_1}{\&}(f_{i_1} > 0) \underset{i_2\in I_2}{\&}(f_{i_2} < 0)\right\}$, where $I \cup I_1 \cup I_2 = \{1, \dots, k\}$. By $\mathscr{U}(\{f_1, \dots, f_k\})$ we denote the partition of the space $(\tilde{\mathbb{Q}}_m)^n$, whose elements are connected components of all $\{f_1, \dots, f_k\}$-cells. A finite set of points $\mathscr{R} \subset (\tilde{\mathbb{Q}}_m)^n$ we call a representative set for the polynomials $f_1, \dots, f_k$ if for every element $W \subset (\tilde{\mathbb{Q}}_m)^n$ of the partition $\mathscr{U}\{f_1, \dots, f_k\})$ $\mathscr{R} \cap W \neq \emptyset$ holds. The algorithm described in [1] yields a representative set $\mathscr{R}$ for $f_1, \dots, f_k$ within time $M^{O(1)}d_0^{O(m)}(kd)^{O(n(m+1))}$. Furthermore, each point from $\mathscr{R}$ satisfied $((kd)^{O(n)}$, $d_0(kd)^{O(n)}$, $(M + md_0)\,(kd)^{O(n)})$-bounds. Applying the algorithm from [15] one can find a subset $\mathscr{R}' \subset \mathscr{R}$ such that for every element $W \in \mathscr{U}(\{f_1, \dots, f_k\})$ the intersection $\mathscr{R}' \cap W$ consists of a single point.

1. Reduction of Finding Connected Components to the Case of a System of Inequalities

Let K be an arbitrary real closed field (see e.g. [8]) and an element $\varepsilon > 0$ be an infinitesimal with respect to the field K (see above). Let us recall some well known facts about real closed fields. A Puiseux series (or fractional-power series) over K is a series of the form $\sum_{i\geq 0} \alpha_i \varepsilon^{\nu_i/\mu}$ where $0 \neq \alpha_i \in K$, the integers $\nu_0 < \nu_1 < \dots$ increase and the integer $\mu \geq 1$. The field $K((\varepsilon^{1/\infty}))$ consisting of all Puiseux series (with zero added) is real closed, hence $K((\varepsilon^{1/\infty} \supset \widetilde{K(\varepsilon)} \supset K(\varepsilon)$. Furthermore, the field $K\left[\sqrt{-1}\right]((\varepsilon^{1/\infty})) = K((\varepsilon^{1/\infty}))$ is algebraically closed.

When $\nu_0 < 0$, the element $a \in K((\varepsilon^{1/\infty}))$ is called infinitely large, if $\nu_0 > 0$ then a is infinitesimal (with respect to the field K). A vector $(a_1, \dots, a_n) \in (K((\varepsilon^{1/\infty})))^n$ is called K-finite if each of its coordinates $a_i (1 \leq i \leq n)$ is not infinitely large. For any K-finite element $a \in K((\varepsilon^{1/\infty}))$ its *standard part* $\mathrm{st}(a) \in K$ is definable (cf. [1, 2]).

Namely, $\mathrm{st}(a) = \alpha_0$ when $\nu_0 = 0$ and $\mathrm{st}(a) = 0$ when $\nu_0 > 0$. Similarly, one can define the standard part of a Puiseux series from $\bar{K}((\varepsilon^{1/\infty}))$. The standard part of a K-finite vector $(a_1, \dots, a_n) \in (K((\varepsilon^{1/\infty})))^n$ is defined componentwise: $\mathrm{st}(a_1, \dots, a_n) = (\mathrm{st}(a_1), \mathrm{st}(a_2), \dots, \mathrm{st}(a_n))$.

Let a system of inequalities [cf. (1)] be given

$$f_1 > 0, \ldots, f_{k_1} > 0, f_{k_1+1} \geq 0, \ldots, f_k \geq 0 \tag{2}$$

where $f_i \in \mathbb{Z}_m[X_1, \ldots, X_n]$. The purpose of the present section is to reduce the proof of the theorem to the case where the formula Ξ has the form (2). Because of this we suppose for the time being that the algorithm required in the theorem is designed already for formulae of type (2) (complexity bounds for this algorithm will be given later).

Applying the theorem to each $\{f_1, \ldots, f_k\}$-cell, the algorithm for every element of the partition $\mathscr{U}(\{f_1, \ldots, f_k\})$ yields a certain quantifier-free formula determining this element. Observe that any connected component of the semialgebraic set Ξ is a union of several elements of the partition and one can select all the elements contained in $\{\Xi\}$. Therefore for finding the connected components of the set $\{\Xi\}$ it suffices (cf. [15]) to test for each pair $V_1, V_2 \in \mathscr{U}(\{f_1, \ldots, f_k\})$, whether $V_1 \cap \overline{V}_2 \neq \varphi$ (here the bar denotes the closure in the topology of the space $(\tilde{\mathbb{Q}}_m)^n$ whose basis consists of open balls).

Let V_1 lie in a certain $\{f_1, \ldots, f_k\}$-cell

$$U_1 = \Big\{ \underset{i \in I}{\&} (f_i = 0) \,\&\, \underset{i_1 \in I_1}{\&} (f_{i_1} > 0) \,\&\, \underset{i_2 \in I_2}{\&} (f_{i_2} < 0) \Big\},$$

thus V_1 is a connected component of U_1. Let $\varepsilon_2 > 0$ be an infinitesimal w.r.t. the field $\mathbb{Q}_m$ and $\varepsilon_2 > 0$ be an infinitesimal w.r.t. the field $\mathbb{Q}_m(\varepsilon_1)$. Denote $F_1 = \widetilde{\mathbb{Q}_m(\varepsilon_1)}$, $F_2 = \widetilde{F_1(\varepsilon_2)}$. By st_2 we denote the standard part w.r.t. ε_2, by st_1 we denote the standard part w.r.t. $\varepsilon_1, \varepsilon_2$ i.e. for an element $a \in F_2$ $\mathrm{st}_2(a) \in F_2$, $\mathrm{st}_1(a) \in \tilde{\mathbb{Q}}_m$ (provided that these standard parts are definable) and the element $(a - \mathrm{st}_2(a))$ is infinitesimal w.r.t. F_1, and the element $(a - \mathrm{st}_1(a))$ is infinitesimal w.r.t. $\mathbb{Q}_m$.

Denote by $V_1^{(\varepsilon_1, \varepsilon_2)} \subset F_2^n$ a semialgebraic set defined in the space F_2^n by the same quantifier-free formula (with the coefficients in the field $\tilde{\mathbb{Q}}_m$) as the set V_1 (an upper index will be utilized in a similar role for other semialgebraic sets below). Introduce a semialgebraic set

$$\mathscr{U}_1 = \Big\{ \underset{i \in I}{\&} (-\varepsilon_2 \leq f_i \leq \varepsilon_2) \,\&\, \underset{i_1 \in I_1}{\&} (f_{i_1} \geq \varepsilon_1) \,\&\, \underset{i_2 \in I_2}{\&} (f_{i_2} \leq -\varepsilon_1) \Big\} \cap \mathscr{D}_0(\varepsilon_1^{-1}) \subset F_2^n$$

(henceforth $\mathscr{D}_x(r)$ denotes the closed ball $\{y : \|x - y\| \leq r\}$). Evidently, $V_1 \subset U_1 \subset \mathscr{U}_1$.

Lemma 1 [15]. a) *There exists a unique connected components* $\mathscr{V}_1$ *of the set* $\mathscr{U}_1$, *which contains* V_1;
b) $\mathrm{st}_2(\mathscr{V}_1) \subset V_1^{(\varepsilon_1, \varepsilon_2)}$;
c) *the relation* $V_1 \cap \overline{V}_2 \neq \varphi$ *is valid iff* $\mathscr{V}_1 \cap \mathscr{V}_2^{(\varepsilon_1, \varepsilon_2)} \neq \varphi$.

Applying the theorem to the set $\mathscr{U}_1$, the algorithm yields a certain quantifier-free formula which determines $\mathscr{V}_1$. In order to test the condition $V_1 \cap \overline{V}_2 \neq \varphi$ it suffices to check, whether $\mathscr{V}_1 \cap \mathscr{V}_2^{(\varepsilon_1, \varepsilon_2)} \neq \varphi$ by virtue of Lemma 1 c). One can test the latter, applying the algorithm from [1] (cf. above) to the yielded quantifier-free formulas, determining the sets $\mathscr{V}_1$ and V_2, respectively.

2. Reduction to the Case of a Nonsingular Hypersurface

We proceed by proving the theorem in the case where the formula Ξ is a system of inequalities (2). In the present section we reduce the proof of the theorem to the case when the semialgebraic set determined by (2) is a nonsingular hyper-surface. Moreover, it suffices to find the connected components situated in a ball with an infinitely large radius (w.r.t. the field $\mathbb{Q}_m$).

Denote by $V_0 \subset (\tilde{\mathbb{Q}}_m)^n$ the semialgebraic set determined by the system (2). Introduce a new variable X_0 and a semialgebraic set $V_1 \subset (\tilde{\mathbb{Q}}_m)^{n+1}$ determined by a system

$$f_1 \geq 0, \ldots, f_k \geq 0, \qquad X_0 \cdot f_1 \cdot \ldots \cdot f_{k_1} = 1 \tag{3}$$

Assume that the connected components of the set V_1 are already found and for each connected component a certain quantifier-free formula $\Xi_i^{(1)}$ determining this component is given. There is a bijective correspondence between the connected components of the sets V_0 and V_1: namely, to a connected component $\{\Xi_i^{(1)}\}$ corresponds a connected component of the set V_0, determined by a formula $\hat{\Xi}_i^{(1)} \& (f_1 > 0) \& \ldots \& (f_{k_1} > 0)$. Where $\hat{\Xi}_i^{(1)}$ is obtained from $\Xi_i^{(1)}$ by substituting $1/(f_1 \cdot \ldots \cdot f_{k_1})(X_1, \ldots, X_n)$ for X_0 and then multiplying all the polynomials occurring in the formula by an appropriate power of $(f_1 \cdot \ldots \cdot f_{k_1})(X_1, \ldots, X_n)$ in order to clear the denominator.

Add one more variable X_{n+1}, introduce the polynomials $f_{k+1} = X_0 f_1 \cdot \ldots \cdot f_{k_1} - 1$, $f_{k+2} = -f_{k+1}$, $f_{k+3} = X_{n+1}$, $f_{k+4} = -f_{k+3}$ and consider the system of inequalities

$$f_1 \geq 0, \ldots, f_{k+4} \geq 0 \tag{4}$$

Then (4) determines a semialgebraic set $V \subset (\tilde{\mathbb{Q}}_m)^{n+2}$ which is isomorphic to V_1 by means of the linear projection $\pi : (X_0, \ldots, X_{n+1}) \to (X_0, \ldots, X_n)$.

Consider a semialgebraic set $\mathscr{V} \subset F_2^{n+2}$ determined by the following system of inequalities ($\varepsilon_1, \varepsilon_2$ have the same meaning as in Sect. 1):

$$f_1 + \varepsilon_1 > 0, \ldots, f_{k+4} + \varepsilon_1 > 0 \tag{5}$$

For a semialgebraic set W by ∂W we denote its boundary in the topology whose basis consists of open balls. Observe that ∂W is also a semialgebraic set. Later on we shall need some lemmas (2–7), whose proofs one can find in [15].

Lemma 2. *For any connected component $\mathscr{W}$ of the set $\mathscr{V}$ its boundary $\partial\mathscr{W}$ is also connected.*

This lemma was the reason for introducing X_{n+1}. Consider a polynomial $g = (f_1 + \varepsilon_1) \cdot \ldots \cdot (f_{k+1} + \varepsilon_1) - \varepsilon_2$.

