Absolute Factoring of Non-holonomic Ideals in the Plane

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ABSTRACT

We study *non-holonomic* overideals of a left differential ideal $J \subset F[\partial_x, \partial_y]$ in two variables where F is a differentially closed field of characteristic zero. One can treat the problem of finding non-holonomic overideals as a generalization of the problem of factoring a linear partial differential operator. The main result states that a principal ideal $J = \langle P \rangle$ generated by an operator P with a separable symbol $symb(P)$ has a finite number of maximal non-holonomic overideals; the symbol is an algebraic polynomial in two variables. This statement is extended to non-holonomic ideals J with a separable symbol. As an application we show that in case of a second-order operator P the ideal $\langle P \rangle$ has an infinite number of maximal non-holonomic overideals iff P is essentially ordinary. In case of a third-order operator P we give sufficient conditions on $\langle P \rangle$ in order to have a finite number of maximal non-holonomic overideals. In the Appendix we study the problem of finding non-holonomic overideals of a principal ideal generated by a second order operator, the latter being equivalent to the Laplace problem. The possible application of some of these results for concrete factorization problems is pointed out.

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1. FINITENESS OF THE NUMBER OF MAXIMAL NON-HOLONOMIC OVER-IDEALS OF AN IDEAL WITH SEPARABLE SYMBOL

Let F be a differentially closed field (or universal differential field in terms of [8], [9]) with derivatives ∂_x and ∂_y; let $P = \sum_{i,j} p_{i,j} \partial_x^i \partial_y^j \in F[\partial_x, \partial_y]$ be a partial differential operator of order n. Considering e.g. the field of rational functions $\mathbb{Q}(x, y)$ as F is a quite different issue. The *symbol* is defined by $symb(P) = \sum_{i+j=n} p_{i,j} v^i w^j$; it is a homogeneous algebraic polynomial of degree n in two variables. The degree of its Hilbert-Kolchin polynomial $ez + e_0$ is called its *differential type*; its leading coefficient is called the *typical differential dimension* [8]. A left ideal $I \subset F[\partial_x, \partial_y]$ is called *non-holonomic* if its differential type equals 1. We study maximal non-holonomic overideals of a principal ideal $\langle P \rangle \subset F[\partial_x, \partial_y]$. Obviously there is an infinite number of maximal *holonomic* overideals of $\langle P \rangle$: for any solution $u \in F$ of $Pu = 0$ we get a holonomic overideal $\langle \partial_x - u_x/u, \partial_y - u_y/u \rangle \supset \langle P \rangle$. We assume w.l.o.g. that $symb(P)$ is not divisible by ∂_y; otherwise one can make a suitable transformation of the type $\partial_x \to \partial_x,\ \partial_y \to \partial_y + b\partial_x,\ b \in F$. In fact choosing b from the subfield of constants of F is possible.

Clearly, factoring an operator P can be viewed as finding principal overideals of $\langle P \rangle$; we refer to factoring over a universal field F as *absolute factoring*. Overideals of an ideal in connection with Loewy and primary decompositions were considered in [6].

Following [4] consider a homogeneous polynomial ideal $symb(I) \subset F[v, w]$ and attach a homogeneous polynomial $g = GCD(symb(I))$ to I. Lemma 4.1 [4] states that $deg(g) = e$. As above one can assume w.l.o.g. that w does not divide g.

We recall that the Ore ring $R = (F[\partial_y])^{-1}\ F[\partial_x, \partial_y]$ (see [1]) consists of fractions of the form $\beta^{-1} r$ where $\beta \in F[\partial_y]$, $r \in F[\partial_x, \partial_y]$, see [3], [4]. We also recall that one can represent $R = F[\partial_x, \partial_y]\ (F[\partial_y])^{-1}$, and two fractions are equal, $\beta^{-1} r = r_1 \beta_1^{-1}$, iff $\beta r_1 = r \beta_1$ [3], [4].

