Analogue of Newton-Puiseux series for non-holonomic D-modules and factoring

Dima Grigoriev
CNRS, Mathématiques, Université de Lille
59655, Villeneuve d'Ascq, France
`dmitry.grigoryev@math.univ-lille1.fr`
http://logic.pdmi.ras.ru/~grigorev

Abstract

We introduce a concept of a fractional-derivatives series and prove that any linear partial differential equation in two independent variables has a fractional-derivatives series solution with coefficients from a differentially closed field of zero characteristic. The obtained results are extended from a single equation to D-modules having infinite-dimensional space of solutions (i. e. non-holonomic D-modules). As applications we design algorithms for treating first-order factors of a linear partial differential operator, in particular for finding all (right or left) first-order factors.

keywords: Newton-Puiseux series for D-modules, fractional derivatives, factoring linear partial differential operators

AMS classification: 35C10, 35D05, 68W30

Introduction

It is well-known that any polynomial equation $t(x,y)=0$ has $deg_y(t)$ (counting with multiplicities) zeroes being Newton-Puiseux series (see e. g. [26])

$$y(x)=\sum_{i_0\le i<\infty} y_i x^{-i/q} \tag{1}$$

for suitable integers $q\ge 1, i_0$ and the coefficients y_i from an algebraically closed field.

In this paper an analogue of Newton-Puiseux series for partial linear differential equations $T=0$ is proposed, and we prove that $T=0$ has a solution of this form. Whereas a Newton-Puiseux series is developed for a (plane) curve, we restrict ourselves with linear partial differential operators T in two derivatives d_x, d_y (in case of 3 or more derivatives there are no solutions of this form in general, see Remark 4.10).

One of the principal features of Newton-Puiseux series is the appearance of fractional exponents. Thus, a question arises, what could be an analogue of fractional powers, so to say "fractional derivatives"? An evident observation shows that in the derivative $y'(x)=\sum_i(-i/q+1)y_{i-q}x^{-i/q}$ the i-th coefficient depends on the $(i-q)$-th coefficient of $y(x)$ itself.

That is why as a differential analogue of Newton-Puiseux series we suggest a *fractional-derivatives series* of the form

$$\sum_{0\leq i<\infty} h_i G^{(-i/q)}$$

where h_i being elements of a differentially closed (or universal in terms of [13]) field F and $G^{(-i/q)}$ is called $(-i/q)$*-th fractional derivative* of G. The symbol $G = G^{(0)} = G_{(s_2,\dots,s_k)}(f_1, f_2, \dots, f_k)$ is defined by rational numbers $1 > s_2 > \cdots > s_k > 0$ and $f_1, \dots, f_k \in F$ (if to continue the analogy with curves, G plays a role of a uniformizing element). For any rational s the s-th fractional derivative $G^{(s)}$ fulfills the identity

$$dG^{(s)} = (df_1)G^{(1+s)} + (df_2)G^{(s_2+s)} + \cdots + (df_k)G^{(s_k+s)}$$

where either a derivative $d = d_x$ or $d = d_y$. The common denominator q of $s_2, \dots, s_k$ plays a role similar to one of the common denominator of the exponents in a Newton-Puiseux series (1). The inequality $k \leq q$ holds.

In a particular case $k = 1$ we have $q = 1$ and as G one can take $g(f_1)$ for any univariate ("undetermined") function g, provided that the composition makes sense, the fractional derivatives $G^{(s)} = g^{(s)}(f_1)$ for integers s. We note that finite sums

$$\sum_{i_0\leq i\leq i_1} h_i G^{(-i)}$$

(so, for $k = q = 1$) appear in the Laplace method as solutions of some second-order equations $T = 0$ (see e. g. [4, 25]).

One can find necessary in the sequel information on D-modules in [2], [16], a survey on their algorithmical aspects in [20]. We mention that there are also applications of Newton polygons over the Weyl algebra $C[x, d_x]$: in [16] to meromorphic connections, in [17] to micro-differential operators and in [18] to the Fourier transform. In case of linear *ordinary* differential operators Newton polygons are employed to produce the canonical form basis of the space of solutions (see e. g. [27], also [7] where an algorithm for this problem with a better complexity bound was designed). A similar form of solutions for linear *partial* differential operators were studied in [1] where, nevertheless, also examples are exhibited of operators without solutions of this form. On the problem of factoring a linear ordinary differential operator one can look in [19], see also [7].

In Section 1 we introduce the principal concept of fractional-derivatives series and give some their basic properties.

In the sequel the crucial role plays the multiplicity m of a linear factor of the *symbol* of the linear partial differential operator T (with coefficients in F) of an order n: the symbol is a homogeneous polynomial in two variables $d_x f_1, d_y f_1$ of the degree n which corresponds to the highest derivatives of T. In Section 2 we develop a method for constructing fractional-derivatives solutions of $T = 0$ and prove the existence of such a solution with $q \leq m$. The method is similar to the Newton-Puiseux expansion, it produces a relevant convex polygon similar to the Newton one, but differs in several aspects. The main of the latter is that the leading equation corresponding to a certain (leading) edge of the polygon is not a univariate polynomial unlike the Newton-Puiseux expansion, but rather a non-linear first-order partial differential equation. This creates difficulties in defining a multiplicity of a solution of the leading equation. Also it is unclear, what could be a differential analogue of the statement (cf. above) that an algebraic equation $t = 0$ has precisely $deg_y(t)$ Newton-Puiseux series solutions of the form (1)? Partially these questions are answered for the introduced in Section 3 generic fractional-derivatives series solutions.

In Section 4 the result of Section 2 is extended from a single partial linear differential equation to a system of equations in several unknown functions having an infinite-dimensional space of solutions (or in other words, to a D-module of a non-zero differential type, one can call it a non-holonomic D-module). To this end for any left ideal $J \subset F[d_x, d_y]$ of the differential type 1 we yield an operator $p \in F[d_x, d_y]$ and show that any fractional-derivatives series solution of the equation $p = 0$ which corresponds to a linear factor (different from $d_y f_1$) of the symbol of p, is a solution of the ideal J as well. In Section 5 we exploit the relation of equivalence of ideals introduced in [10] and establish a kind of duality between equivalence classes of non-holonomic ideals and their sets of fractional-derivatives series solutions. Namely, it is proved that two non-holonomic left ideals $J, J_1 \subset F[d_x, d_y]$ are equivalent if and only if their respective sets of fractional-derivatives series solutions coincide. Also we express the quotient of the spaces of fractional-derivatives series solutions of non-holonomic ideals $J \subset J_1$ via the module of relative syzygies [10] of this pair of ideals.

In Section 6 it is shown that in case of a separable operator T any its power series solution can be obtained as a sum of specifications of its suitable fractional-derivatives series solutions, thereby establishing completeness of the latter. In Section 7 we provide applications of fractional-derivatives series to studying first-order factors of an operator, exploiting that in case of a first-order operator $T = d_y + ad_x + b$ its fractional-derivatives series solutions turn to a single term of the form $hG(f)$ where $(d_y + ad_x)f = 0$ and h being a particular solution of $T = 0$. In Subsection 7.1 an algorithm is designed which finds first-order factors of a given operator, and in Subsection 7.3 an algorithm which constructs the intersection of all the principal ideals generated by the first-order factors of the operator. In Section 8 the possible fractional-derivatives series solutions of a second-order operator obtained by the algorithm from Section 2 are described. This description can help to imagine the shape of fractional-derivatives series solutions and the difficulties which appear while their developing.

1 Fractional-derivatives series

Let F be a differential field of the characteristic 0 with the derivatives $\{d_j\}$ and a subfield of constants $C \subset F$ [13].

Definition 1.1 *Let $f_1, \dots, f_{k_0} \in F$ and rational numbers $1 > s_2 > \cdots s_{k_0} > 0$. We introduce a symbol $G = G^{(0)} = G_{s_2,\dots,s_{k_0}}(f_1, f_2, \dots, f_{k_0})$ together a set $\{G^{(s)}\}_{s\in\mathbb{Q}}$ of its fractional s-th derivatives satisfying the following rule of differentiation for any derivative $d = d_j$:*

$$dG^{(s)} = (df_1)G^{(1+s)} + (df_2)G^{(s_2+s)} + \cdots + (df_{k_0})G^{(s_{k_0}+s)}$$

Clearly, these differentiations commute with each other and one can consider the free F-module with the basis $\{G^{(s)}\}_{s\in\mathbb{Q}}$ as a D-module.

Definition 1.2 *Let q be the common denominator of $s_2, \dots, s_{k_0}$ and $h_i \in F, i \geq 0, s_0 q \in \mathbb{Z}$. Then*

$$H = \sum_{0\leq i<\infty} h_i G^{(s_0 - i/q)} \tag{2}$$

we call a fractional-derivatives series.

For a given G all the fractional-derivatives series (with added 0) constitute a D-module (we study it below in Section 4). Obviously, $k_0 \le q$.

It is easy to see that G satisfies a suitable linear partial differential equation with coefficients in F.

Remark 1.3 *The symbol G plays a role in H similar to the role of the parameter x in a Newton-Puiseux series (1). In particular, specifying the values of x in a certain field one gets points of (a branch of) the curve given by (1). Here one can also provide some specifications of G. Indeed, for an arbitrary family $\{c_{i/q}\}_{i\in\mathbb{Z}}$ where $c_{i/q} \in \mathbb{C}$ the following set*

$$G^{(s)} = \sum_{j_1\ge 0,\dots,j_{k_0}\ge 0} c_{-s-j_1-j_2s_2-\cdots-j_{k_0}s_{k_0}} \frac{f_1^{j_1}}{j_1!}\cdots\frac{f_{k_0}^{j_{k_0}}}{j_{k_0}!}$$

satisfies Definition 1.1.

For example, in case when F is the ring of functions analytic in a certain neighborhood of a given point in the multidimensional complex space and the absolute values $|c_{i/q}|$ are bounded, the latter series also converges in a suitable neighborhood.

From now on let F have two derivatives $\{d_x, d_y\}$. Consider a linear operator

$$T = T_0 + \cdots + T_n \tag{3}$$

of the order n where $T_p = \sum_{0\le j\le p} b_{j,p} d_x^j d_y^{p-j}$ contains the derivatives of the order p and the coefficients $b_{j,p} \in F$. The following lemma holds, in fact, for an arbitrary number of derivatives, nevertheless, the assumption that F has two derivatives simplifies the notations and in the sequel we deal just with operators in two derivatives (one can verify lemma by a direct calculation).

Lemma 1.4 *$d_x^j d_y^{p-j}(hG)$ equals the sum of the terms of the form*

$$\frac{1}{w_1!\cdots w_N!}\binom{j}{l_{1,1},\dots,l_{1,m_1},\dots,l_{k_0,1},\dots,l_{k_0,m_{k_0}},l}\binom{p-j}{r_{1,1},\dots,r_{1,m_1},\dots,r_{k_0,1},\dots,r_{k_0,m_{k_0}},r}$$

$$\prod_{1\le i\le m_1}(d_x^{l_{1,i}} d_y^{r_{1,i}} f_1)\cdots\prod_{1\le i\le m_{k_0}}(d_x^{l_{k_0,i}} d_y^{r_{k_0,i}} f_{k_0})(d_x^l d_y^r h)\cdot G^{(m_1+s_2m_2+\cdots+s_{k_0}m_{k_0})}$$

for all partitions $l_{1,1}+\cdots+l_{1,m_1}+\cdots+l_{k_0,1}+\cdots+l_{k_0,m_{k_0}}+l = j$ of j and $r_{1,1}+\cdots+r_{1,m_1}+\cdots+r_{k_0,1}+\cdots+r_{k_0,m_{k_0}}+r = p-j$ of $p-j$ such that $l_{\kappa,i}+r_{\kappa,i} \ge 1$ for every $1\le\kappa\le k_0$, $1\le i\le m_\kappa$, where $w_1,\dots,w_N$ denote the cardinalities of the partition of the triples $(l_{1,1}, r_{1,1}, 1),\dots,(l_{1,m_1}, r_{1,m_1}, 1),\dots,(l_{k_0,1}, r_{k_0,1}, k_0),\dots,(l_{k_0,m_{k_0}}, r_{k_0,m_{k_0}}, k_0)$ into equal ones, in particular, $w_1+\cdots+w_N = m_1+\cdots+m_{k_0}$.

