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FINDING REAL SOLUTIONS OF SYSTEMS OF ALGEBRAIC INEQUALITIES IN SUBEXPONENTIAL TIME

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1. Suppose polynomials $f_1, \dots, f_k \in \mathbf{Z}[X_1, \dots, X_n]$ are given. In this note we describe an algorithm that determines whether the system of inequalities

$$f_1 \geq 0, \dots, f_l \geq 0, \qquad f_{l+1} > 0, \dots, f_k > 0 \tag{1}$$

has a solution in $\mathbf{R}^n$, and if so, then indicates some of them. Let the degrees $\deg f_i$ be bounded by d, and suppose that the coefficients of the polynomials f_i are bounded by 2^M. Then the size of the system (1) can be bounded (cf. [5] and [6]) by the quantity $L = kMd^n$.

THEOREM. *It is possible to construct an algorithm that determines whether the system* (1) *has a solution in* $\mathbf{R}^n$, *and if so, indicates at least one point on each connected component of the semialgebraic set in* $\mathbf{R}^n$ *defined by this system. The execution time for this algorithm can be bounded by a polynomial in* $M(kd)^{n^2} \leq L^{\log^2 L}$ (*i.e. it is subexponential as a function of input size*).

Concerning the representation of points, see the lemma below.

Previously a bound of order $(Mkd)^{2^n}$ was known for this problem (see, for example, [1] and [2]). We mention further that in the case $\deg f_i = 1$ for $1 \leq i \leq k$ (the problem of linear programming), an algorithm of polynomial complexity was obtained in [3].

Below an algorithm of subexponential complexity for the solution of systems of algebraic equations over an algebraically closed field (see [5] and [6]) will be used essentially.

2. First an important special case will be considered: the system of equations $f_1 = \dots = f_k = 0$, where $f_1, \dots, f_k \in \mathbf{Z}[X_1, \dots, X_{n-1}]$. In (4) it was established that every connected component of the variety $\{f_1 = \dots = f_k = 0\} \subset \mathbf{R}^{n-1}$ consisting of points satisfying the system $f_1 = \dots = f_k = 0$ has nonempty intersection with a closed ball $\mathcal{D}_r \subset \mathbf{R}^{n-1}$ of radius $r \leq \exp(p(L))$ for a suitable polynomial p. Therefore we may restrict ourselves to the intersection $\{f_1 = \dots = f_k = 0\} \cap \mathcal{D}_r$, and locate points on the components of this compact set. Namely, we adjoin an additional equation to the system under consideration, containing a new variable X_n, and we obtain a new system $f_1 = \dots = f_k = X_1^2 + \dots + X_n^2 - r^2 = 0$. Since the set of real roots of the last system coincides with the manifold $V_0 = \{f = 0\}$ of real roots of the polynomial $f = f_1^2 + \dots + f_k^2 + (X_1^2 + \dots + X_n^2 - r^2)^2$, we also consider the manifold V_0.

Let ε be transcendental over $\mathbf{R}$. Then we may consider the field $\mathbf{R}(\varepsilon)$ as a formally real field (see [7], Chapter XI), taking ε to be infinitesimal, i.e. $0 < \varepsilon < a$ for every $0 < a \in \mathbf{R}$. Then the field $F = \mathbf{R}((\varepsilon^{1/\infty})) \supset \mathbf{R}(\varepsilon)$ of Puiseux series, i.e. power series with rational exponents having bounded denominators, is real closed, and the field $\overline{F} = F[\sqrt{-1}] = \mathbf{C}((\varepsilon^{1/\infty}))$ is algebraically closed (see [7], Chapter XI).

Then ε cannot be a critical value [9] of the polynomial f as a function $F^n \to F$; in other words, the system $f - \varepsilon = \partial f/\partial X_1 = \dots = \partial f/\partial X_n = 0$ has no root in F^n, since

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all critical values of a polynomial are algebraic over the field generated by its coefficients. Consequently the variety $V_\varepsilon = \{f - \varepsilon = 0\} \subset F^n$ is a smooth hypersurface by the implicit function theorem [9]. As $V_\varepsilon \subset \mathcal{D}_{r+\varepsilon}$, the variety V_ε is compact. We denote by $V'_\varepsilon \subset V_\varepsilon \subset F^n$ the variety of zeros of the system

$$f - \varepsilon = \partial f/\partial X_2 = \cdots = \partial f/\partial X_n = 0. \tag{2}$$

Observe that if the Jacobian $J(x)$ of the system (2) is nonzero at the point $x = (x_1, \ldots, x_n) \in V'_\varepsilon$, then x is a 0-dimensional irreducible component of the variety of all zeros in $\overline{F}^n$ of the system (2), by the implicit function theorem.

