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RELATION BETWEEN RANK AND MULTIPLICATIVE COMPLEXITY OF A BILINEAR FORM OVER A COMMUTATIVE NOETHERIAN RING

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The concept of multiplicative complexity of a bilinear form is introduced for a commutative Noetherian ring. Rings are described for which the multiplicative complexity coincides with the rank for all forms. It is shown that for regular rings of dimension $\geqslant 3$ the multiplicative complexity can exceed the rank by an arbitrarily large number.

In this article we study a notation which arises in the theory of algebraic complexity of computation (the main concepts and problems of this theory are presented very completely in [1]). One of the problems in algebraic complexity of computation is to estimate the complexity of computing a family of bilinear forms. The tasks of estimating the complexity of computing a product of polynomials or matrices lead to this problem [1]. The complexity of computing a family of bilinear forms is usually estimated over a field (see, e.g., [1, 2]). In this paper we attempt to study the analogous problem for bilinear forms over a commutative ring (a computational interpretation of this problem is discussed below). The problem of complexity of a family of bilinear forms over a ring causes difficulties even in the case of a single form, and we restrict ourselves to this case.

It is shown in [2] that the smallest number of nonlinear operations required to compute a family of bilinear forms is equal to the multiplicative complexity of the family, as defined below (under certain conditions this assertation can also be proved for bilinear forms

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over a ring). The multiplicative complexity $Rg_F(A_1, \dots, A_n)$ of a family of bilinear forms $A_1, \dots, A_n$ over a field F is defined [2] as the smallest N such that there exist bilinear forms B_j $(1 \le j \le N)$ each of rank 1 which contain $A_1, \dots, A_n$ in their F-linear span. We note that the computation of a bilinear form of rank 1 requires a single multiplication of linear forms. It is obvious that $Rg_F A$ is equal to the ordinary rank of the bilinear form A (in what follows we consider the matrices of coefficients instead of the bilinear forms). If F is algebraically closed and we have a pair of matrices, an explicit formula is given in [3] (a closely related result is obtained in [12]) for the multiplicative complexity in terms of the parameters of the canonical Weierstrass–Kronecker form of the pair of matrices.

In this article, we consider matrices over a ring K which is assumed to be commutative Noetherian with identity in what follows. The rank $rg\,A$ is defined as usual as the largest r such that the $u \times v$ matrix A has an $r \times r$ minor different from zero. We define the multiplicative complexity $Rg_K A$ as the smallest N such that $A = \sum_{1 \le j \le N} B_j$ for certain B_j of the form $X_j \cdot Y_j$, i.e., a product of the column vector X_{jv} by the row vector Y_j $(1 \le j \le N)$. More formally, if $A \in Hom_K(K, K^u) = K^u \otimes_K (K^v)^* ((K^v)^* = Hom(K^v, K))$ then

$$Rg_K A = \min \{ N : A = \sum_{1 \le j \le N} z_j \otimes y_j;\ z_j \in K^u, y_j \in (K^v)^*, 1 \le j \le N \}$$

As in the case of matrices over a field, one proves the inequality $Rg_K A \geqslant rg\,A$. As we will see in what follows, the reverse inequality is not always valid.

We give another inequality. Let $A_1, \dots, A_d$ be $u \times v$ matrices over a field F and let $K = F[x_1, \dots, x_d]$ be a ring of polynomials. Then $Rg_K(x_1A_1 + \dots + x_dA_d) \le Rg_F(A_1, \dots, A_d)$. In the situation which we consider, Rg_K can be interpreted as the multiplicative complexity in computing the bilinear form (over F) from some parametric family (depending on d parameters), where it is required that the method of the computation involve all values of the parameters in a "unified" way. The reason for studying Rg_K over a ring (and not over the field of quotients) is that the method used to compute the bilinear form must be suitable for all values of the parameters $x_1, \dots, x_d$. Matrices of the form $x_1A_1 + \dots + x_dA_d$ considered in detail in Sec. 2 correspond to bilinear forms which run over a linear subspace of dimension $\le d$ in the space of bilinear forms over the field F.

In Sec. 1 of this article we describe the class of rings K such that the equality $Rg_K A = rg\,A$ holds for every matrix A over K. The most important assumption is that the global cohomological dimension of K not exceed two. In Sec. 2 we characterize $Rg_K A$ for matrices of the form $x_1A_1 + \dots + x_dA_d$. In particular, this gives the result that the difference $Rg_K A - rg\,A$ can be arbitrarily large for regular rings of dimension greater than two.