For a non-holonomic ideal I denote ideal $\overline{I} = RI \subset R$. Since the ring R is left-euclidean (as well as right-euclidean) with respect to ∂_x over the skew-field $(F[\partial_y])^{-1}\ F[\partial_y]$, we conclude that the ideal $\overline{I}$ is principal. Let $\overline{I} = \langle r \rangle$ for suitable $r \in F[\partial_x, \partial_y] \subset R$ (cf. [4]). Lemma 4.3 [4] implies that $symb(r) = w^m g$ for a certain integer $m \ge 0$ where g is not divisible by w.

Now we expose a construction introduced in [4]. For a family of elements $f_1, \ldots, f_k \in F$ and rational numbers $s_i \in \mathbb{Q}$, $1 > s_2 > \cdots > s_k > 0$ we consider a D-module being a vector space over F with a basis $\{G^{(s)}\}_{s \in \mathbb{Q}}$ where the derivatives of

$$G^{(s)} = G^{(s)}(f_1, \ldots, f_k;\ s_2, \ldots, s_k)$$

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are defined as

$$d_{x_i}G^{(s)} = (d_{x_i}f_1)G^{(s+1)} + (d_{x_i}f_2)G^{(s+s_2)} + \cdots + (d_{x_i}f_k)G^{(s+s_k)}$$

for $i = 1, 2$ using the notations $d_{x_1} = \partial_x, d_{x_2} = \partial_y$.

Next we introduce series of the form

$$\sum_{0\le i<\infty} h_i G^{(s-\frac{i}{q})} \qquad (1)$$

where q is the least common multiple of the denominators of $s_2, \dots, s_k$; one can view (1) as an analogue of Newton-Puiseux series for non-holonomic D-modules. Theorem 2.5 [4] states that for any linear divisor $v + aw$ of $symb(P)$ and any $f_1 \in F$ such that $(\partial_x + a\partial_y)f_1 = 0$ there exists a solution of $P = 0$ of the form (1); conversely, if (1) is a solution of $P = 0$ then $(\partial_x + a\partial_y)f_1 = 0$ for an appropriate divisor $v + aw$ of $symb(P)$. Furthermore, Proposition 4.4 [4] implies that any solution of theform (1) of $r = 0$ such that $(\partial_x + a\partial_y)f_1 = 0$ for suitable $a \in F$ (or equivalently $\partial_y f_1 \ne 0$) is also a solution of the ideal I; then the appropriate linear form $v + aw$ is a divisor of g, and the inverse holds as well.

In [5] we have designed an algorithm for factoring an operator P in case of $symb(P)$ is separable. In particular, in this case there is only a finite number (less than 2^n) of different factorizations of P. Now we show a more general statement for overideals of $\langle P\rangle$.

Theorem 1.1. *Let $symb(P)$ be separable. Then there exists at most $n = ord(P)$ maximal non-holonomic overideals of $\langle P\rangle \subset F[\partial_x, \partial_y]$. Moreover, if there exists a non-holonomic overideal $I \supset \langle P\rangle$ with the attached polynomial $g = GCD(symb(I))$ then there exists a unique non-holonomic overideal, maximal among the ones with the attached polynomial equal g.*

Proof. Let I be a non-holonomic ideal such that $I \supset \langle P\rangle$. Then $\beta P = r_1 r$ for suitable $\beta \in F[\partial_y]$, $r_1 \in F[\partial_x, \partial_y]$ and a polynomial $g = GCD(symb(I))$ attached to I is a divisor of $symb(P)$. We claim that for every pair of non-holonomic ideals $I_1, I_2 \supset \langle P\rangle$ to which a fixed polynomial g is attached, to their sum $I_1 + I_2$ also g is attached. Indeed, any solution of the form (1) of $P = 0$ such that $(v + aw)|g$, is a solution of $r = 0$ as well due to Lemma 4.2 [4] (cf. Proposition 4.4 [4]) taking into account that $symb(P)$ is separable, hence it is also a solution of I as it was shown above and by the same token is a solution of both I_1 and I_2 (in particular $I_1 + I_2$ is also non-holonomic). The claim is established.