We denote by $K \subset V_\varepsilon$ the set of points of V_ε at which the Gauss-Kronecker curvature [10] vanishes. In [8] (see also [4]) it is established that for points $x \in V'_\varepsilon$ the condition $x \notin V'_\varepsilon \cap K$ holds if and only if $J(x) \neq 0$.

We introduce the $n \times n$ matrix

$$\mathcal{M}(X_1, \ldots, X_n) = \left(\prod_{3 \le j \le n} \overbrace{\begin{pmatrix} 1 & & & & & & \\ & 1 & 0 & & \frac{\partial f}{\partial X_j} & & \\ 0 & & \ddots & & & & \\ & -\frac{\partial f}{\partial X_j} & & 1 & 0 & & \\ & & & & 1 & & \\ & & 0 & & & 1 & \\ & & & & & & \ddots \\ & & & & & & & 1 \end{pmatrix}}^{j} \right) \cdot \begin{pmatrix} \frac{\partial f}{\partial X_1} & \frac{\partial f}{\partial X_2} & & & \\ -\frac{\partial f}{\partial X_2} & \frac{\partial f}{\partial X_1} & 0 & & \\ & 0 & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

with entries in $\mathbf{Z}[X_1, \ldots, X_n]$. For a point $x \in V_\varepsilon$ such that $(\partial f/\partial X_1)(x) \neq 0$, the point $\mathcal{M}(x)x \in F^n$ is a zero of the system

$$f(\mathcal{M}^{-1}(x)Y) - \varepsilon = \partial f(\mathcal{M}^{-1}(x)Y)/\partial Y_2 = \cdots = \partial f(\mathcal{M}^{-1}(x)Y)/\partial Y_n \tag{3}$$

in the variables $Y_1, \ldots, Y_n$, where the vector Y equals $(Y_1, \ldots, Y_n)$. If $(\partial f/\partial X_1)(x) \neq 0$, then $\mathcal{M}(x)K$ coincides with the set of points of zero curvature on the variety of zeros in F^n of the polynomial $f(\mathcal{M}^{-1}(x)Y) - \varepsilon \in F[Y_1, \ldots, Y_n]$. We introduce the rational function $g_1(X) = \det(\partial^2 f(\mathcal{M}^{-1}(X)Y)/\partial Y_i \partial Y_j)_{2 \le i,\, j \le n}|_{Y = \mathcal{M}(X)X} \in \mathbf{Q}(X_1, \ldots, X_n)$, where $X = (X_1, \ldots, X_n)$. Then for a point $x \in V_\varepsilon$ with $(\partial f/\partial X_1)(x) \neq 0$ relation $x \in K$ is satisfied, or in other words $\mathcal{M}(x)x \in \mathcal{M}(x)K$, if and only if the Jacobian $(g_1(x))(\partial f(\mathcal{M}^{-1}(x)Y)/\partial Y_1)(\mathcal{M}(x)x) = 0$ for the system (3) at the point $\mathcal{M}(x)x$ (see [8]). Now suppose $g_1 = g_2/g_3$, where the polynomials $g_2, g_3 \in \mathbf{Q}[X_1, \ldots, X_n]$ are relatively prime; we write g for $(\partial f/\partial X_1)g_2$. Then in view of what has been proved, $K \subset \{g = 0\}$.

The set K contains no connected component of the variety V_ε, since the curvature of a compact analytic manifold cannot be equal to zero at each of its points. Hence $\dim(V_\varepsilon \cap \{g = 0\}) \le n - 2$.