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1. In what follows we will use the following reformulation of $Rg\,A$ (we will sometimes omit the subscript K), where A is a $u \times v$ matrix over the field K. Let $M \subset K^v$ be the module (all modules considered over K) given by the rows of A. Then $Rg\,A$ is equal to the

smallest N such that there exists a module M_1, $M \subset M_1 \subset K^v$ which is generated over K by N elements. If K is an integral domain then $\mathrm{rg}\,A$ is equal to the largest number of K-independent elements of the module M. The ring K is called an Rg-ring if $Rg\,A = \mathrm{rg}\,A$ for every matrix A over K.

THEOREM 1. A ring K is an Rg-ring if and only if there exist integral domains $K_1, \ldots, K_n$ such that

1) $K = K_1 \oplus \cdots \oplus K_n$,

2) $\mathrm{gldh}\,K_i \leq 2\ (1 \leq i \leq n)$, where gldh is the global cohomological dimension ([4, Chap. 7]);

3) every projective module over K_i is free $(1 \leq i \leq n)$ (in the case of an integral domain K, this formulation was suggested by A. A. Suslin).

COROLLARY 1.1. Let I be a principal ideal domain. Then

(a) I is an Rg-ring;

(b) $I[x]$ is an Rg-ring.

(In Sec. 2 we will use the fact that $F[x,y]$ is an Rg-ring, where F is a field.)

Part 2) of Theorem 1 in this case follows from the Syzygy theorem [5]; part 3) for case (b) follows from a theorem of Seshadri [6].

We turn to the proof of Theorem 1, which will be broken up into two steps, viz., reducing the general case to the case of integral domains and then proving the result for integral domains.

LEMMA 1. A ring K is an Rg-ring if and only if $K = K_1 \oplus \cdots \oplus K_n$, where every K_i is an integral domain and an Rg-ring $(1 \leq i \leq n)$ (such a decomposition is unique).

The uniqueness of the decomposition (assuming it exists) follows from general arguments. If $K_1 \oplus \cdots \oplus K_s \simeq K'_1 \oplus \cdots \oplus K'_t$ where the $K_i\ (1 \leq i \leq s)$, $K'_j\ (1 \leq j \leq t)$ are integral domains and $s \geqslant t$, then we choose nonzero $b_1 \in K_1, \ldots, b_s \in K_s$, expand $b_i = \sum_{1 \leq j \leq t} b'_{ij}\ (1 \leq i \leq s)(b'_{ij} \in K'_j)$, and consider the sets of indices $I_i = \{j : b'_{ij} \neq 0\}$. Then the sets $I_i\ (1 \leq i \leq s)$ are pariwise disjoint, so $s = t$ and the I_i all consist of a single element: $I_i = \{\pi(i)\}$ (π a permutation of the set $\{1, \ldots, s\}$). In this case, $K_i \simeq K'_{\pi(i)}$.

In one direction, i.e., the fact that the $K_i\ (1 \leq i \leq n)$ are Rg-rings implies that $K = K_1 \oplus \cdots \oplus K_n$ is an Rg-ring, the lemma is proved as follows. Let A be a matrix over K. Then $A = A_1 + \ldots + A_n$, where A_i is a matrix over $K_i\ (1 \leq i \leq n)$ and $r = \mathrm{rg}\,A = \max \mathrm{rg}\,A_i$. Since the K_i are Rg-rings, there exist columns $z_{ij} \in K_i^u$ and rows $y_{ij} \in (K_i^v)^*\ (1 \leq i \leq n, 1 \leq j \leq r)$ such that $A_i = \sum_{1 \leq j \leq r} z_{ij} y_{ij}\ (1 \leq i \leq n)$. Then $A = \sum_{1 \leq j \leq r} (\sum_{1 \leq i \leq n} z_{ij}) (\sum_{1 \leq i \leq n} y_{ij})$, and therefore $\mathrm{rg}\,A = Rg\,A$.

We turn to the proof of the converse, i.e., that every Rg-ring is a direct sum of integral domains. The fact that each of the summands is also an Rg-ring is already obvious.

Let K be an Rg -ring and denote by K_S its complete ring of fractions, i.e., the localization of K relative to the multicatively closed set S of all nondivisors of zero in K. We verify that K_S is an Rg -ring (this holds for any S not containing zero divisors). Let $A=(a_{ij}/s_{ij})$ be a matrix over K_S, where $a_{ij}\in K$, $s_{ij}\in S$. Putting $s=\prod_{ij} s_{ij}$, we then have the matrix $A'=(a_{ij}s/s_{ij})$ over K. Therefore, $r=rgA'=rgA$ and there exist $z_i\in K^u$ and $y_i\in(K^v)^*$ $(1\le i\le r)$ such that $A'=\sum_{1\le i\le r} z_i\otimes y_i$, whence $A=\sum_{1\le i\le r}(z_i/s)\otimes y_i$. Consequently, $Rg\,A=r=rgA$.