2 Constructing fractional-derivatives series solutions

From now on we suppose that the field F is differentially closed (or universal in terms of [13]).

The main purpose of this section is to prove that a linear partial differential equation $T = 0$, see (3), has a solution of the form (2). To simplify the notations we put $s_0 = 0$ and $h = h_0 \ne 0$ in (2).

Denote by $\bar{T}_p(d_x f_1, d_y f_2) = \sum_{0\le j\le p} b_{j,p}(d_x f_1)^j (d_y f_1)^{p-j}$ a homogeneous form of the degree p in $d_x f_1, d_y f_1$. Sometimes, $\bar{T}_n = symb(T)$ is called the symbol of T. Fix a linear factor $a_1 d_x f_1 + a_2 d_y f_1$ of $\bar{T}_n$ having a multiplicity m, the coefficients $a_1, a_2 \in F$.

Expanding $T(H)$ with respect to the fractional derivatives $\{G^{(s)}\}_s$ for $k = 1$ (in other words, assuming for the time being that $dG^{(s)} = (df_1)G^{(1+s)}$, see Definition 1.1), we get that the coefficient at $G^{(n)}$ vanishes, i. e. $h \cdot symb(T) = 0$. Thus, we can suppose that $(a_1 d_x + a_2 d_y) f_1 = 0$. Choose any such f_1 with $grad(f_1) \neq 0$.

For $k \ge 2$ we introduce an auxiliary polygon P_k playing the role similar to the Newton polygon. Now let $k = 2$, in other words, we assume (for the time being) that $dG^{(s)} = (df_1)G^{(1+s)} + (df_2)G^{(s_2+s)}$. The next purpose is to construct s_2 and f_2. It suffices to consider the expansion of the first term $T(hG)$ of $T(H)$ (we'll come back to this issue at the end of the present section). When we talk about the expansion of $T(hG)$ we always refer to Lemma 1.4. If a term $b(\prod_{1\le i\le t}(d_x^{l_i} d_y^{r_i} f_2))G^{(s+s_2 t)}$ occurs in $T(hG)$, where b is a differential polynomial in f_1 and in h (being linear in h), then we place the point (s, t) in P_2 (observe that $s, t \ge 0$ in P_2 are always integral). As P_2 we take the convex hull of these points with the origin $(0, 0)$. If to assign the weight 1 to every derivative $d_x^l d_y^r f_1$ then any term in b gets the weight s due to Lemma 1.4.

One can observe that P_2 lies to the left from the line $\bar{L}_1 = \{s + t = n\}$ with the slope 1 (under the slope of the line $\{s + jt = const\}$ we mean j) again due to Lemma 1.4. Moreover, the point $(n - m, m) \in \bar{L}_1$ belongs to P_2 because the non-zero term

$$\frac{\bar{T}_n}{((a_1 d_x + a_2 d_y) f_1)^m} \cdot ((a_1 d_x + a_2 d_y) f_2)^m \cdot G^{(n-m+s_2 m)}$$

occurs in the expansion of $T(hG)$, taking into account that the factor $(a_1 d_x + a_2 d_y) f_1$ has the multiplicity m in $\bar{T}_n$, and no other term from this expansion gives a contribution in the coefficient at the point $(n - m, m)$. Similarly, one verifies that the points $(n - t, t)$ with $0 \le t \le m - 1$ do not belong to P_2.

Now we assign a (yet unknown) weight s_2 to every derivative $d_x^l d_y^r f_2$. Therefore, to find $s_2 < 1$ we consider the edges of P_2 with the positive slopes less than 1. Choose any such edge L_2 (we call it *leading*) with the endpoints (j_1, t_1), (j_2, t_2), $t_1 > t_2$; we have seen already that $t_1 \le m$. Then the slope of L_2 provides $s_2 = (j_1 - j_2)/(t_1 - t_2)$.

To find f_2 we consider the *leading* differential polynomial $Q_2(f_2)$ which equals the sum of the coefficients at all the points of P_2 which lie on L_2. Then $Q_2(f_2)$ coincides with the coefficient at $G^{(j_1+s_2 t_1)}$ in the expansion of $T(hG)$. As $f_2 \in F$ we take a solution of the leading equation $Q_2(f_2) = 0$. Evidently, $j_1 + s_2 t_1 < n$ since the point of intersection of the line $\bar{L}_2$ (which contains the edge L_2) with j-axis $\{t = 0\}$ is located to the left of the intersection of $\bar{L}_1$ with j-axis.

Thus, we are able to formulate the recursive hypothesis of the procedure under description which constructs $1 > s_2 > s_3 > \cdots$ and $f_1, f_2, f_3, \ldots$. Suppose that $s_2, \ldots, s_k$ and $f_1, f_2, \ldots, f_k$ are already constructed. In addition, a polygon P_k is constructed being a convex hull of the points (j, t) (together with the origin $(0, 0)$) such that a term

$$b(\prod_{1\le i\le t} d_x^{l_i} d_y^{r_i} f_k) G^{(j+s_k t)} \tag{4}$$

occurs in the expansion of $T(hG)$ under the assumption $dG = (df_1)G^{(1)} + (df_2)G^{(s_2)} + \cdots + (df_k)G^{(s_k)}$, see Definition 1.1 (observe that for $k \ge 3$ a rational coordinate $j \ge 0$ can be non-integral, while $t \geq 0$ is always integral). A certain leading edge L_k of P_k is chosen with a slope $s_k > 0$ and with the endpoints (j_3, t_3), (j_4, t_4), $t_3 > t_4$. We name (j_3, t_3) the *pivot* of L_k and t_3 the *multiplicity* of L_k. The leading differential polynomial $Q_k(f_k)$ equals the sum of the coefficients at all the points of P_k which lie on L_k. Then $Q_k(f_k)$ coincides with the coefficient at $G^{(j_3+s_k t_3)}$ in the expansion of $T(hG)$. As $f_k \in F$ a solution of the leading equation $Q_k(f_k) = 0$ is taken. The points of intersections of the lines $\bar{L}_1, \bar{L}_2, \ldots$ with j-axis decrease. Denote by q_k the common denominator of $s_2, \ldots, s_k$, obviously $q_1 = 1$.

To carry out the recursive step, we make the assumption $dG = (df_1)G^{(1)} + (df_2)G^{(s_2)} + \cdots + (df_k)G^{(s_k)} + (df_{k+1})G^{(s_{k+1})}$. The boundary of the polygon P_{k+1} above the pivot of L_k (including the pivot itself) is the same as of P_k.

Let us calculate the points of P_{k+1} located on the line $\bar{L}_k$. Denote by B_t $(t_4 \leq t \leq t_3)$ the coefficient of P_k at the point $(j_3+s_k(t_3-t), t) \in L_k$. Then $Q_k = \sum_{t_3 \leq t \leq t_4} B_t$. One can observe that B_t contains no higher derivative $d_x^l d_y^r f_k$ with $l+r \geq 2$. Indeed, if otherwise B_t contained a term of the form (4) then the coefficient of P_k at the point $(j_3+s_k(t_3-t), t+\sum_{1\leq i\leq t}(l_i+r_i-1))$ would contain the term

$$b(d_x f_k)^{\sum_{1\leq i\leq t} l_i}(d_y f_k)^{\sum_{1\leq i\leq t} r_i} G^{(j_3+s_k(t_3+\sum_{1\leq i\leq t}(l_i+r_i-1)))}$$

due to Lemma 1.4, hence the point $(j_3 + s_k(t_3 - t), t + \sum_{1\leq i\leq t}(l_i + r_i - 1))$ should belong to P_k which leads to a contradiction when $\sum_{1\leq i\leq t}(l_i + r_i - 1) \geq 1$.

Besides, B_t is a linear form in the derivatives of h. We claim that $B_t = h\tilde{B}_t$ for an appropriate differential polynomial $\tilde{B}_t$ in $f_1, \ldots, f_k$. Indeed, if B_t contained a term $(d_x^l d_y^r h)\tilde{b}G^{(j_3+s_k t_3)}$ with $l+r \geq 1$ for a certain $\tilde{b}$ being a differential polynomial in $f_1, \ldots, f_k$, then the coefficient of P_k at the point $(j_3+s_k(t_3-t), t+l+r)$ would contain the term $h\tilde{b}(d_x f_k)^l(d_y f_k)^r G^{(j_3+s_k(t_3+l+r))}$ due to Lemma 1.4, therefore, the point $(j_3 + s_k(t_3 - t), t + l + r)$ should belong to P_k, the achieved contradiction proves the claim.

Thus, B_t will be treated as a homogeneous (of the degree t) polynomial in $d_x f_k, d_y f_k$. For more generality of the auxiliary results below we deem that B_t is a homogeneous polynomial in the variables $v_1, \ldots, v_p$, thereby $p = 2$ and $v_1 = d_x f_k$, $v_2 = d_y f_k$. We denote the corresponding derivatives $\bar{v}_1 = d_x f_{k+1}$, $\bar{v}_2 = d_y f_{k+1}$.

Remark 2.1 *Since the main purpose of the present section is to prove the existence of solutions of the form (2) of an equation* $T = 0$ *(see (3)) it suffices to study only the canonical solutions, namely, when each* s_k *is the slope of a certain edge of* P_k *and* f_k *satisfies a leading equation. Alternatively, one could take* s_k *to be the slope of some line passing through a single vertex, say* (j_3, t_3) *of* P_k*. In this case* $B_{t_3}(f_k) = 0$*, because* B_{t_3} *is a homogeneous polynomial in* $d_x f_k, d_y f_k$*, we get that* f_k *fulfills a certain first-order linear equation* $b_1 d_x f_k + b_2 d_y f_k = 0$*. There is no way to bound the denominators* s_k *for non-canonical solutions (2), the number of steps* k_0*, moreover, the procedure of constructing* $1 > s_2 > s_3 > \cdots$ *and* $f_1, f_2, f_3, \ldots$ *could last infinitely. One might even choose real exponents* s_k *(cf. [11] where an analogue of Newton-Puiseux series solutions with real exponents was studied for non-linear ordinary differential equations).*

Denote by $\bar{B}_t (0 \leq t \leq t_3)$ the coefficient at the point $(j_3 + s_k(t_3 - t), t) \in \bar{L}_k$ of P_{k+1}. Taking into account the assumption on dG and Lemma 1.4, we have

$$\bar{B}_t = \sum_{i_1+\cdots+i_p=t} \frac{1}{i_1! \cdots i_p!} \frac{\partial^t Q_k}{\partial v_1^{i_1} \cdots \partial v_p^{i_p}} \bar{v}_1^{i_1} \cdots \bar{v}_p^{i_p} \tag{5}$$

Therefore, $\bar{B}_t = h\hat{B}_t$ where $\hat{B}_t$ can be treated as a homogeneous polynomial in $\bar{v}_1, \dots, \bar{v}_p$ of the degree t with the coefficients being differential polynomials in $f_1, \dots, f_k$. Let t_0 be the minimal t such that $\bar{B}_t \neq 0$. Then $t_0 \geq 1$ because $Q_k(f_k) = 0$, and $t_0 \leq t_3$ because $\bar{B}_{t_3}$ is obtained from B_{t_3} by means of replacing v_i for $\bar{v}_i, 1 \leq i \leq p$. One can view t_0 as a kind of multiplicity of the solution f_k in Q_k.