Lemma 3. *Any connected component S of the semialgebraic set $\{g = 0\} \subset F_2^{n+2}$ lies in a certain connected component $\mathscr{W}_0$ of a suitable open $\{f_1 + \varepsilon_1, \ldots, f_{k+4} + \varepsilon_1\}$ -cell (i.e. a cell determined by a system of strict inequalities $(-1)^{\sigma_1}(f_1 + \varepsilon_1) > 0, \ldots, (-1)^{\sigma_{k+4}}(f_{k+4} + \varepsilon_1) > 0$ for some $(k+4)$-tuple $\sigma_i \in \{0, 1\}$, $1 \leq i \leq k+4$).*

Lemma 4. *Let $\mathscr{W}_1$ be a connected component of the open cell $\mathscr{V}$, some point $x \in F_1^{n+2} \cap \partial\mathscr{W}_1$ and an element $0 < r \in F_1$. Suppose that $\mathscr{W}_2$ is a connected component of the intersection $\mathscr{W}_1 \cap \mathscr{D}_x(r)$ such that $x \in \partial\mathscr{W}_2$. Then there exists a point $y \in \{g = 0\} \cap \mathscr{W}_2$ for which $\mathrm{st}_2(y) = x$. Conversely, for any point $z \in \{g = 0\} \cap \mathscr{W}_1$, $\mathrm{st}_2(z) \in \partial\mathscr{W}_1$, provided that $\mathrm{st}_2(z)$ is definable.*

Let S be a connected component of the hypersurface $\{g = 0\}$ and $S \subset \mathscr{W}_1$, where $\mathscr{W}_1$ is some connected component of the open cell $\mathscr{V}$ (see Lemma 3). Lemma 2 entails the connectedness of the boundary $\partial\mathscr{W}_1$. Alexander's duality principle [14] implies that S decomposes the set $F_2^{n+2}\backslash S$ into two connected components each having S as its boundary. Here we use the fact that zero is not a critical value of the polynomial g (cf. Lemma 4 [1]) and hence S is a nonsingular hypersurface. In this argument the transfer principle is involved. Thus, it is reasonable to say: two points are situated on the same side of S, or on the contrary, two points are situated on different sides of S.

Lemma 5. a) *The whole boundary $\partial\mathscr{W}_1$ is situated on one side of S;*
b) *Any two points $x, y \in \mathscr{W}_1 \cap F_1^{n+2}$ are situated on the same side of S.*

Lemma 6. *Among all the connected components of the hypersurfaces $\{g = 0\}$ which lie in $\mathscr{W}_1$, there exists exactly one which contains at least one point with definable standard part* st_2.

Lemma 7. *For every connected component $\mathscr{W}_0$ of the set $\mathscr{V}$ which contains at least one point with definable standard part* st_1, *there exists a connected component W_0 of the set V such that $W_0 \subset \mathscr{W}_0$ (if for any point $y \in \mathscr{W}_0$, the standard part $\mathrm{st}_1(y)$ is definable, the connected component W_0 is unique and in this case $\mathrm{st}_1(\mathscr{W}_0) = W_0$). Conversely, for each connected component W of the set V there exists a unique connected component $\mathscr{W}$ of the set $\mathscr{V}$ having a common point with W. Furthermore, an inclusion $\mathscr{W} \supset W$ holds.*

Relying on Lemma 8 [15] and on the transfer principle, one can prove the following lemma, but we give its independent proof. Below $\varepsilon > 0$ is an infinitesimal w.r.t. the field $\mathbb{Q}_m$.

Lemma 8. *The connected components of the set V correspond bijectively to the connected components of the intersection $V_2 = (V^{(\varepsilon)} \cap \mathscr{D}_0(\varepsilon^{-1})) \subset (\widetilde{\mathbb{Q}_m(\varepsilon)})^{n+2}$. Moreover, to each connected component $\hat{W}$ of V_2 corresponds the connected component $\hat{W} \cap (\tilde{\mathbb{Q}}_m)^{n+2}$ of the set V.*

Proof. There exists an element $R_1 \in \mathbb{Q}_m$ such that for all $\tilde{\mathbb{Q}}_m \ni R_2 \geq R_1$ and for each connected component W of the set V the intersection $W \cap \mathscr{D}_0(R_2)$ has the same number of connected components as the intersection $W \cap \mathscr{D}_0(R_1)$ has. One can easily prove it observing that the set of all R_2 for which the intersection $W \cap \mathscr{D}_0(R_2)$ has a given number of connected components, is a semialgebraic subset of a line $\tilde{\mathbb{Q}}_m$, and on the other hand an upper bound on the number of connected components of the intersection $W \cap \mathscr{D}_0(R_2)$ does not depend on R_2 (but only on the number of variables, on degrees and on the number of polynomials occurring in the representation of the set V, cf. [2]).

Consider a semialgebraic set $U \subset (\tilde{\mathbb{Q}}_m)^{n+2}$ consisting of all the points $u \in W_2$ in which the function "square of norm" $(X_0, \ldots, X_{n+1}) \to X_0^2 + \ldots + X_{n+1}^2$ reaches a local minimum on W. Then on every connected component of the set U the square of norm has a constant value. Take R_3^2 to be larger than R_1^2 and than all the values of the square of norm on the connected components of the set U.

Let us prove that for any $R_4 \geq R_3$ the intersection $W \cap \mathscr{D}_0(R_4)$ is connected. Assume the contrary. Let the points $y_1, y_2 \in W \cap \mathscr{D}_0(R_4)$ belong to the different connected components of this intersection. Since W is connected, there exists a bounded connected semialgebraic curve $C \subset W$ containing both y_1, y_2. Then $C \subset \mathscr{D}_0(R_5)$ for a certain R_5, $R_4 < R_5 \in \mathbb{Q}_m$. As the numbers of the connected components of the intersections $W \cap \mathscr{D}_0(R_4)$ and $W \cap \mathscr{D}_0(R_5)$ coincide, there is a connected component W_0 of the set $W \cap \mathscr{D}_0(R_5)$ such that the intersection $W \cap \mathscr{D}_0(R_4) = \emptyset$. Therefore, there exists a point $w_0 \in W_0$ in which the square of norm reaches its minimum on W_0. Hence $w_0 \in U$ and we get a contradiction with the choice of R_3.

Because of the transfer principle, for each $\widetilde{\mathbb{Q}_m(\varepsilon)} \ni R_4 \geq R_3$ the intersection $W^{(\varepsilon)} \cap \mathscr{D}_0(R_4)$ is also connected (cf. [1]), in particular, $W^{(\varepsilon)} \cap \mathscr{D}_0(\varepsilon^{-1})$ is connected. The lemma is proved.

Notice that V_2 can be determined by the following system of inequalities: $f_0 \equiv \varepsilon^{-2} - X_0^2 - \ldots - X_{n+1}^2 \geq 0$, $f_1 \geq 0, \ldots, f_{k+1} \geq 0$. Further in the applications of Lemmas 6, 7 the field $K = \mathbb{Q}_m(\varepsilon)$ will play the role of the field $\mathbb{Q}_m$. The element $\varepsilon_1 > 0$ is infinitesimal w.r.t. K, and the element $\varepsilon_2 > 0$ is infinitesimal w.r.t. $K(\varepsilon_1)$. Let $K_1 = \widetilde{K(\varepsilon_1)}$, $K_2 = \widetilde{K_1(\varepsilon_2)}$ (similar to the above we utilize the notations st_1, st_2). Consider also a semialgebraic set $\mathscr{V}_2 \subset K_2^{n+2}$ determined by a system of inequalities: $f_0 + \varepsilon_1 > 0$, $f_0 + \varepsilon_1 > 0, \ldots, f_{k+4} + \varepsilon_1 > 0$ [cf. (5)] and a polynomial $g_1 = (f_0 + \varepsilon_1)(f_1 + \varepsilon_1) \cdot \ldots \cdot (f_{k+4} + \varepsilon_1) - \varepsilon_2$ (cf. Lemma 3). In consequence of Lemma 6 there is a bijective correspondence between the connected components of the nonsingular hypersurface $\{g_1 = 0\} \subset K_2^{n+2}$ which lie in $\mathscr{V}_2$, and the connected components of the set $\mathscr{V}_2$. By virtue of Lemma 7 there is a bijective correspondence between the connected components of the sets $\mathscr{V}_2$ and V_2, respectively. Finally, applying Lemma 8, we get a bijective correspondence between the connected components of the sets V_2 and V, respectively [and thereby, of V_0, see (2)].

Now we shall describe the reduction of finding the connected components of the set V to finding the connected components of the nonsingular hypersurface $\{g_1 = 0\} \cap \mathscr{D}_0\big(\sqrt{\varepsilon^{-2} - \varepsilon_1}\big)$ (cf. f_0). An algorithm for finding the latter connected components will be described in the next section. Thus, we assume that for each connected component of the hypersurface $\{g_1 = 0\} \cap \mathscr{D}_0\big(\sqrt{\varepsilon^{-2} - \varepsilon_1}\big)$, a certain quantifier-free formula Ω determining it is already given (one can deem w.l.o.g. that all the polynomials $f_1 + \varepsilon_1, \ldots, f_{k+4} + \varepsilon_1$ are positive on this connected component, cf. Lemma 3, otherwise we don't consider this component).

Introduce new variables $\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1}, Y_0, \ldots, Y_{n+1}$ and consider a formula Π of the first-order theory of the field K_2 with free variables $\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1}$, which expresses the condition that the point $(\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1})$ belongs to $V_2^{(\varepsilon_1, \varepsilon_2)}$. Furthermore, any point $(Y_0, \ldots, Y_{n+1})$, being the nearest to the point $(\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1})$ among the points of the hypersurface $\{g_1 = 0\}$, satisfies the formula Ω. Applying Lemma 6, to the connected component $\{\Omega\}$ of the hypersurface $\{g_1 = 0\}$ there corresponds a unique connected component $\mathscr{W}_2$ of the set $\mathscr{V}_2$, such that $\mathscr{W}_2 \supset \{\Omega\}$. By virtue of Lemma 7, to $\mathscr{W}_2$ corresponds a unique connected component W_2 of the set V_2 for which $W_2 \subset \mathscr{W}_2$. It is proved in [15] that for the above formula Π, $W_2 = \{\Pi\} \cap \tilde{K}^{n+2}$ holds. Apply the quantifier elimination procedure from [16] to the formula Π and obtain as a result a quantifier-free formula Π_1 of the theory of the field K_2 equivalent to it with atomic sub-formulae of the type $(h > 0)$, where $h \in \mathbb{Q}_m[\varepsilon]\,[\varepsilon_1, \varepsilon_2, \mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1}]$ and represent $h = \sum_{0 \leq i_1, i_2 \leq \varrho} h_{i_1, i_2} \varepsilon_2^{i_1} \cdot \varepsilon_2^{i_2}$, where $h_{i_1, i_2} \in \mathbb{Q}_m[\varepsilon]\,[\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1}]$.

Thereupon we produce a quantifier-free formula Π_2 with coefficients from the field K, being equivalent to the formula Π_1 for any point $(\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1})$ from the space $\tilde{K}^{n+2}$. Toward this aim, replace every formula $(h > 0)$ in Π_1 by the following formula:

$$
\begin{aligned}
&(h_{0,0} > 0) \vee (h_{0,0} = 0 \,\&\, h_{1,0} > 0) \vee (h_{0,0} = h_{1,0} = 0 \,\&\, h_{2,0} > 0) \vee \ldots \\
&\quad \vee (h_{0,0} = h_{1,0} = \ldots = h_{\varrho-1,0} \,\&\, h_{\varrho,0} > 0) \\
&\quad \vee (h_{0,0} = h_{1,0} = \ldots = h_{\varrho,0} = 0 \,\&\, h_{0,1} > 0) \vee \ldots \\
&\quad \vee (h_{0,0} = \ldots = h_{\varrho,\varrho-1} = 0 \,\&\, h_{\varrho,\varrho} > 0)
\end{aligned}
$$

which is equivalent to the inequality $h > 0$ for any point $(\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1})$ from $\tilde{K}^{n+2}$. The resulting formula we denote by Π_2. Thus, the formula Π_2 determines the set W_2 in the space $\tilde{K}^{n+2}$.