The standard homomorphism $K\to K_S$ is injective. We will show below that K_S is a direct sum of fields and that the canonical projections of the ring K onto the components of this sum (in which we regard K as being embedded) are contained in K. This implies that K is the direct sum of the projections.

LEMMA 1.1. The ring K_S is a direct sum of fields.

We first prove that K_S is a semilocal ring (this does not use the Rg -ring property). Since K is Noetherian, its zero ideal has a minimal primary decomposition $(0)=\bigcap_{1\le i\le l} q_i$, where q_i is a p_i -primary ideal $(1\le i\le l)$ [7, Theorem 7.13]. Then the set of zero divisors $K\setminus S=\bigcup_{i=1}^{l} p_i$ [7, Proposition 4.71], and every ideal contained in $K\setminus S$ is contained in one of the p_i [7, Proposition 1.11(1)]. This and Proposition 3.11(IV) in [7] imply that the list $\{S^{-1}p_i\}_{i=1}^{l}$ contains all the maximal ideals of the ring K_S and $(0)=\bigcap_{i=1}^{l} S^{-1}q_i$ is a minimal primary decomposition ($S^{-1}q_i$ is an $S^{-1}p_i$ -primary ideal) of the zero ideal in K_S [7, Proposition 4.9].

We not prove that every principal ideal of the ring K_S is idempotent, i.e., $(x)=(x^2)$ for every $x\in K_S$. An element $x\in K_S$ is called extremal if $x=yz$ implies that either $(y)=(x)$ or $(y)=(1)$. The fact that the ring K_S is Noetherian implies [7, Proposition 7.3] that every element is a product of extremal elements, and therefore it suffices to prove $(x)=(x^2)$ when x is extremal. If $(x)\ne(1)$, then $xy=0$ for some $y\ne0$. Consider the matrix $A=\begin{pmatrix} x & 0\\ 0 & y\end{pmatrix}$. Then $rgA=1$ and therefore (since K_S is an Rg -ring) there exist elements $u_1,u_2,v_1,v_2\in K_S$ such that $x=u_1v_1$, $0=u_1v_2$, $0=u_2v_1$, $y=u_2v_2$. If $(u_1)=(1)$, then $v_2=0$ and $y=0$. We show analogously that the assumption $(v_1)=(1)$ gives a contradiction. Therefore, $(u_1)=(v_1)=(x)$, i.e., $u_1=\alpha x, v_1=\beta x$ for certain $\alpha,\beta\in K_S$, whence $x=\alpha\beta x^2$, i.e., $(x)=(x^2)$.

We recall that $(0)=\bigcap_{i=1}^{l} S^{-1}q_i$ is the minimal primary decomposition of the zero ideal. Since the ideal $S^{-1}q_i$ is $S^{-1}p_i$ -primary, for every $x\in S^{-1}p_i$ $x^t\in S^{-1}q_i$ for some t, and therefore (since $(x)=(x^t)$) $x\in S^{-1}q_i$, i.e., $S^{-1}q_i=S^{-1}p_i$ $(1\le i\le l)$. Since the decomposition $(0)=\bigcap_{1\le i\le l} S^{-1}p_i$ is minimal, each of the prime ideals $S^{-1}p_i$ is minimal. Hence by the Chinese remainder theorem, $K_S\simeq\bigoplus_{1\le i\le l}(K_S/S^{-1}p_i)$. Lemma 1.1 is proved.

LEMMA 1.2. Let $K\overset{f}{\hookrightarrow}K_S$ be an embedding of the Rg -ring K in K_S, which by Lemma 1.1 is isomorphic to a direct sum of fields $F_1\oplus\cdots\oplus F_n$ and let $F_1\oplus\cdots\oplus F_n\xrightarrow{g_i}F_i$ be the natural projections $(1\le i\le n)$. Then $f(K)\supset g_i f(K)$ $(1\le i\le n)$.

We identify K with its image $f(K)$ and let $1=a_1+\dots+a_n$ be an expansion of the identity $(a_i\in F,\ 1\le i\le n)$. It suffices to verify that $g_i(1)=a_i\in K$ $(1\le i\le n)$ for all representatives a_i having the form of a fraction b/c $(b\in K,\ c\in S)$, in which the ideal $(c)\subset K$ is chosen to be maximal among the principal ideals corresponding to all possible denominators. Considering the matrix $A=\begin{vmatrix} b & 0 \\ c & c-b\end{vmatrix}$, we then have $\det A=b(c-b)=0$ since $b/c\in F_i$, $(c-b)/c \in \sum_{j\ne i}\oplus F_j$. Therefore, $\operatorname{rg} A=1$ and since K is an Rg -ring there exist $u_1,u_2,v_1,v_2\in K$ such that $b=u_1v_1$, $0=u_1v_2$, $c=u_2v_1$, $c-b=u_2v_2$. Since $a_i=u_1/u_2$ ($u_2\in S$ because u_2 is not a zero divisor) and $(u_2)\supset(c)$, by the choice of c we have $(u_2)=(c)$, i.e., $u_2=\alpha c$ for some $\alpha\in K$. Hence $c-b=\alpha c v_2$ and $a_i=1-\alpha v_2\in K$. Lemma 1.2 is proved.