Thus among non-holonomic overideals $I \supset \langle P\rangle$ to which a given polynomial $g|symb(P)$ is attached, there is a unique maximal one. Now take two maximal non-holonomic overideals $I,\ I' \supset \langle P\rangle$ to which polynomials $g,\ g'$ are attached, respectively. Then $g,\ g'$ are reciprocately prime. Indeed, if $v + aw$ divides both $g,\ g'$ then arguing as above one can verify that (1) is a solution of $I + I'$, i.e. the latter ideal is non-holonomic which contradicts to maximality of $I,\ I'$. Theorem is proved. $\square$

Corollary 1.2. *Let $symb(P)$ be separable. Suppose that there exist maximal non-holonomic overideals $I_1, \dots, I_l \supset \langle P\rangle$ such that for the respective attached polynomials $g_1, \dots, g_l$ the sum of their degrees $deg(g_1) + \cdots + deg(g_l) \ge n$. The $\langle P\rangle = I_1 \cap \cdots \cap I_l$.*

Proof. As it was shown in the proof of Theorem 1.1, polynomials $g_j|symb(P)$, $1 \le j \le l$ are pairwise reciprocately prime, hence $g_1 \cdots g_l = symb(P)$. Moreover it was established in the proof of Theorem 1.1 that every solution of $P = 0$ of the form (1) such that $(\partial_x + a\partial_y)f_1 = 0$, is a solution of a unique I_j for which $(u + aw)|g_j$; thus every solution of $P = 0$ of the form (1) is also a solution of $I_1 \cap \cdots \cap I_l$. Therefore the typical differential dimension of ideal the $I_1 \cap \cdots \cap I_l$ equals n (cf. Lemma 4.1 [4]). On the other hand, any overideal of a principal ideal $\langle P\rangle$ of the same typical differential dimension coincides with $\langle P\rangle$; one can verify it by comparing their Janet bases [10]. (We briefly recall that operators $P_1, \dots, P_s \in F[\partial_x, \partial_y]$ form a Janet basis of the ideal $\langle P_1, \dots, P_s\rangle$ if for any element $P \in \langle P_1, \dots, P_s\rangle$ its highest derivative $ld(P)$ is divided by one of $ld(P_i)$, $1 \le i \le s$.) $\square$

Remark 1.3. *One can extend Theorem 1.1 to non-holonomic ideals J such that the homogeneous polynomial $GCD(symb(J))$ is separable: namely, there exists a finite number of maximal non-holonomic overideals $I \supset J$.*

2. NON-HOLONOMIC OVERIDEALS OF A SECOND-ORDER LINEAR PARTIAL DIFFERENTIAL OPERATOR

In this Section we study the structure of overideals of $\langle P\rangle$ when $n = ord(P) = 2$. The case of separable $symb(P)$ is covered by Theorem 1.1.

Proposition 2.1. *Any principal ideal $\langle P\rangle$ for a second-order operator $P = \partial_y^2 + p_1\partial_x + p_2\partial_y + p_3$ with non-separable $symb(P)$ has*

i) no proper non-holonomic overideals in case $p_1 \ne 0$;

ii) an infinite number of maximal non-holonomic overideals in case $p_1 = 0$.