Each polynomial $\hat{h} \in \mathbb{Q}_m[\varepsilon][\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1}]$ occurring in the formula Π_2, we represent as $\hat{h} = \sum_i \varepsilon^i \hat{h}_i$, where $\hat{h}_i \in \mathbb{Q}[\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1}]$. Replace each atomic subformula of the form $\hat{h} > 0$ in the formula Π_2 by the formula $(h_0 > 0) \vee (h_0 = 0 \,\&\, h_1 > 0) \vee \ldots$ (in a similar manner as above), which is equivalent to the inequality $\hat{h} > 0$ for any point $(\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1})$ from $(\tilde{\mathbb{Q}}_m)^{n+2}$. We obtain as a result a formula $\hat{\Pi}_2$ which determines in the space $(\tilde{\mathbb{Q}}_m)^{n+2}$ the semialgebraic set $\hat{W}_2 = W_2 \cap (\tilde{\mathbb{Q}}_m)^{n+2}$ being a connected component of the set V because of Lemma 8. Thus, the algorithm described above produces all the connected components of the set V by force of Lemma 8, thereby all the connected components of the set V_1 by means of substituting zero for X_{n+1} [cf. (4)], and finally, all the connected components of the set V_0 (see the beginning of the section).

3. Finding Connected Components of a Nonsingular Bounded Hypersurface

In the present section we describe an algorithm which finds the connected components (i.e. yields for each of them a certain quantifier-free formula determining it) of the nonsingular hypersurface $\{g_1 = 0\} \subset K_2^{n+2}$ which lie in the set $\mathscr{D}_0\left(\sqrt{\varepsilon^{-2} + \varepsilon_1}\right) \cap \{f_1 + \varepsilon_1 > 0, \ldots, f_{k+1} + \varepsilon_1 > 0\}$ [cf. (5) and Lemmas 3, 6]. Notice that $\{g_1 = 0\} \cap \partial\left(\mathscr{D}_0\left(\sqrt{\varepsilon^{-2} + \varepsilon_1}\right)\right) = \emptyset$ (see Sect. 2). The described algorithm relies essentially on the method from Sect. 3 [15] (we mention also that a similar construction is exposed in [18]) for testing whether two points belong to the same connected component of a nonsingular bounded hypersurface.

The algorithm produces by recursion a rooted tree $\mathscr{T}$ of depth $n+1$ (the depth of a vertex is its distance from the root). Suppose that according to recursive hypothesis, all the vertices of depth at most $n-l$ (where $0 \le n-l \le n$) are already produced, let v be one of the vertices of the depth $n-l$. Suppose also that a certain quantifier-free formula Λ_v in the variables $\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1}$ corresponds to the vertex v.

In the sequel we need to consider a more general situation than in Sect. 2, namely when for a given $0 \le l \le n$ a variety $\{h_v(\eta_1, \ldots, \eta_{n-l}, Y_1, \ldots, Y_{l+2}) = 0\} \subset K_2^{l+2}$ (provided that the point $(\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1})$ satisfies the formula Λ_v) has at most one singular point $\sigma_v = (\sigma_1, \ldots, \sigma_{l+2})$, i.e. a point satisfying the following system of equations:

$$h_v(\eta_1, \ldots, \eta_{n-l}, \sigma_v) = \frac{\partial h_v}{\partial Y_1}(\eta_1, \ldots, \eta_{n-l}, \sigma_v) = \ldots = \frac{\partial h_v}{\partial Y_{l+2}}(\eta_1, \ldots, \eta_{n-l}, \sigma_v) = 0\,.$$

We assume that the polynomial $h_v \in K[\varepsilon_1, \varepsilon_2][Z_1, \ldots, Z_{n-l}][Y_1, \ldots, Y_{l+2}]$ is produced by recursion on $(n-l)$. In addition we suppose that by recursion on $1 \le i \le n-l$ the polynomials $\varphi_v^{(\eta_i)}(Z_i) \in K[\varepsilon_1, \varepsilon_2][\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1}][Z_i]$ are already computed (here and further (η_i) is regarded as an upper index). Apart from that for each $1 \le i \le n-l$ a sequence $\mathscr{S}_v^{(\eta_i)}$ of signs of the derivatives $\partial^j \varphi_v^{(\eta_i)} / \partial Z_i^j$ for all $1 \le j \le \deg_{Z_i}(\varphi_v^{(\eta_i)})$ is given such that η_i is the unique root of the polynomial $\varphi_v^{(\eta_i)}(Z_i)$ satisfying $\mathscr{S}_v^{(\eta_i)}$ (cf. the introduction).

Also we assume that a nonsingular linear transformation $\mathfrak{M}_v$ (defined over the field $\mathbb{Q}$) is produced, which transforms the $(n+2)$-dimensional space with coordinates $X_0, \ldots, X_{n+1}$ into an $(n+2)$-dimensional space with coordinates $Z_1, \ldots, Z_{n-l}$, $Y_1, \ldots, Y_{l+2}$ such that $h_v(\mathfrak{M}_v(X_0, \ldots, X_{n+1})) = g_1(X_0, \ldots, X_{n+1})$.

Finally, let some points be defined:

$$\begin{aligned} &u_v^{(1)}(\eta_1, \ldots, \eta_{n-l}) \\ &\quad = (\eta_1, \ldots, \eta_{n-l}, u_1^{(1)}, \ldots, u_{l+2}^{(1)})^T, \ldots, u_v^{(q)}(\eta_1, \ldots, \eta_{n-l}) \\ &\quad = (\eta_1, \ldots, \eta_{n-l}, u_1^{(q)}, \ldots, u_{l+2}^{(q)})^T \in \{h_w(\eta_1, \ldots, \eta_{n-l}, Y_1, \ldots, Y_{l+2}) = 0\} \\ &\qquad \cap \mathfrak{M}_v \cdot \{f_0 + \varepsilon_1 > 0, \ldots, f_{k+4} + \varepsilon_1 > 0\} \subset K_2^{l+2} \end{aligned}$$

(provided that the point $(\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1})$ satisfies formula Λ_v). Here each coordinate $u_j^{(i)}$ is determined by a polynomial $\varphi_v^{(i,j)} \in K[\varepsilon_1, \varepsilon_2][\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1}][Z]$ and furthermore, by a sequence $\mathscr{S}_v^{(i,j)}$ of the signs of the derivatives of all orders (w.r.t. Z) of this polynomial. In a similar way $\sigma_j (1 \le j \le l+2)$ is determined by a polynomial $\varphi_v^{(\sigma_j)}$ and by a sequence $\mathscr{S}_v^{(\sigma_j)}$ of the signs of the derivatives of this polynomial. We'll require for uniformity that the point $(\eta_1, \ldots, \eta_{n-l}, \sigma_v)$ is contained among the points $u_v^{(1)}(\eta_1, \ldots, \eta_{n-l}), \ldots, u_v^{(q)}(\eta_1, \ldots, \eta_{n-l})$.

We describe now the base $(n = l)$ of the recursion producing the tree $\mathscr{T}$. Namely, to the root v_0 of the tree corresponds a formula $\Lambda_{v_0} = \{g_1 = 0, f_0 + \varepsilon_1 > 0, \ldots, f_{k+4} + \varepsilon_1 > 0\}$, where g_1 plays the role of the polynomial h_{v_0}. For the points $u(1)_{v_0}, \ldots, u_{v_0}^{(g)}$ we take the union of the representative set for the hypersurface $\{g_1 = 0\} \cap \mathscr{D}_0(\sqrt{\varepsilon^{-2} + \varepsilon_1}) \cap \{f_1 + \varepsilon_1 > 0, \ldots, f_{k+1} + \varepsilon_1 > 0\}$ (cf. [1] and the introduction) and the point $(\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1})$.The transformation $\mathfrak{M}_{v_0}$ maps $X_0 \to Y_1$, $X_1 \to Y_2, \ldots, X_{n+1} \to Y_{n+2}$ respectively.

Lemma 9 [15]. *For any point $(\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1})$ satisfying the formula Λ_v, at least one of the integer vectors of the type $(1, t_2, \ldots, t_{l+2})$, where $0 \le t_i \le N' \le (kd)^{n^{O(1)}}$, $2 \le i \le l+2$ satisfies the following property. Every two different points y, $y' \in \{h_v(\eta_1, \ldots, \eta_{n-l}, Y_1, \ldots, Y_{l+2}) = 0\} \subset K_2^{l+2}$ in which the both gradients*

$$\begin{aligned} \operatorname{grad}_y(h_v) &= \left(\frac{\partial h_v(\eta_1, \ldots, \eta_{n-l}, Y_1, \ldots, Y_{l+2})}{\partial Y_1}\right), \ldots, \\ &= \left(\frac{\partial h_v(\eta_1, \ldots, \eta_{n-l}, Y_1, \ldots, Y_{l+2})}{\partial Y_{l+2}}\right) \end{aligned}$$

and $\operatorname{grad}_{y'}(h_v)$ are collinear to the vector $(1, t_2, \ldots, t_{l+2})$, do not lie in the same hyperplane orthogonal to the vector $(1, t_2, \ldots, t_{l+2})$.

The algorithm tests all possible vectors $(1, t_2, \ldots, t_{l+2})$. Fix some $(1, t_2, \ldots, t_{l+2})$ which yields a formula $\Phi_1^{(1,t_2,\ldots,t_{l+2})}$ with coefficients from the field $K(\varepsilon_1, \varepsilon_2)$ of the first-order theory of the field K_2, which expresses the latter condition in Lemma 9 (concerning the points y, y'). So $\Phi_1^{(1,t_2,\ldots,t_{l+2})}$ has free variables $\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1}$ and besides, the variables $Z_1, \ldots, Z_{n-l}, Y_1, \ldots, Y_{l+2}, Y_1', \ldots, Y_{l+2}'$ bounded by the universal quantifier (observe that $\Phi_1^{(1,t_2,\ldots,t_{l+2})}$ does not depend on the formula Λ_v). Apply to $\Phi_1^{(1,t_2,\ldots,t_{l+2})}$ the quantifier elimination procedure from [16] and as a result obtain a quantifier-free formula $\Phi_2^{(1,t_2,\ldots,t_{l+2})}$ equivalent to it.

Consider the $(l+2) \times (l+2)$ matrix

$$\mathscr{B} = \begin{pmatrix} 1 & t_2 \dots t_{l+2} \\ & 1 \quad 0 \\ 0 & \ddots \\ & \quad 1 \end{pmatrix},$$

introduce new variables $\bar{Y}_1, \dots, \bar{Y}_{l+2}$ and define a polynomial $\bar{\bar{h}}_v = h_v(Z_1, \dots, Z_{n-l}, \mathscr{B}^{-1}(\bar{Y}_1, \dots, \bar{Y}_{l+2})^T) \in K[\varepsilon_1, \varepsilon_2][Z_1, \dots, Z_{n-l}, \bar{Y}_1, \dots, \bar{Y}_{l+2}]$. Finally, a polynomial h_v is obtained from $\bar{\bar{h}}_v$ by replacing $\bar{Y}_1, \dots, \bar{Y}_{l+2}$ by $Z_{n-l+1}, Y_1, \dots, Y_{l+1}$ respectively. Thus, we can introduce a linear transformation of $(n+2)$-dimensional spaces $\overline{\mathfrak{M}}_v = \begin{pmatrix} E & 0 \\ 0 & \mathscr{B} \end{pmatrix} \mathfrak{M}_v$, where E denotes the identity matrix.

Based on [15] [see there system (10 a, b)], one can prove that for any point $(\mathscr{Z}_0, \dots, \mathscr{Z}_{n+1})$ satisfying the formula $\Lambda_v \,\&\, \Phi_2^{(1, t_2, \dots, t_{l+2})}$, a system of inequalities

$$h_v = \frac{\partial h_v}{\partial Y_1} = \dots = \frac{\partial \bar{h}_v}{\partial Y_{l+1}} = 0\,, \tag{6a}$$

$$f_i(\mathfrak{M}_v^{-1}(Z_1, \dots, Z_{n-l}, Y_1, \dots, Y_{l+2})) + \varepsilon_1 > 0, \qquad 0 \le j \le K + y$$

$$\varphi_v^{(\eta_i)}(Z_i) = 0, \qquad \mathscr{S}_v^{(\eta_i)}, \qquad 1 \le i \le n - l \tag{6b}$$

has a finite set of solutions in the space K_2^{n+2} with coordinates $Z_1, \dots, Z_{n-l+1}, Y_1, \dots, Y_{l+1}$.