Lemma 2.2 $t_0 \leq t_4 + \frac{(t_3-t_4)q_{k-1}}{q_k}$.

Proof. Suppose the contrary. First we observe that the gap between the ordinates of any pair of consecutive points on L_k is at least q_k/q_{k-1} and that $e = (t_3 - t_4)q_{k-1}/q_k$ is an integer (cf. [26]). Hence L_k contains at most $e + 1$ points. Without loss of generality for the sake of conveniency of notations we assume that L_k contains exactly $e+1$ points (some among them, perhaps, with zero coefficients B_t).

Due to the supposition and the choice of t_0 we have $\bar{B}_t = 0$ for $t_4 \leq t \leq t_4 + e$, i. e. all the derivatives

$$\frac{\partial^t Q_k}{\partial v_1^{i_1} \cdots \partial v_p^{i_p}}$$

of the order t vanish. Fix for the time being non-negative integers $j_1, \dots, j_p$ with the sum $j_1 + \cdots + j_p = t_4$. Then

$$0 = \sum_{i_1 \geq j_1, \dots, i_p \geq j_p; i_1 + \dots + i_p = t} \frac{(t-t_4)!}{(i_1 - j_1)! \cdots (i_p - j_p)!} \frac{\partial^t Q_k}{\partial v_1^{i_1} \cdots \partial v_p^{i_p}} v_1^{i_1 - j_1} \cdots v_p^{i_p - j_p}$$

$$= \sum_{l \geq t} \frac{(l-t_4)!}{(l-t)!} \frac{\partial^{t_4} B_l}{\partial v_1^{j_1} \cdots \partial v_p^{j_p}}$$

due to the Euler's formula. The latter equalities can be treated as a linear $(e+1) \times (e+1)$ system with a non-singular matrix. Its non-singularity is justified by the following result [15]: if $n_1 > \cdots > n_r \geq 0; m_1 > \cdots > m_r \geq 0; n_1 \geq m_1, \dots, n_r \geq m_r$ then the $r \times r$ matrix with the entries $\binom{n_i}{m_j}$ is non-singular. Therefore,

$$\frac{\partial^{t_4} B_l}{\partial v_1^{j_1} \cdots \partial v_p^{j_p}} = 0$$

for any l and any $j_1, \dots, j_p$ with $j_1 + \cdots + j_p = t_4$, in particular B_{t_4} vanishes identically, the obtained contradiction proves the lemma. ∎

Corollary 2.3 *If* $t_0 = t_3$ *then the denominator* $q_{k-1} = q_k$ *does not change.*

Now we are in position to continue the recursive step of the procedure constructing s_{k+1}, f_{k+1}. The polygon P_{k+1} either contains the edge with the slope s_k and with the ordinates $t_0 < t_3$, respectively, of its endpoints, or the edge of P_{k+1} with its above endpoint (j_3, t_3) has the slope less than s_k. In the first case as a leading edge L_{k+1} one takes an edge of P_{k+1} having a positive slope s_{k+1} with the ordinate t_5 of its upper endpoint (j_5, t_5) less or equal to t_0. In this case (j_5, t_5) plays the role of a new pivot with t_5 being the multiplicity of L_{k+1}. As above one produces the leading differential polynomial $Q_{k+1}(f_{k+1})$ and as f_{k+1} chooses a solution of the equation $Q_{k+1}(f_{k+1}) = 0$. In the second case the denominator $q_k = q_{k-1}$ does not increase due to Corollary 2.3, and as L_{k+1} one takes an edge of P_{k+1} having a positive slope s_{k+1} with the ordinate t_5 of its upper endpoint (the pivot) (j_5, t_5) less or equal to t_3. The rest is similar to the first case.

Thus, we have described a recursive procedure constructing $1 > s_2 > s_3 > \cdots$ and $f_1, f_2, f_3, \dots$ which one can view as a tree.

Lemma 2.4 *i) The common denominator q of $s_2, s_3, \dots$ does not exceed 2^{m-1};*

ii) there exists a branch of the tree in which the common denominator q is less or equal to m;

iii) every branch of the tree after at most of q steps arrives to a leading edge with a non-positive slope.

Proof. First we recall that the multiplicity of any leading edge in P_2 is less or equal to m. Therefore, i) follows from Lemma 2.2: if at a certain step the common denominator q_{k-1} is multiplied by q_k/q_{k-1} then the multiplicity decreases at least by $q_k/q_{k-1} - 1$. After the multiplicity reaches 1, the denominator does not change anymore.

ii) Let us take at each step of the described recursive procedure the leading edge with the least possible slope, while the latter is positive. The ordinate of the lower endpoint of this edge $t_4 = 0$. Therefore, Lemma 2.2 entails that $t_0 \le t_3 q_{k-1}/q_k$, this implies ii).

iii) follows from Definition 1.1 because $k_0 \le q$. ∎

Assume now that P_{k+1} in the described procedure contains an edge having a non-positive slope (see Lemma 2.4 ii)). Take such edge $L = L_{k+1}$ with the largest possible non-positive slope in P_{k+1}. We have shown above that the coefficient $\bar{B}_{t_5}$ at the pivot (j_5, t_5) of L_{k+1} equals to $h\hat{B}$ where $\hat{B}$ is a suitable homogeneous polynomial of the degree t_5 in $d_x f_{k+1}, d_y f_{k+1}$ with the coefficients being differential polynomials in $f_1, \dots, f_k$. Denote by $\bar{B}$ the coefficient at the point $(j_5, 0)$ of P_{k+1}, being a linear homogeneous operator in h (one can show that the order of $\bar{B}$ does not exceed t_5 in the same manner as it was shown that $\bar{B}_{t_5}$ has the order 0 in h). If $\hat{B}$ contains a term $b(d_x f_{k+1})^l (d_y f_{k+1})^{t_5 - l}$ for some l and b being a differential polynomial in $f_1, \dots, f_k$, then $\bar{B}$ contains the term $b((d_x)^l (d_y)^{t_5 - l} h)$ due to Lemma 1.4. Hence the order of $\bar{B}$ is greater or equal to t_5 (actually, equals t_5 as we have seen, although we use below only that the order of $\bar{B}$ is positive). In particular, the slope of L equals 0, and P_{k+1} contains no edges with negative slopes. In the construction under description f_{k+1} does not appear and as $h \in F$ we take a solution of the linear homogeneous differential equation $\bar{B}(h) = 0$ (which can be viewed as a leading equation on h).

This completes the construction of the first summand hG of the solution H of the form (2). To obtain the next coefficient h_1 of H we observe that in the expansion of $T(h_1 G^{(-1/q)})$ in the fractional derivatives $\{G^{(i/q)}\}_{-\infty < i < \infty}$ the highest non-zero term equals $\bar{B}(h_1) G^{(j_5 - 1/q)}$, taking into account that this expansion is obtained by means of the shift by $-1/q$ of the expansion of $T(hG)$ while replacing h for h_1. Therefore, for $h_1 \in F$ we get a linear partial differential equation (not necessary, homogeneous) of the form $\bar{B}(h_1) = \bar{f}$ (so, of the same order t_5) for an appropriate $\bar{f} \in F$ being a differential polynomial in $h, f_1, \dots, f_{k_0}$ (in the above notations $k_0 = k$). In a similar way one obtains consecutively $h_2, h_3, \dots$.

Summarizing, the following theorem is proved.

Theorem 2.5 *Any linear partial differential equation $T = 0$ of an order n (see (3)) for each linear factor $(a_1 d_x f_1 + a_2 d_y f_1)$ of a multiplicity m of its symbol $symb(T)$ has a non-zero fractional-derivatives series solution of the form (2) with the denominator $q \le m$.*

One can continue every branch of the tree of the described procedure constructing $1 > s_2 > s_3 > \cdots$ and $f_1, f_2, \dots$ to a solution of the form (2) of $T = 0$, and every solution of the form (2) constructed by a described procedure has the denominator $q \leq 2^{m-1}$.

Corollary 2.6 *If an LPDO T of an order n has no fractional-derivatives series solutions with the denominator $q < n$ then T is irreducible in $F[d_x, d_y]$.*

Remark 2.7 *The bound $q \leq m$ is sharp as shows the following example. Take T (see (3)) such that $(a_1 d_x f_1 + a_2 d_y f_1)$ has the multiplicity m in $\bar{T}_n$, the multiplicity greater or equal to $m - i$ in $\bar{T}_{n-i}$ for every $1 \leq i \leq m-2$ and the multiplicity 0 in $\bar{T}_{n-m+1}$,respectively. Then the polygon P_2 has the edge with the endpoints $(n-m, m)$ and $(n-m+1, 0)$ which being taken as a leading one (actually, there is no other choice for a leading edge), provides the slope $s_2 = 1/m$.*

Remark 2.8 *Theorem 2.5 states the bound $q \leq m$ for a particular solution. It is unclear how sharp is the bound $q \leq 2^{m-1}$ for all constructed solutions. The natural question is whether one can improve it by m (one can verify it for $m \leq 7$ by the direct calculations)? This would be similar to the algebraic situation in which such a bound on the common denominator in all Puiseux series (1) is well known (see e. g. [26]). We also mention that for solutions in the canonical form basis [27] of linear ordinary differential equations a similar to the algebraic situation bound on the common denominator (of the rational exponents) was established in [7].*

3 Multiplicity of generic fractional-derivatives series solutions

In the described recursive construction f_k was chosen as a solution of the equation $Q_k(f_k) = 0$. Different choices of f_k could yield different polygons P_{k+1}. Therefore, the set of (even canonical fractional-derivatives series, see Remark 2.1) solutions of the equation $T = 0$ is quite vast. An interesting open question is whether it is possible to introduce a concept of a multiplicity of a set of fractional-derivatives series solutions and relate it to m? In the present section we give a partial answer to this question for the so-called generic solutions.

We view Q_k as a polynomial in two variables $v_1 = d_x f_k, v_2 = d_y f_k$. Note that this polynomial is not homogeneous, consider its factorization $Q_k = \beta_1^{m_1} \cdots \beta_l^{m_l} \beta$ over F where β is homogeneous and $\beta_1, \dots, \beta_l$ are irreducible non-homogeneous. In the recursive construction from Section 2 we distinguish a case which we call *generic*, namely, when $\beta_i(d_x f_k, d_y f_k) = 0$ for a certain $1 \leq i \leq l$ such that $m_i = \min\{m_1, \dots, m_l\}$, and the point $(d_x f_k, d_y f_k)$ is a non-singular one of the plane curve $Q_k = 0$. In the generic case for the multiplicity of f_k we have $t_0 = m_i$ due to (5). One can assign the multiplicity t_0 to the set of all f_k satisfying the generic case. We call a solution (2) generic if for each of $f_2, \dots, f_{k_0}$ the generic case happens in the construction of (2). When $k_0 = 1$ we call (2) generic as well. At the end of developing any generic solution we arrive to a polygon P_{k+1} having a leading edge L_{k+1} with the slope 0. Let the upper endpoint (pivot) of L_{k+1} be (j_5, t_5), then to this generic solution we assign the multiplicity t_5. Observe that we have assigned the multiplicity to the set of all the generic solutions (2) which follow the same branch in the tree of the construction from Section 2.

Proposition 3.1 *Any linear partial differential equation $T = 0$ of an order n*

i) has a generic solution of the form (2);

ii) the sum of multiplicities of the generic solutions does not exceed n;
iii) the denominator of every generic solution is less than $n^{O(\log n)}$.