Proof. Let $symb(P)$ be non-separable. Then applying a transformation of the type $\partial_x \to b_1\partial_x + b_2\partial_y$, $\partial_y \to b_3\partial_x + b_4\partial_y$ for suitable $b_1, b_2, b_3, b_4 \in F$ one can assume w.l.o.g. that $P = \partial_y^2 + p_1\partial_x + p_2\partial_y + p_3$; it would be interesting to find out when one can carry out these transformations algorithmically. First let $p_1 = 0$. Then P is essentially ordinary, i.e. becomes ordinary after a transformation as above, and for any solution $u \in F$ of the equation $P = 0$ we get a non-holomonic overideal $\langle \partial_y - u_y/u\rangle \supset \langle P\rangle$. Now suppose that $p_1 \ne 0$. Then P is irreducible (see e. g. Corollary 7.1 [4]). Moreover we claim that $\langle P\rangle$ has at most one maximal non-holonomic overideal. Let $I \supset \langle P\rangle$ be a non-holonomic overideal. Choosing arbitrary non-zero elements $b_1, b_2 \in F$ denote the derivation $d = b_1\partial_x + b_2\partial_y$. Similar to the proof of Theorem 1.1 there exists $r \in F[d, \partial_y] = F[\partial_x, \partial_y]$ such that $\langle r\rangle = IR_1 \subset R_1 = (F[d])^{-1}\ F[d, \partial_y]$. Then $\beta P = r_1 r$ for suitable $\beta \in F[d]$, $r_1 \in F[d, \partial_y]$ and $symb(r) = (b_1 v + b_2 w)^m g$ for an integer m and $g|w^2$. If $g = 1$ then I cannot be non-holonomic because of Proposition 4.4 [4] (cf. above). If $g = w^2$ then similar to the proof of Corollary 1.2 one can show that the only non-holonomic overideal of $\langle P\rangle$ among ones to which polynomial w^2 is attached, is just $\langle P\rangle$ itself. It remains to consider the case $g = w$. Applying the Newton polygon construction from [4] to equation $r = 0$ and a divisor w of $symb(r)$, one obtains a solution of the form (1) of $r = 0$ with $G = G(x)$, thereby it is a solution of $P = 0$. On the other hand, applying the Newton polygon construction from [4] to equation $P = 0$, one gets at its first step $f_1 = x$ and at the second step f_2 which fulfils equation $(\partial_y f_2)^2 + p_1 = 0$ and f_2 corresponds to the edge of the Newton polygon with endpoints $(0, 2)$ and $(1, 0)$, so with the slope 1/2. This provides a solution of equation $P = 0$ of the form (1) with $G = G(x, f_2;\ 1/2)$, therefore the equation $P = 0$ has no solutions of the form (1) with $G = G(x)$. The achieved contradiction shows that there are no non-holonomic overideals I with attached polynomial w, this completes the proof of the claim. □

3. ON NON-HOLONOMIC OVERIDEALS OF A THIRD-ORDER OPERATOR

Now we study overideals of $\langle P\rangle$ where the order $n = ord(P) = 3$. Due to Theorem 1.1 it remains to consider non-separable $symb(P)$. In [4] an algorithm has been designed for factoring P; a few explicit calculations for factoring P are provided in [7].

Proposition 3.1. *Let P be a third-order operator with a non-separable $symb(P)$.*

i) When $symb(P)$ has two different linear divisors, one of which of multiplicity 2, then we can assume w.l.o.g. that

$$P = \partial_x\partial_y^2 + p_0\partial_x^2 + p_1\partial_x\partial_y + p_2\partial_y^2 + p_3\partial_x + p_4\partial_y + p_5.$$

If $p_0 \neq 0$ then $\langle P\rangle$ has at most two maximal non-holonomic overideals. Moreover if there exist two different maximal non-holonomic overideals $I_1, I_2 \supset \langle P\rangle$ then $\langle P\rangle = I_1 \cap I_2$;

ii) When $symb(P)$ has a single linear divisor of multiplicity 3 we can assume w.l.o.g. that

$$P = \partial_y^3 + p_0\partial_x^2 + p_1\partial_x\partial_y + p_2\partial_y^2 + p_3\partial_x + p_4\partial_y + p_5.$$

If either $p_0 \neq 0$, either $p_2 \neq 0$ or $p_3 \neq 0$ then $\langle P\rangle$ has at most two maximal non-holonomic overideals. Moreover if there exist two different maximal non-holonomic overideals $I_1, I_2 \supset \langle P\rangle$ then $\langle P\rangle = I_1 \cap I_2$. Otherwise $\langle P = \partial_y^3 + p_2\partial_y^2 + p_4\partial_y + p_5\rangle$ has an infinite number of maximal non-holonomic overideals.

