COMPLEXITY LOWER BOUNDS FOR COMPUTATION TREES WITH ELEMENTARY TRANSCENDENTAL FUNCTION GATES

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Abstract. We consider computation trees which admit as gate functions along with the usual arithmetic operations also algebraic or transcendental functions like exp, log, sin, square root (defined in the relevant domains) or much more general, Pfaffian functions. A new method for proving lower bounds on the depth of these trees is developed which allows to prove a lower bound $\Omega(\sqrt{\log N})$ for testing membership to a convex polyhedron with N facets of all dimensions, provided that N is large enough.

I. Pfaffian Computation Trees.

Definition 1. By a Pfaffian computation tree $\mathcal{T}$ we mean a generalization of an algebraic decision tree (see e.g. [1, 4, 12, 28, 29, 30]) in which at any node v of $\mathcal{T}$ a Pfaffian function f_v in the variables $X_1, \dots, X_n$ (see the definition A2 in the Appendix) is attached, which satisfies the following properties. Let $f_{v_0}, \dots, f_{v_\ell}, f_{v_{\ell+1}} = f_v$ be the functions attached to all the nodes along the branch $\mathcal{T}_v$ of $\mathcal{T}$ leading from the root v_0 to v. We assume that Pfaffian function f_v satisfies the following differential equation

$$df_v = \sum_{1 \le j \le n} g_{v,j}(X_1, \dots, X_n,\ f_{v_0}, \dots, f_{v_\ell}, f_v) dX_j$$

with $g_{v,j} \in \mathbb{R}[X_1, \dots, X_n, U_0, \dots, U_{\ell+1}]$. The tree $\mathcal{T}$ branches at v to its three sons according to the sign of f_v (cf. [1]). Thereby, to each node v one can naturally assign a semi-Pfaffian set $U_v \subset \mathbb{R}^n$ (see the definition A3 in the Appendix) consisting of all the points for which the sign conditions for functions along the branch $\mathcal{T}_v$ are valid. Thus, to three sons of v one assigns the semi-Pfaffian sets $U_v \cap \{f_v > 0\}$,

$U_v \cap \{f_v = 0\}$, $U_v \cap \{f_v < 0\}$, respectively. We assume that f_v is defined on a certain domain (see the definition A2) containing U_v. To any leaf of $\mathcal{T}$ an output either "yes" or "no" is assigned and we say that $\mathcal{T}$ tests the membership problem to the set of all points $(x_1, \dots, x_n) \in \mathbb{R}^n$ for which the outputs of the corresponding leaves of $\mathcal{T}$ are "yes" (see [1]).

Note that a more general notion of a Pfaffian sigmoid was introduced in [10] and a method for obtaining lower bound on the (parallel) complexity was developed.

If we take only arithmetic operations as the gate Pfaffian functions f_v in $\mathcal{T}$ then we come to the algebraic computation trees (see e.g. [1]). As examples of gate Pfaffian functions f_v one could take exp (f_{v_q}), $\log(f_{v_q})$, where log is defined on the positive half-line, $\sqrt{f_{v_q}}$, where square root is defined on the positive half-line, $\sin(f_{v_q})$, where sin is defined on the interval $(-\pi, \pi)$, $\tan(f_{v_q})$, where tan is defined on the interval $(-\pi/2, \pi/2)$, $0 \leq q \leq \ell$. Other examples one can find in section A1 of the Appendix. Trees $\mathcal{T}$ restricted to some special classes of Pfaffian functions (for instance, the mentioned above) can be of a particular interest, but since we are interested in the complexity lower bounds we shall consider arbitrary Pfaffian functions.

Suppose that the degrees $\deg g_{v,j}$ of the polynomials occurring in the definition of the gate functions f_v in $\mathcal{T}$, are less than d.

Now we are able to formulate the main result of the paper. This result was annownced in [15].

Theorem. *Let a Pfaffian computation tree $\mathcal{T}$ test a membership problem to a closed convex polyhedron $P \subset \mathbb{R}^n$, having N facets of all the dimensions. Then the depth K of $\mathcal{T}$ is greater than $\Omega(\sqrt{\log N})$, provided that $N \geq (dn)^{\Omega(n^4 \log d)}$.*

In [11] a particular case of the theorem for $n = 2$, so when P is a polygon, was proved.

Several methods based on topological characteristics are known for obtaining complexity lower bounds for algebraic computation trees testing membership to a semialgebraic set $S \subset \mathbb{R}^n$. In [1], the bound $\Omega(\log C)$ was proved, where C is the number of connected components of S or its complement, in [3,4,28] the bound $\Omega(\log \chi)$ was proved, where χ is the Euler characteristics. The most general (among the listed) bound $\Omega(\log B)$ was proved in [3,29], where B is the sum of Betti numbers of S.

Actually one could directly extend these results to Pfaffian computation trees, replacing in the proofs the references to Milnor's bound [23] on the sum of Betti numbers of a semialgebraic set by the references to Khovanskii's bound [20] for the sum of Betti numbers of a semi-Pfaffian set. This leads to the following proposition [11]. *If a Pfaffian computation tree tests the membership problem to a semi-Pfaffian set W with the sum of Betti numbers $\mathcal{B}$, then the depth of the tree is greater than $\Omega(\sqrt{\log \mathcal{B}})$* [11].

There is a conjecture that the bound in [20] could be improved (see the section A1 in the Appendix). This conjecture implies the lower bounds $\Omega(\log N)$ in the theorem and $\Omega(\log \mathcal{B})$ in the proposition from [11] respectively.

Observe that as the sum of Betti numbers of a convex polyhedron equals to 1, the theorem does not follow, apparently, from the proposition. Note that in [12] the complexity lower bound $\Omega(\log N)$ was proved for testing membership to a polyhedron with N facets by an algebraic decision tree (for large enough N, cf. the theorem). In [30] a similar bound was shown for a weaker model of linear decision trees. The method from [12] cannot be directly generalized to Pfaffian computation trees, since in [12] the efficient quantifier elimination procedure for the first-order theory of reals (see [9, 14, 17, 24]), was essentially used whereas for the theories involving Pfaffian functions (in particular, elementary transcendental), the quantifier elimination does not exist.

We remark that the computations involving other functions, rather than arithmetic, were considered in several papers: in [18] for the computations involving root extractions a lower bound for computing an algebraic function was obtained, in [13] this result was extended for the computations involving exp and log.

We mention that for testing membership to a polyhedron an upper bound $O(\log N)n^{O(1)}$ was shown in [22] even for linear decision trees.

Now we proceed to the proof of the theorem which will continue up to the end of section III.

We start with introducing some necessary concepts and notations. In section II we introduce the notion of i-angle points and prove that the set of i-angle points has the dimension at most i. This notion differs from the concept of sharp points introduced and used in [12], the latter does not work for Pfaffian computation trees. In section III we introduce and study another important technical concept, flat points. All necessary information about Pfaffian functions and sub-Pfaffian sets we included in the Appendix (in which the numbering of all the statements, definitions and sections begins with A).

For an m-plane $Q \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$ denote by $Q(x)$ the m-plane, collinear to Q and containing x. For a facet Π of the polyhedron P denote by $\overline{\Pi}$ the $\dim(\Pi)$-plane, containing Π (we assume a facet to be open, i.e. without its boundary).

Two planes Q_1, Q_2 or arbitrary dimensions are called transversal if

$$\dim(Q_1(0) \cap Q_2(0)) = \max\{0, \dim(Q_1(0)) + \dim(Q_2(0)) - n\}$$

The proofs of the following two easy lemmas one can find in [12] (lemma 1 is also proved in [5]).

Lemma 1. *For each j with $1 \le j \le n$ there exists a family $\mathcal{A}_j$ consisting of $j(n-j)+1$ j-subspaces in $\mathbb{R}^n$ such that for any i-subspace $Q \subset \mathbb{R}^n$, $1 \le i \le n$ there is a j-subspace $R \in \mathcal{A}_j$ which is transversal to Q.*

Lemma 2. *There exists a rotation of coordinates $X_1, \dots, X_n$ such that after this rotation for every j, every $Q \in \mathcal{A}_j$ and for every facet Π of P, the subspace Q and the plane $\overline{\Pi}$ become transversal.*

In what follows we suppose that the coordinate system meets the requirements of lemma 2. Now we reduce consideration to the case when the polyhedron P is bounded. The next construction follows the beginning of the proof of lemma 5 [12].

Let t be the minimal dimension of facets of P. Fix a certain t-facet P_t of P, then t-plane $\overline{P}_t$ is contained in P. On each facet Π of P choose a point $x_\Pi \in \Pi$. Take an arbitrary hyperplane σ transversal to $\overline{P}_t$ and such that the points x_Π for all facets Π of P lie in the same of two open half-spaces of $\mathbb{R}^n \setminus \sigma$ (denote this half-space by Σ). Consider the polyhedron $P \cap (\Sigma \cup \sigma)$, it contains a facet of a dimension less than t. Continuing this process while $t \geq 1$, we come eventually to the case $t = 0$, i.e., polyhedron P' obtained as a result of this process has a vertex.

There exists a linear form $L = \beta_1 X_1 + \cdots + \beta_n X_n$ with $\beta_i \in \mathbb{R}$, $1 \leq i \leq n$ such that for every $\gamma \in \mathbb{R}$ an intersection $P'' = \{L + \gamma \geq 0\} \cap P'$ is compact. Take γ such that $x_\Pi \in \{L + \gamma \geq 0\} \cap P'$ for all Π.

In order to reduce consideration to the compact polyhedron P'', observe that from a Pfaffian computation tree of depth K for the membership problem to P, one can easily produce a Pfaffian computation tree of a depth at most $K+n$ for the membership problem to P''. Assuming that the theorem is valid for the compact P'', and thus $K + n \geq \Omega(\sqrt{\log N})$, we get a similar bound $K \geq \Omega(\sqrt{\log N})$ under the supposed in the hypothesis of the theorem inequality for N. Therefore, in what follows we assume that P is bounded.

In section A2 a sequence

$$\mathbb{R} = \mathbb{R}_0 \subset \mathbb{R}_1 \subset \mathbb{R}_2 \subset \cdots$$

of nonstandard extensions of fields is introduced. One can choose in each $\mathbb{R}_{i+1}$ an element infinitesimal relative to $\mathbb{R}_i$. We denote these elements, respectively, by

$$\epsilon_1 \in \mathbb{R}_1, \{\delta_\ell^{(j)} \in \mathbb{R}_{(\ell-1)(n^2+1)+j+1} : 1 \leq \ell \leq n-1,\ 1 \leq j \leq n^2+1\},$$

$\epsilon_2 \in \mathbb{R}_{n^3-n^2+n+1}$, $\epsilon_3 \in \mathbb{R}_{n^3-n^2+n+2}$ (the reason for these notations would become clear later on). To match the notations denote the fields $\mathbb{R}_1 = \mathbb{R}_{\epsilon_1}$, $\mathbb{R}_{(\ell-1)(n^2+1)+j+1} =$

$\mathbb{R}_{\delta_\ell^{(j)}}$, $1 \le \ell \le n-1$, $1 \le j \le n^2+1$, $\mathbb{R}_{n^3-n^2+n+1} = \mathbb{R}_{\epsilon_2}$, $\mathbb{R}_{n^3-n^2+n+2} = \mathbb{R}_{\epsilon_3}$, respectively. For brevity set also $\mathbb{R}_\delta = \mathbb{R}_{n^3-n^2+n} = \mathbb{R}_{\delta_{n-1}^{(n^2+1)}}$. The completion (see section A2) for any sub-Pfaffian set U (see the definition A4) we denote by $U^{(\delta)} = U^{(n^3-n^2+n)}$, $U^{(\epsilon_3)} = U^{(n^3-n^2+n+2)}$. Analogously we denote the languages (see the section A2) $\mathcal{L}_\delta = \mathcal{L}_{n^3-n^2+n}$, $\mathcal{L}_{\epsilon_2} = \mathcal{L}_{n^3-n^2+n+1}$, $\mathcal{L}_{\epsilon_3} = \mathcal{L}_{n^3-n^2+n+2}$. In the section A2 for each i the standard part st_i is described. Actually, throughout the paper we'll use almost in all the cases $st_{n^3-n^2+n}$ which we'll for brevity denote by st (on occasions we'll use also $st_{n^3-n^2+n+1}$ which we denote by st_{ϵ_2}).

Consider a Pfaffian computation tree $\mathcal{T}$ testing the membership to P with depth K. Fix any its branch with the output "yes", and let $f_{v_0}, \dots, f_{v_K}$ be the Pfaffian functions attached to the nodes along this branch. We rename the functions $\pm f_{v_0}, \dots, \pm f_{v_K}$ by $u_0, \dots, u_K$ in such a way that $u_0, \dots, u_{K_1}$ for a certain $K_1 \leqq K$, correspond to the sign zero, and $u_{K_1+1} > 0, \dots, u_K > 0$ correspond to nonzero signs along the branch. More precisely, consider a semi-Pfaffian set (see the definition A3)

$$W = \{x \in \mathbb{R}^n_{\epsilon_3} : u_0(x) = \cdots = u_{K_1}(x) = 0,\ u_{K_1+1}(x) > 0, \dots, u_K(x) > 0\}.$$

which is the accepting set corresponding to the branch. Then the set $W \cap \mathbb{R}^n$ is the set of points on which $\mathcal{T}$ along the fixed branch outputs "yes", hence $W \cap \mathbb{R}^n \subset P$. Since the functions $u_0, \dots, u_K$ are defined over $\mathbb{R}$, the completion (see the section A2) $(W \cap \mathbb{R}^n)^{(\epsilon_3)} = W$. In the sequel we'll estimate the number of i-facets Π of P such that $\dim(W \cap \Pi \cap \mathbb{R}^n) = i$.

When $K_1 < 0$ the set $(W \cap \mathbb{R}^n)$ lies in the interior of P, so this estimate is trivial. Therefore, we assume that $K_1 \ge 0$ and denote $f = u_0^2 + \cdots + u_{K_1}^2$.

II. Angle Points.

Definition 2. A point $x \in W$ is called a 0-quasiangle if $u_{K_1+1}(x) \ge \epsilon_1, \dots, u_K(x) \ge \epsilon_1$, and there exist points $y_1, \dots, y_n \in \{f - \epsilon_3 = 0\}$ such that the Euclidean distances

$\|y_i - x\| \leq \epsilon_2$, $1 \leq i \leq n$ and

$$\left(\det \begin{vmatrix} \frac{\partial f}{\partial X_1}(y_1) & \dots & \frac{\partial f}{\partial X_n}(y_1) \\ \vdots & & \vdots \\ \frac{\partial f}{\partial X_1}(y_n) & \dots & \frac{\partial f}{\partial X_n}(y_n) \end{vmatrix} \right)^2 > \epsilon_1^2 \Delta(y_1) \cdots \Delta(y_n) \tag{0}$$

where $\Delta = \sum_{1 \leq i \leq n} \left(\frac{\partial f}{\partial X_i} \right)^2$.

Observe that corollary A5 states that for any point $y \in \{f = \epsilon_3\} \subset \mathbb{R}^n_{\epsilon_3}$ the gradient $\mathrm{grad}_y(f) = \left(\frac{\partial f}{\partial X_1}, \cdots, \frac{\partial f}{\partial X_n} \right)(y)$ does not vanish. Notice that the inequality (0) in the definition means that the absolute value of the determinant of the matrix formed by the normalized gradient vectors of f at the points $y_1, \dots, y_n$ is greater than ϵ_1.

Definition 3. A point $x \in W$ is called i-quasiangle ($0 \leq i < n$) if for each $(n-i)$-subspace $\Pi \in \mathcal{A}_{n-i}$ (see lemma 1) the point x is a 0-quasiangle point in the semi-Pfaffian set $W \cap \Pi(x)$ (here we understand 0-quasiangle with respect to a basis in Π whose elements are from $\mathbb{R}^n$, in other words have coordinates from $\mathbb{R}$, the role of f plays the restriction of f on $\Pi(x)$).

The set of i-quasiangle points of W we denote by $\widetilde{A}_i$. Observe that $\widetilde{A}_i$ can be determined by a Pfaffian formula and thus is a sub-Pfaffian set (see the definition A4).