Lemma 1.2 implies that $K=\sum_i\oplus g_i f(K)$; but every $g_i f(K)$ is an integral domain, which completes the proof of Lemma 1. We note that we only used the Rg -ring property for 2×2 matrices in proving the existence of a decomposition of the ring K as a direct sum of integral domains. It remains to describe the Rg -rings which are integral domains.

LEMMA 2. An integral domain K is an Rg -ring if and only if

1) $\operatorname{gldh} K\le 2$;

2) every projective K-module is free.

We prove an intermediate lemma.

LEMMA 1.3. An integral domain K is an Rg -ring if and only if, for every module $M\subset K^n$, $T(K^n/M)=0$ (i.e., K^n/M has no torsion) implies M is free ($T(M_1)$ is the torsion submodule of M_1, cf. [7, Chap. 3, Ex. 12]).

Let K be an integral domain and an Rg -ring, $M\subset K^n$ and $T(K^n/M)=0$. Let $M\subset M_1\subset K^n$, $r=\operatorname{rg} M$ and M_1 be generated by r elements (by the Rg -ring property), so that $\operatorname{rg} M_1=r$. Assume that $M\ne M_1$ and let $a_1,\dots,a_r$ be K-independent elements of M and $a\in M_1\setminus M$. Then for some $\alpha,\alpha_1,\dots,\alpha_r\in K$ $(\alpha\ne 0)$ we have $\alpha a+\sum_{1\le i\le r}\alpha_i a_i=0$, i.e., $\alpha a\in M$; but together with the condition $T(K^n/M)$ this implies $a\in M$. This contradiction shows that $M=M_1$, which means that M is generated by r elements which form a free-module basis.

Conversely, let $M\subset K^n$ and $r=\operatorname{rg} M$. We define the module $M_1=\{a\in K^n$ such that $\alpha a\in M$ for some $0\ne\alpha\in K\}$. Then $\operatorname{rg} M_1=\operatorname{rg} M$. Moreover, $T(K^n/M_1)=0$. Therefore M_1 is a free module of rank r, which completes the proof of Lemma 1.3.

We prove Lemma 2 using Lemma 1.3. Assume the ring K satisfies the condition stated in Lemma 1.3 and let P be a projective module, $P\oplus Q=K^n$. Then $T(K^n/P)=T(Q)=0$, and therefore P is free. Assume that $\operatorname{gldh} K\geqslant 3$. Then by [4, Corollary 1.5 and Chap. VII, Ex. 2] $\operatorname{gldh} K=\operatorname{Sup}\{\operatorname{dh}(K/L)$, where L is an ideal in $K\}=1+\operatorname{Sup}\{\operatorname{dh} L$, where L is an ideal in $K\}$. That is, for some ideal $L\subset K$ we have $\operatorname{dh} L\geqslant 2$. Let L be generated by ℓ elements $\alpha_1,\dots,\alpha_\ell\in K$. Let g be the epimorphism $K^\ell\xrightarrow{g} L$, where $K^\ell=w_1K\oplus\dots\oplus w_\ell K$ and $g(w_i)=\alpha_i$ $(1\le i\le \ell)$. Then the sequence $0\to\operatorname{Ker} g\to K^\ell\xrightarrow{g} L\to 0$ is exact, and since $\operatorname{dh} L\geqslant 2$, the module $\operatorname{Ker} g$ is not projective. On the other hand, $T(K^\ell/\operatorname{Ker} g)=T(L)=0$, so that $\operatorname{Ker} g$ is free.

Conversely, let $M \overset{h}{\hookrightarrow} K^n$ and $T(M_1)=0$ $(M_1=K^n/M)$. There exists a monomorphism $M_1 \overset{f}{\hookrightarrow} K^t$ for some t. Consider the exact sequence

$$0 \to M \xrightarrow{h} K^n \xrightarrow{f_1} K^t \to K^t/M_1 \to 0$$

where f_1 is the composition of the projection of K^n onto M_1 and the monomorphism f. Since $\operatorname{gldh} K \leq 2$, M is projective and therefore free. Lemma 2 is proved.

Theorem 1 follows from Lemmas 1 and 2. We make a few remarks.

COROLLARY 1.2. Let K be a commutative Noetherian local ring and assume $\operatorname{gldh} K \leq 2$. Then K is an Rg-ring.