Proof. Case $i)$ First let $symb(P)$ have two linear divisors; therefore one can assume w.l.o.g. (see above) that w is its divisor of multiplicity 2 and v is its divisor of multiplicity 1. One can write

$$P = \partial_x\partial_y^2 + p_0\partial_x^2 + p_1\partial_x\partial_y + p_2\partial_y^2 + p_3\partial_x + p_4\partial_y + p_5.$$

Suppose that $p_0 \neq 0$. The Newton polygon construction from [4] applied to equation $P = 0$ and to divisor w of $symb(P)$, yields a solution of the form (1) of $P = 0$ with $f_1 = x$ at its first step. At its second step the construction yields f_2 which fulfils equation $(\partial_y f_2)^2 + p_0 = 0$ and which corresponds to the edge of the Newton polygon with endpoints $(1, 2), (2, 0)$, so with the slope 1/2. This provides $G = G(x, f_2;\ 1/2)$ in (1).

Let a non-holonomic ideal $I \supset \langle P\rangle$. Choose $d = b_1\partial_x + b_2\partial_y$ for non-zero $b_1, b_2 \in F$. As in the previous Section there exists $r \in F[d, \partial_y]$ such that $\langle r\rangle = R_1 I \subset R_1 = (F[d])^{-1} F[d, \partial_y]$. Then $\beta P = r_1 r$ for suitable $\beta \in F[d]$, $r_1 \in F[d, \partial_y]$. Rewrite $symb(r) = (b_1 v + b_2 w)^m g$ where $g|(vw^2)$. If either $g = w^2$ or $g = v$, one can argue as in the proof of Theorem 1.1 and deduce that there can exist at most one maximal non-holonomic overideal of $\langle P\rangle$ with the property that the polynomial attached to the overideal is either w^2 or v. Similar to the proof of Corollary 1.2 one can verify that if there exist maximal non-holonomic overideals $I_2, I_1 \supset \langle P\rangle$ with attached polynomials w^2 and v, then $\langle P\rangle = I_1 \cap I_2$. As in Theorem 1.1 the existence of a maximal overideal with the attached polynomial w^2 or v follows from the existence of any non-holonomic overideal with the attached polynomial w^2 or v.

If either $g = w$ or $g = vw$ then applying the Newton polygon construction from [4] to equation $r = 0$ and divisor w of $symb(r)$, one obtains a solution of $r = 0$ (and thereby, of $P = 0$ due to Lemma 4.2 [4]) of the form (1) with $G = G(x)$ which contradicts to the supposition $p_0 \neq 0$ (see above). Thus, in case $p_0 \neq 0$ the ideal $\langle P\rangle$ has a finite number, less or equal than 2, of maximal non-holonomic overideals (similar to Theorem 1.1).

When $p_0 = 0$ this is not always true, say for $P = (\partial_x + b)(\partial_y^2 + b_3\partial_y + b_4)$ (cf. case $n = 2$ in the previous Section). It would be interesting to clarify for which P this is still true.

Case $ii)$ Now we consider the last case when $symb(P)$ has a unique linear divisor with multiplicity 3. As above one can assume w.l.o.g. that $symb(P) = w^3$, so

$$P = \partial_y^3 + p_0\partial_x^2 + p_1\partial_x\partial_y + p_2\partial_y^2 + p_3\partial_x + p_4\partial_y + p_5.$$

Keeping the notations we get $\langle r\rangle = R_1 I$ and $\beta P = r_1 r$. Then $symb(r) = (b_1 v + b_2 w)^m g$ where $g|w^3$. If $g = w^3$ then arguing as in the proof of Corollary 1.2 we deduce that the only non-holonomic overideal of $\langle P\rangle$ to which polynomial w^3 is attached, is just $\langle P\rangle$ itself.