Proof. Each $Q_k, k \geq 2$ is non-homogeneous, that is why i) is justified taking into account Theorem 2.5.

ii) follows (similar to the algebraic Newton-Puiseux series [26]) by inverse induction along the tree of the procedure described in Section 2 due to the inequality $m_1 + \cdots + m_l \leq t_3 - t_4$.

The latter inequality together with Lemma 2.2 imply that $t_0 \leq t_3 \frac{q_k}{2q_k - q_{k-1}}$. Therefore, in developing a generic solution by means of the procedure from Section 2 there are at most $\log_{3/2} n$ steps at which the denominator augments. At each such step the denominator grows less than in n times (cf. the proof of Lemma 2.2), this entails iii). ∎

Remark 3.2 *In a particular case* $m = 1$ *we have* $q = k_0 = 1$*, all the solutions of the form (2) are canonical, the polygon* P_2 *contains a single edge with a slope less than 1, namely, the edge with the endpoints* $(n-1, 1)$ *and* $(n-1, 0)$ *having the slope 0. It provides the leading linear equation on* h *of the first order, the leading equation* $(a_1 d_x + a_2 d_y) f_1 = 0$ *on* f_1 *is linear and of the first order as well, thus, the multiplicity 1 is assigned to the set of (generic) solutions in case* $m = 1$.

Remark 3.3 *While in (2) we consider series with decreasing orders of derivatives of* G*, one can easily verify that an equation* $T = 0$ *for an arbitrary* $f \in F$ *has a solution of the form*

$$\sum_{0 \leq i < \infty} h_i G^{(i)}$$

where $G = G(f)$*, with increasing orders of derivatives of* G*. Thus, continuing the analogy with plane curves, the latter series could be viewed as corresponding to expanding at finite points all being regular (so, without proper fractional derivatives, i. e.* $k_0 = 1$ *in Definition 1.1), while (2) corresponds to expanding at the infinity.*

4 Fractional-derivatives series solutions of non-holonomic D-modules

First let $J = \langle p_1, \ldots, p_l \rangle \subset F[d_x, d_y]$ be a differential (non-holonomic) left ideal of the differential type 1 [13, 14]. This means that the Hilbert-Kolchin polynomial $K_J(z) = ez + e_0$ of J has the degree 1. Denote by $symb(J) \subset F[d_x f_1, d_y f_1]$ a homogeneous ideal generated by the symbols of elements of J (cf. Section 2). Then K_J coincides with the Hilbert polynomial $K_{symb(J)}$ [2, 21] (one can also deduce this from the Janet base of J [22, 10], we mention that the concept of Janet bases was a differential historical predecessor of the one of Groebner bases). Denote $g = GCD(symb(J)) \in F[d_x f_1, d_y f_1]$.

Lemma 4.1 *The degree* e *of the ideal* $symb(J)$ *coincides with* $deg(g)$.

Proof. Since $symb(J) \subset \langle g \rangle$ it suffices to verify that $dim_F(\langle g \rangle / symb(J)) < \infty$. Nullstellensatz entails that $(symb(J)/g) \supset (d_x f_1, d_y f_1)^s$ for a suitable s, therefore, the homogeneous component

$$\langle g \rangle_{deg(g)+s} = g \cdot (d_x f_1, d_y f_1)^s \subset g \cdot (symb(J)/g) = symb(J) \quad ∎$$

Remark 4.2 *If ideal* $J \subset F[d_x, d_y]$ *is holonomic (so, with differential type* 0*) then* $GCD(symb(J)) = 1$.

The degree e (being the leading coefficient of the Hilbert-Kolchin polynomial) is called the typical differential dimension of J [13, 14].

For any homogeneous polynomial $g_0 \in F[d_x f_1, d_y f_1]$ and $a \in F$ denote by $mult_a(g_0)$ the multiplicity of the linear form $d_x f_1 + a d_y f_1$ in g_0. Also for any $p \in F[d_x f_1, d_y f_1]$ we denote for brevity $mult_a(p) = mult_a(symb(p))$. W.l.o.g. assume that $d_y f_1$ does not divide g (otherwise, one can perform a suitable C-linear transformation of d_x, d_y). We have $mult_a(symb(J)) = mult_a(g)$ and $e = \sum_a mult_a(g)$ (cf. Lemma 4.1). For the time being fix $a \in F$ such that $mult_a(g) \geq 1$.

Now we introduce the ring $R = F[d_x, d_y](F[d_y])^{-1}$ of *partial-fractional differential operators* [8]. Its elements has the form $p_0 b^{-1}$ where $p_0 \in F[d_x, d_y]$, $b \in F[d_y]$. One can verify (see [8]) that R is an Ore ring [2], any element of R can be written in a form $\bar{b}^{-1}\bar{p}$ for appropriate $\bar{p} \in F[d_x, d_y], \bar{b} \in F[d_y]$. Thereby $R = (F[d_y])^{-1} F[d_x, d_y]$ and $p_0 b^{-1} = \bar{b}^{-1}\bar{p}$ if and only if $\bar{b}p_0 = \bar{p}b$. Also in [8] one can find the algorithms for addition and multiplication of elements in R. Any element from R can be written in the form $b^{-1}\sum_{0 \leq i \leq w} b_i d_x^i$ for suitable $b, b_i \in F[d_y]$ (because a finite family of elements from R has a common denominator which belongs to $F[d_y]$, see [8]).

For the time being fix $G = G_{(s_2,\dots,s_k)}(f_1, \dots, f_k)$ such that $(d_x + a d_y) f_1 = 0$, $d_y f_1 \neq 0$ (cf. Section 1). Denote by $V = V_G$ the $F[d_x, d_y]$-module which consists of all fractional-derivatives series of the form (2) added by 0.

Lemma 4.3 *V is an R-module*

Proof. For any $0 \neq H \in V$ and $0 \neq b \in F[d_y]$ we claim that $bH \neq 0$. Indeed, let

$$H = hG^{(s)} + \sum_{i \geq 1} h_i G^{(s-i/q)}, h \neq 0; b = t_n d_y^n + \sum_{0 \leq i \leq n-1} t_i d_y^i, t_n \neq 0,$$

then

$$bH = h t_n (d_y f_1)^n G^{[s+n)} + \sum_{i \geq 1} \hat{h}_i G^{(s+n-i/q)} \neq 0$$

For any $H_1 \in V$ we need to prove the existence of $\bar{H} \in V$ such that $b^{-1}H_1 = \bar{H}$, i. e. $H_1 = b\bar{H}$ (the claim above implies that $\bar{H}$ is unique). Let $H_1 = h_{1,0}G^{(s)} + h_{1,1}G^{(s-1/q)} + \cdots ; h_{1,0} \neq 0$. Then we look for $\bar{H} = \bar{h}G^{(s-n)} + \bar{h}_1 G^{(s-n-1/q)} + \cdots$. Comparing the coefficients of H_1 and $b\bar{H}$ at $G^{(s)}$, we get $h_{1,0} = \bar{h} t_n (d_y f_1)^n$ which yields $\bar{h}$. Comparing the coefficients at $G^{(s-1/q)}$ yields $\bar{h}_1$ and so on. ∎

Remark 4.4 *By the same token multiplying by* $0 \neq p \in F[d_x, d_y]$ *on* V *is an isomorphism, provided that* $(d_x + a d_y) f_1$ *does not divide* $symb(p)$.

The ring R is left-euclidean (as well as right-euclidean) with respect to d_x over the skew-field $F[d_y](F[d_y])^{-1}$, cf. Lemma 1.3 [8]. Hence the ideal $\overline{J} = \langle p_1, \dots, p_l \rangle \subset R$ is principal, let $\overline{J} = \langle p \rangle$ for an appropriate $p \in J \subset F[d_x, d_y]$. Then for any $p_0 \in J$ (actually, moreover for $p_0 \in \overline{J}$) the equalities

$$\overline{p_0} p = \overline{b_0} p_0, \overline{b} p = \sum_{1 \leq j \leq l} \overline{p_j} p_j \tag{6}$$

hold for suitable $\overline{p_j} \in F[d_x, d_y]$; $0 \neq \overline{b}, \overline{b_0} \in F[d_y]$.

According to (6) we have $symb(\overline{p_0})symb(p) = symb(\overline{b_0})symb(p_0)$, whence

$$mult_a(p_0) = mult_a(\overline{b_0}) + mult_a(p_0) = mult_a(\overline{p_0}) + mult_a(p) \geq mult_a(p),$$

therefore, $mult_a(p) \leq mult_a(J)$ since

$$mult_a(J) = mult_a(g) = \min_{p_0 \in J} mult_a(p_0).$$

On the other hand, from (6) we get

$$mult_a(p) = mult_a(\overline{b}) + mult_a(p) = mult_a(\sum_{1 \leq j \leq l} \overline{p_j} p_j) \geq mult_a(J)$$

Thus, the following lemma is proved.

Lemma 4.5 *For any $a \in F$ we have $mult_a(p) = mult_a(J)$.*

Proposition 4.6 *A fractional-derivatives series H with $d_y f_1 \neq 0$ (see (2)) is a solution of the linear partial differential equation $p = 0$ if and only if H is a solution of the ideal J.*

Proof. If $pH = 0$ then from (6) we have $0 = \overline{p_0} pH = \overline{b_0} p_0 H$. Hence $p_0 H = 0$ due to Lemma 4.3. The inverse statement follows again from (6) and Lemma 4.3. ■

Corollary 4.7 *For any $a_1, a_2 \in F$ such that $mult_{(a_1 d_x f_1 + a_2 d_y f_1)}(symb(J)) \geq 1$, the ideal $J \subset F[d_x, d_y]$ has a solution of the form (2) with a denominator $q \leq mult_{(a_1 d_x f_1 + a_2 d_y f_1)}(symb(J))$ and $a_1 d_x f_1 + a_2 d_y f_1 = 0, grad(f_1) \neq 0$.*

Proof. It follows from Lemma 4.5, Proposition 4.6, Lemma 2.4 and Theorem 2.5. ■

Remark 4.8 *If for every $a \in F$ the ideal J has the multiplicity $mult_a(J) \leq 1$ then all the solutions of J of the form (2) are canonical, and J has precisely $e = \sum_{a \in F} mult_a(J)$ (which equals the typical differential dimension of J, cf. Lemma 4.1) families of fractional-derivatives series solutions. Moreover, to each of these families a multiplicity 1 can be naturally assigned (cf. Remark 3.2).*

Finally, let $U \subset (F[d_x, d_y])^l$ be a (non-holonomic) $F[d_x, d_y]$-module of the differential type at least 1, obviously, the differential type does not exceed 2 (recall that the differential type equals the degree of the Hilbert-Kolchin polynomial of U [13, 14]). Denote by $u_1, \ldots, u_l$ a free base of $(F[d_x, d_y])^l$. For any $1 \leq r \leq l$ consider the submodule $U_r = \{\sum_{r \leq i \leq l} p_i u_i \in U\}$ where $p_i \in F[d_x, d_y]$. Denote by $J_r = \{p_r\} \subset F[d_x, d_y]$ the left ideal being the projection of U_r on the r-th component. Then the differential type of U coincides with the maximum of the differential types of $\{J_r\}_{1 \leq r \leq l}$ (one can verify this, e. g. using the Janet bases of $\{J_r\}_{1 \leq r \leq l}$ which provide a triangular Janet base of U). Take the minimal r_0 such that J_{r_0} has the differential type at least 1.

One has the natural action $U \times V^l \to V$ on the free $F[d_x, d_y]$-module $V^l = \{\sum_{1 \leq i \leq l} H_i v_i\}$ where $H_i \in V$ (cf. Lemma 4.3) and $v_1, \ldots, v_l$ is a free base of V^l. If $\sum_{1 \leq i \leq l} p_i H_i = 0$ then we call $\sum_{1 \leq i \leq l} H_i v_i$ a solution of $\sum_{1 \leq i \leq l} p_i u_i$ (we shall choose G and thereby, $V = V_G$ later). We are looking for a solution of the form $\sum_{1 \leq i \leq l} H_i v_i$ of the module U.