Definition 4. The points of the set $A_i = st(\widetilde{A}_i) \subset \mathbb{R}^n_\delta$ are called i-angles.

Lemma A7 implies that A_i is sub-Pfaffian and definable over $\mathbb{R}_1$. Due to lemma A4, $A_i \subset W$.

Lemma 3. *Let P_i be an i-facet of P with dimension (see definition A5)* $\dim(W \cap P_i) = i$. *If for two points $\widetilde{x} \in W \cap \mathbb{R}^n \cap P_i$, and $x \in P_i^{(\delta)}$ the distance $\|x - \widetilde{x}\|$ is infinitesimal relative to $\mathbb{R}$, then $x \in \widetilde{A}_i$.*

Remark. Actually, the lemma states that $x \in A_i$ since $x = st(x) \in st(\widetilde{A}_i) = A_i$.

Proof of the lemma. Since $\dim(W \cap P_i) = i$, the lemma A1 implies that f vanishes on $\overline{P}_i$. Throughout this paper $B_z(s) \subset \mathbb{R}^n_{\epsilon_3}$ denotes the open ball centered at z with radius s. There exists $0 < c \in \mathbb{R}$ such that $u_j(\widetilde{x}) > c$, $K_1 + 1 \leq j \leq K$, therefore there exists $0 < r \in \mathbb{R}$ such that $u_j(y) > \frac{c}{2}$, $K_1 + 1 \leq j \leq K$ for any $y \in B_{\widetilde{x}}(2r) \cap \mathbb{R}^n$, taking into account that the Pfaffian functions u_j are defined over $\mathbb{R}$. According to the transfer principle (see the section A2) $u_j(y) > \frac{c}{2}$, $K_1 + 1 \leq j \leq K$ for any $y \in B_{\widetilde{x}}(2r) \cap \mathbb{R}^n_{\epsilon_3}$. In particular, $u_j(x) > \frac{c}{2} > \epsilon_1$, $K_1 + 1 \leq j \leq K$.

Fix an arbitrary subspace $\Pi \in \mathcal{A}_{n-i}$. Our purpose is to show that x is 0-quasiangle in the set $W \cap \Pi(x)$, which will imply the lemma (see definition 3). Since Π is transversal to P_i, the point x is a vertex of the polyhedron $\mathcal{P} = (P \cap \Pi(x))^{(\epsilon_3)}$ (see lemma 2 and the supposition just after it). The vertex x belongs to at least $n - i$ of $(n - i - 1)$-facets (of the maximal dimension) of $\mathcal{P}$. Observe that for each of these facets the normalized orthogonal (in $\Pi(x)$) vector has the coordinates in $\mathbb{R}$. Choose any $T_1, \ldots, T_{n-i}$ among them.

Notice that for any point $y \in cl(B_x(r)) \cap \mathbb{R}^n_{\epsilon_3}$ (where cl denotes the closure in the topology with a base of all open balls) the inequalities $u_j(y) > \epsilon_1$, $K_1 + 1 \leq j \leq K$ hold since $cl(B_x(r)) \subset B_{\widetilde{x}}(2r)$. Hence, $\bigcap_{K_1+1 \leqq j \leqq K} \{f = 0, \ \& \ u_j \geqq \epsilon_1\}^{(\epsilon_3)}$ $\cap cl(B_x(r)) = \{f = 0\}^{(\epsilon_3)} \cap cl(B_x(r))$.

Denote by $\mathcal{D} \subset \Pi(x)$ the intersection of the unique closed cone $\mathcal{K}$ with the vertex at x formed in $\Pi(x)$ by $(n - i - 1)$-planes $\overline{T}_1, \ldots, \overline{T}_{n-i}$ containing $\mathcal{P}$, with the ball $cl(B_x(r))$. For any point $z \in \{f = \epsilon_3\} \cap \Pi(x) \cap cl(B_x(r))$ we have $st_{\epsilon_2}(z) \in \{f = 0\} \cap \Pi(x) \cap cl(B_x(r))$, due to lemma A4. Therefore, $st_{\epsilon_2}(z) \in W \cap cl(B_x(r)) \cap \Pi(x) \subset \mathcal{P} \cap cl(B_x(r)) \subset \mathcal{D}$, in particular the distance $\rho(z, \mathcal{D})$ from the point z to the set $\mathcal{D}$ is infinitesimal relative to $\mathbb{R}_{\epsilon_2}$. Since the set $\{f = \epsilon_3\} \cap \Pi(x) \cap cl(B_x(r))$ is closed in the topology with a base of all open balls, and bounded, the maximum value ρ_0 of $\rho(z, \mathcal{D})$ over all the points $z \in \{f = \epsilon_3\} \cap \Pi(x) \cap cl(B_x(r))$ exists (here we use the transfer principle), and is infinitesimal relative to $\mathbb{R}_{\epsilon_2}$.

Shift (in $\Pi(x)$) each of $(n - i - 1)$-planes $\overline{T}_1, \ldots, \overline{T}_{n-i}$ parallel to itself outward from $\mathcal{D}$ to the distance ρ_0. Denote the resulting shifted $(n - i - 1)$-planes by $\overline{T}'_1, \ldots, \overline{T}'_{n-i}$, respectively. Denote by x' the (unique) common point of $\overline{T}'_1 \cap \cdots \cap$

$\overline{T}'_{n-i}$. Denote by $\mathcal{D}'$ the intersection of the closed cone $\mathcal{K}'$ formed by $\overline{T}'_1, \dots, \overline{T}'_{n-i}$, with the vertex in x' with the ball $cl(B_x(r))$. Then $\{f = \epsilon_3\} \cap \Pi(x) \cap cl(B_x(r)) \subset \mathcal{D}'$. Observe that the distance $\|x - x'\|$ is infinitesimal relative to $\mathbb{R}_{\epsilon_2}$.

We replace $(n-i-1)$-planes $\overline{T}'_j$, $1 \le j \le n-i$ (in $\Pi(x)$) by some $(n-i-1)$-planes $\overline{T}''_j$, $1 \le j \le n-i$, respectively, in the following way. Take any hyperplane Ω (in $\Pi(x)$), defined over $\mathbb{R}_\delta$, such that the intersection $C_1 = \Omega \cap K \subset \mathcal{D}$. Then C_1 is a $(n-i-1)$-dimensional simplex, let its $(n-i-2)$-facets which are the intersections of Ω with $T_1, \dots, T_{n-i}$, respectively, be determined in Ω by the equations $\{L_j = 0\}$, $1 \le j \le n-i$ for some linear polynomials L_j defined over $\mathbb{R}_\delta$. Thus $C_1 = \{L_1 \ge 0, \dots, L_{n-i} \ge 0\} \cap \Omega$. Consider now $(n-i-1)$-dimensional simplex $C_2 = \{L_1 + \epsilon_2 \ge 0, \dots, L_{n-i} + \epsilon_2 \ge 0\} \cap \Omega \supset C_1$. The facets of C_2 are $\{L_j = -\epsilon_2\} \cap \Omega$, $1 \le j \le n-i$, and therefore, they are parallel to the corresponding facets of C_1. Denote by $\overline{T}^{(3)}_j$, $1 \le j \le n-i$ the hyperplane (in $\Pi(x)$) containing x and $\{L_j = -\epsilon_2\} \cap \Omega$. Denote by $\mathcal{K}^{(3)} \subset \Pi(x)$ the cone formed by $\overline{T}^{(3)}_j$, $1 \le j \le n-i$ containing C_2; observe that $\mathcal{K}^{(3)} \supset \mathcal{K}$.

We claim that the sine of the angle α between the hyperplanes $\overline{T}_j$ and $\overline{T}^{(3)}_j$, (i.e., between vectors, orthogonal to $\overline{T}_j$ and $\overline{T}^{(3)}_j$ respectively) is infinitesimal relative to $\mathbb{R}_\delta$. Indeed, consider the unique 2-plane ω_j, $1 \le j \le n-i$ passing through x and orthogonal to $\{L_j = 0\} \cap \Omega$. It intersects $(n-i-2)$-plane $\{L_j = 0\} \cap \Omega$ (respectively, $(n-i-2)$-plane $\{L_j = -\epsilon_2\} \cap \Omega$) at the unique point y_j (respectively $y^{(3)}_j$). Observe that the vector in ω_j orthogonal to the line ℓ_j passing through x and y_j (respectively, the line $\ell^{(3)}_j$ passing through x and $y^{(3)}_j$) is orthogonal to $\overline{T}_j$ (respectively, $\overline{T}^{(3)}_j$). The segment $(y_j, y^{(3)}_j)$ lies on the line $\omega_j \cap \Omega$ and is orthogonal to $\{L_j = 0\} \cap \Omega$. Hence the distance between $(n-i-2)$-planes $\{L_j = 0\} \cap \Omega$ and $\{L_j = -\epsilon_2\} \cap \Omega$, which is equal to the length of the segment $[y_j, y^{(3)}_j]$, is infinitesimal relative to $\mathbb{R}_\delta$. Since the angle α equals to the angle between the lines ℓ_j and $\ell^{(3)}_j$, we conclude that $\sin \alpha$ is infinitesimal relative to $\mathbb{R}_\delta$ taking into account that in the triangle $(x, y_j, y_j^{(3)})$ the vertices x and y_j are defined over $\mathbb{R}_\delta$, therefore the sides (x, y_j) and $(x, y_j^{(3)})$ are not infinitesimal relative to $\mathbb{R}_\delta$ and $(y_j, y_j^{(3)})$ is infinitesimal relative to $\mathbb{R}_\delta$. This proves the claim.

Let us show that there exists an element $0 < \beta \in \mathbb{R}_{\epsilon_2}$ such that for any two points $z_1 \in \partial C_1$, $z_2 \in \partial C_2$ from the boundaries, (see definition A8 (here we mean the boundary in the hyperplane Ω)), the sine of the angle between the lines (x, z_1) and (x, z_2) is greater or equal to β. Since both points z_1, z_2 range over bounded closed sets, there exists (due to the transfer principle) the minimum β of these sines. Observe that $\beta > 0$ since $\partial C_1 \cap \partial C_2 = \emptyset$. One could define the element β by a formula of the language $\mathcal{L}_{\epsilon_2}$. Therefore, $\beta \in \mathbb{R}_{\epsilon_2}$ by the transfer principle, as was to be shown.

Note that the cones $\mathcal{K}$ and $\mathcal{K}'$ are isometric. We define the desired $(n-i-1)$-planes $\overline{T}_j''$, $1 \le j \le n-i$ as the images of $\overline{T}_j^{(3)}$, respectively, under the shift mapping the cone $\mathcal{K}$ onto $\mathcal{K}'$, then the cone $\mathcal{K}''$ formed by $\overline{T}_j''$, $1 \le j \le n-i$, is the image of the cone $\mathcal{K}^{(3)}$.

For every $1 \le j \le n-i$, pick a point $x_j \in \{f = \epsilon_3\} \cap \Pi(x) \cap cl\, B_x(r)$ with the property that x_j is the nearest to $\overline{T}_j''$ on the (bounded and closed) set $\{f = \epsilon_3\} \cap \Pi(x) \cap cl\, B_x(r)$. Lemma A4 entails that there exists a point $y \in \{f = \epsilon_3\} \cap \Pi(x) \cap B_x(r)$ such that $\|x - y\|$ is infinitesimal relative to $\mathbb{R}_{\epsilon_2}$, therefore $\|x' - y\|$ is infinitesimal relative to $\mathbb{R}_{\epsilon_2}$ as well, hence the distance from x_j to $\overline{T}_j''$ is also infinitesimal relative to $\mathbb{R}_{\epsilon_2}$. Denote by $x_j'' \in \overline{T}_j''$ the orthogonal projection of x_j on $\overline{T}_j''$. Let us prove that $\|x_j - x'\|$ is infinitesimal relative to $\mathbb{R}_{\epsilon_2}$. Since $x_j \in (\{f = \epsilon_3\} \cap \Pi(x) \cap cl(B_x(r))) \subset \mathcal{D}'$, the segment (x_j, x_j'') intersects $\partial\mathcal{K}'$ (here we mean the boundary in $\Pi(x)$) at the unique point x_j'. Since the sine of the angle γ between the lines (x', x_j) and (x', x_j'') is greater than or equal to the sine of the angle between the lines (x', x_j'') and (x', x_j') which, in its turn, is greater or equal to β (see above), we conclude that $\sin\gamma \ge \beta \in \mathbb{R}_{\epsilon_2}$. Therefore, $\|x' - x_j\| = \frac{\|x_j - x_j''\|}{\sin\,\gamma}$ is infinitesimal relative to $\mathbb{R}_{\epsilon_2}$, $1 \le j \le n-i$, which was to be proved. Hence, $\|x - x_j\|$

is infinitesimal relative to $\mathbb{R}_{\epsilon_2}$ as well, in particular $x_j \in B_x(r)$.

Observe that the gradient $grad_{x_j}(\hat{f})$ (where $\hat{f}$ denotes the restriction of f on $\Pi(x)$, cf. Definition 3) does not vanish because $x_j \in \{f = \epsilon_3\} \cap \Pi(x)$ (see corollary A5) and it is orthogonal to the hyperplane $\overline{T}''_j$ (in $\Pi(x)$), as x is the nearest to $\overline{T}''_j$ on the set $\{f = \epsilon_3\} \cap \Pi(x) \cap cl\, B_x(r)$. Since the sines of the angles between any pair of hyperplanes $\overline{T}_{j_1}, \overline{T}_{j_2}$ (in $\Pi(x)$) is greater than a certain c, $0 < c \in \mathbb{R}$, we conclude that the sines of the angles between any pair of hyperplanes $\overline{T}''_{j_1}$, $\overline{T}''_{j_2}$ is greater than $c/2$ according to the claim proved above (stating that the sine of the angle between $\overline{T}_{j_1}$ and $\overline{T}''_{j_1}$ is infinitesimal relative to $\mathbb{R}_\delta$). Therefore

$$\det \left(\frac{grad_{x_1}(\hat{f})}{\|grad_{x_1}(\hat{f})\|}, \dots, \frac{grad_{x_{n-i}}(\hat{f})}{\|grad_{x_{n-i}}(\hat{f})\|} \right) > c_1 > 0$$

for a suitable $c_1 \in \mathbb{R}$.

Taking the points $x_1, \dots, x_{n-i}$ as the points $y_1, \dots, y_{n-i}$ in the definition 2 we get that x is 0-quasiangle in the semi-Pfaffian set $W \cap \Pi(x)$, whence x is i-quasiangle because $(n-i)$-plane $\Pi \in \mathcal{A}_{n-i}$ was chosen arbitrarily. □

Corollary. *Let a point $\widetilde{x} \in W \cap P_i \cap \mathbb{R}^n$ and the dimension in the point $\widetilde{x}$ $\dim_{\widetilde{x}}(W \cap P_i) = i$, then*

(a) $\dim_{\widetilde{x}}(A_i \cap P_i^{(\delta)}) = i$;

(b) $\dim(A_i \cap P_i^{(\epsilon_3)}) = i$.

Proof. Lemma 3 and the remark following this lemma imply that for any $0 < \rho \in \mathbb{R}_\delta$ which is infinitesimal relative to $\mathbb{R}$, we have the inclusion $(B_{\widetilde{x}}(\rho) \cap P_i^{(\delta)}) \subset A_i$, this provides a).

Moreover, lemma 3 and the remark imply that $(B_{\widetilde{x}}(\rho) \cap P_i^{(\delta)}) \subset st(\widetilde{A}_i \cap P_i^{(\epsilon_3)})$. Thus, b) follows from lemma A8. □

Lemma 4. $\dim(A_i) \leq i$

Proof. First let us reduce the proof to the case $i = 0$, in which $\tilde{A}_0, A_0$ are defined for a set W given by Pfaffian functions $u_1, \dots, u_K$ defined over $\mathbb{R}_\delta$ (rather than $\mathbb{R}$), see section A2.