In the sequel we have to solve several times some systems of polynomial inequalities with coefficients which are polynomials in $\mathscr{Z}_0, \dots, \mathscr{Z}_{n+1}$. Moreover a quantifier-free formula Λ in $\mathscr{Z}_0, \dots, \mathscr{Z}_{n+1}$ is given such that for any point $(\mathscr{Z}_0, \dots, \mathscr{Z}_{n+1})$ satisfying the formula Λ, the system under consideration has a finite number of solutions (in the space K_2^{n+2} with coordinates $Z_1, \dots, Z_{n-l+1}, Y_1, \dots, Y_{l+1}$). Our next task is to describe a subroutine for solving systems of this sort. Using the subroutine, we produce a partition $\mathscr{U}_1$ of the semialgebraic set $\{\Lambda\} = \bigcup_j \{\Lambda^{(j)}\}$ into semialgebraic sets $\{\Lambda^{(j)}\}$, determined by some quantifier-free formulae $\Lambda^{(j)}$. Furthermore, the subroutine yields, for each $\Lambda^{(j)}$, a family of polynomials $\Psi^{(j)}_{i_1,1,\mathscr{S}}$, $\Psi^{(j)}_{i_2,l,\mathscr{S}} \in K[\varepsilon_1, \varepsilon_2][\mathscr{Z}_0, \dots, \mathscr{Z}_{n+1}][Z]$, $1 \le i_1 \le n - l + 1$, $1 \le i_2 \le l + 1$, $1 \le \mathscr{S} \le t^{(j)}$ for a suitable $t^{(j)}$. Finally, the subroutine yields the sequences $\mathscr{S}^{(j)}_{i_1,1,\mathscr{S}}$, $\mathscr{S}^{(j)}_{i_2,2,\mathscr{S}}$ of signs of the derivatives of all orders of the polynomials $\Psi^{(j)}_{i_1,1,\mathscr{S}}$, $\Psi^{(j)}_{i_2,2,\mathscr{S}}$, respectively. For any point $(\mathscr{Z}_0, \dots, \mathscr{Z}_{n+1})$ satisfying $\Lambda^{(j)}$ a formula

$$\bigvee_{1 \le \mathscr{S} \le t^{(j)}} \Big(\Psi^{(j)}_{i_1,1,\mathscr{S}} = 0,\, \mathscr{S}^{(j)}_{i_1,1,\mathscr{S}},\, 1 \le i_1 \le n - l + 1; \quad \Psi^{(j)}_{i_2,2,\mathscr{S}} = 0,\, \mathscr{S}^{(j)}_{i_2,2,\mathscr{S}},\, 1 \le i_2 \le l + 1\Big) \tag{7}$$

determines the (finite) set of solutions (in the space K_2^{n+2}) of the system under consideration. For a fixed $1 \le \mathscr{S} \le t^{(j)}$ the disjunctive term in (7) determines one of the solutions of the system.

We describe this subroutine in applying it to the system (6a, b), with the formula $\Lambda_v \,\&\, \Phi_2^{(1, t_2, \dots, t_{l+2})}$ playing the role of Λ (in other situations the subroutine would be applied in similar ways). Denote by $U \subset K_2^{2(n+2)}$ the semialgebraic set determined by system (6a, b). Applying the algorithm from [16] for each index $1 \le i_2 \le l+1$ or $i_1 = n - l + 1$, we obtain a projection $U_1 \subset K_2^{n+3}$ of the set U onto the space K_2^{n+3}

with coordinates $\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1}, Y_{i_2}$ or $\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1}, Z_{n-l+1}$, respectively. The algorithm from [16] represents U_1 as a union of conjunctive semialgebraic sets of the kind $\hat{U} = \{P_0 = 0, P_1 > 0, \ldots, P_\chi > 0\}$ where $P_i \in K[\varepsilon_1, \varepsilon_2][\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1}, Z]$. Applying again the algorithm from [16], we find the projection on $\hat{U}$ onto the space K_2^{n+2} with coordinates $\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1}$ in the form $\{\hat{\Lambda}\}$ for an appropriate quantifier-free formula $\hat{\Lambda}$. For any point $(\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1})$ satisfying $\hat{\Lambda} \,\&\, \Lambda_v \,\&\, \Phi_2^{(1,t_2,\ldots,t_{l+2})}$ there is a finite number of points being projected into this point from the set $\hat{U}$, because of the property of system (6a, b). Using [1] we find all the possible feasible sequences of signs of the derivatives $\dfrac{\partial^j P_0}{\partial Z^j}$ of all orders j for the points from $\hat{U}$. As a result we get a partition $\{\hat{\Lambda}\} = \bigcup_i \{\hat{\Lambda}^{(i)}\}$ into semialgebraic sets, moreover for each $\hat{\Lambda}^{(i)}$ a family $\{\mathscr{S}_j^{(i)}\}$ of sequences of signs of the derivatives of the polynomial P_0 is produced such that

$$\hat{U} \cap (\{\hat{\Lambda}^{(i)}\} \times K_2) \cap (\{\Lambda_v \,\&\, \Phi_2^{(1,t_2,\ldots,t_{l+2})}\} \times K_2)$$
$$= \left\{(P_0 = 0) \,\&\, \left(\bigvee_j \mathscr{S}_j^{(i)}\right)\right\} \cap (\{\Lambda_v \,\&\, \Phi^{(1,t_2,\ldots,t_{l+2})}\} \times K_2).$$

Thereupon considering different semialgebraic sets $\hat{U} \subset K_2^{n+3}$ (from the partition of the set U_1) the subroutine yields a partition $\mathscr{U}_2$ of the projection of the set U_1 onto the space K_2^{n+2} such that each set $\{\hat{\Lambda}^{(i)}\}$ is a union of several elements of the partition $\mathscr{U}_2$. The subroutine constructs $\mathscr{U}_2$ by applying [1] to the family of polynomials occurring in formulae $\hat{\Lambda}^{(i)}$ for all partitions corresponding to different sets $\hat{U}$. The subroutine yields for every semialgebraic set $U_2 \subset K_2^{n+2}$ (being an element of the partition $\mathscr{U}_2$) a quantifier-free formula Λ_2 of the kind (7) (with the difference that Λ_2 is defined in the space K_2^{n+3}) such that

$$U_1 \cap (U_2 \times K_2) \cap (\{\Lambda_v \,\&\, \Phi_2^{(1,t_2,\ldots,t_{l+2})}\} \times K_2)$$
$$= \{\Lambda_2\} \cap (\{\Lambda_v \,\&\, \Phi_2^{(1,t_2,\ldots,t_{l+2})}\} \times K_2) \subset K_2^{n+3}.$$

Actually, Λ_2 determines the values of the coordinate under consideration (among Y_{i_2}, $1 \le i_2 \le l+2$ or Z_{n-l+1}) for the solutions of system (6a, b) for the points $(\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1})$ from the set $U_2 \cap (\{\Lambda_v \,\&\, \Phi^{(1,t_2,\ldots,t_{l+2})}\} \times K_2)$.

Next the subroutine produces the partition $\mathscr{U}_1'$ being finer than all the partitions of type $\mathscr{U}_2$ for various coordinates Y_{i_2}, $1 \le i_2 \le l+1$ and Z_{n-l+1} again involving [1] (cf. above). Fix some element of $\mathscr{U}_1'$. Our next task is to clarify which of the values of the coordinates Y_{i_2}, $1 \le i_2 \le l+1$ and Z_{n-l+1} determined by the formulae $\Lambda_2 \cap (\{\Lambda_v \,\&\, \Phi^{(1,t_2,\ldots,t_{l+2})}\} \times K_2)$ constitute the solutions of system (6a, b). In order to do this, for each coordinate among Y_{i_2}, $1 \le i_2 \le l+1$ and Z_{n-l+1} we choose the corresponding formula determining it and apply the algorithm from [1] to the conjunction of the chosen formulae and of (6a, b) and as a result obtain a formula of type (7). Finally, as the required partition $\mathscr{U}_1$ of the semialgebraic set $\{\Lambda_v \,\&\, \Phi^{(1,t_2,\ldots,t_{l+2})}\}$ we take the restriction on the latter set of the partition $\mathscr{U}_1'$.

This completes the description of the subroutine for solving systems of polynomial inequalities with coefficients which are polynomials in $\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1}$.

We remark that the polynomials $\psi_{i_1,1,\mathscr{S}}^{(j)}, \psi_{i_2,2,\mathscr{S}}^{(j)}$ occurring in formula (7), do not depend on the initial formula Λ (but only on a system of inequalities to which the subroutine is applied). Moreover, a formula $\Lambda^{(j)}$ which determined an element of the partition $\mathscr{U}_1$ is a conjunction of the form $(\Lambda^{(j)})' \& \Lambda$, where the formula $(\Lambda^{(j)})'$ (it determined an element of the partition $\mathscr{U}_1'$, see above) does not depend on Λ. We shall make use of these remarks in Sect. 4.

Now we return to the recursive step of an algorithm finding connected components of the hypersurface under description. The recursive step includes three procedures for producing the options for the values η_{n-l+1} of the coordinate Z_{n-l+1} (see above). The algorithm will test (branching in the tree $\mathscr{T}$ under construction) all different elements of the partition $\mathscr{U}_1$. Fix a certain element $\{\Lambda^{(j)}\}$ of this partition, and denote $\Lambda_3 = \Lambda^{(j)}$. The algorithm looks over various formulae Λ_2, given above for the space K_2^{n+3} with coordinates $\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1}, Z_{n-l+1}$, such that $K_2^{n+3} \supset \{\Lambda_2\} \cap (\{\Lambda_3\} \times K_2) = U_1 \cap (\{\Lambda_3\} \times K_2)$ (in other words, the projection of $\{\Lambda_2\}$ contains $\{\Lambda_3\}$, see above). For any point $(\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1})$ satisfying formula Λ_3, formula Λ_2 determines a finite set of options for η_{n-l+1}. This completes the description of the first procedure for producing the options for η_{n-l+1}.

Recall that a formula of type (7) was found such that for any point $(\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1})$ satisfying formula Λ_3, there exists a finite number of points satisfying (7) [thereby satisfying system (6a, b)], moreover for every option produced (according to the first procedure) for η_{n-l+1} there exists a unique point with the value of coordinate Z_{n-l+1} equal to η_{n-l+1}, by virtue of Lemma 9. It would be the only singular point of the variety $\{\bar{h}_v(\eta_1, \ldots, \eta_{n-l}, \eta_{n-l+1}, Y_1, \ldots, Y_{l+1}) = 0\} \subset K_2^{l+1}$ and we denote it by $\sigma^{(\eta_1, \ldots, \eta_{n-l}, \eta_{n-l+1})}$ cf. above.

According to the second procedure the algorithm takes as an option for η_{n-l+1} and expression $u_1^{(i)} + \sum_{2 \le j \le l+2} t_j u_j^{(i)}$ (for various $1 \le i \le q$), i.e. $(n-l+1)$-th coordinate of the vector $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ & & 1 \\ 0 & & \mathscr{B} \end{pmatrix} (u_v^{(i)}(\eta_1, \ldots, \eta_{n-l}))$.