We note that $\deg(g) < (2dn)^3$; we write N for $(2d)^{n+4}n^3$. The algorithm constructs a family Γ of $(N+1)^{n-1}$ vectors, $\Gamma = \{\gamma = (\gamma_2, \ldots, \gamma_n)\}$, where each γ_i for $2 \le i \le n$ runs independently over the values $0, 1, \ldots, N$. We show that there is a $\gamma \in \Gamma$ for which the polynomial g is nonzero at all solutions in F^n to the system of equations

$$f - \varepsilon = (\partial f/\partial X_2) - \gamma_2(\partial f/\partial X_1) = \cdots = (\partial f/\partial X_n) - \gamma_n(\partial f/\partial X_1) = 0. \tag{4}$$

Indeed, we consider the regular mapping $\varphi\colon \overline{F}^n \to \overline{F}^n$ defined by $\varphi = (\partial f/\partial X_1, \ldots, \partial f/\partial X_n)$. Then the degree of the Zariski closure, $\deg \overline{\varphi(\{f - \varepsilon = g = 0\})}$, is less than or equal to $\deg(\{f-\varepsilon=g=0\})\cdot(2d)^n < N$ [11]. Therefore for some $\gamma\in\Gamma$ the point $(1,\gamma)\in\overline{F}^n$ does not lie on the cone of the variety $\overline{\varphi(\{f-\varepsilon=g=0\})}$; this is the desired γ.

Thus, there is a vector $\gamma\in\Gamma$ such that in every connected component of the variety $V_\varepsilon\subset F^n$ there are at least two points which are solutions of (4), and (4) has no solution in K.

The algorithm runs over the elements $\gamma\in\Gamma$; we fix one such γ. We construct the nonsingular $n\times n$ matrix

$$\mathcal{B}=\prod_{2\le j\le n}\left(\begin{array}{ccc} \overbrace{1 \qquad\quad}^{j} & & \gamma_j \\ -\gamma_j & & 0 \\ & 0 & 1 \end{array}\right)\Bigg\} j.$$

We set $f^{(1)}=f(\mathcal{B}^{-1}X)\in\mathbf{Z}[X_1,\dots,X_n]$, and we consider the system (see (2))

$$f^{(1)}-\varepsilon=\partial f^{(1)}/\partial X_2=\dots=\partial f^{(1)}/\partial X_n=0. \tag{5}$$

We denote its root variety by $V^{(1)}\subset\overline{F}^n$.

Applying the algorithms of [5] and [6] to the system (5), we decompose the variety $V^{(1)}=\bigcup_q V_q^{(1)}$ into irreducible components over $\mathbf{Q}(\varepsilon)$. Then, applying the method of [6], Chapter II, §1, to each component $V_q^{(1)}$ for which $\dim V_q^{(1)}=0$, the algorithm finds zero-dimensional varieties $V^{(2)}\subset\mathbf{C}^n$, irreducible over $\mathbf{Q}$, that are contained in the variety of zeros of the system $f^{(1)}=\partial f^{(1)}/\partial X_2=\dots=\partial f^{(1)}/\partial X_n=0$.

Here the zero-dimensional varieties $V_m^{(2)}$ are presented in the following form. For each variety $V_m^{(2)}$ the algorithm constructs an irreducible (over $\mathbf{Q}$) polynomial $\Phi\in\mathbf{Q}[Z]$, such that for every point $(\xi_1,\dots,\xi_n)\in V_m^{(2)}$ the field $\mathbf{Q}(\xi_1,\dots,\xi_n)$ is isomorphic to $\mathbf{Q}[Z]/(\Phi)\simeq\mathbf{Q}[\theta]$, where $\Phi(\theta)=0$ and the primitive element $\theta=\sum_1^n\lambda_i\xi_i$ for suitable natural numbers $1\le\lambda_i\le\deg\Phi\le(2d)^n$. Furthermore, the algorithm finds explicitly

$$\xi_i=\sum_{0\le j<\deg\Phi}\beta_i^{(j)}\theta^j,$$

with $\beta_i^{(j)}\in\mathbf{Q}$, for $1\le i\le n$, $0\le j<\deg\Phi$.

For each component $V_m^{(2)}$ the algorithm checks whether or not it contains at least one of the real points $(\xi_1,\dots,\xi_n)\in V_m^{(2)}\cap\mathbf{R}^n$. This is equivalent to $\theta=\sum_1^n\lambda_i\xi_i\in\mathbf{R}$. Thus, it is sufficient for the algorithm to check whether the polynomial Φ has at least one real zero, which may be accomplished using the Sturm sequence [7]. If $\Phi(\theta)=0$ and $\theta\in\mathbf{R}$, then the vector $\mathcal{B}^{-1}(\xi_1,\dots,\xi_n)\in\mathbf{R}^n$ is a real zero of the polynomial f.

On the basis, e.g., of [12] the algorithm can find a rational approximation $\theta^{(s)}\in\mathbf{Q}$ such that $|\theta-\theta^{(s)}|<2^{-s}$ for any fixed natural number s, in a time that is polynomial in s and the size of the polynomial Φ. This completes the description of the algorithm for finding zeros of f.