Thus, let $i \geq 1$ and suppose that $e = \dim(A_i) \geq i+1$. Due to corollary A1, there exists a nonsingular point $y \in A_i$ such that $\dim_y(A_i) = e$. Denote by T_y the tangent plane to A_i at the point y. Since $\dim(T_y) = e$ one can find $(n-i)$-subspace $\Pi \in \mathcal{A}_{n-i}$ such that $\dim(T_y \cap \Pi(y)) = e - i$. Take any $(n-e)$-subspace $R \subset \Pi$ defined over $\mathbb{R}_\delta$ for which $(T_y \cap R(y)) = \{y\}$. Consider the linear orthogonal projection $\pi : \mathbb{R}^n_{\epsilon_3} \to \mathbb{R}^e_{\epsilon_3}$ onto e-subspace along R. Then $\dim \pi(T_y) = e$. Therefore, $\pi(A_i) \subset \mathbb{R}^e_\delta$ contains e-dimensional ball $B_{\pi(y)}(r)$ for a certain $0 < r \in \mathbb{R}_\delta$ (by the implicit function theorem and the transfer principle).

For any point $x \in A_i$ there is a point $x' \in \widetilde{A}_i$ such that $st(x') = x$, hence $st(\pi(\widetilde{A}_i)) \supset B_{\pi(y)}(r)$.

By assumption that the lemma is valid for the case $i = 0$.

Then for any point $z \in \mathbb{R}^n_\delta$ applying this assumption to the set of 0-angle points of the intersection $\Pi(z) \cap W$ we conclude that the sub-Pfaffian set $st(\Pi(z) \cap \widetilde{A}_i)$ has the dimension at most 0 (taking into account the definition 3 of i-quasiangle points and that $\Pi(z)$ is defined over $\mathbb{R}_\delta$).

Let us show that $\pi(\widetilde{A}_i)$ does not contain a ball $B_w(r_1)$ for any $0 < r_1 \in \mathbb{R}_\delta$ and $w \in \mathbb{R}^e_{\epsilon_3}$. Assume the contrary, then there exists a point $w_1 \in B_w(r_1) \cap \mathbb{R}^e_\delta$. Let $z_1 \in \mathbb{R}^n_\delta$ be a point such that $\pi(z_1) = w_1$. Denote $\Pi_1 = \pi(\Pi)$, then $\dim \Pi_1 = e - i$, $\Pi = \pi^{-1}(\Pi_1)$. Then the following inequalities hold:

$$\dim st(\Pi_1(w_1) \cap \pi(\widetilde{A}_i)) \geq \dim st(\Pi_1(w_1) \cap B_w(r_1)) = e - i \geq 1.$$

On the other hand, $\Pi_1(w_1) \cap \pi(\widetilde{A}_i) = \pi(\widetilde{A}_i \cap \Pi(z_1))$, and, therefore,

$$\dim st(\Pi_1(w_1) \cap \pi(\widetilde{A}_i)) \leq \dim st(\widetilde{A}_i \cap \Pi(z_1)) \leq 0,$$

(the latter inequality was proved above). The obtained contradiction shows that $\pi(\widetilde{A}_i)$ does not contain a ball $B_w(r_1)$ for any $0 < r_1 \in \mathbb{R}_\delta$.

We claim that for any ball $B_{z_2}(r_2) \subset B_{\pi(y)}(r)$ defined over $\mathbb{R}_{\epsilon_3}$ such that $0 < r_2 \in \mathbb{R}_\delta$, the intersection $B_{z_2}(r_2) \cap \partial\pi(\widetilde{A}_i) \neq \emptyset$. Assume the contrary. Then either $B_{z_2}(r_2) \subset \pi(\widetilde{A}_i)$ or $B_{z_2}(r_2) \cap \pi(\widetilde{A}_i) = \emptyset$. The inclusion $B_{z_2}(r_2) \subset \pi(\widetilde{A}_i)$ is impossible as was shown above. If $B_{z_2}(r_2) \cap \pi(\widetilde{A}_i) = \emptyset$, then $st(z_2) \notin st(\pi(\widetilde{A}_i))$, the latter contradicts to the inclusions $st(\pi(\widetilde{A}_i)) \supset B_{\pi(y)}(r) \supset B_{st(z_2)}(r_2/2)$ of the sets in the space $\mathbb{R}_\delta^e$. This proves the claim.

Because of lemma A3, $\dim(\partial(\pi(\widetilde{A}_i))) \leq e - 1$. Applying lemma A8, we get $\dim(st(\partial(\pi(\widetilde{A}_i)))) \leq e - 1$.

On the other hand we shall now prove that $st(\partial(\pi(\widetilde{A}_i))) \supset B_{\pi(y)}(r)$. This contradiction completes the proof of the reduction of the lemma to the case $i = 0$. Indeed, let $z_3 \in B_{\pi(y)}(r)$. Observe that the set $D = \{\|z - z_3\|^2 : z \in \partial(\pi(\widetilde{A}_i))\}$ is sub-Pfaffian. Due to Corollary A4, D is a finite union of points and intervals. Let ω be the minimal among these points and the endpoints of these intervals. Suppose that $z_3 \notin st(\partial(\pi(\widetilde{A}_i)))$, i.e. there does not exist $z \in \partial(\pi(\widetilde{A}_i))$ such that $st(z) = z_3$. Thus, $\omega > r_3^2$ for an element $0 < r_3 \in \mathbb{R}_\delta$. It follows that $B_{z_3}(r_3) \cap \partial(\pi(\widetilde{A}_i)) = \emptyset$. This contradicts the claim just proved.

Now let $i = 0$. Suppose the statement of the lemma is wrong and $\dim(A_0) = s \geq 1$. There is a linear projection $\pi : \mathbb{R}_\delta^n \to \mathbb{R}_\delta^s$ onto a certain coordinate s-subspace, such that $\pi(A_0) \supset B_z(r)$ for some $z \in \mathbb{R}_\delta^s$, $0 < r \in \mathbb{R}_\delta$. Choose an open interval $L \subset B_z(r)$ of the length $2r$ passing through z.

Our nearest purpose is to prove the existence of a sub-Pfaffian curve (i.e. one-dimensional sub-Pfaffian set) $C_0 \subset A_0$ such that $\pi(C_0) = L$ and the mapping $\pi : C_0 \to L$ is bijective. This follows from the next, a more general construction.

Let $V \subset F^n$, $U \subset F^m$ be sub-Pfaffian sets where F is one of the fields $\mathbb{R}_j$ defined in the section A2 and $\varphi : V \to U$ be a sub-Pfaffian mapping (i.e. a mapping with a sub-Pfaffian graph). Let us describe one of the possible ways to construct a sub-Pfaffian set $V_0 \subset V$ such that the restriction $\varphi : V_0 \to \varphi(V)$ of φ is bijective.

For every point $u \in \varphi(V)$ take the (unique) point $v_u \in V$ such that $\varphi(v_u) = u$ according to the following rule (actually, this rule is quite flexible).

A projection $\pi_1(\varphi^{-1}(u))$ of $\varphi^{-1}(u)$ onto the axis X_1 is a union of a finite number of intervals (with or without endpoints) since $\pi_1(\varphi^{-1}(u))$ is sub-Pfaffian (see Corollary A4). Let $\mathfrak{a}_1, \mathfrak{a}_2$ be the endpoints of the leftmost among these intervals (note that a sub-Pfaffian set is always bounded, see Definition A4). Then $\frac{\mathfrak{a}_1+\mathfrak{a}_2}{2} \in \pi_1(\varphi^{-1}(u))$. Consider the projection $\pi_2(\varphi^{-1}(u) \cap \{X_1 = \frac{\mathfrak{a}_1+\mathfrak{a}_2}{2}\})$ onto the axis X_2. Continuing in the similar way, after n steps we obtain a point $v_u = (\frac{\mathfrak{a}_1+\mathfrak{a}_2}{2}, \dots) \in \varphi^{-1}(u)$. We define V_0 as a set of all the obtained points v_u for all $u \in \varphi(V)$. One can easily prove that V_0 is sub-Pfaffian and the mapping $\varphi : V_0 \to \varphi(V)$ is bijective.

Applying this construction to the mapping $\pi\big|_{\pi^{-1}(L)\cap A_0} : \pi^{-1}(L) \cap A_0 \to L$ we get a required sub-Pfaffian curve $C_0 \subset A_0$. Since there is only a finite number of connected components of C_0 (see Corollary A3), there exists a connected component C such that $\pi(C)$ is an interval of a length $r_0 > 0$ for a certain $r_0 \in \mathbb{R}_\delta$. Then the completion $C^{(\epsilon_3)} \subset \mathbb{R}^n_{\epsilon_3}$ is a connected component of the curve $C_0^{(\epsilon_3)} \subset \mathbb{R}^n_{\epsilon_3}$ (see the section A3).

Fix a nonsingular point $x \in C$ (due to corollaries A1, A4 C has only a finite number of singular points). Denote by $\tau \subset \mathbb{R}^n_\delta$ the tangent line to C at x, then its completion $\tau^{(\epsilon_3)} \subset \mathbb{R}^n_{\epsilon_3}$ is tangent to $C^{(\epsilon_3)}$. After a suitable linear coordinate transformation (defined over $\mathbb{R}_\delta$) one can assume that $x = 0$ and τ coincides with the axis X_n. Denote by γ the projection mapping on the axis X_n.

There exists $0 < \mu \in \mathbb{R}_\delta$ satisfying the following properties:

(i) the unique connected component c of the intersection $C \cap \{-\mu < X_n < \mu\} \subset \mathbb{R}^n_\delta$, containing 0, is a nonsingular curve and the mapping $\gamma^{-1} : (-\mu, \mu) \to c$ is definable and doubly differentiable;

(ii) there exists $0 < \lambda \in \mathbb{R}_\delta$ such that for any $y \in (-\mu, \mu)$ the inequality $\|\gamma^{-1}(0, \dots, 0, y) - (0, \dots, 0, y)\| \le \lambda |y|^2$ holds.

One can prove the existence of μ for the curves in $\mathbb{R}^n$ using Taylor formula, and then for C applying the transfer principle.

The transfer principle also implies that (i), (ii) hold for the completions $c^{(\epsilon_3)} \subset C^{(\epsilon_3)}$ and any $y \in (-\mu, \mu)^{(\epsilon_3)}$.

The angle between a line ℓ and a hyperplane $\mathcal{P}$ in $\mathbb{R}^n_{\epsilon_3}$ is defined as a difference between $\pi/2$ and the angle between ℓ and the vector orthogonal to $\mathcal{P}$. Observe that there exists $0 < \nu \in \mathbb{R}_{\epsilon_1}$ such that if n normalized vectors $v_1, \dots v_n \in \mathbb{R}^n_{\epsilon_3}$ satisfy the inequality $|det(v_1, \dots v_n)| > \epsilon_1$, then for any hyperplane $\mathcal{P}$ there is i, $1 \leq i \leq n$ for which the sine of the angle between v_i and $\mathcal{P}$ is greater than ν (actually, one could take $\nu = \epsilon_1/2$ but we will not use this particular value).

Introduce the sub-Pfaffian set $V \subset \mathbb{R}^n_{\epsilon_3}$ consisting of all the points $z = (z_1, \dots, z_n) \in \mathbb{R}^n_{\epsilon_3}$ such that

(1) $z \in \{f = \epsilon_3\}$, $|z_n| < \mu$;
(2) sine of the angle between $grad_z(f - \epsilon_3)$ and the hyperplane $\{X_n = 0\}$ is greater than ν;
(3) for a given z_n the minimum of the distance to the axis X_n (i.e., of the function $(X_1^2 + \dots + X_{n-1}^2)^{1/2}$) on the set of all the points satisfying 1), 2) is attained at z.

Let us apply the above construction to the projection $\gamma : V \to (-\mu, \mu)$. The construction supplies us with a sub-Pfaffian subset $V_0 \subset V$ such that each nonempty preimage $\gamma^{-1}(y)$ contains exactly one point from V_0. Therefore $\dim(V_0) \leq 1$.

We claim that, actually, $\dim(V_0) = 1$. Suppose the contrary, then V_0 would consist of a finite number of points (see corollary A4). We show, however, that V_0 contains infinitely many points.

Indeed, take an arbitrary point $y \in \mathbb{R}_\delta \cap (-\mu, \mu)$ and the (unique) point $w \in c$ such that $\gamma(w) = (0, \dots, 0, y)$. Since $c \subset A_0$ there exists (see the definition 4 of 0-angle points) a point $w_1 \in \widetilde{A}_0$ such that $st(w_1) = w$, therefore (see definition 2 of 0-quasiangle points) there exists a point $w_2 \in \{f = \epsilon_3\}$ for which $\|w_1 - w_2\| \leq \epsilon_2$ and the sine of the angle between the vector $grad_{w_2}(f - \epsilon_3)$ and the hyperplane $\{X_n = 0\}$ is greater than ν (see (0)). Because $st\|w_2 - w\| = 0$ and for the orthogonal projection $\|\gamma(w_2) - \gamma(w)\| \leq \|w_2 - w\|$, we deduce that $st(\gamma(w_2)) = st(\gamma(w)) = \gamma(w)$. Since the point w_2 satisfies the conditions 1), 2) in the definition of V, there exists a point $w_3 \in \{f = \epsilon_3\}$ such that $\gamma(w_3) = \gamma(w_2)$, the sine of the angle between $grad_{w_3}(f - \epsilon_3)$ and the hyperplane $\{X_n = 0\}$ is greater than ν, and w_3 has the minimal distance to the axis X_n among the points with these properties. Then $w_3 \in V$.

Thus, we have shown that for each point $y \in \mathbb{R}_\delta \cap (-\mu, \mu)$ there exists a point $w_3 \in V$ such that $st(\gamma(w_3)) = (0, \dots, 0, y)$. Because of the above construction, there exists the unique point $w_4 \in V_0$ for which $\gamma(w_4) = \gamma(w_3)$. Hence V_0 contains an infinite number of points, i.e. $\dim(V_0) = 1$.

Let $V_0 = \bigcup_i \mathcal{V}_i$ be the decomposition of V_0 into the connected components. Since V_0 is sub-Pfaffian, it has only a finite number of singular points and a finite number of points at which the tangent to the curve V_0 is orthogonal to the axis X_n (i.e. of the critical points of the mapping γ), here we invoke corollaries A1, A4. It follows that each $\mathcal{V}_i$ admits a finite partition $\mathcal{V}_i = \bigcup_j \mathcal{V}_{ij} \cup \bigcup_{j_1} v_{ij_1}$, where every $\mathcal{V}_{ij}$ is a nonsingular connected sub-Pfaffian curve (without the endpoints) not containing the critical points of γ, and every v_{ij_1} is a set consisting of a single point.

We have shown above that $st(\gamma(V_0)) = [-\mu, \mu]$. Since $\gamma(\mathcal{V}_{ij}) \subset (-\mu, \mu)^{(\epsilon_3)}$ is connected (as an image of a connected curve), it is an interval, hence $st(\gamma(\mathcal{V}_{ij})) \subset [-\mu, \mu]$ is a closed interval. Therefore, there are i_0, j_0 for which an interval $I = st(\gamma(\mathcal{V}_{i_0 j_0}))$ has a positive length $|I| \in \mathbb{R}_\delta$, besides I contains 0 and does not lie entirely to the left of 0.

Due to the implicit function theorem, one may represent the curve $\mathcal{V}_{i_0 j_0}$ in a parametrical form: $(X_1(X_n), \dots, X_{n-1}(X_n), X_n)$ where $X_1, \dots, X_{n-1}$ are smooth functions. Observe that for any point $z = (X_1(z_n), \dots, X_{n-1}(z_n), z_n) \in \mathcal{V}_{i_0 j_0}$ the tangent vector $(\dot{X}_1(z_n), \dots, \dot{X}_{n-1}(z_n), 1)$ at this point to the curve $\mathcal{V}_{i_0 j_0}$ has a sine of the angle with the axis X_n greater than ν, since this tangent vector is orthogonal to $grad_z(f - \epsilon_3)$, taking into account inclusions $V_0 \subset V \subset \{f = \epsilon_3\}$. In other words $\sum_{1 \le i \le n-1} (\dot{X}_i(z_n))^2 > \nu^2/(1 - \nu^2)$.