First we put $H_{r_0+1} = \cdots = H_l = 0$ and as $H_{r_0} \neq 0$ take a fractional-derivatives series being a solution of the ideal J_{r_0} according to Corollary 4.7 in case when the differential type of J_{r_0} equals 1. When the differential type of J_{r_0} equals 2, in other words, $J_{r_0} = 0$, we take as $H_{r_0} \neq 0$ an arbitrary fractional-derivatives series. In both cases $H_{r_0}v_{r_0}$ is a solution of the submodule U_{r_0}. Thus, we have chosen $G = G_{(s_2,\dots,s_k)}(f_1,\dots,f_k)$ and thereby, $V = V_G$. As above we can assume w.l.o.g. that in the equation $(a_1d_x + a_2d_y)f_1 = 0$ we have $a_1 \neq 0$, so $(d_x + ad_y)f_1 = 0$ (performing if necessary a suitable C-linear transformation of d_x, d_y).

Now we construct H_r by recursion on $r_0 - r \geq 0$. Suppose that we have already constructed an element $\sum_{r+1\leq i\leq l} H_iv_i$ being a solution of U_{r+1} for some $r+1 \leq r_0$. Since J_r has the differential type 0 (due to the choice of r_0), J_r contains a certain element $0 \neq b \in F[d_y]$. Consider a corresponding element $u = bu_r + \sum_{r+1\leq i\leq l} p_iu_i \in U_r$. According to Lemma 4.3 one can find $H_r \in V$ such that $bH_r + \sum_{r+1\leq i\leq l} p_iH_i = 0$. For any element $\bar{u} = \sum_{r\leq i\leq l} \bar{p}_iu_i \in U_r$ applying the left euclidean division in R one can represent $\bar{u} = \bar{p}_rb^{-1}u + \hat{u}$ for an appropriate $\hat{u} \in U_{r+1}$. Then $\hat{u}(\sum_{r\leq i\leq l} H_iv_i) = 0$ by the recursive hypothesis. Besides, $\bar{p}_rb^{-1}u(\sum_{r\leq i\leq l} H_iv_i) = 0$ because of Lemma 4.3. Hence $\bar{u}(\sum_{r\leq i\leq l} H_iv_i) = 0$ which completes the recursive step.

Summarizing, the following main theorem of the paper is proved.

Theorem 4.9 *Any (non-holonomic) module in $(F[d_x, d_y])^l$ of the differential type at least 1 has a fractional-derivatives series non-zero solution.*

Remark 4.10 *One could consider an ideal $J \subset F[d_{x_1},\dots,d_{x_t}]$ still of the differential type 1 with a number of derivatives $t \geq 3$ and ask whether J has always a fractional-derivatives series solution? The answer to this question is negative already for $t = 3$ and an ideal $J = \langle p_1, p_2\rangle \subset F[d_{x_1}, d_{x_2}, d_{x_3}]$ (being generic of the differential type 1) generated by an operator p_1 of the first order and p_2 of the second order.*

5 Duality between non-holonomic ideals and fractional-derivatives series solutions

There is a well-known duality [13] between (left) differential ideals and their spaces of solutions (being an analogue of the duality between radical ideals and varieties in algebraic geometry). To establish a similar duality for non-holonomic ideals in $F[d_x, d_y]$ (so, of the differential type 1) we need to make use of the equivalence relation on ideals introduced in [10]. We say that non-holonomic ideals $0 \neq J, J_0 \subset F[d_x, d_y]$ are *equivalent* if the leading coefficients of degree 1 (see Section 4) Hilbert-Kolchin polynomials of three ideals $J, J_0, J \cap J_0$ coincide (denote these leading coefficients by e), then moreover, $GCD(symb(J)) = GCD(symb(J_0)) = GCD(symb(J \cap J_0))$ and the degree of the latter polynomial equals e (see Lemma 4.1). In this case ideal $\langle J, J_0\rangle$ is also non-holonomic and $GCD(symb(\langle J, J_0\rangle)) = GCD(symb(J))$ as well [10, 3, 24] and moreover, clearly four ideals J, J_0, $J \cap J_0$, $\langle J, J_0\rangle$ are equivalent. Equivalence classes of ideals play a similar role to classes (in algebraic geometry) of plane curves with the same sets of 1-dimensional components. In this Section we prove that the sets of fractional-derivatives series solutions of equivalent non-holonomic ideals coincide and that there is a duality between the equivalence classes of ideals and their respective sets (which basically means that to distinct classes correspond distinct sets).

In this Section we keep the notations from Section 4. The next lemma states that the multiplication by $0 \neq p \in F[d_x, d_y]$ on D-module $V = V_G$ is an epimorphism (the conditions on its injectivity follow from Theorem 2.5, cf. also Remark 4.4).

Lemma 5.1 *For any* $0 \neq p \in F[d_x, d_y]$ *we have* $pV = V$.

Proof. Let $0 \neq H_1 = \overline{h}G^{(s_0)} + \sum_{i\geq 1} \overline{h_i} G^{(s_0-i/q)} \in V$ (see (2)). We search for $H = hG^{(s)} + \sum_{i\geq 1} h_i G^{(s-i/q)} \in V$ such that $pH = H_1$. Treating h, s as indeterminates we get from Lemma 1.4 that

$$p(hG^{(s)}) = \eta G^{(s+\kappa/q)} + \sum_{1\leq i\leq \kappa} \eta_i G^{(s+\kappa/q-i/q)} \tag{7}$$

for certain integer $\kappa \geq 0$ and linear ordinary differential operators $\eta \neq 0, \eta_i$ in h (with coefficients being differential polynomials in $f_1, \dots, f_k$ which we recall are assumed to be fixed). Indeed, such κ with non-zero η exists since in the expansion of $p(hG^{(s)})$ the coefficient at $G^{(s)}$ in (7) equals $p(h)$, so is a non-zero linear ordinary differential operator. Therefore, we put $s = s_0 - \kappa/q$, and there exists $h \in F$ for which $\eta(h) = \overline{h}$ (because F is differentially closed). At the next step comparing the coefficients of pH and H_1 at $G^{(s+\kappa/q-1/q)}$ one can find h_1 from an equation of the form $\eta(h_1) = \tilde{h}$ for certain $\tilde{h} \in F$, and so on one can find $h_2, h_3, \dots$ consecutively. ∎

First consider two equivalent non-holonomic ideals $J, J_0 \subset F[d_x, d_y]$. We claim that four sets of all fractional-derivatives series solutions of $J, J_0, J \cap J_0, \langle J, J_0\rangle$, respectively, coincide. As in Section 4 one can suppose w.l.o.g. that $d_y f_1$ does not divide $g = GCD(symb(J))$ and consider left ideals $\overline{J}, \overline{J_0} \subset R$ being principal. Let $\overline{J} = \langle p\rangle$, $\overline{J \cap J_0} = \langle p_0\rangle$ for suitable generators $p, p_0 \in F[d_x, d_y]$. Then $bp_0 = p_2 p$ for appropriate $p_2 \in F[d_x, d_y]$, $0 \neq b \in F[d_y]$. Lemma 4.5 implies that $symb(p_0)$ coincides with $symb(p)$ up to a power of $d_y f_1$, whence $symb(p_2)$ is a power of $d_y f_1$. Therefore, Proposition 4.6 and Remark 4.4 entail the required claim on coincidence of the sets of fractional-derivatives series solutions of ideals J, $J \cap J_0$ (and in a similar way also of J_0, $\langle J, J_0\rangle$).

Now let non-holonomic ideals $J \subsetneq J_1 \subset F[d_x, d_y]$ be non-equivalent. Our purpose is to find a fractional-derivatives series solution of J being not a solution of J_1. Denote $g_1 = GCD(symb(J_1))$, clearly $g_1 | g$. Again we suppose w.l.o.g. that $d_y f_1$ does not divide g. Let $\overline{J} = \langle p\rangle$, $\overline{J_1} = \langle p_1\rangle \subset R$. Then $b_1 p = p_3 p_1$ for suitable $p_3 \in F[d_x, d_y]$, $0 \neq b_1 \in F[d_y]$. Since J, J_1 are not equivalent we have $\deg(g_1) < \deg(g)$ (see Lemma 4.1) and because of that Lemma 4.5 implies that $symb(p_3)$ has a divisor of the form $d_x f_1 + a_3 d_y f_1$ for a certain $a_3 \in F$. Due to Theorem 2.5 there exists a fractional-derivatives series solution $H_1 = \sum_{i\geq i_0} h_i G^{(-i/q)}$ (see (2)) of equation $p_3 H = 0$ for appropriate $G = G_{s_2,\dots,s_k}(f_1, f_2, \dots, f_k)$ where $d_x f_1 + a_3 d_y f_1 = 0$. Now we apply Lemma 5.1 to p_1 and obtain a fractional-derivatives series $H \in V_G$ such that $p_1 H = H_1$. Therefore, $b_1 p H = 0$ and hence $pH = 0$ in view of Lemma 4.3. Thus, H is a desired solution of J being not a solution of J_1.

Finally, consider non-equivalent non-holonomic ideals $J, J_1 \subset F[d_x, d_y]$ and assume that their respective sets of fractional-derivatives series solutions coincide. Then ideal $\langle J, J_1\rangle$ has also the same set of fractional-derivatives series solutions, in particular $\langle J, J_1\rangle$ is non-holonomic by virtue of Remark 4.2 and of Theorem 2.5. Therefore, due to the proved above three ideals J, $\langle J, J_1\rangle$, J_1 are equivalent which contradicts to the assumption.

We summarize the proved duality in the following

Proposition 5.2 *Non-holonomic left ideals* $J, J_0 \subset F[d_x, d_y]$ *are equivalent if and only if they have the same sets of all fractional-derivatives series solutions.*

For a non-holonomic ideal $I \subset F[d_x, d_y]$ denote by $[I]$ the equivalence class of non-holonomic ideals which contains I and by $V(I)$ the set of all fractional-derivatives series solutions of I. In [10] we define the following partial ordering on the classes: $[J]$ is *subordinated* to $[I]$ if there exist ideals $J_1 \in [J]$, $I_1 \in [I]$ such that $J_1 \subset I_1$.

Corollary 5.3 *$[J]$ is subordinated to $[I]$ if and only if $V(I) \subset V(J)$.*

Proof. Let $V(I) \subset V(J)$, then $V(\langle I, J\rangle) = V(I)$. Proposition 5.2 entails that $\langle I, J\rangle$ is equivalent to I, hence $[J]$ is subordinated to $[I]$.

The inverse implication is evident. ■

Now we connect the subordination relation with localizations of ideals in the ring $R = F[d_x, d_y](F[d_y])^{-1}$ (see Section 4).

Proposition 5.4 *If $V(I) \subset V(J)$ then $\overline{J} \subset \overline{I} \subset R$ (provided that $d_y f_1$ does not divide $GCD(symb(I))$).*

Proof. Let $\overline{J} = \langle p\rangle$, $\overline{I} = \langle q\rangle$ for suitable $p, q \in F[d_x, d_y]$ (cf. above). Then $p = p_0 q$ for appropriate $p_0 \in R$. Whence $V(I) \subset V(J)$ relying on Proposition 4.6. ■

For a pair of left ideals $J \subset J_1 \subset F[d_{x_1}, \dots, d_{x_m}]$ we have introduced in [10] a concept of relative syzygies. Namely, let $J_1 = \langle p_1, \dots, p_t\rangle$, then we define the left *module of relative syzygies*

$$Syz(J, J_1) = \{(q_1, \dots, q_t) : \sum_{1\le i\le t} q_i p_i \in J;\ q_i \in F[d_{x_1}, \dots, d_{x_m}],\ 1 \le i \le t\}.$$

Making use of [20] one can verify [10] that module $Syz(J, J_1)$ is independent of a choice of generators $p_1, \dots, p_t$. Let us denote by $U(J) \subset F$ the space of solutions of J which can be treated as a C-vector space. It was proved in [10] that the quotient $U(J)/U(J_1)$ is isomorphic to $U(Syz(J, J_1)) \subset F^t$.