We make a few remarks concerning the justification of our algorithm. We recall that there is a $\gamma\in\Gamma$ such that in each connected component of the variety V_ε there is a zero $x=(x_1,\dots,x_n)\in F^n\backslash K$ of the system (4), and here the gradient $(\operatorname{grad}f)(x)$ is proportional to the vector $(1,\gamma)$. Then $(\operatorname{grad}f^{(1)})(\mathcal{B}x)$ is proportional to $(1,0,\dots,0)$, and consequently the vector $(\chi_1,\dots,\chi_n)=\mathcal{B}x\in F^n\backslash K_1$ satisfies (5), where $K_1=\mathcal{B}K$ is the set of points of zero curvature of the variety $\{f^{(1)}-\varepsilon=0\}$. The vector $(\chi_1,\dots,\chi_n)\in V_q^{(1)}$ for an appropriate zero-dimensional component $V_q^{(1)}\subset\overline{F}^n$ (see (2) and the remark thereafter). Since every element $0\ne\chi\in F$ is uniquely representable in the form $\chi=\varepsilon^\alpha(\chi^{(0)}+\omega)$, where $\alpha\in\mathbf{Q}$, $0\ne\chi^{(0)}\in\mathbf{R}$ and the element ω is infinitesimal, on taking into account the fact that $\|(\chi_1,\dots,\chi_n)\|\le(Nn)^n\|x\|\le(Nn)^n(r+\varepsilon)$, we find the representation $(\chi_1, \ldots, \chi_n) = (\varepsilon^{\alpha_1}(\chi_1^{(0)} + \omega_1), \ldots, \varepsilon^{\alpha_n}(\chi_n^{(0)} + \omega_n))$, with $\alpha_1 \geq 0, \ldots, \alpha_n \geq 0$. Then the vector

$$(\chi_1^{(1)}, \ldots, \chi_n^{(1)}) = \lim_{\delta \to 0}(\delta^{\alpha_1}\chi_1^{(0)}, \ldots, \delta^{\alpha_n}\chi_n^{(0)}) \in \mathbf{R}^n$$

lies on an appropriate 0-dimensional component $V_m^{(2)} \subset \mathbf{C}^n$. We call the number $\chi_j^{(1)} \in \mathbf{R}$ the *real part* of the element $\chi_j \in F$. Hence, the vector $(\xi_1, \ldots, \xi_n) = B^{-1}(\chi_1^{(1)}, \ldots, \chi_n^{(1)}) \in V_0$, and in view of the properties of $\gamma \in \Gamma$, in each connected component of the variety V_0 the algorithm finds at least one point of this type, bearing in mind that each connected component of V_0 coincides with the set of real parts of all points of the corresponding connected component of V_ε. Thus the following lemma is proved (the bounds here use [5] and [6]):

LEMMA. a) *It is possible to construct an algorithm that produces a finite set of real solutions $(\xi_1, \ldots, \xi_n) \in \mathbf{R}^n$ for the equation $f = 0$, so that in each connected component of the variety $V_0 = \{f = 0\} \subset \mathbf{R}^n$ there is at least one point of this set. Moreover, for each point $(\xi_1, \ldots, \xi_n)$ the algorithm constructs expressions*

$$\xi_i = \sum_{0 \leq j < \deg \Phi} \beta_i^{(j)} \theta^j \in \mathbf{Q}[\theta]$$

which give a field isomorphism $\mathbf{Q}(\xi_1, \ldots, \xi_n) = \mathbf{Q}[\theta] \simeq \mathbf{Q}[Z]/(\Phi)$, where Φ is the minimal polynomial for θ over the field $\mathbf{Q}$, and further $\theta = \sum_{1 \leq i \leq n} \lambda_i \xi_i$ with natural number coefficients $1 \leq \lambda_i \leq \deg \Phi \leq (2d)^n$. The execution time for the algorithm is bounded by a polynomial in $kMd^{n^2} \leq L^n \leq L^{\log L}$.

b) *Furthermore, for every zero $(\xi_1, \ldots, \xi_n)$ and for each natural number s, it is possible to find a rational approximation $(\xi_1^{(s)}, \ldots, \xi_n^{(s)}) \in \mathbf{Q}^n$ so that $|\xi_i - \xi_i^{(s)}| < 2^{-s}$, in time which is polynomial in $Md^{n^2}s$.*

REMARK. The proof of the lemma in fact also goes through when the initial polynomials f_1 (see (1)) have coefficients in the ring $\mathbf{Z}[\varepsilon]$, where the role of ε in the proof is played by an element $\varepsilon_1 > 0$ which is infinitesimal with respect to ε.