For each pair of indices $1 \le i < j \le n-1$ either there are at most a finite number of the tangent vectors $(\dot{X}_1(z_n), \dots, \dot{X}_{n-1}(z_n), 1)$ at the points of the curve $\mathcal{V}_{i_0 j_0}$ such that $\dot{X}_i(z_n) = \pm \dot{X}_j(z_n)$ or all these vectors satisfy one of the two conditions: $\dot{X}_i(z_n) = \dot{X}_j(z_n)$ or $\dot{X}_i(z_n) = -\dot{X}_j(z_n)$, because $\mathcal{V}_{i_0 j_0}$ is sub-Pfaffian. Therefore, there exists a connected sub-Pfaffian curve $\mathcal{V} \subset \mathcal{V}_{i_0 j_0}$ for which the length of the interval $st(\gamma(\mathcal{V})) \in \mathbb{R}_\delta$ is positive, besides $st(\gamma(\mathcal{V}))$ contains 0 and does not lie entirely to the left of 0. Apart from that, either $|\dot{X}_i(z_n)| \neq |\dot{X}_j(z_n)|$, for any pair $1 \leq i < j \leq n-1$ and any point $(X_1(z_n), \dots, X_{n-1}(z_n), z_n) \in \mathcal{V}$, or for a certain pair $1 \leq i < j \leq n-1$, one of the two conditions $\dot{X}_i(z_n) = \dot{X}_j(z_n)$ or $\dot{X}_i(z_n) = -\dot{X}_j(z_n)$ holds for any point from $\mathcal{V}$. Let us assume that $|\dot{X}_i(z_n)| \neq |\dot{X}_j(z_n)|$ for any pair $1 \leq i < j \leq n-1$ (the case $|\dot{X}_i(z_n)| = |\dot{X}_j(z_n)|$ can be treated in a similar way). There exists s, $1 \leq s \leq n-1$ such that $|\dot{X}_s(z_n)| > |\dot{X}_j(z_n)|$, $1 \leq j \leq n-1$, $s \neq j$ for all the points for $\mathcal{V}$. Moreover, $\dot{X}_s(z_n)$ has a constant nonnegative sign for all the points from $\mathcal{V}$. For definiteness suppose that $\dot{X}_s(z_n) > 0$ for all the points from $\mathcal{V}$ (the case $\dot{X}_s(z_n) < 0$ can be considered in a similar manner). Then $\dot{X}_s(z_n) > \nu/((n-1)(1-\nu^2))^{1/2} = \nu_0 \in \mathbb{R}_{\epsilon_1}$ and $\nu_0 > 0$.

Let an interval $[0, \mu_2] \subset st(\gamma(\mathcal{V})) \subset [-\mu, \mu]$ where $0 < \mu_2 \in \mathbb{R}_\delta$. Then for any μ_3, $\mu_4 \in \mathbb{R}_\delta$, such that $0 < \mu_3 < \mu_4 < \mu_2$, the completion of the interval $[\mu_3, \mu_4]^{(\epsilon_3)} \subset \gamma(\mathcal{V})$. Since $\dot{X}_s(z_n) > \nu_0$ for any $z_n \in [\mu_3, \mu_4]^{(\epsilon_3)}$, for any point $\eta \in [\mu_3, \mu_4]^{(\epsilon_3)}$ the inequality

$$X_s(\eta) - X_s(\mu_3) \geq \nu_0(\eta - \mu_3)$$

holds. Indeed, the latter statement could be written as a formula of the first-order theory of real closed fields, in the case of the field $\mathbb{R}$ it is true because

$$X_s(\nu) - X_s(\mu_3) = \int_{\mu_3}^{\nu} \dot{X}_s \geq \nu_0(\nu - \mu_3),$$

then use the transfer principle.

Let $y \in (-\mu, \mu) \cap \mathbb{R}_\delta$. We have proved above that for the unique point $w = \gamma^{-1}(0, \dots, 0, y) \in c \subset A_0$ there exists a point $w_1 \in \widetilde{A}_0$ such that $st(w_1) = w$, besides there exists a point $w_2 \in \{f = \epsilon_3\}$ such that $\|w_1 - w_2\| \leq \epsilon_2$ and the sine of the angle between the vector $grad_{w_2}(f - \epsilon_3)$ and the hyperplane $\{X_n = 0\}$ is greater than ν. Then the distance from w_2 to the axis X_n does not exceed $\|w_2 - w_1\| + \|w_1 - w\| + \|w - (0, \dots, 0, y)\| \leq \epsilon_2 + \|w_1 - w\| + \lambda y^2 \leq \lambda_0 y^2$ for $\lambda \in \mathbb{R}_\delta$, introduced in (ii) above, and any $\lambda < \lambda_0 \in \mathbb{R}_\delta$. So the distance to the axis X_n from the unique point $w_4 \in V_0$, for which $\gamma(w_4) = \gamma(w_2)$, also does not exceed $\lambda_0 y^2$. Note that $st(\gamma(w_4)) = (0, \dots, 0, y)$.

On the other hand if $y \in [\mu_3, \mu_4] \cap \mathbb{R}_\delta$, then applying the above arguing to the point $(y + \mu_3)/2$ instead of y we prove the existence of a point $w_5 \in V_0$ such that $st(\gamma(w_5)) = (0, \dots, 0, (y + \mu_3)/2)$ and the distance to the axis X_n from the point w_5 does not exceed $\lambda_0((y + \mu_3)/2)^2$. Arguing as above, we get

$$X_s(w_4) - X_s(w_5) \geq \nu_0 \|\gamma(w_4) - \gamma(w_5)\| > \nu_1 (y - \mu_3)/2$$

for arbitrary $\nu_1 \in \mathbb{R}_{\epsilon_1}$, $0 < \nu_1 < \nu_0$. Then either the distance from the point w_4 to the axis X_n or the distance from the point w_5 to X_n is greater than $\nu_1(y - \mu_3)/4$, on the other hand both distances do not exceed $\lambda_0 y^2$. Taking any y, $0 < y \in \mathbb{R}_\delta$, such that $y < \nu_1/(\nu_1 + 4\lambda_0)$ and $\mu_3 = y^2$, we get a contradiction because $\nu_1(y - y^2)/4 > \lambda_0 y^2$. $\square$

III. Flat Points.

Definition 4. Let $0 \leq i \leq n - 1$. A point $x \in A_i$ is i-flat if there exists an i-plane Π, passing through x such that $\dim(\Pi \cap A_i) = i$.

Denote by $\phi_i \subset A_i$ the set of i-flat points. Note that for $i = 0$ lemma 4 implies that $\dim A_0 \leq 0$, i.e. A_0 consists of at most finite set of points (see corollary A4), therefore $\phi_0 = A_0$.

Lemma 5. *a) There is at most a finite number of i-planes Π such that* $\dim(\Pi \cap \phi_i) = i$;

b) ϕ_i is contained in the union of all i-planes described in a).

Proof. If $\phi_i = \emptyset$, the lemma is trivial, so suppose that $\phi_i \neq \emptyset$. Since $\phi_0 = A_0$

consists of a finite number of points, the lemma for the case $i = 0$ is obvious. So, in what follows we assume that $i \geq 1$.

b) is evident. Note that if Π satisfies a) then $\dim(\Pi \cap A_i) = i$ since $\phi_i \subset A_i$.

Introduce a set $\widehat{\phi}_i \subset \phi_i$ consisting of all the points $y \in \phi_i$ for which there exists an i-plane Π passing through y, such that for suitable $0 < r \in \mathbb{R}_\delta$ we have $B_y(r) \cap \Pi \subset \phi_i$. The set $\widehat{\phi}_i$ is obviously sub-Pfaffian.

Besides, $\dim \widehat{\phi}_i = i$. Indeed, lemma 4 implies that $\dim \widehat{\phi}_i \leq i$. On the other hand as $\phi_i \neq \emptyset$, there exists i-plane Π such that $\dim(\Pi \cap A_i) = i$, hence $\Pi \cap A_i \supset \Pi \cap B_{y_1}(r_1)$ for some $y_1 \in \Pi$, $0 < r_1 \in \mathbb{R}_\delta$. Then $\Pi \cap B_{y_1}(r_1) \subset \widehat{\phi}_i$, i.e. $\dim \widehat{\phi}_i \geq i$.

If suffices to prove that there exists only a finite number of i-planes Π for which $\dim(\Pi \cap \widehat{\phi}_i) = i$. This would imply the item a) of the lemma since for any i-plane Π such that $\dim(\Pi \cap \phi_i) = i$ we have $\dim(\Pi \cap \widehat{\phi}_i) = i$.

Denote by $\hat{\hat{\phi}}_i \subset \widehat{\phi}_i$ the set of all nonsingular points of $\widehat{\phi}_i$. The set $\widehat{\phi}_i \setminus \hat{\hat{\phi}}_i$ of all singular points is sub-Pfaffian and $\dim(\widehat{\phi}_i \setminus \hat{\hat{\phi}}_i) \leq i - 1$ (see corollary A1 and lemma A2). For any point $y_2 \in \hat{\hat{\phi}}_i$ there is the unique i-plane Π' such that for an appropriate $0 < r_2 \in \mathbb{R}_\delta$ we have $B_{y_2}(r_2) \cap \Pi' \subset \widehat{\phi}_i$. Then for a suitable $0 < r_3 \in \mathbb{R}_\delta$, a certain neighbourhood of y_2 in $\widehat{\phi}_i$ coincides with $B_{y_2}(r_3) \cap \Pi'$, moreover $B_{y_2}(r_3) \cap \Pi'$ is a neighbourhood of y_2 in $\hat{\hat{\phi}}_i$.

If $\dim(\Pi \cap \widehat{\phi}_i) = i$ for i-plane Π then $\Pi \cap \widehat{\phi}_i$ contains a nonsingular point $y_3 \in \hat{\hat{\phi}}_i$ (since $\dim(\widehat{\phi}_i \setminus \hat{\hat{\phi}}_i) \leq i - 1$); moreover a neighbourhood of y_3 in $\hat{\hat{\phi}}$ coincides with $B_{y_3}(r_4) \cap \Pi$ for a suitable $0 < r_4 \in \mathbb{R}_\delta$. Thus, it is sufficient to show that there is only a finite number of i-planes Π such that $\dim(\Pi \cap \hat{\hat{\phi}}_i) = i$.

Each connected component of $\hat{\hat{\phi}}_i$ is contained in an i-plane Π, since for any point $y_4 \in \hat{\hat{\phi}}_i$ its certain neighbourhood in $\hat{\hat{\phi}}_i$ coincides with $B_{y_4}(r_5) \cap \Pi''$ for some $0 < r_5 \in \mathbb{R}_\delta$ and i-plane Π''. Because the number of connected components of $\hat{\hat{\phi}}_i$ is finite (see corollary A3), the number of i-planes Π such that $\dim(\Pi \cap \hat{\hat{\phi}}_i) = i$ is also finite. $\square$

Lemma 6. *If a connected component φ of ϕ_i has a nonempty intersection $\varphi \cap P_i \neq \emptyset$*

with a i-facet P_i of P, then $\varphi \subset P_i$.

Proof. First we prove for a connected component φ_0 of ϕ_i the following statement: if $\varphi_0 \cap cl(P_i) \neq \emptyset$ then $\varphi_0 \subset cl(P_i)$. Assume the contrary. Then there exists a point $y \in \varphi_0 \cap \overline{P}_i$ such that $y \in cl(\varphi_0 \setminus \overline{P}_i) \subset cl(\Phi_i \setminus \overline{P}_i)$. Due to lemma 5, there is a finite family $\mathcal{P}$ of i-planes Π such that $\dim(\Pi \cap \phi_i) = i$ and ϕ_i lies in the union of all these i-planes. Let us show that there exists $\Pi' \in \mathcal{P}$, $\Pi' \neq \overline{P}_i$ such that $y \in \Pi'$. Indeed, let $y_j \underset{j\to\infty}{\longrightarrow} y$, where $y_j \in \varphi_0 \setminus \overline{P}_i$. For each j there is $\Pi'' \in \mathcal{P}$ such that $y_j \in \Pi''$ (obviously $\Pi'' \neq \overline{P}_i$). Since $\mathcal{P}$ is finite there exists an infinite subsequence y_{j_ℓ}, $1 \leq \ell < \infty$ and $\Pi''' \in \mathcal{P}$ for which $y_{j_\ell} \in \Pi'''$, $1 \leq \ell < \infty$. Thus $y \in \Pi''' \neq \overline{P}_i$.

Since $\phi_i \subset A_i \subset W \cap \mathbb{R}^n_\delta$ (see the remark following the definition 4) the function f vanishes on the intersection of Π''' with the domain of f (see the lemma A1), taking into account that $\dim(\Pi''' \cap \phi_i) = i$. Besides $u_{K_1+1}(y) > 0, \dots, u_K(y) > 0$, therefore $u_{K_1+1}, \dots, u_K$ are positive also in $B_y(\rho)$ for an appropriate $0 < \rho \in \mathbb{R}_\delta$. Hence $\Pi''' \cap B_y(\rho) \subset W \cap \mathbb{R}^n_\delta$. This contradicts to the inclusion $W \cap \mathbb{R}^n_\delta \subset P^{(\delta)}$ because y belongs to the closure $cl(P_i)$ of i-facet of the convex polyhedron $P^{(\delta)}$. Thus $\varphi_0 \subset cl(P_i)$, and the statement is proved.

To complete the proof of the lemma it suffices to show that $\varphi \cap (cl(P_i) \setminus P_i) = \emptyset$. If $z \in \varphi \cap (cl(P_i) \setminus P_i)$ then there is another i-facet P_i' of P such that $z \in cl(P_i')$. Then, by the proved above, $\varphi \subset cl(P_i')$, this contradicts to $\varphi \cap P_i \neq \emptyset$. $\square$

Our next purpose is to explicitly describe (see lemma 7 below) the sufficient condition for i-flatness of a point $x \in A_i$ by means of Pfaffian formula with purely existential quantifier prefix.

Let Π be an i-plane containing x and for some points $v_1, \dots, v_i \in \Pi \cap A_i$, the vectors $v_1 - x, \dots, v_i - x$ be linearly independent. Denote by $\gamma_1, \dots, \gamma_{(i+1)n}$ the coordinates of the vectors $x, v_1, \dots, v_i$. Due to lemma A9, 1) the degree of sub-Pfaffian transcendency $[\gamma_1, \dots, \gamma_{(i+1)n}]_{\mathbb{R}_1} \leq (i+1)n \leq n^2$. Introduce the points $w^{(j)} = x + \sum_{1 \leq \ell \leq i} \delta_\ell^{(j)}(v_\ell - x) \in \Pi$, $1 \leq j \leq n^2 + 1$.

Lemma 7. *Let the points* $x, v_1, \dots, v_i \in A_i \cap \Pi$. *If* $w^{(1)}, \dots, w^{(n^2+1)} \in A_i \cap \Pi$, *then* x *is* i*-flat and moreover* $\dim(A_i \cap \Pi) = i$.