Here we establish a similar result for non-holonomic ideals $J \subset J_1 \subset F[d_x, d_y]$ and their spaces of fractional-derivatives series solutions $V_G(J) \subset V_G$ of the form (2) for any G fixed for the time being (see Section 4), again we treat $V_G(J)$ as a C-vector space. As in Section 4 one can assume w.l.o.g. that $d_y f_1 \ne 0$.

Mapping $\psi : v \to (p_1, \dots, p_t)^T v$ assures a monomorphism $V_G(J)/V_G(J_1) \hookrightarrow V_G(Syz(J, J_1))$. To show that it is an epimorphism take an arbitrary vector $(w_1, \dots, w_t) \in V_G(Syz(J, J_1)) \subset V_G^t$. The following property holds: for any $q_1, \dots, q_t \in F[d_x, d_y]$ such that $\sum_{1\le i\le t} q_i p_i = 0$ (moreover, one can suppose that $\sum_{1\le i\le t} q_i p_i \in J$) we have $\sum_{1\le i\le t} q_i w_i = 0$. Clearly, this property holds also for any $q_1 \dots, q_t \in R$ (see Section 4). Consider principal ideal $\overline{J_1} = \langle p_1, \dots, p_t\rangle = \langle p\rangle \subset R$ for suitable $p \in F[d_x, d_y]$. Then $p = b^{-1} \sum_{1\le i\le t} \eta_i p_i$ for appropriate $0 \ne b \in F[d_y]$, $\eta_1, \dots, \eta_t \in F[d_x, d_y]$. Denote $w = \sum_{1\le i\le t} \eta_i w_i \in V_G$. Due to Lemma 5.1 there exists $v \in V_G$ such that $(\sum_{1\le i\le t} \eta_i p_i)v = w$. For each $1 \le i_0 \le t$ one can find $\lambda_{i_0} \in R$ for which $p_{i_0} = \lambda_{i_0} p$, the mentioned above property implies that $w_{i_0} = \lambda_{i_0} b^{-1}(\sum_{1\le i\le t} \eta_i w_i)$, hence $w_{i_0} = p_{i_0} v$, i. e. $\psi(v) = (w_1, \dots, w_t)$. Finally, we check that $v \in V_G(J)$. Indeed, an arbitrary $q \in J$ can be represented as $q = \sum_{1\le i\le t} q_i p_i$ for certain $q_1, \dots, q_t \in F[d_x, d_y]$, then $(q_1, \dots, q_t) \in Syz(J, J_1)$, whence $qv = \sum_{1\le i\le t} q_i w_i = 0$. Thus, in the introduced notations we have proved the following

Proposition 5.5 *For any non-holonomic ideals $J \subset J_1 \subset F[d_x, d_y]$ and G there is an isomorphism of C-vector spaces $V_G(J)/V_G(J_1)$ and $V_G(Syz(J, J_1))$.*

One can deduce Proposition 5.2 from the latter Proposition invoking Theorem 4.9.

It would be interesting to clarify, whether for non-holonomic ideals $I, J \subset F[d_x, d_y]$ the equality

$$GCD(symb(I)) \cdot GCD(symb(J)) = GCD(symb(\langle I, J \rangle)) \cdot GCD(symb(I \cap J))$$

holds? Observe that the degrees of the polynomials in both sides of the latter equality coincide in view of [3, 24] taking into account Lemma 4.1. A more subtle question is whether for any G the equality $V_G(I \cap J) = V_G(I) + V_G(J)$ is true?

6 Completeness of fractional-derivatives solutions for separable linear partial differential operators

Let $T = T_n + \cdots + T_0 \in F[d_x, d_y]$ be a *separable* LPDO, i.e. its symbol $symb(T) = \overline{T_n} = \prod_{1 \le i \le n}(d_x f - a_i d_y f)$ is the product of n pairwise distinct homogeneous linear forms in $d_x f, d_y f$. One can always bring $symb(T)$ to this form monic with respect to $d_x f$ making, if necessary, a C-linear transformation of d_x, d_y in case when $symb(T)$ has a divisor $d_y f$.

For each $1 \le i \le n$ the equation $T = 0$ has a fractional-derivatives series solution of the form (due to Theorem 2.5)

$$h_{0,i} G^{(0)}(f_i) + h_{1,i} G^{(-1)}(f_i) + \cdots \tag{8}$$

where $d_x f_i - a_i d_y f_i = 0$ and $h = h_{0,i}$ satisfies the first-order LPDE

$$\frac{\overline{T_n}(f_i)}{(d_x f_i - a_i d_y f_i)} (d_x h - a_i d_y h) + \overline{T_{n-1}}(f_i) h = 0 \tag{9}$$

We observe that $h_{j,i}; j = 1, 2, \ldots$ satisfy similar to (9) equations with the highest (first-order) form $\frac{\overline{T_n}(f_i)}{(d_x f_i - a_i d_y f_i)} (d_x h_{j,i} - a_i d_y h_{j,i})$, being not necessary homogeneous.

From now on throughout this section we assume that F is the field of meromorphic functions in a certain domain $M \subset \mathbb{C}^2$, thus the coefficients of T belong to F. For a suitable point $(x_0, y_0) \in M$ the series (8) can be rewritten as a formal power series in $x - x_0, y - y_0$. Our goal is to find a point (x_0, y_0) and look for solutions of $T = 0$ as power series in $x - x_0, y - y_0$.

We choose a point $(x_0, y_0) \in M$ such that all the coefficients of T at this point are defined and in addition, the values $a_i(x_0, y_0)$ are pairwise distinct for $1 \le i \le n$. The latter is equivalent to that the discriminant of $symb(T)$ does not vanish at this point. Therefore, all the points of M out of an appropriate analytic subvariety of M of the dimension 1 satisfy these requirements.

One takes a solution f_i (being a power series in $x - x_0, y - y_0$) of the equation $d_x f_i - a_i d_y f_i = 0$ with a vanishing free coefficient (which we denote by $f_i(x_0, y_0) = 0$) and with a non-vanishing vector of coefficients at the first powers of $x - x_0, y - y_0$ (which we denote by $(d_x f_i, d_y f_i)(x_0, y_0)$, thereby $d_y f_i(x_0, y_0) \neq 0$). We observe that this LPDE has always a solution with arbitrary chosen free coefficient and non-vanishing vector of the coefficients at the first powers of $x - x_0, y - y_0$ since the vector of the coefficients $(1, -a_i)$ at its highest (first) derivatives does not vanish at the point (x_0, y_0). Hence the free coefficient of the power series $d_x f_i - a_j d_y f_i$ does not vanish when $j \neq i$ due to the requirement on the discriminant. Therefore, by the same token one can find a solution h of the equation (9) with a non-zero free coefficient which we denote by $h(x_0, y_0) \neq 0$.

We take an arbitrary solution of $T = 0$, being a power series in $x - x_0, y - y_0$ and intend to represent it as a sum of n solutions of the form (8) (for $1 \leq i \leq n$) in which $G^{(0)}(f_i)$ is replaced by its specialization (see Remark 1.3)

$$\sum_{j \geq 0} c_{j,i} \frac{f_i^j}{j!}$$

with indeterminate coefficients $c_{j,i} \in \mathbb{C}$. Then

$$G^{(-l)} = \sum_{j \geq 0} c_{j,i} \frac{f_i^{j+l}}{(j+l)!}.$$

Suppose that by recursion on k the coefficients $c_{j,i}$ for $j \leq k-1, 1 \leq i \leq n$ are already produced. Our purpose is to produce $c_{k,i}, 1 \leq i \leq n$. Clearly, any solution of $T = 0$ being a power series of the form $\sum_{p,q \geq 0} b_{p,q}(x - x_0)^p (y - y_0)^q$ is determined by the coefficients $b_{p,q}$ with $0 \leq p \leq n-1$.

For each $0 \leq p \leq n-1$ the contribution of the term at $c_{k,i}$ (see (8)) into $b_{p,k-p}$ equals to

$$h(x_0, y_0)(d_x^p d_y^{k-p} \frac{f_i^k}{k!})(x_0, y_0) = h(x_0, y_0)(a_i^p (d_y f_i)^k)(x_0, y_0)$$

taking into account that $f_i(x_0, y_0) = 0, d_y f_i(x_0, y_0) \neq 0, h(x_0, y_0) \neq 0$.

Therefore, we obtain a linear (algebraic) system (in general, not necessary homogeneous) on $c_{k,i}; 0 \leq i \leq n-1$ with the matrix being of the van-der-Monde type $(a_i^p(x_0, y_0))$. This allows one to find uniquely $c_{k,i}; 0 \leq i \leq n-1$ and thereby, carry out the recursive step.

Theorem 6.1 *For a separable LPDO T of the order n with the coefficients being meromorphic in a certain complex domain M the sum of n spaces of specialisations of fractional-derivatives series solutions of $T = 0$ of the form (8) (for fixed f_i and $h_{j,i}$) coincides with the space of all the solutions of $T = 0$ as formal power series in $x - x_0, y - y_0$ for any point (x_0, y_0) from M out of a suitable analytic subvariety of the dimension 1.*

It would be interesting to extend this theorem to a non-separable LPDO. Let us also mention that in [9] an algorithm for factoring a separable LPDO was produced.

7 Applications to studying first-order factors of a linear partial differential operator

7.1 Finding first-order factors of a linear partial differential operator

Let $T = T_n + \ldots + T_0$ be a LPDO of an order n in 2 independent variables, where $T_j = \sum_i a_{i,j-i} d_x^i d_y^{j-i}$ is a sum of the derivatives of the order j. We assume that the coefficients $a_{i,j}$ are taken from the field $\mathbb{Q}(x, y)$ in order to design algorithms, while f is taken from a universal field F (cf. Section 6).

As we are looking for the first-order factors of T of the form $L = d_x + ad_y + b \in F[d_x, d_y]$ we need to study the solutions of $L = 0$ (w.l.o.g. one can assume that the coefficient at d_x of L does not vanish, otherwise one can change the roles of x and y). Take any solution f of the symbol $(d_x + ad_y)f = 0$ of L such that $d_y f \neq 0$ and consider $G = G^{(0)}(f)$ (cf. Section 6). For any $h \in F$ being a "particular" solution of $L = 0$, we have that hG is a fractional-derivatives series solution of $L = 0$.

Lemma 7.1 *An operator T has a right first-order factor L if and only if the equation $T = 0$ has a solution of the form hG.*

Proof. If T has a right factor L then T has a solution hG.