Denote by $\mathscr{P}^{(\eta_1, \ldots, \eta_{n-l}, \eta_{n-l+1})}$ the plane of the form $\{Z_1 = \eta_1, \ldots, Z_{n-l} = \eta_{n-l}, Z_{n-l+1} = \eta_{n-l+1}\}$ (for a point $(\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1})$ satisfying formula Λ_3, the plane $\mathscr{P}^{(\eta_1, \ldots, \eta_{n-l}, \eta_{n-l+1})} \subset K_2^{n+2}$). By $\mathscr{P} = \bigcup_{\eta_{n-l+1}} \mathscr{P}^{(\eta_1, \ldots, \eta_{n-l}, \eta_{n-l+1})}$ we denote the union of these planes which ranges over all options for η_{n-l+1} according to the first procedure. Denote $N_2 = (2kd)^{l+2}$ and

$$\bar{\Delta} = \sum_{1 \le i \le l+1} \left(\frac{\partial \bar{h}_v(\eta_1, \ldots, \eta_{n-l}, Z_{n-l+1}, Y_1, \ldots, Y_{l+1})}{\partial Y_i} \right)^2 .$$

Lemma 10 [15]. *For any point $\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1})$ satisfying formula Λ_3, there exist $0 \le \lambda_1, \ldots, \lambda_l \le N_2$ such that for each $0 \le i \le l$ and every component $W \subset \bar{K}_2^{l+2}$ irreducible over the field $\bar{K}_2$ (here $\bar{K}_2 = K_2[\sqrt{-1}]$ denotes the algebraic closure) of a variety determined by the following system of equations*

$$\begin{aligned} &\bar{h}_v(\eta_1, \ldots, \eta_{n-l}, Z_{n-l+1}, Y_1, \ldots, Y_{l+1}) \\ &= \left(\frac{\partial \bar{h}_v(\eta_1, \ldots, \eta_{n-l}, Z_{n-l+1}, Y_1, \ldots, Y_{l+1})}{\partial Y_1} \right) = \frac{\lambda_1}{N_2(l+2)} \bar{\Delta} = \ldots \\ &= \left(\frac{\partial \bar{h}_v(\eta_1, \ldots, \eta_{n-l}, Z_{n-l+1}, Y_1, \ldots, Y_{l+1})}{\partial Y_i} \right)^2 - \frac{\lambda_i}{N_2(l+2)} \bar{\Delta} = 0 \qquad (8)_i \end{aligned}$$

for which an intersection $W \cap (K_2^{l+2} \backslash \mathscr{P}) \neq \emptyset$ is nonempty, the dimension $\dim_{\bar{K}_2}(W) = l - i + 1$.

Lemma 11 [15]. *For any point* $\mathscr{Z}_0, \dots, \mathscr{Z}_{n+1})$ *satisfying formula* Λ_3 *and for every connected component* U *of the hypersurface* $\{g_1=0\} \subset K_2^{n+2}$ *such that* $U \subset \mathscr{D}_0\left(\sqrt{\varepsilon^{-2}+\varepsilon_1}\right)$ *(see the beginning of the section) and any* $\alpha \in K_2$ *differing from all the options for* η_{n-l+1} *produced according to the first procedure, if an intersection* $U_0 = \mathfrak{M}_v U \cap \{Z_1 = \eta_1, \dots, Z_{n-l} = \eta_{n-l}, Z_{n-l+1} = \alpha\} \neq \emptyset$ *is nonempty then each connected component of the set* U_0 *has a point which satisfies system* $(8)_l$.

The algorithm tests all possible $0 \le \lambda_1, \dots, \lambda_l \le N_2$ and for each of them (let us fix some $\lambda_1, \dots, \lambda_l$) yields a quantifier-free formula Λ_4 in the variables $\mathscr{Z}_0, \dots, \mathscr{Z}_{n+1}$ such that for any point $(\mathscr{Z}_0, \dots, \mathscr{Z}_{n+1})$ satisfying formula $\Lambda_3 \,\&\, \Lambda_4$, the variety determined in the space K_2^{l+2} by system $(8)_l$, is a semialgebraic curve. Namely, firstly the algorithm yields a formula Λ_4', with free variables $\mathscr{Z}_0, \dots, \mathscr{Z}_{n+1}$, which expresses a requirement that a projection of the set difference $C^{(\eta_1, \dots, \eta_{n-l})}$ of the semialgebraic set determined by formula $(8)_l \,\&\, (f_0+\varepsilon_1 > 0, \dots, f_{k+4}+\varepsilon_1 > 0)$ and the set $\mathscr{P}$ into each two-dimensional coordinate plane (i.e. a plane spanned by some two coordinates among Y_{i_2}, $1 \le i_2 \le l+1$ and Z_{n-l+1}) does not contain any disk (with a positive radius). We suppose here that the elements $\eta_1, \dots, \eta_{n-l}$ in $(8)_l$ are given by formulae of type (7). The requirement means that the dimension of the set difference is at most 1. Observe that formula Λ_4' has three quantifier alternations. Apply to formula Λ_4' the quantifier elimination algorithm [16] and as a result obtain an equivalent quantifier-free formula Λ_4 in the variables $\mathscr{Z}_0, \dots, \mathscr{Z}_{n+1}$.

Lemma 11 implies that the dimension of the variety $C^{(\eta_1, \dots, \eta_{n-l})}$ is at least 1. On the other hand, Lemma 10 entails the existence of an l-tuple $0 \le \lambda_1, \dots, \lambda_l \le N_2$ for which the mentioned dimension is at most 1, thus $C^{(\eta_1, \dots, \eta_{n-l})}$ is a curve.

Our next task is to represent the local extrema of the coordinate function Z_{n-l+1} on the curve $C^{(\eta_1, \dots, \eta_{n-l})}$. One can easily produce a formula Λ_5 determining local extrema (provided that the point $(\mathscr{Z}_0, \dots, \mathscr{Z}_{n+1})$ satisfies the formula $\Lambda_3 \,\&\, \Lambda_4$). The formula Λ_5' has free variables $\mathscr{Z}_0, \dots, \mathscr{Z}_{n+1}, Z_1, \dots, Z_{n-l+1}, Y_1, \dots, Y_{l+1}$ and for its description, two quantifier alternations suffice. Apply the quantifier elimination algorithm [16] to Λ_5' and as a result obtain an equivalent quantifier-free formula Λ_5''.

Thereupon produce the projection of the set $\{\Lambda_5''\} \subset K_2^{2(n+2)}$ into the space K_2^{n+3} with coordinates $\mathscr{Z}_0, \dots, \mathscr{Z}_{n+1}, Z_{n-l+1}$ again invoking the algorithm [16]. The algorithm determines the projection of the set $\{\Lambda_5''\}$ by a quantifier-free formula Λ_5''' which is represented as a disjunction of systems of polynomial inequalities. To each system of inequalities apply the subroutine exposed above for solving systems of polynomial inequalities with coefficients polynomial in $\mathscr{Z}_0, \dots, \mathscr{Z}_{n+1}$. As a result, for each system of inequalities, we get a partition of the semi-algebraic set $\{\Lambda_3 \,\&\, \Lambda_4\} \subset K_2^{n+2}$ and for every element of the partition we get a formula of type (7). After that the algorithm yields a partition which is finer than the partitions of the set $\{\Lambda_3 \,\&\, \Lambda_4\}$ for all systems of inequalities under consideration (using [1], see above). The algorithm tests all the elements of the latter partition. Fix a certain element of this partition, and let it be determined by a quantifier-free formula Λ_5. Moreover the algorithm yields formula $(7)_{\Lambda_5}$ of the kind (7) corresponding to Λ_5 such that by restricting on the cylinder $\{\Lambda_5\} \times K_2$ formula $(7)_{\Lambda_5}$ determines the set $\{\Lambda_5'''\} \cap (\{\Lambda_5\} \times K_2)$ (cf. above the exposition of the subroutine).

Observe that for any point $(\mathscr{Z}_0, \dots, \mathscr{Z}_{n+1})$ satisfying $\Lambda_3 \,\&\, \Lambda_4 \,\&\Lambda_5$ there is a finite number of values of the coordinate Z_{n-l+1} which satisfy formula $(7)_{\Lambda_5}$. Namely, the values of the local extrema of the coordinate function Z_{n-l+1} on the curve $C^{(\eta_1, \dots, \eta_{n-l})}$. We take these values as the values of η_{n-l+1} being produced according to the third procedure (cf. above), and this completes the description of the third procedure.

The algorithm finds all the feasible orderings of the produced values of η_{n-l+1} in the following way. Take any pair of values of η_{n-l+1} determined by quantifier-free formulae Ψ_1, Ψ_2 respectively, of type (7) with variables $Z^{(1)}, Z^{(2)}$ (apart from the variables $\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1}$) respectively. Invoking [16] the algorithm produces the projections of the following 3 semialgebraic sets $\{\Psi_1 \& \Psi_2 \& Z^{(1)} > Z^{(2)}\}$, $\{\Psi_1 \& \Psi_2 \& Z^{(1)} < Z^{(2)}\}$, $\{\Psi_1 \& \Psi_2 \& Z^{(1)} = Z^{(2)}\} \subset K_2^{n+4}$ into the space K_2^{n+2} with coordinates $\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1}$. Let their quantifier-free formulae be $\Lambda_6^{(1)}, \Lambda_6^{(2)}, \Lambda_6^{(3)}$, respectively. Collect all the polynomials occurring in formulae of types $\Lambda_6^{(1)}, \Lambda_6^{(2)}, \Lambda_6^{(3)}$ for all pairs of values of η_{n-l+1} and find all feasible families of signs of these polynomials invoking [1]. Observe that every feasible family of signs determined some ordering of the produced values of η_{n-l+1} (provided that a point $(\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1})$ satisfies formula $\Lambda_3 \& \Lambda_4 \& \Lambda_5$). Fix some family of signs and denote the corresponding quantifier-free formula with variables $\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1}$ by Λ_6.

Thereupon the algorithm produces the endpoints of the curve $C^{(\eta_1, \ldots, \eta_{n-l})}$, i.e. the points lying in the intersections of planes of type $\{Z_1 = \eta_1, \ldots, Z_{n-l} = \eta_{n-l}, Z_{n-l+1} = \eta_{n-l+1}\}$ (for different values of η_{n-l+1} produced according to the three procedures exhibited above) with the closure of the curve $C^{(\eta_1, \ldots, \eta_{n-l})}$ (in the topology with base of open balls. Notice that there is a finite number of endpoints. For this purpose one yields a formula determining the endpoints with free variables $\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1}, Z_1, \ldots, Z_{n-l+1}, Y_1, \ldots, Y_{l+1}$ (provided that the point $(\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1})$ satisfies formula $\Lambda_3 \& \Lambda_4 \& \Lambda_5 \& \Lambda_6$) and having two quantifier alternations. Using [16] the algorithm yields an equivalent quantifier-free formula Λ_7'. Apply to Λ_7' the subroutine exposed above for solving systems of polynomial inequalities with polynomial coefficients. As a result we get a partition of the semialgebraic set $\{\Lambda_3 \& \Lambda_4 \& \Lambda_5 \& \Lambda_6\}$. The algorithm tests all the elements of this partition. Fix one of the elements of the partition, let it be determined by a quantifier-free formula Λ_7 in the variables $\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1}$. Moreover the algorithm yields a formula $(7)'_{\Lambda_7}$ of the form (7), which by restricting on the cylinder $\{\Lambda_7\} \times K_2^{n+2}$ determines the set $\{\Lambda_7'\} \cap (\{\Lambda_7\} \times K_2^{n+2})$ (cf. above).

One can easily yield a formula $(7)_{\Lambda_7}$ of the form (7) such that for any point $(\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1})$ satisfying formula $\Lambda_3 \& \Lambda_4 \& \Lambda_5 \& \Lambda_6 \& \Lambda_7$, the set of points which satisfy formula $(7)_{\Lambda_7}$ coincides with the union of the points satisfying formula $(7)'_{\Lambda_7}$, as well as the points produced earlier $u_v^{(1)}(\eta_1, \ldots, \eta_{n-l}), \ldots, u_v^{(q)}(\eta_1, \ldots, \eta_{n-l})$ and lastly the points of type $\sigma^{(\eta_1, \ldots, \eta_{n-l}, \eta_{n-l+1})}$ for various values of η_{n-l+1} produced according to the first procedure. All the points which satisfy formula $(7)_{\Lambda_7}$ we denote by $\bar{u}_v^{(1)} = \bar{u}_v^{(1)}(\eta_1, \ldots, \eta_{n-l}), \ldots, \bar{u}_v^{(\bar{q})} = \bar{u}_v^{(\bar{q})}(\eta_1, \ldots, \eta_{n-l})$.

Our next task is to specify what points among $\bar{u}_v^{(1)}, \ldots, \bar{u}_v^{(\bar{q})}$ are linked by the curve $C^{(\eta_1, \ldots, \eta_{n-l})}$. For this purpose one needs (similar constructions were exposed in [10, 18]) a projection of the space K_2^{n+2} onto a plane K_2^2 with the property that only a finite number of the points of the plane have more than one inverse images of the curve $C^{(\eta_1, \ldots, \eta_{n-l})}$ (provided that a point $(\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1})$ satisfies the formula $\Lambda_3 \& \Lambda_4 \& \Lambda_5 \& \Lambda_6 \& \Lambda_7$). Observe that a projection with this property does exist (and, even "almost" any projection fits) because of "general position" argument.