Let $g|w^2$. Applying the Newton polygon construction from [4] to equation $r = 0$ and linear divisor w of $symb(r)$ one gets a solution of $r = 0$ (and thereby of $P = 0$) with either $G = G(x)$ or $G = G(x, f_2;\ 1/2)$ where $\partial_y f_2 \neq 0$ (cf. above).

Application of the Newton polygon construction from [4] to equation $P = 0$ (and unique linear divisor w of $symb(P)$) at its first step provides $f_1 = x$. The second step requires a trial of cases. First let $p_0 \neq 0$. Then the second step yields f_2 which fulfils equation $(\partial_y f_2)^3 + p_0 = 0$ and which corresponds to the edge of the Newton polygon with endpoints $(0, 3)$, (2.0), so with the slope 2/3. Thus we obtain a solution of the form (1) with $G = G(x, f_2, \dots;\ 2/3, \dots)$, hence $\langle P\rangle$ in case $p_0 \neq 0$ has no non-holonomic overideals with attached polynomial g being a divisor of w^2 (see above).

Now assume that $p_0 = 0$ and $p_1 \neq 0$. Then the second step provides solutions of $P = 0$ of the form (1) with two different possibilities. Either the Newton polygon construction chooses the vertical edge with endpoints $(1, 1)$, $(1, 0)$ as a leading edge at the second step, then it terminates at the second step yielding a solution of the form (1) with $G = G(x)$; we recall that in the construction from Section 2 [4] only edges with non-negative slopes are taken as leading ones and the construction terminates while taking a vertical edge, so with the slope 0, as a leading one, in particular the edge with endpoints $(1, 1)$, $(1, 0)$ is taken as a leading one regardless of whether the coefficient at point $(1, 0)$ vanishes. As the second possibility the construction yields a solution of the form (1) with $G = G(x, f_2, \dots;\ 1/2, \dots)$ where $f_2 \neq 0$ fulfils equation $(\partial_y f_2)^3 + p_1\partial_y f_2 = 0$ corresponding to the edge of the Newton polygon with endpoints $(0, 3)$, $(1, 1)$, so with the slope 1/2. One can suppose w.l.o.g. that the Newton polygon construction terminates at its third step (thereby $G = G(x, f_2;\ 1/2)$), otherwise $\langle P\rangle$ cannot have a non-holonomic overideal to which a divisor g of w^2 is attached (see above).

If $g = w^2$ then any solution H_2 of $P = 0$ of the form (1) with $G = G(x, f_2;\ 1/2)$ is a solution of $r = 0$ because otherwise $rH_2 \neq 0$, being also of the form (1) with $G = G(x, f_2;\ 1/2)$, cannot be a solution of $r_1 = 0$ taking into account that $symb(r_1)$ does not divide on w^2 (cf. Lemma 4.2 [4]). Else if $g = w$ then $rH_2 \neq 0$ (again taking into account that $symb(r)$ does not divide on w^2) and therefore $r_1(rH_2) = 0$. Hence for a solution H_1 of $P = 0$ of the form (1) with $G = G(x)$ (see above) we have $rH_1 = 0$ since otherwise rH_1 being also of the form (1) with $G = G(x)$ cannot be a solution of $r_1 = 0$ (again cf. Lemma 4.2 [4]). Then arguing as in the proof of Theorem 1.1 one concludes that in case $p_0 = 0$ and $p_1 \neq 0$ ideal $\langle P \rangle$ can have at most two maximal non-holonomic overideals with attached polynomials w and w^2. Similar to the proof of Corollary 1.2 (cf. the preceding Subsection) one can verify that if there exist maximal non-holonomic overideals $I_1, I_2 \supset \langle P \rangle$ with attached polynomials w and w^2, then $\langle P \rangle = I_1 \cap I_2$. As in Theorem 1.1 the existence of a maximal overideal with the attached polynomial w (or respectively, w^2) follows from the existence of any non-holonomic overideal with the attached polynomial w or w^2.