Conversely, assume that $T = 0$ has a solution hG. Dividing T with remainder by L one can represent $T = SL + \sum_{0 \le i \le n} b_i d_y^i$ for a suitable operator S. Consider the largest k such that $b_k \neq 0$. Then in the expansion of $(\sum_{0 \le i \le k} b_i d_y^i) hG$ in $\{G^{(s)}\}$ the coefficient at $G^{(k)}$ equals $b_k h (d_y f)^k \neq 0$. The obtained contradiction shows that $T = SL$. ∎

Thus, we are looking for a solution of $T = 0$ of the form hG. Expanding $T(hG) = A_0 G^{(0)} + \cdots + A_n G^{(n)}$, we get first that $A_n = symb(T)$. Therefore, we fix for the time being a linear divisor of the form $d_x f + ad_y f$ of $symb(T)$ and assume that this divisor vanishes. Thereby, the calculations below (arthmetic manipulations and polynomial factoring) will be carried out over the field $\mathbb{Q}(x,y)[a]$. This can be fulfilled representing $\mathbb{Q}(x,y)[a] \simeq \mathbb{Q}(x,y)[z]/(g)$ where $g \in \mathbb{Q}(x,y)[z]$ is the minimal polynomial of a (see [5]). So, we obtain n equations $A_0 = \cdots = A_{n-1} = 0$ treated as LPDO in h with the coefficients being non-linear differential polynomials in f. We denote the ring of all these polynomials by $P = \mathbb{Q}(x,y)[a]\{d_x f, d_y f\}$. Applying to $A_0 = \cdots = A_{n-1} = 0$ the procedure of constructing a Janet base [22] one gets the conditions of solvability in h of $A_0 = \cdots = A_{n-1} = 0$ expressed as a disjunction of systems of the form

$$p_1 = \cdots = p_l = 0, p_0 \neq 0 \tag{10}$$

where $p_i \in P$. Using the relation $d_x f + ad_y f = 0$ one can reduce each p_i to an (ordinary) differential polynomial $\bar{p}_i$ in $d_y f$. Denote the ring of ordinary differential polynomials by $R = \mathbb{Q}(x,y)[a]\{d_y f\}$.

Applying to the formula $\bar{p}_1 = \cdots = \bar{p}_l = 0, \bar{p}_0 \neq 0$ the subroutine of the elimination procedure in the theory of ordinary differentially closed fields from [23] (see also [6] where its improvement with a better complexity bound was designed) one obtains an equivalent disjunction of systems of the form

$$r = 0, r_0 \neq 0 \tag{11}$$

for suitable differential polynomials $r, r_0 \in R$. Briefly, this subroutine consists in alternative executing 2 types of steps while there are more than one equality of (ordinary) differential polynomials. The first type of steps is executed when all the highest derivatives occurring in these polynomials are equal, in this case the algorithm calculates their GCD viewing them as (algebraic) polynomials in this highest derivative (and branching depending on vanishing the leading coefficients). Else, if not all the highest derivatives are equal, as the second type of steps one can diminish the highest derivative. Moreover, if r contains the $d_y^k f$ as its highest derivative then r considered as an (algebraic) polynomial in the ring $K = \mathbb{Q}(x,y)[a][f, d_y f, \ldots, d_y^k f]$ is irreducible. In addition, r_0 is less than r with respect to the term ordering, i. e. if r_0 contains $d_y^{k_0} f$ as its highest derivative then either $k_0 < k$ or $k_0 = k$ and the degree of r_0 with respect to $d_y^k f$ is less than the similar degree of r.

Replace $d_x f$ by $-ad_y f$ in $d_x r$. This yields a differential polynomial $\hat{r} \in R$ of the order at most $k+1$ (its role is similar to an S-pair in Janet type algorithm [22]). If $\hat{r}$ does not belong to the differential ideal $\langle r \rangle \subset R$, we again apply to the system $r = \hat{r} = 0, r_0 \neq 0$ the used above subroutine from the elimination procedure and get an equivalent disjunction of systems of the form (11) with less term ordering than of r and continue as above.

Now assume that $\hat{r}$ belongs to $\langle r \rangle$. Then we claim that any solution of (11) provides a solution of (10). Indeed, otherwise, the ideal $\langle r, d_x f + ad_y f \rangle \subset P$ would contain an appropriate power r_0^s [13], p.146-148. This yields a relation of the form

$$r_0^s = \sum_{i,j} A_{i,j} d_x^i d_y^j r + \sum_{i,j} B_{i.j} d_x^i d_y^j (d_x f + ad_y f)$$

for suitable $A_{i,j}, B_{i,j} \in P$. Replacing in this relation $d_x f$ for $-ad_y f$ and taking into account that $\hat{r}$ belongs to $\langle r \rangle$, we deduce that

$$r_0^s = \sum_j \hat{A}_j d_y^j r \tag{12}$$

for certain $\hat{A}_j \in R$. From the equation $d_y r = 0$ we express

$$d_y^{k+1} f = \hat{B}_{k+1} / \frac{\partial r}{\partial (d_y^k f)}$$

for an appropriate $\hat{B}_{k+1} \in K$. After that express successively

$$d_y^{k+2} f = \hat{B}_{k+2} / \frac{\partial r}{\partial (d_y^k f)}, d_y^{k+3} f = \hat{B}_{k+3} / \frac{\partial r}{\partial (d_y^k f)}, \dots .$$

Substitute these expressions in (12), this results in the equality

$$r_0^s \left(\frac{\partial r}{\partial (d_y^k f)}\right)^t = Ar$$

for some t and $A \in K$. But r is irreducible in K and r_0 is less than r with respect to the term ordering. The obtained contradiction proves the claim and the following theorem.

Theorem 7.2 *There is an algorithm which tests whether an operator $T \in \mathbb{Q}(x,y)[d_x, d_y]$ has a first-order factor with the coefficients in a universal field F. The algorithm invokes two subroutines: the elimination of an unknown function in a system of LPDO's (in other words, a parametric Janet base), and a subroutine from the elimination procedure in the theory of ordinary differentially closed fields.*

Remark 7.3 *If one uses a direct method of finding the coefficients of a first-order operator L and of an $(n-1)$-th order Q such that $T = QL$, then one has to apply an elimination in the theory of partial differentially closed fields whose complexity is unclear how to estimate in a reasonable way (cf. [23, 6]).*

Remark 7.4 *One can also search for left first-order factors of an LPDO (by means of considering an adjoint operator).*

Corollary 7.5 *There is an algorithm to factor LPDO's of the orders at most 3.*

7.2 Intersection of principal first-order ideals

In this subsection by F we denote a differential field with derivatives d_x, d_y.

First consider the ideals $I_i = \langle d_x + ad_y + b_i \rangle$ with the same highest (first-order) forms where $a, b_i \in F, 1 \leq i \leq n$.

Proposition 7.6 *The ideal* $I_1 \cap \cdots \cap I_n$ *is principal*

Proof. Denote $E = d_x + ad_y$. The ring $F[E]$ is left-euclidean, therefore, the intersection $\hat{I_1} \cap \ldots \cap \hat{I_n} = \langle Q \rangle \subset F[E]$ is principal where we denote $\hat{I_i} = \langle d_x + ad_y + b_i \rangle \subset F[E]$ and $Q = q_s E^s + \cdots + q_0$ for certain $q_0, \ldots, q_s \in F, q_s \neq 0$ and $s \leq n$.

Our aim is to prove by induction on n that $I_1 \cap \cdots \cap I_n = \langle Q \rangle \subset F[d_x, d_y]$. Assume that it is already proved and consider the intersection $I_1 \cap \cdots \cap I_n \cap I_{n+1}$. There can occur two cases. Either $\hat{I_1} \cap \ldots \cap \hat{I_n} \cap \hat{I}_{n+1} = \hat{I_1} \cap \ldots \cap \hat{I_n}$, in this case $Q = N(d_x + ad_y + b_{n+1})$ for a suitable $N \in F[E]$, therefore, $I_{n+1} \supset \langle Q \rangle$ and $I_1 \cap \cdots \cap I_n \cap I_{n+1} = \langle Q \rangle$.

Or else $\hat{I_1} \cap \ldots \cap \hat{I_n} \supsetneq \hat{I_1} \cap \ldots \cap \hat{I_n} \cap \hat{I}_{n+1} = \langle M \rangle$ for an appropriate $M = m_{s+1}E^{s+1} + \cdots + m_0 \in F[E]$ with $m_0, \ldots, m_{s+1} \in F$. Clearly, $M \in I_1 \cap \cdots \cap I_n \cap I_{n+1}$. It is necessary to show that for any $V \in I_1 \cap \cdots \cap I_n \cap I_{n+1}$ we have $V \in \langle M \rangle$. Since the highest derivative with respect to d_x which occurs in M is d_x^{s+1}, one can divide V by M with remainder and get $V = WM + U$ where $W, U \in F[d_x, d_y]$ for a certain $U \in I_1 \cap \cdots \cap I_n \cap I_{n+1}$ such that $s_0 = ord_{d_x}(U) \leq s$. If $U = 0$ we are done, so suppose that $U \neq 0$. We have

$$U = ZQ = T(d_x + ad_y + b_{n+1}) \tag{13}$$

for suitable $Z, T \in F[d_x, d_y]$, hence $s_0 = s$ and $ord_{d_x}(Z) = 0, ord_{d_x}(T) = s - 1$. One can expand $T = t_{s-1}E^{s-1} + \cdots + t_0$ for appropriate $t_0, \ldots, t_{s-1} \in F[d_y]$. Thus, the equation (13) one rewrite with respect to the powers of E:

$$Z(q_s E^s + \cdots + q_0) = (t_{s-1}E^{s-1} + \cdots + t_0)(E + b_{n+1})$$

which is equivalent to a system of the following $s+1$ equalities:

$$Zq_j = t_{j-1} + t_j b_{j,j} + t_{j+1} b_{j,j+1} + \cdots + t_{s-1} b_{j,s-1} \tag{14}$$

for suitable $b_{j,j}, \ldots, b_{j,s-1} \in F; 1 \leq j \leq s$ and

$$Zq_0 = t_0 b_{n+1} + t_1 b_{0,1} + t_2 b_{0,2} + \cdots + t_{s-1} b_{0,s-1} \tag{15}$$

Viewing the right-hand sides of the equations (14), (15) as a linear system in $t_0, \ldots, t_{s-1}$ we get that there is a unique linear combination (from the right) of s expressions in the right-hand sides of (14) which equals (15), the coefficients $f_1, \ldots, f_s$ of this combination belong to F. Therefore, the solvability of (14), (15) in $Z \neq 0, t_0, \ldots, t_{s-1}$ entails the equality

$$q_1 f_1 + \cdots + q_s f_s = q_0 \tag{16}$$

Thus (13) implies (16). Hence as a solution of the system (14), (15) one can take $Z = 1$ and consecutively express $t_{s-1} \in F$ from the equation (14) with $j = s$, after that express $t_{s-2} \in F$ from the equation (14) with $j = s - 1$ and so on, finally express $t_0 \in F$ from (14) with $j = 1$. The last equation (15) of the system is fulfilled due to (16). As a result we obtain (cf. (13)) $Q = (t_{s-1}E^{s-1} + \cdots + t_0)(E + b_{n+1})$ with $t_i \in F$, in other words $\hat{I_1} \cap \ldots \cap \hat{I_n} = \langle Q \rangle \subset \hat{I}_{n+1} \subset F[E]$.

This leads to contradiction with the assumption $\hat{I}_1 \cap \ldots \cap \hat{I}_n \supsetneq \hat{I}_1 \cap \ldots \cap \hat{I}_n \cap \hat{I}_{n+1}$, which shows that the supposition $U \neq 0$ was wrong, thus $I_1 \cap \ldots \cap I_n \cap I_{n+1} = \langle M \rangle$. The proposition is proved. ■

Corollary 7.7 *The ideal $I_1 \cap \ldots \cap I_n$ is generated by an element from $F[E]$.*

Now let the ideals $I_i = \langle d_x + a_i d_y + b_i \rangle \subset F[d_x, d_y]$ be given, where $a_i, b_i \in F, 1 \leq i \leq k$. Our goal is to study their intersection $I = I_1 \cap \ldots \cap I_k$. Combining together all the classes of the ideals with the same a_i and making use of Corollary 7.7 we replace the intersection from one class by $\langle Z_i \rangle$ for a certain $Z_i \in F[E_i]$ where $E_i = d_x + a_i d_y$. Then $I = I_1 \cap \ldots \cap I_k = \langle Z_1 \rangle \cap \cdots \cap \langle Z_l \rangle$ for some l. Denote $s_i = ord(Z_i); 1 \leq i \leq l$ and $s = s_1 + \cdots + s_l$.