The fact that K is an integral domain follows from [5, Chap. IV, Theorem 5 and its Corollary 4]. For the proof that every projective module over a local ring is free, cf. [6].

In connection with Theorem 1, the question arises whether it is possible to estimate $Rg_K A$ in terms of $rg\, A$ and, perhaps, certain other characteristics of the ring K. It turns out that this is easy to do if $\operatorname{gldh} K \leq 2$. This shows that the condition $\operatorname{gldh} K \leq 2$ is very important in Lemma 2 and Theorem 1.

COROLLARY 1.3. Let the integral domain K be a commutative Noetherian local ring and $\operatorname{gldh} K = d \leq 2$, A a matrix over K. Then $Rg\, A \leq rg\, A + d$ (in particular, if K is a Dedekind domain [7, Chap. 9] then $Rg\, A \leq rg\, A + 1$).

Let $M \subset K^n$ and $rg\, M = r$. As in the proof of Lemma 1.3, we construct the module $M_1 = \{w \in K^n : \alpha w \in M$ for some $0 \neq \alpha \in K\}$. Then $rg\, M_1 = rg\, M$ and $T(K^n/M_1) = 0$. As in the proof of Lemma 2, we construct a monomorphism $M_2 \to K^t$, where $M_2 = K^n/M_1$, and an exact sequence $0 \to M_1 \to K^n \to K^t \to K^t/M_2 \to 0$. Since $\operatorname{gldh} K \leq 2$, M_1 is projective.

We use a result due to Swan [8]. Writing $gen(M')$ for the smallest number of generating elements of the module M', the main result in [8] and the Syzygy theorem [5] imply that $gen(M') \leq \sup_p gen((M_1)_p) + d$, where $(M_1)_p$ is the localization of M' by the prime ideal $p \subset K$. The module $(M_1)_p$ is free since it is projective over the local ring K_p. Moreover, $rg(M_1)_p = rg\, M_1 = r$, and therefore $gen((M_1)_p) = r$. Finally, $gen(M_1) \leq r + d$. Corollary 1.3 is proved.

As in the case of matrices over a field, the function Rg can be extended to families of matrices over a ring K (here we assume that K is an integral domain) and the question posed in Theorem 1 can be considered, viz., what are the conditions on K in order for $Rg_K(A_1,\dots,A_n) = Rg_F(A_1,\dots,A_n)$ to hold for every $n \geq 1$ and all matrices $A_1,\dots,A_n$ over K, where F is the field of quotient of K? It can be shown by carrying the proof of the main result in [9] (for the case of an exterior product) over to the case of tensor products that the above equality holds only if $K = F$.

2. In this second section we consider rings of dimension greater than two. As follows from Theorem 1, $Rg\, A > rg\, A$ for certain matrices A over such rings. We will strengthen this inequality below and show in particular that the difference $Rg\, A - rg\, A$ is unbounded from above for an extremely large class of rings of dimension greater than two (it is of interest

to compare this result with Corollary 1.3).

Let K be a commutative Noetherian regular ring and an integral domain and assume $d = \operatorname{gldh} K$ (in our problem, the important example is the case $K = F[x_1, \ldots, x_d]$, which we keep in mind in developing our arguments). We also impose a not very burdensome constraint of a geometric character on the ring K, i.e., the ring K embeds in residue field $F = K/m$ modulo its a maximal ideal m of height d (in this section we consider only rings satisfying this restriction). We denote by $x_1, \ldots, x_d$ elements of the ideal m which project in the F-vector space m/m^2 to form a basis (the existence of such elements follows from results in [7, Chap. 11, Sec. 3]). The above restriction is satisfied, e.g., when the ring K is the coordinate ring of a variety of dimension greater than two.

In this section we limit ourselves to considering matrices of the following form and characterize Rg_K for them. A matrix over the ring K is said to be square-free if its elements are F-linear forms in $x_1, \ldots, x_d$. In what follows we will use the following reformulation of the definition of $Rg\,A$ (A $u \times v$ matrix): $Rg\,A$ is equal to the smallest N such that for a certain $u \times N$ matrix B and $N \times v$ matrix C we have $A = BC$.

LEMMA 2.1. Let the matrix A be square-free. Then $Rg\,A$ is equal to the smallest N such that A can be transformed into the form

$$\begin{array}{c|c|c|} & q & v-q \\ \hline p & & \\ \hline u-p & & O \\ \hline \end{array}\;, \quad \text{where } N = p+q \; (p, q \geqslant 0),$$

using elementary transformations over the field F (here and below the displayed matrices are broken up into blocks, and if the form of the submatrices in any block is known, this is specifically indicated).