Furthermore, let $p_0 = p_1 = 0,\ p_3 \neq 0$. Then as in case $p_0 \neq 0$ we argue that the second step of the Newton polygon construction applied to equation $P = 0$ yields f_2 which fulfils equation $(\partial_y f_2)^3 + p_4 = 0$ and which corresponds to the leading edge of the Newton polygon with endpoints $(0, 3),\ (1, 0)$, so with the slope $1/3$. Thus the Newton polygon construction yields a solution of $P = 0$ of the form (1) with $G = G(x, f_2, \dots;\ 1/3, \dots)$ and again $\langle P \rangle$ in case $p_0 = p_1 = 0,\ p_3 \neq 0$ under consideration has no non-holonomic overideals with an attached polynomial being a divisor of w^2.

Finally, when $p_0 = p_1 = p_3 = 0$ the ideal $\langle P = \partial_y^3 + p_1\partial_y^2 + p_3\partial_y + p_5 \rangle$ has an infinite number of maximal non-holonomic overideals; this is similar to the second-order case $P = \partial_y^2 + p_4\partial_y + p_5$, see above. □

A few examples applying the preceding result are given next.

EXAMPLE 1. The operator

$$L \equiv \partial_{yy} + x\partial_x + \partial_y + y$$

is immediately recognized as absolutely irreducible by case *i*) of PROPOSITION 2.1 because $p_1 \neq 0$.

EXAMPLE 2. Consider the operator

$$L \equiv \partial_{xyy} + \partial_{xx} + y\partial_{yy} + (y + 1)\partial_x + 2\partial_y + y.$$

Due to $p_0 = 1$, case *i*) of the above proposition applies. In fact, there is only a single first-order right factor as may be seen from

$$L = (\partial_{yy} + \partial_x + 1)(\partial_x + y);$$

this decomposition may be obtained by using the function `FirstOrderRightFactors` provided on the website `www.alltypes.de` [11].

EXAMPLE 3. Case *ii*) of PROPOSITION 3.1 applies to the operator

$$L \equiv \partial_{yyy} + \frac{x}{y^2}\partial_{xy} + \left(1 + \frac{2}{y}\right)\partial_{yy} + \frac{x(y-2)}{y^3}\partial_x + \frac{2y-3}{y^2}\partial_y - \frac{y-2}{y^3};$$

although $p_0 = 0$, due to $p_2 \neq 0$ and $p_3 \neq 0$ the operator can have at most two different right factors. It turns out that there are no first-order right factors at all.

It is a challenge to design an algorithm which produces non-holonomic overideals of a given differential ideal $J \subset F[\partial_x, \partial_y]$ in general. If the goal is solving linear pde's attached to these operators, $F = \mathbb{Q}(x, y)$ is of particular interest. Some of the results reported in this article may be applied for obtaining a partial answer; e.g. by case *i*) of PROPOSITION 2.1 it may be possible to exclude the existence of any factor very efficiently.

Appendix. Explicit formulas for Laplace transformation

We exhibit a short exposition and explicit formulas for the Laplace transformation [2]. Let $Q = \partial_{xy} + a\partial_x + b\partial_y + c$ be a second-order operator which has its *Laplace divisor* $L_n = \sum_{0 \le i \le n} l_i \partial_x^i$ of order n, i. e. $Q,\ L_n$ form a Janet basis of ideal $\langle Q,\ L_n \rangle$. Hence

$$PQ = (\partial_y + a)L_n \tag{2}$$

for a suitable $P = \sum_{0 \le i \le n-1} p_i \partial_x^i$. (This form of P is obtained by comparing the highest terms which divide on ∂_x^n in (2).)