Lemma 7.8 *For any $Q \in I$ we have $ord_{d_x}(Q) \geq s$.*

Proof. Observe that $symb(Q)$ is divided by $\prod_{1 \leq i \leq l} (d_x f + a_i d_y f)^{s_i}$ treated as a homogeneous polynomial in $d_x f, d_y f$. ■

Theorem 7.9 *a) The ideal I is principal if and only if I contains Q with the order $ord(Q) \leq s$;*

b) in this case $ord(Q) = s$ and $I = \langle Q \rangle$.

Proof. Obviously, the typical differential dimension $dim(\langle Z_i \rangle) = s_i; 1 \leq i \leq l$ [13] and $dim(I) \leq s$ due to [3, 24]. Hence if $I = \langle L \rangle$ is principal then $ord(L) = dim(I) \leq s$.

Conversely, let $Q \in I$ and $ord(Q) \leq s$, by virtue of Lemma 7.8 we have $ord(Q) = s$ and the derivative d_x^s occurs in Q. Our purpose is to show that $I = \langle Q \rangle$. Indeed, take any $V \in I$ and divide V by Q with remainder, we get $V = WQ + U$ where $ord_{d_x}(U) < s$, therefore, $U = 0$ due to Lemma 7.8. Thus, $I = \langle Q \rangle$. ■

Corollary 7.10 *Let the differential field $F = \mathbb{Q}(x, y)$. There is a polynomial-time algorithm which tests whether I is principal.*

Proof. First the algorithm produces $Z_i; 1 \leq i \leq l$ by finding a non-zero solution of a linear (algebraic) homogeneous system on the coefficients from F of $T_1, \ldots, T_n \in F[E_i]$ such that $T_1(d_x + a_i d_y + b_1) = \cdots = T_n(d_x + a_i d_y + b_n)$ with the minimal possible order $ord(T_1) = \cdots = ord(T_n)$ (trying consecutively the orders 1,2...). Denote $Z_i = T_1(d_x + a_i d_y + b_1), s_i = ord(Z_i)$, then Z_i is a generator of the ideal $\langle d_x + a_i d_y + b_1 \rangle \cap \cdots \cap \langle d_x + a_i d_y + b_n \rangle$, see Corollary 7.7.

Thereupon the algorithm looks for $V_1, \ldots, V_l \in F[d_x, d_y]$ with $ord(V_i) \leq s - s_i; 1 \leq i \leq l$ such that $V_1 Z_1 = \cdots = V_l Z_l$. The latter we treat as a linear (algebraic) homogeneous system in the coefficients from F of $V_1, \ldots, V_l$. Theorem 7.9 entails that this system has a non-zero solution if and only if I is principal. ■

Remark 7.11 *Observe that the usual method of finding the intersection of ideals invoking Groebner bases, runs in double-exponential time.*

7.3 Constructing intersection of all first-order factors

In this subsection F denotes a universal field [13] with two derivatives d_x, d_y.

The purpose of this subsection is to construct the intersection $U \subset F[d_x, d_y]$ of all the principal ideals $\langle L\rangle$ for the first-order factors $L \in F[d_x, d_y]$ of $T \in \mathbb{Q}(x, y)[d_x, d_y]$. Evidently, $U \supset \langle T\rangle$. We mention that in [10] a radical of a module of a differential type τ was defined as the intersection of the maximal classes of τ-equivalent modules, and a question was posed whether one can calculate the radical. Here U (which could be called a *first-order radical*) is defined as an ideal (rather than a class of equivalent ideals) and moreover, we calculate U.

Observe that the construction from the Subsection 7.1 represents the family V of all the solutions of the form hG (and which correspond to first-order factors of T due to Lemma 7.1) as follows (we use the notations from Subsections 7.1, 7.2). We assume that a is fixed, while f just satisfies the equality $d_x f + a d_y f = 0$. The family V is a union of subfamilies of the form V_0 where V_0 is given by means of a Janet base

$$\{\sum_{i_1,i_2} v_{i_1,i_2,l} d_x^{i_1} d_y^{i_2} h\}_l \tag{17}$$

for h where $v_{i_1,i_2,l} \in R$ together with a system (11) for f.

For each element $hG \in V$ consider the first-order LPDO $L_{hG} = d_x + a d_y + b_{hG}$ such that $L_{hG}(hG) = 0$ (see Lemma 7.1). We claim that one can extend Proposition 7.6 from a finite to an infinite number of principal ideals and conclude that the ideal $\cap_{hG\in V}\langle L_{hG}\rangle$ is principal and moreover, is generated by a suitable element $Q = \sum_{0\le i\le s} q_i E^i \in F[E]$ (see Corollary 7.7). Indeed, one add consecutively the ideals $I_1 = \langle L_{h_1G_1}\rangle, I_2 = \langle L_{h_2G_2}\rangle, \dots$ for $h_jG_j \in V$, while the intersection $\hat{I}_1 \cap \cdots \cap \hat{I}_{j-1} \cap \hat{I}_j \subsetneq \hat{I}_1 \cap \cdots \cap \hat{I}_{j-1}$ decreases (cf. the proof of Proposition 7.6). Then $\hat{I}_1 \cap \cdots \cap \hat{I}_j = \langle Q_j = \sum_{0\le i\le j} q_{i,j} E^i\rangle$ for appropriate $q_{i,j} \in F$ (cf. the proof of Proposition 7.6). Hence $\langle T\rangle \subset I_1 \cap \cdots \cap I_j = \langle Q_j\rangle$ due to Corollary 7.7. Thus, $j \le n$ and $\cap_{hG\in V}\langle L_{hG}\rangle = I_1 \cap \cdots \cap I_j$ which proves the claim.

To produce $Q = Q_j = \sum_{0\le i\le j} q_i E^i$ the algorithm successively tries $j = 0, 1, \dots$, treating q_i as indeterminates. The aim is to find Q such that $Q(hG) = 0$ for any $hG \in V_0$ (for each subfamily V_0 of V). The algorithm expands $Q(hG) = A_0 G^{(0)} + \cdots + A_j G^{(j)}$ (cf. Subsection 7.1). One can view each A_i as an LPDO in h with the coefficients being linear forms in $q_0, \dots, q_j$ over R. The algorithm divides every $A_i, 0 \le i \le j$ with the remainder by the Janet base (17), as a result we obtain LPDO $\bar{A}_i = \sum_{i_1,i_2} a_{i,i_1,i_2} d_x^{i_1} d_y^{i_2}$. Thus, Q vanishes at any $hG \in V_0$ if and only if $a_{i,i_1,i_2} = 0$ for all $0 \le i \le j; i_1, i_2$ under condition (11).

Denote by $\mathcal{S}$ the conjunction of the systems $a_{i,i_1,i_2} = 0$ for all $0 \le i \le j; i_1, i_2$ and for all subfamilies of the form V_0 of V. One can treat $\mathcal{S}$ as a homogeneous linear over $q_0, \dots, q_j$ system with parameters being derivatives $f, d_y f, \dots, d_y^l f$ for a certain l. Solving this parametric linear system (see e.g. [7]) the algorithm finds the (algebraic) conditions on $f, d_y f, \dots, d_y^l f$ under which the system is solvable and in addition, finds the expressions for solutions (being rational functions in the parameters). After that the algorithm tests whether these conditions are compatible with (11), applying the subroutine from the elimination procedure which yields formula (11) in Subsection 7.1. If yes then the algorithm produces a solution $q_0, \dots, q_j \in R$ of the parametric linear system. Else, the algorithm proceeds from the current value j to the next value $j + 1$.

Thus, the algorithm for each a such that $d_x f + a d_y f$ is a (linear) divisor of $symb(T)$, produces applying the described above construction a generator $Q_a \in F[d_x, d_y]$ of the (principal) ideal being the intersection of all the principal ideals generated by the divisors of the form

$d_x + ad_y + b$ of T for varying b. Finally, the algorithm finds the intersection $U = \cap_a \langle Q_a \rangle$ over all the divisors $d_x f + ad_y f$ of *symb*(T) making use of Janet bases (cf. [10]). Thus, the following theorem is proved.

Theorem 7.12 *For any LPDO $T \in \mathbb{Q}(x,y)[d_x, d_y]$ one can construct the intersection of all the principal ideals generated by the first-order factors of T.*

8 Fractional-derivatives series solutions of a second-order operator and factoring

In this section we study a particular case of a second-order LPDO $T = T_0 + T_1 + T_2$ and describe its possible fractional-derivatives series solutions being outputs of the algorithm from Section 2. First, if the symbol *symb*(T) is separable then for each of its two different linear divisors $d_x f_1 + ad_y f_1$ the algorithm provides a fractional-derivatives series solution of T of the form (cf. (8))

$$\sum_{0 \le i < \infty} h_i G^{(-i)}$$

where $G = G(f_1)$ and $d_x f_1 + ad_y f_1 = 0, \quad f_1 \neq const.$ The vertices of the Newton polygon P_2 are $(0,2),\ (0,0),\ (1,1),\ (1,0)$ and its only edge with a slope less than 1 is the edge with endpoints $(1,1),\ (1,0)$ having slope 0. Thus, the construction of G terminates after the first step of the algorithm.

From now on let us assume that *symb*(T) is non-separable and write $T = d_x^2 + 2ad_x d_y + a^2 d_y^2 + b_{0,1} d_x + b_{1,0} d_y + b_{0,0}$. The first step of the algorithm from Section 2 yields f_1 such that $d_x f_1 + ad_y f_1 = 0$. Introduce the discriminant of T as follows:

$$(-T + b_{0,0}) f_1 = (d_x a + 2ad_y a + ab_{0,1} - b_{1,0})(d_y f_1) =: Disc \cdot (d_y f_1).$$

If $Disc \neq 0$ we take any f_2 which satisfies the following (non-homogeneous) first-order LPDE:

$$d_x f_2 + ad_y f_2 = \sqrt{Disc \cdot (d_y f_1)}$$

and $G = G_{1/2}(f_1, f_2)$ (see Definition 1.1), then the algorithm constructs a fractional-derivatives series

$$\sum_{0 \le i < \infty} h_i G^{(-i/2)}$$

being a solution of $T = 0$. Each of two values of the sign of the square root provides a generic solution of the multiplicity 1 (see Section 3). It corresponds to the leading edge of the Newton polygon P_2 (whose vertices are $(0,2),\ (0,0),\ (1,0)$) with endpoints $(0,2),(1,0)$ having the slope $1/2$ at the second step of the algorithm. After the second step the vertices of the Newton polygon P_3 are $(0,2),\ (0,0),\ (1/2,1),\ (1/2,0)$ and its only edge with a slope less than $1/2$ is the edge with endpoints $(1/2,1),\ (1/2,0)$ having slope 0. Thus, the construction of G terminates after the second step of the algorithm.

When $Disc = 0$ the algorithm yields a (fractional-derivatives series) solution $hG(f_1)$ of $T = 0$ for an arbitrary particular h such that $T(h) = 0$. It corresponds to the leading edge with endpoints $(0,2),(0,0)$ having the slope 0 at the second step of the construction (the Newton polygon P_2 coincides with the same edge), thus the construction of G terminates after the first step and the algorithm provides a generic solution of the multiplicity 2. Relying on Lemma 7.1 one obtains the following corollary (cf. [9]).

Corollary 8.1 *A second-order LPDO with a non-separable symbol is irreducible if and only if* $Disc \neq 0$.

Acknowledgements. The author is grateful to the Max-Planck Institut fuer Mathematik, Bonn where the paper was written, to Y.Manin for his attention to the work and to F.Schwarz who has pointed to the Laplace method.

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