Since elementary transformations do not change Rg, the inequality $Rg\,A \leqslant N$ is obvious. Conversely, assume that $A = BC$, where B is $u \times N$ matrix, C an $N \times v$ matrix $(N = Rg\,A)$. Since $K = F + m$, there exist nonsingular F-matrices G_1 and G_2 such that

$$G_1 B G_2 = \begin{array}{c|c|c|} & p & N-p \\ \hline p & \begin{matrix} \beta_1 & & O \\ & \ddots & \\ O & & \beta_p \end{matrix} & O \\ \hline u-p & O & O \\ \hline \end{array} + B',$$

where $0 \neq \beta_i \in F\,(1 \leqslant i \leqslant p)$ and all the elements of the matrix B' belong to the ideal m. Since $G_1 A = (G_1 B G_2)(G_2^{-1} C)$ and $G_1 A$ is square-free,

$$G_2^{-1} C = \begin{array}{|c|c} \multicolumn{1}{c}{v} & \\ \hline C_1 & p \\ \hline C_2 & N-p \\ \hline \end{array}\;,$$

where all the elements of the $p\times v$ submatrix C_1 belong to $\mathfrak{m}$.

Analogously, there exist nonsingular F-matrices H_1 and H_2 such that

$$H_1=\begin{array}{cc} \begin{array}{cc} p & N-p \end{array} & \\ \left|\begin{array}{c|c} \begin{matrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{matrix} & 0 \\ \hline 0 & \end{array}\right| & \begin{array}{l} p \\ N-p\end{array}\end{array} \quad\text{and}\quad H_1G_2^{-1}CH_2=\begin{array}{cc} & \begin{array}{cc} q & v-q\end{array} \\ \begin{array}{r} p \\ q \\ N-p-q\end{array} & \left|\begin{array}{c|c} 0 & 0 \\ \hline \begin{matrix}\gamma_1 & & 0\\ & \gamma_2\ddots & \\ 0 & & \gamma_q\end{matrix} & 0 \\ \hline 0 & 0\end{array}\right|\end{array} + C',$$

where all the elements of C' belong to $\mathfrak{m}$; $0\neq\gamma_i\in F$ $(1\le i\le q)$. The matrix equality A = BC then takes the form $(G_1BG_2H_1^{-1})(H_1G_2^{-1}CH_2)=G_1AH_2$, and if

$$G_1AH_2=\begin{array}{cc} \begin{array}{cc} q & v-q\end{array} & \\ \left|\begin{array}{c|c} & \\ \hline & A'\end{array}\right| & \begin{array}{l} p \\ u-p\end{array}\end{array}$$

then using the form of the matrices $G_1BG_2H_1^{-1}$ and $H_1G_2^{-1}CH_2$ we obtain that all the elements of the $(u-p)\times(v-q)$ matrix A' belong to $\mathfrak{m}^2$; but on the other hand, they are F-linear forms in $x_1,\dots,x_d$. Since the $x_1,\dots,x_d$ are F-linearly independent, $A'=0$. We conclude the proof of Lemma 2.1 on the basis of the inequality $\operatorname{Rg}A=N\geq p+q$.

Remark 2.1. The lemma just proved permits a reduction of the calculation of square-free Rg matrices over rings which satisfy the restriction stated above to solving a system of equations and inequalities over the field F. Therefore, in the situation which we consider it is enough to study Rg over the ring $F[x_1,\dots,x_d]$.

The following reformulation of Lemma 2.1 will be used in the sequel. Namely, $\operatorname{Rg}A=\min\limits_{\tau_1,\tau_2}(u+v-\tau_1-\tau_2)$, where $BAC=0$ for certain F-matrices B and C such that $\operatorname{rg}B=\tau_1$, $\operatorname{rg}C=\tau_2$.

LEMMA 2.2. Let the matrix A be square-free and have the form $\begin{vmatrix}A_1 & A_3\\ 0 & A_2\end{vmatrix}$, where A_1 and A_2 are $u_1\times v_1$ and $u_2\times v_2$ matrices. Then $\operatorname{Rg}A\geq\operatorname{Rg}A_1+\operatorname{Rg}A_2$.

Proof. Let $BAC=0$ and $\operatorname{rg}B=\tau_1$, $\operatorname{rg}C=\tau_2$ (with dimensions $\tau_1\times u$ and $v\times\tau_2$). We write down the above matrix equality in block form:

$$0=|B_1B_2|\cdot\begin{vmatrix}A_1 & A_3\\ 0 & A_2\end{vmatrix}\cdot\begin{vmatrix}C_1\\ C_2\end{vmatrix}=B_1A_1C_1+B_1A_3C_2+B_2A_2C_2.$$

By our assumption, the intersections of the left and right kernels satisfy

$$\operatorname{Ker}_\ell B_1\cap\operatorname{Ker}_\ell B_2=\{0\},\quad \operatorname{Ker}_r C_1\cap\operatorname{Ker}_r C_2=\{0\}. \tag{1}$$