If a Laplace divisor exists then $\langle Q, L_n \rangle$ is a proper non-holonomic overideal of $\langle Q \rangle$. Conversely, one can show (cf. [2]) that if $\langle Q \rangle$ has a proper non-holonomic overideal then there exists either a Laplace divisor L_n (for a suitable n) or a Laplace divisor of the form $\sum_{0 \le i \le n} t_i \partial_y^i$ with respect to ∂_y. That is why the problem of searching for a Laplace divisor is equivalent to finding non-holonomic proper overideals of $\langle Q \rangle$.

Open question: is there an algorithm which decides for a given Q whether it has a Laplace divisor? In particular, an upper bound on n would suffice for an algorithm.

Comparing the highest terms in (2) which divide on ∂_y, we get that $L_n = P(\partial_x + b)$. Thus

$$PQ = (\partial_y + a)P(\partial_x + b). \tag{3}$$

We have $Q \neq (\partial_y + a)(\partial_x + b)$ iff $0 \neq ab + b_y - c \equiv K_0$.

LEMMA 3.2. *If $K_0 \neq 0$ then there are unique $B,\ C$ such that*

$$(\partial_x + B)Q = (d_{xy} + a\partial_x + B\partial_y + C)(\partial_x + b) \tag{4}$$

PROOF. (4) is equivalent to an algebraic linear system in B, C,

$$aB - C = b_y + ab - a_x - c, \tag{5}$$

$$(c - b_y)B - bC = b_{xy} + ab_x - c_x \tag{6}$$

□

Therefore (3) holds iff $P = P_1(\partial_x + B)$ by means of dividing P by $\partial_x + B$ with remainder. Substituting the latter equality to (3) and making use of (4) we obtain the equality

$$P_1(\partial_{xy} + a\partial_x + B\partial_y + C) = (\partial_y + a)P_1(\partial_x + B). \tag{7}$$

Now (7) is similar to (3) but with the order $ord(P_1) = ord(P) - 1 = n - 1$ and a new second-order operator $Q_1 = \partial_{xy} + a\partial_x + B\partial_y + C$. Continuing this way we get the Laplace transformation with $K_1 = aB + B_y - C$ etc.

More uniformly denote $b_0 \equiv b,\ c_0 \equiv c$, then $b_1 \equiv B$, $c_1 \equiv C$, b_2, c_2 etc. obtained from Lemma 3.2. Denote

$$K_i \equiv ab_i + (b_i)_y - c_i,\ \ Q_i \equiv \partial_{xy} + a\partial_x + b_i\partial_y + c_i.$$

Corollary 3.3. *There exists L_n satisfying (2) iff for the minimal m such that $K_m = 0$ we have $m \leq n$. In this case*

$$L_n = P_{n-m}(\partial_x + b_{m-1}) \cdots (\partial_x + b_0) \qquad (8)$$

where $P_{n-m} = \sum_{0 \leq i \leq n-m} p_i \partial_x^i$ is an arbitrary operator of the order $n - m$ which fulfils

$$P_{n-m}(\partial_y + a) = (\partial_y + a)P_{n-m}. \qquad (9)$$

For any order $n-m \geq 0$ such an operator P_{n-m} exists. The pair Q, L_n constitutes a Janet basis of the ideal $\langle Q,\ L_n \rangle$. The ideal $\langle Q,\ L_m \rangle$ is the unique maximal non-holonomic overideal of $\langle Q \rangle$ which corresponds to a divisor y of $symb(Q) = xy$ (see Theorem 1.1).

Proof. Applying Laplace transformations as above, if $m > n$ we don't get a solution of (2) after n steps since (3) with $PQ_n = (\partial_y + a)P(\partial_x + b_n)$ would not have a solution with P of the order 0. If $m \leq n$ then successively following Laplace transformations we arrive to (8) in which (9) is obtained from equality $PQ_m = (\partial_y + a)P(\partial_x + b_m)$ (see (3)) and taking into account that $K_m = 0$. □

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