Proof. Suppose that on the contrary, $\dim(A_i \cap \Pi) \leq i-1$. Consider the sub-Pfaffian set $\mathcal{A} \subset \mathbb{R}_\delta^{(i+1)n+i}$ consisting of all the points

$$(y_1, \dots, y_n, y_{1,1}, \dots, y_{1,n}, y_{2,1}, \dots, y_{2,n}, \dots, y_{i,1}, \dots, y_{i,n}, z_1, \dots, z_i)$$

for which $(y_1, \dots, y_n) + \sum_{1 \leq \ell \leq i} z_\ell((y_{\ell,1}, \dots, y_{\ell,n}) - (y_1, \dots, y_n)) \in A_i$ (cf. expressions for $w^{(j)}$). Then $\mathcal{A}$ is definable over $\mathbb{R}_1$ since A_i is definable over $\mathbb{R}_1$ (see the remark following the definition 4). Besides $\dim(\mathcal{A} \cap \{(y_1, \dots, y_n, y_{1,1}, \dots, y_{1,n}, \dots, y_{i,1}, \dots, y_{i,n}) = (\gamma_1, \dots, \gamma_{(i+1)n})\}) = \dim(A_i \cap \Pi) \leq i-1$ by the supposition. According to the definition A10, this means that $[(\delta_1^{(j)}, \dots, \delta_i^{(j)}) : (\gamma_1, \dots, \gamma_{(i+1)n})]_{\mathbb{R}_1} \leq i-1$, for each $1 \leq j \leq n^2+1$ since $w^{(j)} \in A_i \cap \Pi$. Applying several times lemma A10 proceeding by induction on j, and taking into account that $[(\delta_1^{(j)}, \dots, \delta_i^{(j)}) : (\gamma_1, \dots, \gamma_{(i+1)n}, \delta_1^{(1)}, \dots, \delta_i^{(1)}, \dots, \delta_1^{(j-1)}, \dots, \delta_i^{(j-1)})]_{\mathbb{R}_1} \leq [(\delta_1^{(j)}, \dots, \delta_i^{(j)}) : (\gamma_1, \dots, \gamma_{(i+1)n})]_{\mathbb{R}_1}$ we obtain the inequality $[\gamma_1, \dots, \gamma_{(i+1)n}, \delta_1^{(1)}, \dots, \delta_i^{(1)}, \dots, \delta_1^{(j)}, \dots, \delta_i^{(j)}]_{\mathbb{R}_1} \leq n^2 + j(i-1)$ for each $0 \leq j \leq n^2+1$.

Putting $j = n^2+1$ leads to a contradiction since $[\gamma_1, \dots, \gamma_{(i+1)n}, \delta_1^{(1)}, \dots, \delta_i^{(1)}, \dots, \delta_1^{(n^2+1)}, \dots, \delta_i^{(n^2+1)}]_{\mathbb{R}_1} \geq [\delta_1^{(1)}, \dots, \delta_i^{(1)}, \dots, \delta_1^{(n^2+1)}, \dots, \delta_i^{(n^2+1)}]_{\mathbb{R}_1} = i(n^2+1)$ because of lemma A9, 2). $\Box$

Definition 6. A point $y \in \widetilde{A}_i$ is called i-pseudoflat if there exist the points $v_1, \dots, v_i \in \widetilde{A}_i$ such that $|\det(v_1 - y, \dots, v_i - y)^T(v_1 - y, \dots, v_i - y)| > \epsilon_1$ (where $(v_1 - y_1, \dots, v_i - y)^T$ denotes the transposition of $n \times i$ matrix with the columns $v_1 - y, \dots, v_i - y$) and the points $y + \sum_{1 \leq \ell \leq i} \delta_\ell^{(j)}(v_\ell - y) \in \widetilde{A}_i$, $1 \leq j \leq n^2+1$.

The sub-Pfaffian set of all i-pseudoflat points denote by $\widetilde{\phi}_i$.

Lemma 8. *If* $\dim(W \cap P_i) = i$ *then* $W \cap P_i \cap \mathbb{R}^n \subset \widetilde{\phi}_i$.

Proof. Let $\widetilde{x} \in W \cap P_i \cap \mathbb{R}^n$. Take arbitrary points $v_1, \dots, v_i \in W \cap P_i \cap \mathbb{R}^n$ such that the vectors $v_1 - \widetilde{x}, \dots, v_i - \widetilde{x}$ are linearly independent, then

$$\mathbb{R} \ni |\det(v_1 - \widetilde{x}, \dots, v_i - \widetilde{x})^T(v_1 - \widetilde{x}, \dots, v_i - \widetilde{x})| > 0,$$

obviously

$$|\det(v_1 - \widetilde{x}, \dots, v_i - \widetilde{x})^T (v_1 - \widetilde{x}, \dots, v_i - \widetilde{x})| > \epsilon_1.$$

The distance from a point $\widetilde{w}^{(j)} = \widetilde{x} + \sum_{1 \leq \ell \leq i} \delta_\ell^{(j)} (v_\ell - \widetilde{x}) \in \mathbb{R}_\delta^n$ to $\widetilde{x}$ is infinitesimal relative to $\mathbb{R}_1$ for each $1 \leq j \leq n^2 + 1$. Lemma 3 implies that $\widetilde{w}^{(j)} \in \widetilde{A}_i$, $1 \leq j \leq n^2 + 1$, hence $\widetilde{x} \in \widetilde{\phi}_i$ by definition 6. $\square$

Lemma 9. $st(\widetilde{\phi}_i) \subset \phi_i$

Proof. Let $\widetilde{y} \in \widetilde{\phi}_i$ and $v_1, \dots, v_i \in \widetilde{A}_i$ satisfy definition 6. Observe that $|\det(st(v_1) - st(\widetilde{y}), \dots, st(v_i) - st(\widetilde{y}))^T (st(v_1) - st(\widetilde{y}), \dots, st(v_i) - st(\widetilde{y}))| > \epsilon_1/2$, taking into account lemma A4 and that the points $\widetilde{y}, v_1, \dots, v_i \in \widetilde{A}_i \subset W \subset P$ are $\mathbb{R}$-finite (see section A2). Furthermore, $st(\widetilde{y}) + \sum_{1 \leq \ell \leq i} \delta_\ell^{(j)} (st(v_\ell) - st(\widetilde{y})) \in st(\widetilde{A}_i) = A_i$, $1 \leq j \leq n^2 + 1$. Denote by Π the unique i-plane passing through the points $st(\widetilde{y}), st(v_1), \dots, st(v_i)$. Lemma 7 entails that $st(\widetilde{y}) \in \phi_i$ and $\dim(\Pi \cap A_i) = i$. $\square$

Let $\widetilde{\phi}_i = \bigcup_j \widetilde{\varphi}_j$, $\phi_i = \bigcup_j \varphi_\ell$ be the representations of $\widetilde{\phi}_i$ and ϕ_i, respectively, as the unions of (necessarily sub-Pfaffian, see the section A3) connected components. Lemmas A6, A7 imply that $st(\widetilde{\varphi}_j)$ is a sub-Pfaffian connected set. Hence due to lemma 9, for each j there is ℓ such that $st(\widetilde{\varphi}_j) \subset \varphi_\ell$. For any i-facet P_i of P such that $\dim(W \cap P_i \cap \mathbb{R}^n) = i$, lemma 8 entails that $W \cap P_i \cap \mathbb{R}^n \subset \widetilde{\phi}_i$. Take a point $x \in W \cap P_i \cap \mathbb{R}^n$, then $x \in \widetilde{\varphi}_j$ for a certain j. It follows that $st(\widetilde{\varphi}_j) \subset \varphi_\ell$ for a suitable ℓ, thus $x = st(x) \in st(\widetilde{\varphi}_j) \subset \varphi_\ell$. Due to lemma 6, $\varphi_\ell \subset P_i$. So, to any facet P_i such that $\dim(W \cap P_i \cap \mathbb{R}^n) = i$, corresponds (not necessary unique) connected component $\widetilde{\varphi}_j$, and to different such i-facets P_i, P_i' correspond different connected components, respectively. Thus, we obtain the following lemma.

Lemma 10. *The number of i-facets P_i such that $\dim(W \cap P_i \cap \mathbb{R}^n) = i$, does not exceed the number of connected components of $\widetilde{\phi}_i$.*

Observe that $\widetilde{\phi}_i$ can be defined by a Pfaffian formula ψ having a prefix with only existential quantifiers. Moreover, the prefix contains $O(n^4)$ quantifiers, since for each of $O(n^2)$ points $v_1, \ldots, v_i$, $y + \sum_{1 \le \ell \le i} \delta_\ell^{(j)}(v_\ell - y)$, $1 \le j \le n^2 + 1$, the formula ψ expresses the condition of membership to the set $\widetilde{A}_i$ (see definition 6), which, in its turn, requires $O(n^2)$ existential quantifiers (see definitions 2, 3), namely for the coordinates of the points $y_1, \ldots, y_n$. The polynomials, occurring in ψ, and the polynomials of the type $g_{v,j}$, occurring in the definition of Pfaffian functions $u_0, \ldots, u_K$ (see the beginning of the section 1), have the degrees less than $O(dn)$ (cf. (0)). The number of all these polynomials (i.e., the number of atomic subformulas of ψ) can be bounded by $n^{O(1)}K$ (see lemma 1 and definitions 2, 3). Therefore, the number of all connected components of the sub-Pfaffian set $\widetilde{\phi}_i$ does not exceed $2^{K^2}(dn\,K)^{O(K+n^4)}$, due to corollary A2. Together with lemma 10 this implies the following lemma.

Lemma 11. *The number of i-facets P_i such that* $\dim(W \cap P_i \cap \mathbb{R}^n) = i$, *does not exceed* $2^{K^2}(dn\,K)^{O(K+n^4)}$.

In order to complete the proof of the theorem one observes that the Pfaffian computation tree $\mathcal{T}$ contains at most 3^K branches and for each $0 \le i \le n-1$ for each i-facet P_i there is a branch of $\mathcal{T}$ such that $\dim(W' \cap P_i \cap \mathbb{R}^n) = i$ where W' is the accepting set corresponding to this branch. Hence $N \le 3^K 2^{K^2}(dn\,K)^{O(K+n^4)}$. Together with the assumption $N \ge (dn)^{\Omega(n^4 \log d)}$, this entails the inequality $K \ge \Omega(\sqrt{\log N})$. $\square$

APPENDIX. Sub-Pfaffian sets

A1. Gabrielov's theorem and Khovanskii's bound.

In this section we give definitions and describe some properties of concepts related to Pfaffian functions and to the subsets of $\mathbb{R}^n$ defined by these functions. We skip all the proofs which could be found elsewhere.

The concept of Pfaffian function was introduced by Khovanskii [19, 20], who had established their fundamental properties.

Definition A1. *A subset $A \subset \mathbb{C}^n$ is called complex analytic variety if any point of $\mathbb{C}^n$ has a neighbourhood U such that the intersection $A \cap U$ coincides with the set $\{g_i = \cdots = g_k = 0\} \cap U$ where $g_i, \dots, g_k$ are complex analytic (holomorphic) functions on U (see e.g.* [21]*).*

We say that a real analytic function f has a domain $G \subset \mathbb{R}^n$, if there is an open subset $\mathcal{G} \subset \mathbb{R}^n$ such that f is defined on $\mathcal{G}$ and $G \subset \mathcal{G}$.

Definition A2. *(a) A Pfaffian chain of the length r and degree $d_1 \geqq 1$ is a sequence of real analytic functions $f_1, \dots, f_r$ with the following properties.*

1. For each $1 \leqq j \leqq r$ there exists a complex analytic function $\tilde{f}_j$ defined in a subset $\widetilde{G}_j \subset \mathbb{C}^n$, such that $\mathbb{C}^n \setminus \widetilde{G}_j$ is a complex analytic variety , and f_j is the restriction of $\tilde{f}_j$ on $\mathbb{R}^n$.

   Observe that as real analytic function f_j has a domain $G_j \subset \widetilde{G}_j \cap \mathbb{R}^n$. Let $\widetilde{G} = \bigcap_{1 \leqq j \leqq r} \widetilde{G}_j$ and $G = \bigcap_{1 \leqq j \leqq r} G_j$.
2. Every f_j, $1 \leqq j \leqq r$ satisfies a Pfaffian equation
$$df_j(X) = \sum_{1 \leqq i \leqq n} g_{ij}(X, f_1(X), \dots, f_j(X)) dX_i$$
   for $1 \leqq j \leqq r$. Here $X = (X_1, \dots, X_n)$, $g_{ij} \in \mathbb{R}[X, Y_1, \dots, Y_j]$, $\deg_{X, Y_1, \dots, Y_j}(g_{ij}) \leqq d_1$.

(b) A function $f(X) = P(X, f_1(X), \dots, f_r(X))$, where $P \in \mathbb{R}[X, Y_1, \dots, Y_r]$, $\deg_{X, Y_1, \dots, Y_r}(P) \leqq d_2$ is called a Pfaffian function (with a Pfaffian chain $f_1, \dots , f_r$) of length r and degree $d = d_1 + d_2$.

Note that our definition of a Pfaffian function is more restrictive than a usual one (see [19, 20]) due to the requirement of existence of $\tilde{f}_j$.

Examples. (the exposition follows [8])

(1) Pfaffian function of the length 0 and degree $d+1$ are polynomials of degree not exceeding d.

(2) The exponential function $f(X) = e^{aX}$ is Pfaffian of the length 1 and degree 2, with $\widetilde{G} = \mathbb{C}$, $G = \mathbb{R}$, due to the equation

$$df(X) = af(X)dX.$$

(3) The function $f(X) = 1/X$ is Pfaffian of the length 1 and degree 3 with $\widetilde{G} = \{X \neq 0\} \subset \mathbb{C}$, $G = \{X \neq 0\} \subset \mathbb{R}$, due to the equation $df(X) = -f^2(X)dX$.

(4) Logarithm $f(X) = ln(X)$ is Pfaffian of length 2 and degree 3 with $\widetilde{G} = \{X \neq 0\} \subset \mathbb{C}$, $G = \{X > 0\} \subset \mathbb{R}$,

$$df(X) = g(X)dX, \quad dg(X) = -g^2(X)dX$$

where $g(X) = 1/X$.

(5) Tangent $f(X) = \tan(X)$ is Pfaffian of the length 1 and degree 3 with

$$\widetilde{G} = \bigcap_{k \in \mathbb{Z}} \{X \neq \frac{\pi}{2} + k\pi\} \subset \mathbb{C}, \quad G = \widetilde{G} \cap \mathbb{R},$$

due to the equation $df(X) = (1 + f^2(X))dX$.

(6) Cosine $cos(X)$ is Pfaffian of the length 2 and degree 3 with

$$\widetilde{G} = \bigcap_{k \in \mathbb{Z}} \{X \neq \pi + 2k\pi\} \subset \mathbb{C}, \quad G = \widetilde{G} \cap \mathbb{R},$$

due to the equations

$$\cos(X) = 2f(X) - 1, \;\; df(X) = -f(X)g(X)dX, \;\; dg(X) = 1/2(1 + g^2(X))dX,$$

where $f(X) = \cos^2(X/2)$ and $g(X) = \tan(X/2)$.

(7) Sine $f(X) = \sin(X)$ is Pfaffian of the length 3 and degree 3 in

$$\widetilde{G} = \bigcap_{k \in \mathbb{Z}} \{X \neq \pi + 2k\pi\} \subset \mathbb{C}, \quad G = \widetilde{G} \cap \mathbb{R},$$

due to the equations $df = g(X)dx$ where $g(X) = \cos(X)$.

Let us now list some elementary properties of Pfaffian functions, describing the behaviour of their parameters under the basic operations (proofs are simple, see e.g. [8]).

(1) The sum and the product of two Pfaffian functions f_1 and f_2 of length r_1 and r_2, degrees d_1 and d_2, with $\widetilde{G} = \widetilde{H}_1$, and $\widetilde{G} = \widetilde{H}_2$, $G = H_1, G = H_2$ respectively, are Pfaffian functions of the lengths $r_1 + r_2$, degree $d_1 + d_1$ and with $\widetilde{G} = \widetilde{H}_1 \cap \widetilde{H}_2$, $G = H_1 \cap H_2$ for both the sum and the product. If two Pfaffian functions are defined by the same Pfaffian chain of the length r, the length of the sum and the product is also r.

(2) A partial derivative of a Pfaffian function of the length r and the degree d is a Pfaffian function of the length r and degree $2d$.

(3) Let $X = (X_1, \dots, X_n)$, $Z = (Z_1, \dots, Z_\ell)$ be tuples of variables and f be a Pfaffian function in X, Z of the length r_1, degree d_1 and with $\widetilde{G} = \widetilde{H}_1 \subset \mathbb{C}^{n+\ell}$, $G = H_1 \subset \mathbb{R}^{n+\ell}$.