Consider a projection of K_2^{n+2} onto a plane K_2^2 in parametrical form

$$\left\{T_1 = \sum_{1 \le i \le n-l+1} \alpha_{i,1} Z_i + \sum_{1 \le i \le l+1} \beta_{i,1} Y_i,\ T_2 = \sum_{1 \le i \le n-l+1} \alpha_{i,2} Z_i + \sum_{1 \le i \le l+1} \beta_{i,2} Y_i\right\}$$

where T_1, T_2 are the coordinates of the plane and α, β (with subindices) are the parameters. For a point on a closed (in the topology with base of open balls) semialgebraic curve we shall call its ramification degree the number of points of intersection of the curve with a circle of an arbitrary sufficiently small positive radius with the center in this point. The projection satisfies the desired property iff firstly, any point of the projection of the closure $\bar{C}^{(\eta_1,\dots,\eta_{n-l})}$ with the ramification degree 2 has a unique inverse image on the curve $\bar{C}^{(\eta_1,\dots,\eta_{n-l})}$ and secondly, for any point y of the curve $\bar{C}^{(\eta_1,\dots,\eta_{n-l})}$ for all sufficiently close points z to y on the curve $\bar{C}^{(\eta_1,\dots,\eta_{n-l})}$ the points y, z have distinct projections into the plane. The conjunction of the two latter conditions can be written as a formula having coefficients in the field K_2 and free variables $\alpha_{1,1}, \dots, \beta_{l+1,1}, \alpha_{1,2}, \dots, \beta_{l+1,2}$ and with 6 quantifier alternations (provided that a point $(\mathscr{Z}_0, \dots, \mathscr{Z}_{n+1})$ satisfies the formula $\Lambda_3 \,\&\, \Lambda_4 \,\&\Lambda_5 \,\&\, \Lambda_6 \,\&\, \Lambda_7$). Invoking [16] one can yield an equivalent quantifier-free formula which contains at most $(kd)^{n^{O(1)}}$ atomic subformulae of the kind $(p_i \geq 0)$, moreover $\deg_{\alpha_{1,1},\dots,\beta_{l+1,1},\alpha_{1,2},\dots,\beta_{l+1,2}}(p_i) \leq (kd)^{n^{O(1)}}$. Thus, if the product of all polynomials p_i does not vanish in a point $\alpha_{1,1}, \dots, \beta_{l+1,1}, \alpha_{1,2}, \dots, \beta_{l+1,2}$ then the projection with these values of the parameters $\alpha_{1,1}, \dots, \beta_{l+1,1}, \alpha_{1,2}, \dots, \beta_{l+1,2}$ satisfied the desired property. Therefore for a suitable $N_4 \leq (kd)^{n^{O(1)}}$ there exist integers $0 \leq \alpha_{1,1}, \dots, \beta_{l+1,1}, \alpha_{1,2}, \dots, \beta_{l+1,2}$ for which the projection satisfies the desired property.

The algorithm tests all integer values $0 \leq \alpha_{1,1}, \dots, \beta_{l+1,1}, \alpha_{1,2}, \dots, \beta_{l+1,2} \leq N_4$. Fix one of them and write down the condition that the corresponding projection satisfies the desired property as a quantifier-free formula Λ_8 with free variables $\mathscr{Z}_0, \dots, \mathscr{Z}_{n+1}$ (cf. above).

Denote by $\mathscr{U} \subset K_2^{2(n+2)}$ a semialgebraic set $\left\{(8)_l \,\&\, \underset{3\leq i\leq 8}{\&} \Lambda_i)\right\} \backslash \mathscr{P}$ (cf. Lemma 10).

Using [16] the algorithm produces the projection of $\mathscr{U}$ into the space K_2^{n+4} with coordinates $\mathscr{Z}_0, \dots, \mathscr{Z}_{n+1}, T_1, T_2$. To this projection we add the projections $\hat{u}_v^{(1)}(\eta_1, \dots, \eta_{n-l}), \dots, \hat{u}_v^{(\bar{q})}(\eta_1, \dots, \eta_{n-l})$ of the earlier produced points $\bar{u}_v^{(1)}(\eta_1, \dots, \eta_{n-l}), \dots, \bar{u}_v^{(\bar{q})}(\eta_1, \dots, \eta_{n-l})$, and let the resulting semialgebraic set be determined by a quantifier-free formula Ω_1.

Observe that for any point $(\mathscr{Z}_0, \dots, \mathscr{Z}_{n+1})$ satisfying the formula $\underset{3\leq i\leq 8}{\&} \Lambda_i$ every line of the type $\{T_1 = \text{const}\} \subset K_2^2$ has only a finite number of common points with the set $\{\Omega_1\}$ taking into account that the curve $\bar{C}^{(\eta_1,\dots,\eta_{n-l})}$ is situated in the ball $\mathscr{D}_0\left(\sqrt{\varepsilon^{-2} + \varepsilon_1}\right)$ being a union of some connected components of an appropriate algebraic curve (cf. Lemma 3 in Sect. 2) and the main property of the projection.

Thereupon the algorithm produces all the points in the projection of the curve $\bar{C}^{(\eta_1,\dots,\eta_{n-l})}$ of one of the following two sorts. Firstly, the points with ramification degree at least 3. Secondly, the points in which the coordinate J_1 reaches a local extremum on the projection of the curve $\bar{C}^{(\eta_1,\dots,\eta_{n-l})}$. Denote by $\hat{u}_v^{(1)}(\eta_1, \dots, \eta_{n-l}), \dots, \hat{u}_v^{(\bar{q})}(\eta_1, \dots, \eta_{n-l})$ the union of all the points of the first and the second sorts with the points $\hat{u}_v^{(1)}(\eta_1, \dots, \eta_{n-l}), \dots, \hat{u}_v^{(\bar{q})}(\eta_1, \dots, \eta_{n-l})$. For producing these points the algorithm expresses the condition of membership of a point in either to the first or second set, or to the family of points $\{\hat{u}_v^{(1)}(\eta_1, \dots, \eta_{n-l}), \dots, \hat{u}_v^{(\bar{q})}(\eta_1, \dots, \eta_{n-l})\}$, as a formula with free variables $T_1, T_2, \mathscr{Z}_0, \dots, \mathscr{Z}_{n+1}$ and 3 quantifier alternations. With the aid of [16] the algorithm yields an equivalent quantifier-free formula Ω_2. Apply to Ω_2 the subroutine for solving systems of polynomial inequalities with polynomial coefficients (cf. above), as a result obtain a partition of the semialgebraic set $\left\{\underset{3\leq i\leq 8}{\&} \Lambda_i)\right\} \subset K_2^{n+2}$. The algorithm tests all the elements of the partition. Fix a certain element, let it be determined by a formula Λ_9, then the algorithm yields a formula $(7)_{\Lambda_9}$ of the kind (7) such that $\left\{(7)_{\Lambda_9}) \,\&\, \underset{3\le i\le 9}{\&} \Lambda_i\right\} = \left\{\Omega_2 \,\&\, \underset{3\le i\le 9}{\&} \Lambda_i\right\}$.

Our next goal is to specify how the points $\hat{u}_v^{(1)}(\eta_1, \ldots, \eta_{n-l}), \ldots, \hat{u}_v^{(\bar{q})}(\eta_1, \ldots, \eta_{n-l})$ are connected, by means of the projection of the curve $\bar{C}^{(\eta_1,\ldots,\eta_{n-l})}$. Introduce an element $\varepsilon_3 > 0$ being an infinitesimal w.r.t. the field $\widetilde{K_2}$ and $\varepsilon_4 > 0$ being an infinitesimal w.r.t. the field $K_2(\varepsilon_3)$, denote the field $K_4 = \widetilde{K_2(\varepsilon_3, \varepsilon_4)}$. Let us draw the lines of type $\{T = \text{const}\}$ through the points $\hat{u}_v^{(1)}(\eta_1, \ldots, \eta_{n-l}), \ldots, \hat{u}_v^{(\bar{q})}(\eta_1, \ldots, \eta_{n-l})$. The algorithm finds (see below) all the points of intersections of these lines with the projection of the curve $\bar{C}^{(\eta_1,\ldots,\eta_{n-l})}$ and unites these points with the points $\hat{u}_v^{(1)}(\eta_1, \ldots, \eta_{n-l}), \ldots, \hat{u}_v^{(\bar{q})}(\eta_1, \ldots, \eta_{n-l})$ denote the obtained points by $\hat{u}_v^{(1)}(\eta_1, \ldots, \eta_{n-l}), \ldots, \hat{u}_v^{(\bar{q})}(\eta_1, \ldots, \eta_{n-l})$. Consider one of the mentioned lines and denote it by $L = \{T_1 = \kappa\}$. Furthermore, consider also the lines $L^+ = \{T_1 = \kappa + \varepsilon_4\}$, $L^- = \{T_1 = \kappa - \varepsilon_4\}$ and the algorithm finds (see below) the points of intersection of these lines with the projection of the curve $(\bar{C}^{(\eta_1,\ldots,\eta_{n-l})})^{(\varepsilon_3,\varepsilon_4)}$ (recall, according to the notation of Sect. 1, that the latter curve is determined in the space K_4^{n+2} by the same formula as the curve $\bar{C}^{(\eta_1,\ldots,\eta_{n-l})}$. Let a certain point $\hat{u}^{(j)}(\eta_1, \ldots, \eta_{n-l})$ lie on the line L, observe that the number of points of intersections situated in a disc $\mathscr{D}_{\hat{u}_v^{(j)}(\eta_1,\ldots,\eta_{n-l})}(\varepsilon_3)$, coincides with the ramification degree of the point $\hat{u}_v^{(j)}(\eta_1, \ldots, \eta_{n-l})$ provided that $\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1}) \in \left\{\underset{3\le i\le 9}{\&} \Lambda_i\right\} \cap K_2^{n+2}$.

The algorithm finds the points of intersection with the lines of type L, L^+, L^- invoking the subroutine for solving systems of inequalities with polynomial coefficients.

The subroutine produces a partition of the set $\left\{\underset{3\le i\le 9}{\&} \Lambda_i\right\}^{(\varepsilon_3,\varepsilon_4)} \subset K_4^{n+2}$ into semialgebraic subsets. The algorithm tests all the elements of the partition. Fix one of the elements, and let it be determined by a quantifier-free formula Δ'_{10} with coefficients from the ring $K_2[\varepsilon_3, \varepsilon_4]$. Moreover, the algorithm yields a formula $(7)_{\Delta'_{10}}$ (with coefficients from $K_2[\varepsilon_3, \varepsilon_4]$ of the type (7) which determined the points of intersection of the projection of the curve $\bar{C}^{(\eta_1,\ldots,\eta_{n-l})(\varepsilon_3,\varepsilon_4)}$ with the lines of type L, L^+, L^- in the cylinder $\left\{\underset{3\le i\le 9}{\&} \Lambda_i \,\&\, \Lambda'_{10}\right\} \times K_4^2$.

Thereupon the algorithm produces the inverse images on the curve $\bar{C}^{(\eta_1,\ldots,\eta_{n-l})(\varepsilon_3,\varepsilon_4)}$ of the points of intersections with lines of type L, L^+, L^- again invoking the subroutine for solving systems of polynomial inequalities with polynomial coefficients. As a result the algorithm obtains a partition of the semialgebraic set $\left\{\underset{3\le i\le 9}{\&} \Lambda_i \,\&\, \Lambda'_{10}\right\} \subset K_4^{n+2}$. As above the algorithm tests all the elements of the partition. Fix some element, let it be determined by a quantifier-free formula Λ'_{11} with free variables $\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1}$ moreover the soubroutine yields a formula $(7)_{\Lambda'_{11}}$ of type (7), which determined the inverse images of the points of intersections on the curve $\bar{C}^{(\eta_1,\ldots,\eta_{n-l})(\varepsilon_3,\varepsilon_4)}$ in the cylinder $\left\{\underset{3\le i\le 9}{\&} \Lambda_i \,\&\, \Lambda'_{10} \,\&\, \Lambda'_{11}\right\} \times K_4^{n+2}$.