Writing $b=\operatorname{rg}\operatorname{Ker}_\ell B_1$, $c=\operatorname{rg}\operatorname{Ker}_r C_2$, we have

$$((\operatorname{Ker}_\ell B_1)B_2)A_2C_2=\{0\},\quad B_1A_1(C_1(\operatorname{Ker}_r C_2))=\{0\}. \tag{2}$$

It follows from (1) that $\operatorname{rg}((\operatorname{Ker}_\ell B_1)B_2)=b$, $\operatorname{rg}(C_1(\operatorname{Ker}_r C_2))=c$. Then (2) and Lemma 2.1 give the inequalities

$$\operatorname{Rg} A_1 \leq u_1 + v_1 - (\tau_1 - b + c), \quad \operatorname{Rg} A_2 \leq u_2 + v_2 - (b + \tau_2 - c)$$

Adding these and again applying Lemma 2.1, we conclude the proof of Lemma 2.2.

COROLLARY 2.2. $\operatorname{Rg}\begin{vmatrix} A & 0 \\ 0 & B \end{vmatrix} = \operatorname{Rg} A + \operatorname{Rg} B$ (A, B square-free).

We introduce the following functions defined on the positive integers and study their properties. Put $R_F^d(\tau) = \sup_{\operatorname{rg} A = \tau} \operatorname{Rg}_K A$, where the $\sup$ is taken over square-free matrices A and rings K satisfying the conditions $\operatorname{gldh} K = d$, $F = K/m \subset K$ for some maximal ideal m of height d. Since the elements $x_1, \ldots, x_d$ are algebraically independent over F (cf. [7, Corollary 11.21]), we may by Remark 2.1 take K to be $F[x_1, \ldots, x_d]$. We also define $R_F(\tau) = \sup_d R_F^d(\tau)$. Since the subsequent results are valid for any field F, we omit the subscript F in R_F^d (I conjecture that in fact R_F and R_F^d R_F do not depend on F). Corollary 2.2 implies

COROLLARY 2.3. $R^d(\tau_1 + \tau_2) \geq R^d(\tau_1) + R^d(\tau_2)$, $R(\tau_1 + \tau_2) \geq R(\tau_1) + R(\tau_2)$.

We now estimate the function $R^d(\tau)$ from above.

LEMMA 2.3. For every $\tau \geq 1$ we have the inequalities $R^d(\tau) \leq \tau + \left[\frac{\tau}{2}\right] + \left[\frac{\left[\frac{\tau}{2}\right]}{2}\right] + \cdots < 2\tau$, where the number of terms in the sum is $d-1$ $(d \geq 2)$ ($[e]$ is the integer part of e).

The proof is by induction on d and τ. The induction is started using the equalities $R^2(\tau) = \tau$ (Corollary 1.1) and $R^d(1) = 1$ (which follows from the fact that the ring $F[x_1, \ldots, x_d]$ is factorial and Remark 2.1).

In the general case $(d \geq 3, \tau \geq 2)$ we carry out the following F-elementary transformations with a $u \times v$ square-free matrix A. (We say that a linear form contains the variable x_i $(1 \leq i \leq d)$ if the coefficient of x_i in the form is nonzero.) If some element of A contains x_1, then we move it into the upper left corner and then arrange that no other element in the first row or first column contains x_1. Consider the $(u-1) \times (v-1)$ submatrix A' of A obtained by crossing out the first row and first column. If some element of the matrix A' contains x_1, we treat it just as we did A', etc. Assume that after t_1 steps ($t_1 \geq 0$) the elements of the $(u - t_1) \times (v - t_1)$ submatrix of A obtained by crossing out the first t_1 rows and columns does not contain x_1.

We then move some element of this submatrix containing x_2 (if it exists) into the upper left corner and arrange that no other element in the first row or column contains x_2, etc. After t_2 such steps $(t_2 \geq 0)$ the elements of the remaining $(u - t_1 - t_2) \times (v - t_1 - t_2)$ submatrix do not contain x_1, x_2; we carry out the procedure analogously with x_3, etc.

As a result of the F-elementary transformations, the matrix A is reduced to the form

$$\left|\begin{array}{c|c} L_1^1 \cdots L_{t_1}^1 L_1^2 \cdots L_{t_2}^2 \cdots L_1^l \cdots L_{t_l}^l & C \\ \hline B & 0 \end{array}\right|$$

where only the forms $L_1^1, \ldots, L_{t_1}^1$ contain the variable x_1; only forms $L_1^2, \ldots, L_{t_2}^2, \ldots$ contain

x_2 in the $(u-t_1)\times(v-t_1)$ submatrix obtained from A by crossing out the first t_1 rows and columns; and only the forms $L^{\ell}_1,\dots,L^{\ell}_{t_\ell}$ $(1\le \ell\le d;\, t_1\ge 0,\dots,t_{\ell-1}\ge 0, t_\ell\ge 1)$ contain the variable x_ℓ in the submatrix obtained by crossing our the first $t_1+\dots+t_{\ell-1}$ rows and columns.