Let $h = (h_1, \dots, h_\ell)$ be an ℓ-tuple of Pfaffian functions in X of length r_2, degree d_2, with a common Pfaffian chain, with $\widetilde{G} = \widetilde{H}_2 \subset \mathbb{C}^n$, $G = H_2 \subset \mathbb{R}^n$, such that $(x, h(x)) \in H_1$ for all $x \in H_2$. Then the complex analytic function $\tilde{g} \equiv \tilde{f}(X, \tilde{h}(X))$ (see (a), 1 of the Definition A2) is defined in a subset $\widetilde{H}_3 \subset \mathbb{C}^n$ such that $\mathbb{C}^n \setminus \widetilde{H}_3$ is a complex analytic variety of a dimension smaller than n. Indeed, the preimage of the complex analytic variety $\mathbb{C}^{n+\ell} \setminus \widetilde{H}_1$ in $\mathbb{C}^n \setminus \widetilde{H}_2$, under the map $\tilde{h}$, is also a complex analytic variety different from $\mathbb{C}^n$ since $\tilde{g}$ is a composition of analytic functions. Therefore, the dimension of this preimage is less than n (see [21]). An easy computation (see [8]) shows that $g \equiv f(X, h(X))$ is a Pfaffian function in G_2 of the length $r_1 + r_2$ and degree $d_1 d_2$.

Lemma A1. *Let f be a Pfaffian function with $G \subset \mathbb{R}^n$ and $L \subset \mathbb{R}^n$ be a p-plane. If there exist $x \in G \cap L$ and r, $0 < r \in \mathbb{R}$ such that f vanishes in the intersection $L \cap B_x(r)$ then f vanishes in $G \cap L$ (here $B_x(r)$ denotes an open n-dimensional ball centered at x with radius r).*

Proof. Consider complex analytic function $\tilde{f}$ corresponding to f as in the Definition A2, and the complex p-plane $\tilde{L}$, defined in $\mathbb{C}^n$ by the same system of linear equations as L. Since $\tilde{L}$ is an irreducible complex analytic variety, either it is contained in the variety $\mathbb{C}^n \setminus \widetilde{G}$ or the complex dimension $\dim_{\mathbb{C}}(\tilde{L} \cap (\mathbb{C}^n \setminus \widetilde{G})) < \dim_{\mathbb{C}}(\tilde{L})$ (by the dimension of intersection theorem, see [21]). The first alternative is impossible because $x \in L \subset \tilde{L}$. Since $\dim(L \cap B_x(r)) = p$, the second alternative implies that the complex analytic function $\tilde{f}$ is defined on p-plane $\tilde{L}$ everywhere except a subset $\tilde{L} \setminus \widetilde{G}$ of a dimension less than p, and vanishes on a subset of complex dimension p. Since $\widetilde{G} \cap \tilde{L}$ is connected in the topology with the base of all open balls of $\tilde{L}$, treated as $2p$-dimensional real space, we conclude that $\tilde{f}$ vanishes on $\tilde{L} \cap \widetilde{G}$. Hence $\tilde{f}$ is identically zero on $\tilde{L}$. It follows that the restriction f of $\tilde{f}$ vanishes on $G \cap L \subset \mathbb{R}^n$. $\square$

Next we define by induction two closely linked notions: quantifier-free Pfaffian formula, semi-Pfaffian set. Again our definitions will be more restrictive than the original ones (see [19, 20, 7]).

Definition A3. Let h_0 be a Pfaffian chain of length 1, with h_0 defined in $\mathbb{R}^n$. A quantifier-free formula of rank 0 is an expression of the form

$$\Phi^{(0)} = \bigvee_{1 \leqq i \leqq s_0} (f_{i1}^{(0)} = \cdots = f_{ik_i^{(0)}}^{(0)} = 0 \ \& \ g_{i1}^{(0)} > 0 \ \& \ \cdots \ \& \ g_{i\ell_i^{(0)}}^{(0)} > 0), \qquad (1)$$

where $f_{ij}^{(0)}, g_{ij}^{(0)}$ are Pfaffian functions (called atomic functions), with h_0 as a common Pfaffian chain (see Definition A2(b)), thus, in particular, $f_{ij}^{(0)}, g_{ij}^{(0)}$ are defined in $\mathbb{R}^n$. Suppose that we had already defined a concept of a quantifier-free Pfaffian formula $\chi^{(\ell)}$ of rank ℓ, $0 \leqq \ell \in \mathbb{Z}$. A semi-Pfaffian set $W \subset \mathbb{R}^n$, determined by

$\chi^{(\ell)}$ is the set of all points $x \in \mathbb{R}^n$, satisfying $\chi^{(\ell)}$. We write $W = \{\chi^{(\ell)}\}$. A quantifier-free Pfaffian formula of the rank $\ell + 1$ is of the form

$$\Phi^{(\ell+1)} = \bigvee_{1 \leqq i \leqq s_{\ell+1}} (f_{i1}^{(\ell+1)} = \cdots = f_{ik_i^{(\ell+1)}}^{(\ell+1)} = 0 \ \& \ g_{i1}^{(\ell+1)} > 0 \ \& \ \cdots \ \& \ g_{i\ell_i^{(\ell+1)}}^{(\ell+1)} > 0),$$

where $f_{ij}^{(\ell+1)}, g_{ij}^{(\ell+1)}$ are Pfaffian functions with the common Pfaffian chain $h_0, \ldots, h_{\ell+1}$. Here the function $h_{\ell+1}$ is defined in a domain G which is a closure of a semi-Pfaffian set of the kind $\{\chi^{(\ell)}\}$, where $\chi^{(\ell)}$ is a quantifier-free Pfaffian formula of the rank ℓ. Functions $f_{ij}^{(\ell+1)}, g_{ij}^{(\ell+1)}$, together with all atomic functions occuring in the description of $\chi^{(\ell)}$ are called atomic functions of $\Phi^{(\ell+1)}$.

Example.

The set $\{\tan(X) = 0 \ \& \ a < X < b\} \subset \mathbb{R}$, where $-\pi/2 < a < b < \pi/2$, is semi-Pfaffian, defined by a quantifier-free Pfaffian formula. On the other hand, the set $\{\tan(X) = 0\} \cap \bigcup_{k \in \mathbb{Z}} \{a + k\pi < X < b + k\pi\} \subset \mathbb{R}$ for $-\pi/2 < a < b < \pi/2$ (cf. Example (5) above) is not semi-Pfaffian.

Definition A4. Fix a certain R, $0 < R \in \mathbb{R}$ and let $\mathcal{K}^n \subset \mathbb{R}^n$ be the n-dimensional cube centered at the origin and having an edge with length $2R$. A Pfaffian formula is an expression of the form $\psi = Q_1 Y_1 Q_2 Y_2 \ldots Q_t Y_t (\Phi)$ where Φ is a quantifier-free Pfaffian formula of arbitrary rank (called quantifier-free part of ψ) with atomic functions in $n + t$ variables $Y_1, \ldots, Y_t, X_1, \ldots, X_n$ and Q_j, $1 \leqq j \leqq t$ are quantifiers $\exists$ or $\forall$, each restricted on the interval $(-R, R) \subset \mathbb{R}$. A sub-Pfaffian set $V \subset \mathbb{R}^n$, determined by ψ, is the set of all points $x \in \mathcal{K}^n$, satisfying ψ. We write $V = \{\psi\}$.

We say that two Pfaffian formulas ψ, ψ' are equivalent if $\{\psi\} = \{\psi'\}$.

Definition A5. The local dimension $dim_x(V)$ of a set V at a point $x \in V$ is the maximal p, $0 \leqq p \in \mathbb{Z}$ such that the linear projection of a neighbourhood of x in V onto a coordinate p-subspace (along all the rest of coordinates) contains a p-dimensional ball. The dimension $\dim(V)$ of V is the maximal value $\dim_x(V)$ for all $x \in V$.

Definition A6. A point x of a set $V \subset \mathbb{R}^n$, with $\dim(V) = p$, is called analytically nonsingular (or nonsingular) if a neighbourhood of x in V is analytically diffeomorphic (respectively, C^1-diffeomorphic) to an open p-dimensional ball. Denote by V_a^0 (or, by V^0) the set of all analytically nonsingular (respectively, nonsingular) points of V. The points of the set $V_a^* = V \setminus V_a^0$ (respectively $V^* = V \setminus V^0$) are called analytically singular (respectively, singular).

For a set $V \subset \mathbb{R}^n$ denote by $cl(V)$ its closure in the topology with a base of all open balls in $\mathbb{R}^n$.

Definition A7. For a set $V \subset \mathbb{R}^n$ the disjoint family $\{V_i\}$ of subsets $V_i \subset V$ is called a smooth stratification of V if

1. $V = \bigcup\limits_i V_i$
2. each V_i, called a stratum, is an analytic manifold in $\mathbb{R}^n$
3. if $V_i \cap cl(V_j) \neq \emptyset$, then $V_i \subset cl(V_j)$ and $\dim(V_i) < \dim V_j$.

Proposition A1. ([16, 26]) *For any sub-Pfaffian set $V \subset \mathbb{R}^n$ there exists a finite smooth stratification.*

Corollary A1. $\dim(V^*) < \dim(V)$.

Proof. The inequality $\dim(V_a^*) < \dim(V)$ directly follows from Proposition A1, the inequality $\dim(V^*) \leqq \dim(V_a^*)$ follows from the obvious inclusion $V^* \subset V_a^*$. □

Lemma A2. *For a sub-Pfaffian set $V \subset \mathbb{R}^n$ the subsets V^0 and V^* are sub-Pfaffian.*

Proof. The sets V^0 and V^* can be described by appropriate Pfaffian formulas involving a Pfaffian formula defining V. □

Definition A8. For a set $V \subset \mathbb{R}^n$ the boundary ∂V is a subset of all points $x \in \mathbb{R}^n$ such that for every r, $0 < r \in \mathbb{R}$, the intersections $B_x(r) \cap V \neq \emptyset$, $B_x(r) \cap (\mathbb{R}^n \setminus V) \neq \emptyset$.

Lemma A3. *For a sub-Pfaffian set $V \subset \mathbb{R}^n$ the dimension* $\dim(\partial V) \leqq n-1$.

Proof. Let $\{V_i\}$ be a finite smooth stratification of V, see Proposition A1. Suppose first that $\dim(V) < n$. Then, the closure $cl(V) = \bigcup_i cl(V_i) = \bigcup_i \partial V_i = \partial V$. On the other hand, $\dim(cl(V)) = \dim(V)$ [7], hence the lemma is valid in this case.

Now let $\dim(V) = n$. The set V is representable as $V = V_{\max} \cup V_{\min}$, where $V_{\max}$ is the union of all n-dimensional strata of V, and $V_{\min}$ is the union of the remaining strata (of smaller dimensions). Then

$$\dim(\partial V) \leqq \dim(\partial V_{\max} \cup \partial V_{\min}) = \dim((cl(V_{\max}) \setminus V_{\max}) \cup \partial V_{\min})$$
$$= \max\{\dim(cl(V_{\max}) \setminus V_{\max}),\ \dim(\partial V_{\min})\}.$$

According to [7], $\dim(cl(V_{\max}) \setminus V_{\max}) < \dim(V_{\max})$. The inequality $\dim(\partial V_{\min}) \leqq n-1$ was proved before. $\square$

Definition A9. Let ψ be a Pfaffian formula having N atomic Pfaffian functions in n variables with the same Pfaffian chain of length r and degrees less than d. The 4-tuple (N, n, r, d) is called the format of ψ.

Proposition A2. *([7], theorem 2) For a Pfaffian formula ψ of a format (N, n, r, d) there exists an equivalent formula ψ' having only existential quantifiers, and of the format (N', n', r', d'), where the values N', n', r', d' are bounded from above by the value of a suitable function in N, n, r, d.*

Proposition A3. *([7], theorem 1) For a sub-Pfaffian set $\{\psi\} \subset \mathbb{R}^n$ with a Pfaffian formula ψ of a format (N, n, r, d), any of its connected components can be defined by a Pfaffian formula of a format (N', n', r', d'), where the values N', n', r', d' are bounded from above by the value of an appropriate function in N, n, r, d.*

Proposition A4. *([19, 20]) The number of the connected components of a semi-Pfaffian set $\{\Phi\}$ defined by a quantifier-free formula Φ of the format (N, n, r, d) does not exceed $2^{r^2} n^{O(r)} (Nd)^{O(r+n)}$.*

There is a generally adopted conjecture that under the hypothesis of the Proposition A4 the bound $n^{O(r)}(Nd)^{O(r+n)}$ is actually true.

Corollary A2. *The number of the connected components of a sub-Pfaffian set $\{\psi\}$, defined by a formula ψ of the format (N, n, r, d) in which only existential quantifiers can occur, does not exceed $2^{r^2}n^{O(r)}(Nd)^{O(r+n)}$.*

Proof. It is sufficient to note that the number of the connected components of a projection of a set does not exceed the number of the connected components of the set itself. □

Corollary A3. *The number of the connected components of an arbitrary sub-Pfaffian set $\{\psi\}$, defined by a formula ψ of a format (N, n, r, d) is finite, moreover, is bounded from above by the value of a certain function in N, n, r, d.*

Proof. Apply to ψ successively the Proposition A3 and the Corollary A2. □

Corollary A4. *Zero-dimensional sub-Pfaffian set in $\mathbb{R}^n$ is finite. A sub-Pfaffian set in $\mathbb{R}^1$ is a finite union of points and (open, closed or semi-closed) intervals. In each case the number of the points or the intervals is bounded from above by the value of a certain function in the format of a formula representing the sub-Pfaffian set.*

Proof. Directly follows from Lemma A2 and Corollary A3. □

A2. Sub-Pfaffian sets over nonstandard extensions of reals.

In the main text of the paper we consider the extensions of the field $\mathbb{R}$ with "nonstandard" (in particular infinitesimal) elements. The following digest from nonstandard analysis is taken from [27] , for a detailed exposition see [6].

There exists a sequence of ordered fields

$$\mathbb{R}_0 = \mathbb{R} \subset \mathbb{R}_1 \subset \mathbb{R}_2 \subset \cdots \subset \mathbb{R}_k \subset \dots$$

in which the field $\mathbb{R}_k$, $k \geqq 1$ contains an element $\varepsilon_k > 0$ infinitesimal relative to the elements of $\mathbb{R}_{k-1}$ (i.e., for every positive element $a \in \mathbb{R}_{k-1}$ the inequaltiy $\varepsilon_k < a$

is true). In addition, for every function $\varphi : \mathbb{R}^n_{k-1} \longrightarrow \mathbb{R}_{k-1}$ there exists a natural extension, being a function φ from $\mathbb{R}^n_k$ to $\mathbb{R}_k$. It follows, invoking characteristic functions, that each subset $S \subset \mathbb{R}^n_{k-1}$ has a natural extension to $\mathbb{R}^n_k$. We say that $\mathbb{R}_j$ is a nonstandard extension of $\mathbb{R}_i$ for $0 \leqq i < j$.

Consider the language $\mathcal{L}_k$, $k \geqq 0$ of the first order predicate calculus, in which the set of all function symbols is in a bijective correspondence with the set of all functions of several arguments from $\mathbb{R}_k$ taking values in $\mathbb{R}_k$ and the only predicate is the equality relation. We shall say that the closed (i.e., containing no free variables) formula Φ of the language $\mathcal{L}_k$ is true in $\mathbb{R}_k$, $k \geqq 0$, if and only if the statement expressed by this formula with respect to $\mathbb{R}_k$ is true. The following "transfer principle" is valid: for all integers $0 \leqq i < j$ the closed formula Φ of $\mathcal{L}_i$ is true in $\mathbb{R}_i$ if and only if it is true in $\mathbb{R}_j$.

An element $z \in \mathbb{R}_k$, $k \geqq 1$ is called infinitesimal relative to $\mathbb{R}_j$, $j < k$, if for every $0 < w \in \mathbb{R}_j$ the inequality $|z| < w$ is valid. An element $z \in \mathbb{R}_k$ is called infinitely large, if $z = 1/z'$, where z' is infinitesimal. If $z \in \mathbb{R}_k$ is not infinitely large relative to $\mathbb{R}_j$, z is called $\mathbb{R}_j$-finite.

One can prove [6] that if an element $z \in \mathbb{R}_k$ is $\mathbb{R}_j$-finite then there exist unique elements $z_1 \in \mathbb{R}_j$ and $z_2 \in \mathbb{R}_k$, where z_2 is infinitesimal relative to $\mathbb{R}_j$, such that $z = z_1 + z_2$. In this case z_1 is called the standard part of z (relative to $\mathbb{R}_j$) and is denoted by $z_1 = st_j(z)$. One can extend the operation st_j (componentwise) to vectors from $\mathbb{R}^n_k$ and (elementwise) to subsets of $\mathbb{R}^n_k$.