Provided that a point $(\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1})$ belongs to se set $\left\{\underset{3\le i\le 9}{\&} \Lambda_i \,\&\, \Lambda'_{10}\right\} \cap K_2^{n+2}$, consider a line of type L_1^- corresponding to one of the computed lines L_1 (provided that it does exist) such that between the parallel lines L^+ and L_1^- there are no computed lines (we call L^+ and L_1^- adjacent lines). Observe that the lines L^+ and L_1^- contain both the same number of points of intersection and the projection of the curve

$\bar{C}^{(\eta_1,\ldots,\eta_{n-l})(\varepsilon_3,\varepsilon_4)}$ links the respective points according to their ordering on the lines L^+ and L_1^-. Hence, the standard parts of these points w.r.t. ε_3, ε_4 (lying on the lines L and L_1, respectively) are also linked by the projection of the curve. The structures of links for all pairs of adjacent lines determine, in particular, the structure of links of the points $\hat{u}_v^{(1)}(\eta_1, \ldots, \eta_{n-l}), \ldots, \hat{u}_v^{(\bar{q})}(\eta_1, \ldots, \eta_{n-l})$.

After that applying a procedure similar to the one that was used for finding all the feasible orderings of the values of η_{n-l+1} (see above) the algorithm finds, firstly, all the feasible orderings of the computed lines of the type $\{T_1 = \text{const}\}$, secondly, all the feasible orderings of the computed points of intersection on lines of type L^+, L^-; thirdly, the algorithm tests pairs x, y of computed points of intersection on lines of type L^+, L^- with some inverse images x_1, y_1 respectively, on the curve $\bar{C}^{(\eta_1,\ldots,\eta_{n-l})(\varepsilon_3,\varepsilon_4)}$ [recall that the inverse images are determined by the formulae of the type $(7)_{\Lambda'_{11}}$], such that the distance between x_1, y_1 is less than ε_3 (i.e. their standard parts w.r.t. ε_3, ε_4 coincide). Fourthly, the algorithm tests for which points of intersections x and for which points $\bar{u}_v^{(j)}(\eta_1, \ldots, \eta_{n-l})$ (where $1 \le j \le \bar{q}$) some inverse image of x on the curve $\bar{C}^{(\eta_1,\ldots,\eta_{n-l})(\varepsilon_3,\varepsilon_4)}$ and the point $\bar{u}_v^{(j)}(\eta_1, \ldots, \eta_{n-l})$ are situated at a distance less than ε_3. Recall that for a point $(\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1})$ from the set $\left\{ \underset{3\le i\le 9}{\&} \Lambda_i \,\&\, \Lambda'_{10} \,\&\, \Lambda'_{11} \right\} \cap K_4^{n+2}$ the inverse images under consideration are unique. The algorithm tests all the possibilities for the four described tests. Fix one of the possibilities. Observe (provided that a point $(\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1})$ belongs to the set $\left\{ \underset{3\le i\le 9}{\&} \Lambda_i \,\&\, \Lambda'_{10} \,\&\, \Lambda'_{11} \right\} \cap K_4^{n+2}$) that the possibility determines also the structure of links of the points $\bar{u}_v^{(1)}(\eta_1, \ldots, \eta_{n-l}), \ldots, \bar{u}_v^{(\bar{q})}(\eta_1, \ldots, \eta_{n-l})$ by the curve $\bar{C}^{(\eta_1,\ldots,\eta_{n-l})}$ (the mentioned structure can be represented by a graph in which the indicated points are vertices and are adjacent if they are connected by the curve $\bar{C}^{(\eta_1,\ldots,\eta_{n-l})}$. The algorithm yields also a quantifier-free formula Λ'_{12} with free variables $\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1}$ such that for any point $(\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1})$ from the set $\left\{ \underset{3\le i\le 9}{\&} \Lambda_i \,\&\, \Lambda'_{10} \,\&\, \Lambda'_{11} \,\&\, \Lambda'_{12} \right\} \cap K_2^{n+2}$ the fixed possibility is valid.

For each atomic subformula of the type $p>0$ occurring in the formula $\Lambda'_{10} \,\&\, \Lambda'_{11} \,\&\, \Lambda'_{12}$ where the polynomial $p \in K_2[\varepsilon_3,\varepsilon_4]\ [\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1}]$, write $p = \sum_{0\le i,\, j\le \mathscr{S}} p_{ij}\varepsilon_3^i\varepsilon_4^j$, where $p_{i,j} \in K_2[\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1}]$. Then the formula $p > 0$ for the points $(\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1}) \in K_2^{n+2}$ is equivalent to a formula

$$\begin{aligned}(p_{0,0} > 0) &\vee (p_{0,0} = 0 \,\&\, p_{1,0} > 0) \vee \ldots \\ &\vee (p_{0,0} = p_{1,0} = \ldots = p_{\mathscr{S}-1,0} = 0 \,\&\, p_{\mathscr{S},0} > 0) \\ &\vee (p_{0,0} = \ldots = p_{\mathscr{S},0} = 0 \,\&\, p_{\mathscr{S},1} > 0) \vee \ldots \\ &\vee (p_{0,0} = \ldots = p_{\mathscr{S},\mathscr{S}-1} = 0 \,\&\, p_{\mathscr{S},\mathscr{S}} > 0)\end{aligned}$$

(cf. the end of Sect. 2). Replace in the formula $\Lambda'_{10} \,\&\, \Lambda'_{11} \,\&\, \Lambda'_{12}$ each atomic subformula of the type $p > 0$ by the indicated formula, as a result we obtain a formula $\Lambda_{10} \,\&\, \Lambda_{11} \,\&\, \Lambda_{12}$.

Thus, we can complete the description of the recursive step for constructing the tree $\mathscr{T}$ (see the beginning of the section). Namely, to each pair consisting of a certain computed formula of the form $\underset{3\le i\le 12}{\&} \Lambda_i$ and of a certain produced value of η_{n-l+1}, there corresponds some vertex w of the tree $\mathscr{T}$ which is an immediate descendant of the vertex v (different pairs of this type correspond to different immediate descendants of v). $\bar{h}_v(\eta_1, \ldots, \eta_{n-l}, \eta_{n-l+1}, Y_1, \ldots, Y_{l+1})$ plays the role of the polynomial h_w. $\sigma(\eta_1, \ldots, \eta_{n-l}, \eta_{n-l+1})$ plays the role of the point σ_w (provided that the value of η_{n-l+1} under consideration was produced according to the first procedure). The points among the produced points $\bar{u}_v^{(1)}(\eta_1, \ldots, \eta_{n-l}), \ldots, \bar{u}_v^{(\bar{q})}(\eta_1, \ldots, \eta_{n-l})$ whose coordinate Z_{n-l+1} equals to the value of η_{n-l+1} under consideration, play the role of the points $u_w^{(1)}(\eta_1, \ldots, \eta_{n-l+1}), \ldots, u_w^{(p)}(\eta_1, \ldots, \eta_{n-l+1})$.

As a linear transformation $\mathfrak{M}_w$ we take $\overline{\mathfrak{M}_v}$. Finally, the algorithm outputs a graph representing the structure of links between the points $\bar{u}_v^{(1)}(\eta_1, \ldots, \eta_{n-l}), \ldots, \bar{u}_v^{(\bar{q})}(\eta_1, \ldots, \eta_{n-l})$ defined by the curve $\bar{C}^{(\eta_1, \ldots, \eta_{n-l})}$. This completes the description of the recursive step (for $n-l$).

To each leaf w_1 of the tree $\mathscr{T}$ corresponds the value $l=-1$ and a quantifier-free formula Λ_i. The variety $\{h_{w_1}(\eta_1, \ldots, \eta_{n+1}Y_1) = 0\}$ consists of a finite number of points on a line.

Collect all the polynomials occurring in the formula Λ_{w_1} for all the leaves w_1 and denote them by $g^{(1)}, \ldots, g^{(t)} \in K[\varepsilon_1, \varepsilon_2]\,[\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1}]$. Relying on [1,2] the algorithm enumerates all the $\{g^{(1)}, \ldots, g^{(t)}\}$-cells (see the introduction). Fix for the time being a certain cell $\mathscr{C} \subset K_2^{n+1}$. Select (again involving [1]) all the vertices v_1 of the three $\mathscr{T}$ such that $\{\Lambda_{v_1}\} \supset \mathscr{C}$. For every selected vertex v_1 the algorithm yields by recursion on its depth a graph $G_{v_1}^{(\mathscr{C})}$. Base case of the recursion: For a selected leaf w_0 the vertices of $G_{w_0}^{(\mathscr{C})}$ correspond bijectively to the points of the variety $\{h_{w_0}(\eta_1, \ldots, \eta_{n+1}Y_1) = 0\}$ and $G_{w_0}^{(\mathscr{C})}$ has no edges. For considering a recursive step assume w.l.o.g. that a vertex v of $\mathscr{T}$ is selected, i.e. $\{\Lambda_v\} \supset \mathscr{C}$, and we utilize the notations introduced above. Consider all the selected immediate descendants w in the tree $\mathscr{T}$ of v, observe that for all of them the formulae Λ_w are the same, namely, $\{\Lambda_w\}$ is the unique element of the partition yielded above of the set $\{\Lambda_v\}$ such that $\{\Lambda_w\} \supset \mathscr{C}$ so the same are also the graphs representing the structure of links of the points of the kind $\bar{u}_v^{(1)}(\eta_1, \ldots, \eta_{n-l}), \ldots, \bar{u}_v^{(\bar{q})}(\eta_1, \ldots, \eta_{n-l})$. In fact different vertices w of this type correspond (for the fixed formula Λ_w) to different values of η_{n-l+1} (see above). Thus, the set of vertices of the graph $G_v^{(\mathscr{C})}$ is the union of the sets of vertices of the graphs $G_w^{(\mathscr{C})}$ for all the selected immediate descendants of v. The set of edges of $G_v^{(\mathscr{C})}$ is obtained as the union, firstly, of all the sets of edges of $G_w^{(\mathscr{C})}$, and secondly, the edges of the graph representing the structure of links of the points of type $\bar{u}_v^{(1)}(\eta_1, \ldots, \eta_{n-l}), \ldots, \bar{u}_v^{(\bar{q})}(\eta_1, \ldots, \eta_{n-l})$ which corresponds to the formula Λ_w.

At the end of this process the algorithm yields the graph $G_{v_0}^{(\mathscr{C})}$ corresponding to the root v_0 of the tree $\mathscr{T}$. One can prove (following the proof of correctness of the algorithm in Sect. 4 [15]) that the number of the connected components of the graph $G_{v_0}^{(\mathscr{C})}$ coincides with the number of the connected components of the hypersurface $\{g_1 = 0,\ f_0 + \varepsilon_1 > 0, \ldots, f_{k+4} + \varepsilon_1 > 0\}$, moreover each connected component of $G_{v_0}^{(\mathscr{C})}$ contains a unique representative point among $u_{v_0}^{(1)}, \ldots, u_{v_0}^{(q)}$ for the hypersurface $\{g_1 = 0,\ f_0 + \varepsilon_1 > 0, \ldots, f_{k+4} + \varepsilon_1 > 0\}$ (produced at the beginning of the present section). Hence there is a unique representative point $u^{(\mathscr{C})} = u_{v_0}^{(i)}$ for a suitable i such that $u^{(\mathscr{C})}$ and the point $(\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1})$ belong to the same connected component of the graph $G_{v_0}^{(\mathscr{C})}$. Then the connected component of the hypersurface $\{g_1 = 0,\ f_0 + \varepsilon_1 > 0, \ldots, f_{k+4} + \varepsilon_1 > 0\}$ which contains a representative point $u_{v_0}^{(i_0)}$, coincides with the union on all the $\{g^{(1)}, \ldots, g^{(t)}\}$-cells $\mathscr{C}_0$ such that $u^{(\mathscr{C}_0)} = u_{v_0}^{(i_0)}$ (it follows from Sect. 4 [15]).