3. We now turn to the consideration of the case of weak inequalities $f_1 \geq 0, \ldots, f_k \geq 0$ in the system (1), which determine a semialgebraic set $V \subset \mathbf{R}^n$. Just as in the case of equalities, we may assume that $V \subset \mathcal{D}_r$. We consider the polynomial

$$g = (f_1 + \varepsilon)(f_2 + \varepsilon) \cdots (f_k + \varepsilon) - \varepsilon^k \in \mathbf{Z}[\varepsilon][X_1, \ldots, X_n],$$

and we apply the lemma to it, bearing in mind the last remark.

As a result the algorithm finds points in each connected component of the variety $\{g = 0\} \subset F^n$, and then, as above, the real parts $(\xi_1, \ldots, \xi_n) \in \mathbf{R}^n$ of these points. Then for each point $(\xi_1, \ldots, \xi_n)$ the algorithm checks for which i, $1 \leq i \leq k$, the equality $f_i(\xi_1, \ldots, \xi_n) = h_i(\theta) = 0$ holds, where $h_i \in \mathbf{Q}[Z]$, on the basis of assertion a) of the lemma (cf. [5] and [6]). If $h_1(\theta) \neq 0$, then for any zero $\theta_1 \in \mathbf{C}$ of the polynomial h_1, the inequality $|\theta - \theta_1| > \exp(-p_1(L))$ holds for a suitable polynomial p_1 (cf. [12]). Further, the algorithm finds a rational approximation $\theta^{(s)} \in \mathbf{Q}$ such that $|\theta - \theta^{(s)}| < \exp(-p_1(L))/2$, using e.g. [12] (cf. assertion b) of the lemma). Then $h_i(\theta) \in \mathbf{R}$ has the same sign as $h_i(\theta^{(s)}) \in \mathbf{Q}$. The algorithm computes $h_i(\theta^{(s)})$ for all $1 \leq i \leq k$ for which $h_i(\theta) \neq 0$. If for all $1 \leq i \leq k$ either $f_i(\xi_1, \ldots, \xi_n) = 0$ or $h_i(\theta^{(s)}) > 0$ holds, then the point $(\xi_1, \ldots, \xi_n) \in V$. The set of solutions of (1) required in the theorem coincides with the set of all points $(\xi_1, \ldots, \xi_n)$ which satisfy these conditions.

Bearing in mind that V coincides with the set of real parts of the closed (in the Euclidean topology) semialgebraic set $V^{(\varepsilon)} \subset F^n$ of all solutions of the system of inequalities $g \geq 0$, $f_1 + \varepsilon > 0, \ldots, f_k + \varepsilon > 0$, and in addition that the boundary of each connected component of the set $V^{(\epsilon)}$ lies in the variety $\{g = 0\} \subset F^n$ (cf. [8]), we find that there is at least one point $(\xi_1, \dots, \xi_n)$ constructed by the algorithm on each connected component of the set V.

We turn finally to the consideration of the system (1). We introduce a new variable Z and we consider the system $f_1 \geq 0, \dots, f_l \geq 0$, $f_{l+1} \geq 0, \dots, f_k \geq 0$, $Z \cdot f_{l+1} \cdots f_k = 1$. Applying the algorithm described in the case of weak inequalities to this system, we complete the proof of the theorem.

4. We will call nonempty semialgebraic point set in $\mathbf{R}^n$ an $(f_1, \dots, f_k)$-*cell* if it consists of points satisfying the conditions

$$\{f_{i_1} = 0\}_{i \in I_1}, \quad \{f_{i_2} > 0\}_{i_2 \in I_2}, \quad \{f_{i_3} < 0\}_{i_3 \in I_3},$$

for some partition $I_1 \cup I_2 \cup I_3 = \{1, \dots, k\}$. Then $\mathbf{R}^n$ is decomposed into $(f_1, \dots, f_k)$-cells.

REMARK. It is possible to construct an algorithm which enumerates all the $(f_1, \dots, f_k)$-cells and indicates at least one point on every connected component of each cell. The execution time of this algorithm is polynomial in $M(kd)^{n^2}$.

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3. We consider th
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