In what follows, all the functions φ we shall consider in $\mathbb{R}^n_k$, $k \geqq 0$, will be Pfaffian. By this we mean that for each φ there exists a Pfaffian function φ' definable over $\mathbb{R}$ (i.e., in the sense of the Definition A2) such that φ is the result of a replacement of some variables in φ' by some elements of $\mathbb{R}_k$.

Moreover, we assume that the domain $G \subset \mathbb{R}^n_k$ of φ is a sub-Pfaffian set, defined by a Pfaffian formula Π with atomic functions definable over $\mathbb{R}$ and some variables replaced by elements from $\mathbb{R}_k$. We say that φ is definable over $\mathbb{R}_k$.

For any $\ell > k$, the same function φ', formula Π and the replacements determine the function $\varphi^{(\ell)} : G^{(\ell)} \longrightarrow \mathbb{R}_\ell$ which coincides with φ in $\mathbb{R}_k^n$ and is called the completion of φ over $\mathbb{R}_\ell$, similarly $G^{(\ell)} \subset \mathbb{R}_\ell^n$ (determined by Π) is called the completion of G over $\mathbb{R}_\ell$.

Basic notions, introduced in section A1 can be naturally extended to a nonstandard field $\mathbb{R}_k$ for $k > 0$. Thus, we shall consider semi-Pfaffian sets, sub-Pfaffian sets, Pfaffian formulas, determined in $\mathbb{R}_k^n$ by Pfaffian functions definable over $\mathbb{R}_k$. In this case we say that the sets and formulas are definable over $\mathbb{R}_k$.

If a sub-Pfaffian set $W \subset \mathbb{R}_k^n$ is determined in $\mathbb{R}_k^n$ by a Pfaffian formula Φ with atomic subformulas definable over $\mathbb{R}_k$ then the sub-Pfaffian set in $\mathbb{R}_\ell^n$, $\ell > k$ determined by the same formula in which the atomic functions are replaced by their completions is called the completion of W and is denoted by $W^{(\ell)}$.

Some of the basic statements proved earlier in this Appendix can be extended (using the transfer principle) to the fields $\mathbb{R}_k$ for $k > 0$. This obviously concerns the statements: lemma A1, corollary A1, lemma A2, lemma A3, proposition A2, corollary A4. Propositions A3, A4, Corollaries A2, A3, about the estimates of the connected components are also extendable (see below).

The following lemma illustrates a use of the transfer principle and the notion of the standard part.

Lemma A4. *Let $f : S \to \mathbb{R}_k$ be a Pfaffian function defined in a sub-Pfaffian bounded set $S \subset \mathbb{R}_k^n$. Denote by $S^{(k+1)}$ the completion of S over $\mathbb{R}_{k+1}$ and by $f^{(k+1)}$ the completion of f. Then for any point $x \in S^{(k+1)}$ such that $B_x(r) \subset S^{(k+1)}$ for some r, $0 < r \in \mathbb{R}_k$, the standard part $st_k(f^{(k+1)}(x)) = f(st_k(x))$. If in addition, there do not exist $y \in S$ and R, $0 < R \in \mathbb{R}_k$ such that $f(z) = 0$ for all $z \in B_y(R)$, and besides $f(w) \geq 0$ for all $w \in S$, then*

$$st_k(\{f^{(k+1)} = \varepsilon_{k+1}\}) = \{f = 0\}.$$

Proof. First, observe that any Pfaffian function is continuous. This is true for a Pfaffian function φ definable over $\mathbb{R}$ (since φ is analytic, see definition A2), then the

Pfaffian formula of the language $\mathcal{L}_0$ expressing continuity, is valid for the completion $\varphi^{(\ell)}$, $\ell \geq 0$, due to the transfer principle, and hence, it is valid as well for Pfaffian functions definable over arbitrary $\mathbb{R}_\ell$. The equality $st_k(f^{(k+1)}(x) = f(st_k(x))$ and thereby the inclusion $st_k(\{f^{(k+1)} = \varepsilon_{k+1}\}) \subset \{f = 0\}$ follows from continuity of f and $f^{(k+1)}$.

Now let $x \in \{f = 0\}$. Take r, $0 < r \in \mathbb{R}_k$ such that $B_x(r) \subset S$ (cf. definition A2). Consider a sub-Pfaffian set $D = \{\|x - z\|^2 : z \in S^{(k+1)},\ f^{(k+1)}(z) = \varepsilon_{k+1}\} \subset \mathbb{R}_{k+1}$. If it is empty, then $f^{(k+1)}$ is less than ε_{k+1} everywhere on the ball $B_x(r)$, by virtue of the theorem on intermediate values of continuous functions which holds for Pfaffian functions by the transfer principle, hence f vanishes everywhere on the ball $B_x(r) \cap \mathbb{R}^n_k$ and we get a contradiction. Due to the Corollary A4 the set D consists of a finite union of points and intervals. Denote by u the minimum of these points and endpoints of these intervals. If $st_k(u) > 0$ then the function $f^{(k+1)}$ on the ball $B_x(\sqrt{u}) \cap B_x(r)$ takes the values less than ε_{k+1} because of continuity of $f^{(k+1)}$. Therefore, f vanishes everywhere on the ball

$$B_x(\sqrt{u}) \cap B_x(r) \cap \mathbb{R}^n_k \supset B_x(st_k(\sqrt{u})/2) \cap B_x(r) \cap \mathbb{R}^n_k$$

with a positive radius from $\mathbb{R}_k$ (sf. above). The obtained contradiction shows that $st_k(u) = 0$. Take any point w such that $f^{(k+1)}(w) = \varepsilon_{k+1}$ and $\|w - x\|^2 \leq u + \varepsilon_{k+1}$, then $st_k(w) = x$. $\square$

Lemma A5. *Let a sub-Pfaffian set $W \subset \mathbb{R}^n_k$, defined by a Pfaffian formula Π, be finite. Then the completion $W^{(\ell)} \subset \mathbb{R}^n_\ell$, $\ell > k$ of W coincides with W.*

Proof. Let $W = \{x^{(1)}, \dots, x^{(t)}\}$. Then the following formula of the language $\mathcal{L}_k$ is true over $\mathbb{R}_k$:

$$\mathop{\&}_{1 \leqq i \leqq t} \Pi(x^{(i)}) \ \&\ \forall X_1 \cdots \forall X_n \left(\mathop{\&}_{1 \leqq i \leqq t} ((X_1, \dots, X_n) \neq x^{(i)}) \Rightarrow \neg \Pi(X_1, \dots, X_n) \right).$$

By the transfer principle, this formula is also true over $\mathbb{R}_\ell$. $\square$

For a Pfaffian function $f : G \longrightarrow \mathbb{R}_k$, $G \subset \mathbb{R}_k^n$ a point $x \in G$ is called the critical point of f if the gradient vector $\left(\frac{\partial f}{\partial X_1}, \dots, \frac{\partial f}{\partial X_n}\right)(x) = 0$. The value $f(x)$ is called, in this case, the critical value of f. The value which is not critical is called regular.

Corollary A5. *For a Pfaffian function f definable over $\mathbb{R}_k$, any element $\alpha \in \mathbb{R}_\ell \setminus \mathbb{R}_k$ for $\ell > k$ cannot be a critical value of f.*

Proof. Observe that the set $\Gamma_k \subset \mathbb{R}_k$ of all critical values of f is sub-Pfaffian and definable over $\mathbb{R}_k$.

Suppose first that $k = 0$. Then Corollary A4 implies that Γ_0 consists of a finite number of points and segments. Moreover, by Sard's theorem, Γ_0 actually consists of a finite number of points. For all sub-Pfaffian sets of the form Γ_0 and having a fixed format the latter statement can be expressed by a formula of the language $\mathcal{L}_0$ (taking into the account that the number of points is bounded via the format). Hence, by the transfer principle the statement is true for any $k \geqq 0$, i.e., Γ_k is finite.

According to lemma A5, the completion $\Gamma_k^{(\ell)} = \Gamma_k \subset \mathbb{R}_k$, and, therefore $\alpha \notin \Gamma_k^{(\ell)}$. $\square$

Corollary A6. *Let a Pfaffian function*

$$f : G \longrightarrow \mathbb{R}_k, \quad G \subset \mathbb{R}_k$$

be definable over $\mathbb{R}_k$ and $f \not\equiv 0$ on G. If $\alpha \in \mathbb{R}_\ell \setminus \mathbb{R}_k$ for $\ell > k$ then $f(\alpha) \neq 0$.

Proof. According to lemma A1 and corollary A4, the set W of roots of f is finite. Apply lemma A5 to W. $\square$

A3. Connected components of sub-Pfaffian sets over non-standard fields.

Now we are going to extend the notion of the connected component to the sub-Pfaffian sets definable over $\mathbb{R}_k^n$, $k \geqq 1$. Observe that a direct way to do this, starting with the topology on $\mathbb{R}_k^n$ with the base of all open balls, would lead to unnatural objects, e.g., the segment $[0, 1] \subset \mathbb{R}_k$ is not connected in this topology. The analogous construction of connected components for semialgebraic sets over non-standard fields was described in [14].

Let $V = \{\Pi\}$ be a sub-Pfaffian set in $\mathbb{R}^n$ determined by a Pfaffian formula Π. The Proposition A3 and the Corollary A3 imply the existence of a function $\omega : \mathbb{N} \longrightarrow \mathbb{N}$ such that if the elements of the 4-tuple format of Π are bounded from above by some $\mathcal{N} \in \mathbb{N}$, then:

1. The number of the connected components does not exceed $\omega(\mathcal{N})$;
2. For each connected component V_i of V there exists a Pfaffian formula Π_i of a format with components not exceeding $\omega(\mathcal{N})$, such that $V_i = \{\Pi_i\}$.

It follows that for a given positive integer $\mathcal{N}$, there exists a Pfaffian formula $\Omega_{\mathcal{N}}$ of the language $\mathcal{L}_0$, expressing the existence of a decomposition of any sub-Pfaffian set $V = \{\Pi\}$ of the format of Π less than $\mathcal{N}$ into its connected components

$$V = \bigcup_i \{\Pi_i\}$$

such that the format of every Π_i, and the number of Π_i, are less than $\omega(\mathcal{N})$. Moreover, the formula $\Omega_{\mathcal{N}}$ states that for each pair of indices $i_1 \neq i_2$ the components $\{\Pi_{i_1}\}$ and $\{\Pi_{i_2}\}$ are "separated", i.e. the following Pfaffian formula of the language $\mathcal{L}_0$ is valid:

$$\forall\, (x \in \{\Pi_{i_1}\}) \exists z > 0 \forall\, (y \in \{\Pi_{i_2}\})(\|x - y\| \geqq z).$$

Besides, the formula $\Omega_{\mathcal{N}}$ claims the connectedness of every component $\{\Pi_i\}$, this means that there do not exist two "separated" sub-Pfaffian subsets of $\{\Pi_i\}$, each determined by a Pfaffian formula with format less than $\omega(\omega(\mathcal{N}))$.

Apart from that, for given positive integers $\mathcal{N}$, $\mathcal{M}$ one can verify a formula $\Omega_{\mathcal{N},\mathcal{M}}$ of language $\mathcal{L}_0$ expressing the following statement. If a sub-Pfaffian set $\{\Pi\}$ (where the format of Π is less than $\mathcal{N}$) can be represented as a union of more than one and less than $\mathcal{M}$ pairwise "separated" sub-Pfaffian sets, each being determined by a Pfaffian formula of $\mathcal{L}_0$ of a format less than $\mathcal{M}$, then $\{\Pi\}$ can be represented as a union of more than one and less than $\omega(\mathcal{N})$ pairwise "separated" connected sub-Pfaffian sets, each being determined by a Pfaffian formula of $\mathcal{L}_0$ of a format less than $\omega(\mathcal{N})$.

Applying the transfer principle to the formulas $\Omega_{\mathcal{N}}$, $\Omega_{\mathcal{N},\mathcal{M}}$ for all positive integers $\mathcal{N}, \mathcal{M}$, we conclude that any sub-Pfaffian set, defined over $\mathbb{R}_k$, $k \geqq 0$, can be uniquely represented as a union of its pairwise "separated" connected components, moreover, each component is sub-Pfaffian and is connected, i.e. cannot be represented as a union of more than one pairwise "separated" sub-Pfaffian sets.

Having defined the connected components of a sub-Pfaffian set definable over $\mathbb{R}_k$, $k \geqq 0$, one can use the transfer principle to extend to this set Propositions A3, A4 and Corollaries A2 and A3.

Lemma A6. *Let $V \subset \mathbb{R}_k^n$, $W \subset \mathbb{R}_{k+t}^n$ be two sub-Pfaffian sets and $V = st_k(W)$. Let*

$$V = \bigcup_m V_m, \qquad W = \bigcup_\ell W_\ell$$

be the decompositions of the sets V, W into their connected components. Then, for every index m there exist such indices $\ell_1, \dots, \ell_s$ that $st(W_{\ell_1} \cup \dots \cup W_{\ell_s}) = V_m$. Moreover, for each ℓ there exists the unique index m such that $st(W_\ell) \subset V_m$.

Proof. Is almost verbatim repetition of the proof of the lemma 1 in [14]. $\square$

For a sub-Pfaffian set $W \subset \mathbb{R}_k^n$, $k \geqq 0$, we denote by $cl(W)$ its closure in the topology in $\mathbb{R}_k^n$ with the base of all open balls.

Lemma A7. *(cf. [25]) Let $W_Y = \{\psi_Y\} \subset \mathbb{R}_k^{n+t}$ be a sub-Pfaffian set determined by a Pfaffian formula ψ_Y in which the atomic Pfaffian functions are in variables $X_1, \dots, X_n$, $Y_1, \dots, Y_t$, $Z_1, \dots, Z_s$, where first $n+t$ variables occur free. Let, for the sequence of fields $\mathbb{R}_k \subset \mathbb{R}_{k+1} \subset \cdots \subset \mathbb{R}_t$, the element ε_{k+i+1} be infinitesimal relative to $\mathbb{R}_{k+i}$ for $0 \leqq i \leqq t-1$. Denote by ψ_ε the Pfaffian formula which is the result of the replacement of Y_ℓ by $\varepsilon_{k+\ell}$ for every $1 \leqq \ell \leqq t$; let $W_\varepsilon = \{\psi_\varepsilon\} \subset \mathbb{R}_{k+t}^n$. Then the set $V = st_k(W_\varepsilon) \subset \mathbb{R}_k^n$ is sub-Pfaffian.*

Proof. It is sufficient, due to Proposition A2, to prove the lemma for the case $\psi_Y = \exists Z_1, \dots \exists Z_s(\Phi_Y)$ with quantifier-free Φ_Y. Observe that $W_\varepsilon = \pi\{\Phi_\varepsilon\}$ where Φ_ε is quantifier-free formula, being the result of the replacement of Y_ℓ by $\varepsilon_{k+\ell}$,

$1 \leqq \ell \leqq t$ in Φ_Y, and π is the linear projection map on the subspace of coordinates $X_1, \dots, X_n$ along the coordinates $Z_1, \dots, Z_s$.

The proof can be conducted by induction on t, in which an ith induction step proves that the set $st_{k+t-i}(W_\varepsilon)$ is sub-Pfaffian. It will be obvious from the formula (4) below that the output of the inductive step, namely, the set $st_{k+t-i}(W_\varepsilon)$, satisfies the requirements for the set W_ε of the lemma, i.e., there exists a sub-Pfaffian set W'_Y, determined by a Pfaffian formula ψ'_Y in variables $X_1, \dots, X_n$, $Y_1, \dots, Y_{t-i}$, $Z'_1, \dots, Z'_{s'}$, where first $n+t-i$ variables occur free, such that $st_{k+t-i}(W_\varepsilon) = \{\psi'_\varepsilon\}$, where ψ'_ε is the result of the replacement of Y_ℓ by $\varepsilon_{k+\ell}$ for every $1 \leqq \ell \leqq t-i$.