This completes the description of the algorithm which finds the connected components of a nonsingular bounded hypersurface of the form $\{g_1 = 0, f_0 + \varepsilon_1 > 0, \ldots, f_{k+4} + \varepsilon_1 > 0\}$, and thereby the algorithm which finds the connected components of an arbitrary semialgebraic set (see Sects. 1, 2).

4. Complexity Analysis of the Algorithm for Finding Connected Components

First of all we turn ourselves to complexity analysis of the algorithm exposed in Sect. 3. Firstly, we shall obtain a priori bounds on the degrees of the polynomials $\varphi_v^{(i,j)}$ determining the points $u_v^{(1)}(\eta_1, \ldots, \eta_{n-l}), \ldots, u_v^{(g)}(\eta_1, \ldots, \eta_{n-l})$ and also of the polynomials $\varphi^{(\eta_{\mathscr{S}})_v}$ defining $\eta_{\mathscr{S}}$, then we shall obtain a priori bounds on the bit-sizes of the coefficients of these polynomials. After that we shall bound the degrees, the bit-sizes of the coefficients and the number of the polynomials occurring in the formulae $\Lambda_3, \ldots, \Lambda_{12}$, finally, we shall estimate the running time of the algorithm. We say that a formula satisfies a $(D, D_0, \mathscr{M})$-bound if all the polynomials occurring in this formula satisfy this bound and the number of the atomic subformulae in it does not exceed D.

By recursion on $(n - l)$ we yield some family of S formulae of the first-order theory of the field K_2, with the aid of which later on we get a priori bounds on the degrees and on the bit-sizes of the coefficients of the polynomials $\varphi_v^{(i,j)}$, $\varphi_v^{(\eta_{\mathscr{S}})}$ which correspond to the vertex v of the tree $\mathscr{T}$ (cf. Sect. 5 [15]). We'll utilize the notation introduced by describing the recursive hypothesis at the beginning of Sect. 3. For each of the points $u_v^{(i)}(\eta_1, \ldots, \eta_{n-l})$ (we fix i for the time being) assume that a suitable formula of the first-order theory of the field K_2 (with 3 quantifier alternations) of the following form is already computed:

$$\Phi_v^{(i)} = \exists T_{1,1} \ldots \exists T_{1,\tau_1} \forall T_{2,1} \ldots \forall T_{2,\tau_2} \exists T_{3,1} \ldots \forall T_{3,\tau_3} (\Psi_v^{(i)}) ,$$

here $\Psi_v^{(i)}$ is a quantifier-free formula with atomic subformulae of the kind $(f \geq 0)$, where $f \in K[\varepsilon_1, \varepsilon_2]\ [T_{1,1}, \ldots, T_{3,\tau_3}, Z_1, \ldots, Z_{n-l}, Y_1, \ldots, Y_{l+2}, \mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1}]$. Moreover for any point $(\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1})$ (fix it for some time) satisfying the formula Λ_v, every point of a semialgebraic set $\{\Phi_v^{(i)}\} \cap \{Z_1 = \eta_1, \ldots, Z_{n-l} = \eta_{n-l}\} \subset K_2^{n+2}$ is a zero-dimensional connected component of the semialgebraic set $\{\Phi_v^{(i)}\}$, besides the point $u_v^{(i)}(\eta_1, \ldots, \eta_{n-l}) \in (\{\Phi_v^{(i)}\} \cap \{Z_1 = \eta_1, \ldots, Z_{n-l} = \eta_{n-l}\})$.

Thereupon one yields a formula $\Phi_{v,I}^{(i)}$ (cf. formula $(15)_I$ 9n [15]) of the form $\Phi_v^{(i)}$ with free variables $Z_1, \ldots, Z_{n-l}, Z_{n-l+1}, \mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1}$ such that for any point $(\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1})$, satisfying formula Λ_3, exactly the values of η_{n-l+1} produced according to the first procedure (see Sect. 3), satisfy a formula $\Phi_{v,I}^{(i)} \,\&\, Z_1 = \eta_1 \,\&\, \ldots \,\&\, Z_{n-l} = \eta_{n-l}$. Similarly a formula $\Phi_{v,II}^{(i)}$ can be computed (cf. formula $(15)_{II}$ in [15]) corresponding to the second procedure for producing values of η_{n-l+1}. Lastly, one can find a formula $\Phi_{v,III}^{(i)}$ (cf. formulae $(15)_{III\max}$, $(15)_{III\min}$ in [15]) such that exactly the values of η_{n-l+1} produced according to the third procedure, satisfy a formula $\Phi_{v,III}^{(i)} \,\&\, Z_1 = \eta_1 \,\&\, \ldots \,\&\, Z_{n-l} = \eta_{n-l}$, provided that a point $(\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1})$ satisfies a formula of the kind Λ_5 (see Sect. 3).

For each of the produced values of $\hbar_{n-l+1}$ (satisfying one of the formulae $\Phi_{v,I}^{(i)}$, $\Phi_{v,II}^{(i)}$, $\Phi_{v,III}^{(i)}$) one can yield a formulae $\Phi_w^{(i)}$ (again with 3 quantifier alternations) having free variables $Z_1, \ldots, Z_{n-l+1}, Y_1, \ldots, Y_{l+1}, \mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1}$, such that for any point $(\mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1})$ satisfying formula $\underset{3\leq i\leq 7}{\&} \Lambda_i$ every point of a intersection $\{\Phi_w^{(i)}\} \cap \{Z_1 = \eta_1, \ldots, Z_{n-l} = \eta_{n-l}, Z_{n-l+1} = \eta_{n-l+1}\}$ is a zero-dimensional connected component of a semialgebraic set $\{\Phi_w^{(i)}\}$, as well as every point among $\bar{u}_v^{(1)}, \ldots, \bar{u}_v^{(q)}$ (see Sect. 3) for which the coordinates $Z_1 = \eta_1, \ldots, Z_{n-l} = \eta_{n-l}$, $Z_{n-l+1} = \eta_{n-l+1}$ satisfies formula $\Phi_w^{(i)}$ (cf. Sect. 5 [15]). On the next step of recursion, the formulae of form $\Phi_w^{(i)}$ play a role similar to the role of the formula $\Phi_v^{(i)}$ (by constructing the tree $\mathscr{T}$).

Applying the bounds for the quantifier elimination algorithm from [16] to the formulae of the form $\Phi_v^{(i)}$ we obtain the bounds:

$$\deg_{Z, \mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1}}(\varphi_v^{(i,j)}), \quad \deg_{Z_{\mathscr{S}}, \mathscr{Z}_0, \ldots, \mathscr{Z}_{n+1}}(\varphi_v^{(\eta_{\mathscr{S}})}) \leq (kd)^{n^{O(1)}};$$

$$\deg_{\delta_1, \ldots, \delta_m, \varepsilon, \varepsilon_1, \varepsilon_2}(\varphi_v^{(i,j)}), \quad \deg_{\delta_1, \ldots, \delta_m, \varepsilon, \varepsilon_1, \varepsilon_2}(\varphi_v^{(\eta_{\mathscr{S}})}) \leq d_0(kd)^{n^{O(1)}},$$

for $1 \leq j \leq l+2$, $1 \leq \mathscr{S} \leq n-l$ [cf. (1) in the introduction] taking into account that in formula of the form $\Phi_v^{(i)}$, there occur polynomials of type h_v, but neither polynomials $\varphi_v^{(i,j)}$, $\varphi_v^{(\eta_{\mathscr{S}})}$ nor formular of the type Λ_l, $l \leq 12$ (cf. the remark after the description of the subroutine for solving systems of polynomial inequalities with polynomial coefficients and the bounds in Sect. 5 [15]). Using these bounds one can estimate N' from Lemma 9 as in Sect. 5 [15]. After that, one can bound the bit-sizes of the coefficients of the polynomials of type h_v (and thereby the bit-sizes of the coefficients of the polynomials occurring in formulae of the form $\Phi_v^{(i)}$) as $(M + m \log d_0)(nkd)^{O(1)}$. Therefore applying again [16] to the formulae $\Phi_v^{(i)}$, we obtain the bounds for bit-sizes of the coefficients: $l(\varphi_v^{(i,j)})$, $l(\varphi^{(\eta_{\mathscr{S}})}) \leq (M + md_0)(kd)^{n^{O(1)}}$ [cf. (1)].

Now we proceed to the estimates for the quantifier-free formulae of type Λ_l. Recall (see the remark just after the description of the subroutine for solving systems of polynomial inequalities with the polynomial coefficients) that the formula Λ_w is obtained on the recursive step as a conjunction $\Lambda_v \,\&\, \Lambda'$ for an appropriate formula Λ' and moreover Λ' does not depend on Λ_v. Namely, Λ' is the conjunction of several formulae yielded at the recursive step. The first yielded formula has a form $\Phi_2^{(1,t_2,\ldots,t_{l+2})}$ (see Lemma 9). The bounds ascertained above for $\varphi_v^{(i,j)})$, $\varphi^{(\eta_{\mathscr{S}})}$ and the bounds from [16] imply that the formula $\Phi_2^{(1,t_2,\ldots,t_{l+2})}$ satisfies a

$$((kd)^{n^{O(1)}}, d_0(kd)^{n^{O(1)}}, (M + md_0)(kd)^{n^{O(1)}})\text{-bound}. \qquad (*)$$

Similarly, the formulae $\Lambda_3, \ldots, \Lambda_{12}$ satisfy $(*)$. Hence each formula of the type Λ_v also satisfies $(*)$. Because of that the yielded quantifier-free formula which determines a connected component (see the end of Sect. 3) also satisfies $(*)$.

Consider now the formulae which determine the connected components of the semialgebraic set $V_0 \subset (\tilde{\mathbb{Q}}_m)^n$ being determined by system of inequalities (2) (see Sect. 2). Recall that in Sect. 2, the quantifier elimination algorithm [16] is applied to a formula of the type Π; applying the bounds ascertained above we conclude that the quantifier-free formula Π_1 and thereby the formulae which determine the connected components, again satisfy $(*)$. Taking into account the construction in Sect. 1 and the bound $(kd)^{O(n)}$ on the number of the elements of the partition $\mathscr{U}(\{f_1, \ldots, f_k\})$ (see [1, 2]), we deduce a bound of the form $(*)$ on the formulae which determine the connected components of a semialgebraic set $\{\Xi\}$ (see the theorem in the introduction).

To complete the proof of the theorem, it remains to estimate the running time of the algorithm. First, we estimate the running time of the algorithm in Sect. 3. The algorithm in Sect. 3 has a recursive structure which is represented by means of the rooted tree $\mathcal{T}$ with depth $n-1$. The branching degree at every vertex of $\mathcal{T}$ does not exceed $(kd)^{n^{O(1)}}$ by virtue of the bounds in [16], hence the whole number of vertices of the three $\mathcal{T}$ can be bounded by a similar value. Besides, the recursive step (producing all the immediate descendants of the vertex v in the tree $\mathcal{T}$) requires at most $M^{O(1)} \cdot (kd)^{n^{O(1)}}(m+1) \cdot d_0^{O(n+m)}$ time, since the algorithms from [1] or [16] deal with systems or with formulae which contain polynomials h_v, $\varphi_v^{(i,j)}$, $\varphi_v^{(\eta\mathcal{S})}$, and into the applications of [1] or [16] are iterated a fixed number of times (at most 11 times corresponding to the formulae $\Phi_2^{(1,t_2,\ldots,t_{l+2}}$, $\Lambda_3, \ldots, \Lambda_{12}$). Here we take into account the bounds ascertained aboves of the form $(*)$ for the polynomials h_v, $\varphi_v^{(i,j)}$, $\varphi_v^{(\eta\mathcal{S})}$ and the time-bounds in [1, 16]. Apart from that the algorithm at the beginning produces a representative set for the variety $\{g_1 = 0,\ f_0+\varepsilon_1 > 0, \ldots, f_{k+4}+\varepsilon_1 > 0\}$ such that each of its connected components contains a unique representative point, also within a similar time-bound (see [1, 15]). Thus, a similar value bounds the running time of the algorithm in Sect. 3.

The running time of the reduction in Sect. 2 can be also bounded by a similar value in force of the time-bound of the algorithm in [16]. A similar time-bound is valid also for the reduction in Sect. 1, relying on [1]. This completes the proof of the theorem.

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