Thus, we assume that $t = 1$.

We can identify the sets $\{\Phi_\varepsilon\}$ and $\{\Phi_Y \ \& \ (Y_1 = \varepsilon_{k+1})\}$.

Let us prove that

$$st_k(\{\Phi_Y \ \& \ (Y_1 = \varepsilon_{k+1})\}) = cl(\{\Phi_Y \ \& \ (Y_1 > 0)\}) \cap \{Y_1 = 0\}. \qquad (2)$$

Observe that the right side of the equality (2) is a sub-Pfaffian set.

Let $x \in st_k(\{\Phi_Y \ \& \ (Y_1 = \varepsilon_{k+1})\})$, then there exists $z \in \{\Phi_Y \ \& \ (Y_1 = \varepsilon_{k+1})\}$ such that $x = st_k(z)$. Hence, $x \in \{Y_1 = 0\}$. Suppose that $x \notin cl(\{\Phi_Y \ \& \ (Y_1 > 0)\})$. Then there exists an element r, $0 < r \in \mathbb{R}_k$ such that $B_x(r) \cap \{\Phi_Y \ \& \ (Y_1 > 0)\} = \emptyset$. This contradicts to the inclusion $z \in \{\Phi_Y \ \& \ (Y_1 = \varepsilon_{k+1})\} \subset \{\Phi_Y \ \& \ (Y_1 > 0)\}$.

Suppose now that

$$x \in cl(\{\Phi_Y \ \& \ (Y_1 > 0)\}) \cap \{Y_1 = 0\},$$

i.e. x belongs to the right side of (2).

Let us prove the following claim: for any element R, $0 < R \in \mathbb{R}_k$, there exists an element α, $0 < \alpha \in \mathbb{R}_k$, such that for every β, $0 < \beta \in \mathbb{R}_k$, $\beta < \alpha$ the intersection

$$B_x(R) \cap \{\Phi_y \ \& \ (Y_1 = \beta)\}$$

is nonempty. Indeed, since the set $B_x(R) \cap \{\Phi_Y \ \ \& \ \ (Y_1 > 0)\}$ is sub-Pfaffian, and, thus has a finite number of the connected components (see the considerations preceding the lemma), there exists a connected component U of this set such that $x \in cl(U)$. One can take as α the Y_1-coordinate of any point from U and the claim is proved.

It follows (with a help of the transfer principle) that for every fixed R, $0 < R \in \mathbb{R}_k$ the intersection

$$B_x(R) \cap \{\Phi_Y \ \ \& \ \ (Y_1 = \varepsilon_{k+1})\} \neq \emptyset. \tag{3}$$

Observe that the set $A = \{\|z - x\|^2 : z \in \{\Phi_Y \ \ \& \ \ (Y_1 = \varepsilon_{k+1})\}\} \subset \mathbb{R}_{k+1}$ is sub-Pfaffian. Due to Corollary A4, A is a finite union of points and intervals. Let $w \in \mathbb{R}_{k+1}$ be the minimal among these points and the endpoints of these intervals.

Suppose that $x \notin st_k(\{\Phi_Y \ \ \& \ \ (Y_1 = \varepsilon_{k+1})\})$, i.e. there does not exist $z \in \{\Phi_Y \ \ \& \ \ (Y_1 = \varepsilon_{k+1})\}$ such that $st_k(z) = x$. Thus, $w > r_1^2$ for an element $0 < r_1 \in \mathbb{R}_k$. It follows that $B_x(r_1) \cap \{\Phi_Y \ \ \& \ \ (Y_1 = \varepsilon_{k+1})\} = \emptyset$. This contradicts (3) for $R = r_1$, and the equality (2) is proved.

We have:

$$st_k(W_\varepsilon) = st_k(\pi(\{\Phi_Y \ \ \& \ \ (Y_1 = \varepsilon_{k+1})\})) =$$

$$\pi(st_k(\{\Phi_Y \ \ \& \ \ (Y_1 = \varepsilon_{k+1})\})) = \pi(cl(\{\Phi_Y \ \ \& \ \ (Y_1 > 0)\}) \cap \{Y_1 = 0\}). \tag{4}$$

The latter set is obviously sub-Pfaffian, this proves the lemma. $\square$

Lemma A8. *Let $W \subset \mathbb{R}^n_{k+t}$ be a sub-Pfaffian set, $V = st_k(W) \subset \mathbb{R}^n_k$. Then $\dim(V) \leq \dim(W)$.*

Proof. Suppose the contrary, let $\dim W = \ell - 1$, $\dim(V) \geq \ell$. There exists a linear projection $\pi : \mathbb{R}^n_{k+t} \to \mathbb{R}^\ell_{k+t}$ definable over $\mathbb{R}$ such that $\dim(\pi(W)) = \dim(W), \dim(\pi(V)) = \ell$, here $\pi(V) \subset \mathbb{R}^\ell_k$ (actually "almost any" linear projection satisfies these properties). Using the obvious identity $st_k(\pi(W)) = \pi(st_k(W))$ one can assume without loss of generality that $\dim(W) = n-1$, $\dim(V) = n$. Hence V contains a ball of a certain radius $0 < r \in \mathbb{R}_k$.

Fix some integer M which we'll specify later. Making a suitable affine transformation of the coordinates (definable over $\mathbb{R}_k$), we can assume that the following requirements are fulfilled (cf. lemma 2). The set V contains n-dimensional cube $\mathcal{K}$ with a side $0 < r_1 \in \mathbb{R}_k$, contained in the nonnegative ortant and having the origin as one of its nodes. Moreover, we require that for each $1 \le j \le n$ and a j-plane P being the intersection of any $(n-j)$ hyperplanes of the form $P_s^{(m)} = \{X_s = \frac{m}{M} r_1\}$, $1 \le s \le n$, $0 \le m \le M$, the dimension $\dim(W \cap P) \le j - 1$.

Observe that the hyperplanes $P_s^{(m)}$ divide $\mathcal{K}$ in M^n small cubes with sides r_1/M. Moreover for each $0 \le j \le n$ and each j-plane P the intersection $P \cap \mathcal{K}$ is divided by the same way in M^j j-facets being j-dimensional cubes with sides r_1/M (we assume here that a facet contains its boundary). Note that the boundary of j-facet is the union of $(j-1)$-facets. Denote by ν_j the number of j-facets which have common points with W. Denote by $\mathcal{A}_j$, $0 \le j \le n$ the intersection of the set $W \cap \mathcal{K}$ with the union of all j-planes of the described form. Obviously, $\mathcal{A}_j$ is a sub-Pfaffian set. Denote by α_j the number of connected components of $\mathcal{A}_j$.

We claim that $\nu_j \le 2(n-j+1)\nu_{j-1} + \alpha_j$, $1 \le j \le n$. Indeed, $\nu_j \le \nu_j^{(0)} + \nu_j^{(1)}$, where $\nu_j^{(0)}$ is the number of j-facets $Q^{(0)}$ which have common points with the connected components $C^{(0)}$ of $\mathcal{A}_j$ such that $C^{(0)}$ has no common points with j-facets other than $Q^{(0)}$, and $\nu_j^{(1)}$ is the number of j-facets $Q^{(1)}$ not satisfying this property and $Q^{(1)} \cap W \ne \emptyset$. Obviously, $\nu_j^{(0)} \le \alpha_j$. For j-facet $Q^{(1)}$ take any connected component $C^{(1)}$ of $\mathcal{A}_j$ such that $C^{(1)}$ has common points with some j-facet different from $Q^{(1)}$, then $C^{(1)}$ has a common point with a certain $(j-1)$-facet R from the boundary of $Q^{(1)}$, attach to $Q^{(1)}$ any such $(j-1)$-facet R. Since any $(j-1)$-facet R lies in the boundary of at most $2(n-j+1)$ j-facets, R can be attached to at most $2(n-j+1)$ j-facets. Hence $\nu_j^{(1)} \le 2(n-j+1)\nu_{j-1}$ that proves the claim.

Corollary A3 implies that there exists an integer c which depends only on the format of a Pfaffian formula defining the set W such that the number of connected components of the intersection of $W \cap \mathcal{K}$ with any j-plane does not exceed c. Therefore, $\alpha_j \leq c(M+1)^{n-j}$.

Clearly, $\nu_n = M^n$ since $st_k(W) \supset \mathcal{K}$ (indeed, if some n-facet does not intersect with W then its center does not belong to $st_k(W)$). Using the bound on α_j and the proved above claim we prove by induction on $0 \leq j \leq n-1$ the existence of integers c_j such that $\nu_{n-j} \geq \frac{1}{c_j} M^n$ for large enough arbitrary M.

On the other hand, $\mathcal{A}_1$ consists of a finite number of points (since $\dim(\mathcal{A}_1) = 0$), hence $\nu_0 \leq \alpha_1$, then the proved claim (for $j = 1$) entails $\nu_1 \leq (2n+1)\alpha_1 \leq c' M^{n-1}$ for an appropriate integer c', that leads to a contradiction for large enough $M > c' c_{n-1}$. $\square$

A4. Degree of sub-Pfaffian transcendency.

Let $1 \leq j_1 < j_2$ and the elements $\gamma_1, \dots, \gamma_k,\ \theta_1, \dots, \theta_\ell \in \mathbb{R}_{j_2}$. Denote the coordinates in $\mathbb{R}_{j_2}^{k+\ell}$ by $Y_1, \dots, Y_{k+\ell}$.

Definition A10. The degree of sub-Pfaffian transcendency $[(\theta_1, \dots, \theta_\ell) : (\gamma_1, \dots, \gamma_k)] = [(\theta_1, \dots, \theta_\ell) : (\gamma_1, \dots, \gamma_k)]_{\mathbb{R}_{j_1}}$ is the minimal integer $s \geqq 0$ such that there exists a sub-Pfaffian set $S \subset \mathbb{R}_{j_2}^{k+\ell}$ definable over $\mathbb{R}_{j_1}$ such that $(\gamma_1, \dots, \gamma_k, \theta_1, \dots, \theta_\ell) \in S$ and $\dim(S \cap \{Y_1 = \gamma_1, \dots, Y_k = \gamma_k\}) = s$.

When $k = 0$ we write simply $[\theta_1, \dots, \theta_\ell]$.

Observe that the definition correlates with the usual notion of degree of transcendency of the fields extension $[F(\theta_1, \dots, \theta_\ell, \gamma_1, \dots, \gamma_k) : F(\gamma_1, \dots, \gamma_k)]$ replacing $\mathbb{R}_{j_1}$ by a field F and taking as S an algebraic variety.

Lemma A9. *1)* $[\theta_1, \dots, \theta_{\ell+1}] \leq [\theta_1, \dots, \theta_\ell] + 1$;

2) $[\varepsilon_{j_1+1}, \dots, \varepsilon_{j_2}] = j_2 - j_1$ *(the infinitesimals* ε_j *were introduced in section A2).*

Proof. 1) Let $S \subset \mathbb{R}_{j_2}^{\ell}$ be as in the definition, then the point $(\theta_1, \dots, \theta_{\ell+1})$ belongs to the cylinder $S \times \mathbb{R}_{j_2} \subset \mathbb{R}_{j_2}^{\ell+1}$.

2) Conduct the proof by induction on $(j_2 - j_1)$. The base of induction for $j_2 - j_1 = 0$ is trivial. For the inductive step assume the contrary and let $S \subset \mathbb{R}_{j_2}^{j_2 - j_1}$ be as in

the definition A10 such that $(\varepsilon_{j_1+1}, \dots, \varepsilon_{j_2}) \in S$ and $\dim(S) = s \le j_2 - j_1 - 1$. Let $Y_1, \dots, Y_{j_2-j_1}$ be the coordinates in $\mathbb{R}_{j_2}^{j_2-j_1}$. Consider the sub-Pfaffian set $S_0 = \{y : \dim(\{Y_1 = y\} \cap S) = s\} \subset \mathbb{R}_{j_2}$. Then $\dim(S_0) = 0$, since $\dim(S) = s$. Observe that S_0 is defined over $\mathbb{R}_{j_1}$, hence, due to corollary A4, S_0 consists of a finite number of points all belonging to $\mathbb{R}_{j_1}$. Denote $S_1 = \{Y_1 = \varepsilon_{j_1+1}\} \cap S \subset \{Y_1 = \varepsilon_{j_1+1}\} \simeq \mathbb{R}_{j_2}^{j_2-j_1-1}$. Then $\dim(S_1) \le s - 1$, and one can apply the inductive hypothesis to the set S_1, taking into account that $(\varepsilon_{j_1+2}, \dots, \varepsilon_{j_2}) \in S_1$. $\square$

The following lemma is an analogy of the additivity of the usual degree of transcendency: $[F_3 : F_1] = [F_3 : F_2] + [F_2 : F_1]$ for fields extensions $F_1 \subset F_2 \subset F_3$.

Lemma A10. $[\gamma_1, \dots, \gamma_k, \theta_1, \dots, \theta_\ell] = [\gamma_1, \dots, \gamma_k] + [(\theta_1, \dots, \theta_\ell) : (\gamma_1, \dots, \gamma_k)]$.

Proof. Denote $[\gamma_1, \dots, \gamma_k, \theta_1, \dots, \theta_\ell] = m$, $[\gamma_1, \dots, \gamma_k] = p$, $[(\theta_1, \dots, \theta_\ell) : (\gamma_1, \dots, \gamma_k)] = s$. First prove

1) $m \le p + s$.

Let a sub-Pfaffian set S be as in the definition A10. Consider the sub-Pfaffian set $U_1 \subset \mathbb{R}_{j_2}^k$ consisting of all the points $(y_1, \dots, y_k)$ for which $\dim(S \cap \{Y_1 = y_1, \dots, Y_k = y_k\}) \le s$. Then U_1 is definable over $\mathbb{R}_{j_1}$. Due to the definition A10 there exists a sub-Pfaffian set $U \subset \mathbb{R}_{j_2}^k$ definable over $\mathbb{R}_{j_1}$ such that $(\gamma_1, \dots, \gamma_k) \in U$ and $\dim U = p$.

Denote by $\pi : \mathbb{R}_{j_2}^{k+\ell} \to \mathbb{R}_{j_2}^k$ the natural projection onto the subspace with the coordinates $Y_1, \dots, Y_k$. Consider the sub-Pfaffian set $\mathcal{U} = S \cap ((U \cap U_1) \times \mathbb{R}_{j_2}^\ell) \subset \mathbb{R}_{j_2}^{k+\ell}$. Then $\mathcal{U}$ is definable over $\mathbb{R}_{j_1}$, besides $(\gamma_1, \dots, \gamma_k, \theta_1, \dots, \theta_\ell) \in \mathcal{U}$. The dimension $\dim(\mathcal{U}) \le p + s$, since $\dim(\pi(\mathcal{U})) \le \dim(U) = p$ and for any point $y \in \pi(\mathcal{U})$ we have $\dim(\mathcal{U} \cap \pi^{-1}(y)) \le s$.

2) $m \ge p + s$.

According to the definition A10 there exists a sub-Pfaffian set $\mathcal{V} \subset \mathbb{R}_{j_2}^{k+\ell}$ definable over $\mathbb{R}_{j_1}$ such that $(\gamma_1, \dots, \gamma_k, \theta_1, \dots, \theta_\ell) \in \mathcal{V}$ and $\dim(\mathcal{V}) = m$. Denote $\dim(\mathcal{V} \cap \{Y_1 = \gamma_1, \dots, Y_k = \gamma_k\}) = s_1$. Obviously $s_1 \ge s$. Consider the sub-Pfaffian set $V_1 \subset \mathbb{R}_{j_2}^k$ consisting of all the points $(y_1, \dots, y_k)$ for which $\dim(\mathcal{V} \cap \{Y_1 =$

$y_1, \dots, Y_k = y_k\}) \geq s_1$. Then V_1 is definable over $\mathbb{R}_{j_1}$ and $(\gamma_1, \dots, \gamma_k) \in V_1$, therefore $\dim V_1 \geq p$. Arguing similarly as in 1), we get $m \geq s_1 + \dim V_1 \geq s+p$. $\square$

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