We put $t=t_1+\dots t_\ell$. Then the $t\times t$ submatrix in the upper left-hand corner of A is nonsingular. Indeed, the monomial $x_1^{t_1}x_2^{t_2}\cdot\ldots\cdot x_\ell^{t_\ell}$ appears in its determinant with a nonzero coefficient. Therefore, $\tau=\operatorname{rg} A\ge t$.

On the other hand, $\operatorname{rg}B+\operatorname{rg}C\le\tau$. Without loss of generality we may assume that $\operatorname{rg}B\le[\tau/2]$. We also remark that the elements of the matrices B and C do not contain variable x_1. Therefore, $\operatorname{Rg}A\le t+R^{d-1}([\tau/2])$, which completes the proof of Lemma 2.3.

We now use the following well-known fact [10, Problem 98]: if $y_1, y_2,\dots$ are real numbers with $y_{i+j}\le y_i+y_j$ for all i, j and $\left\{\frac{y_i}{i}\right\}$ is bounded from below, then the limit $\lim\limits_{i\to\infty}\frac{y_i}{i}$ exists. Using Corollary 2.3 and Lemma 2.3, we apply this result to the sequences $\{-R^d(\tau)\}_{\tau\ge 1}$ and $\{-R(\tau)\}_{\tau\ge 1}$. We calculate the limit $\lim\limits_{\tau\to\infty}\frac{R(\tau)}{\tau}$.

THEOREM 2. $\lim\limits_{\tau\to\infty}\frac{R(\tau)}{\tau}=2$.

Remark 2.4. The following sharpened form of the theorem is obtained from the proof given below and Corollary 2.3: $\lim\limits_{\tau\to\infty}\frac{R^{2d}(\tau)}{\tau}\ge\lim\limits_{\tau\to\infty}\frac{R^{2d-1}(\tau)}{\tau}\ge\frac{2d-1}{d}$. This gives in particular the result promised above, saying that the difference $\operatorname{Rg}A-\operatorname{rg}A$ is unbounded for matrices A over a ring of dimension greater than two. Moreover, the properties of the matrix $A_{2,2}$ constructed below, together with Lemma 2.3 and Corollary 2.3 give $R^3(\tau)=\left[\frac{3}{2}\tau\right]$.

In what follows we construct a sequence of square free matrices $\{A_n\}$ (A_n over the ring $F[x_1,\dots,x_{2n-1}]$, $n=1,2,\dots$) satisfying $\tau_n=\operatorname{rg}A=\binom{2n-2}{n-1}$, $R_n=\operatorname{Rg}A=\binom{2n-1}{n-1}$. By Lemma 2.3, this will imply the theorem.

We construct the family of matrices $A_{s,t}$ $(s,t\ge 1)$ by induction on s and t in the following six steps:

1) $A_{s,t}$ has dimension $u_{s,t}\times v_{s,t}$ $u_{s,t}=\binom{s+t-1}{s-1}$, $v_{s,t}=\binom{s+t-1}{s}$;

2) every nonzero element of the matrix $A_{s,t}$ is equal to $\pm x_i$ $(1\le i\le s+t-1)$;

3) every variable $x_i\,(1\le i\le s+t-1)$ appears in exactly $\tau_{s,t}=\binom{s+t-2}{s-1}$ elements of the matrix $A_{s,t}$, and these $\tau_{s,t}$ elements lie in different rows and columns;

4) every row of the matrix $A_{s,t}$ contains t nonzero elements and every column contains s nonzero elements;

5) $A_{s,t+1}\cdot A_{s+1,t}=0$ $(s,t\ge 1)$;

6) $\operatorname{rg}A_{s,t}=\tau_{s,t}$, $\operatorname{Rg}A_{s,t}=\min(u_{s,t}, v_{s,t})$

The matrix $A_{s,1}$ is the $s\times 1$ column $\boxed{\begin{matrix}x_s\\-x_{s-1}\\\vdots\\(-1)^{s-1}x_1\end{matrix}}$, $A_{1,t}$ is the $1\times t$ row $\boxed{x_1\cdots x_t}$ $(s,t\ge 1)$. Conditions 1)-4) and 6) hold for the matrices $A_{s,1}$ and $A_{1,t}$ and $A_{1,2}\cdot A_{2,1}=0$.

We now construct the matrix $A_{s+1,t+1}$ $(s,t\ge 1)$:

$$A_{s+1,t+1} = \begin{vmatrix} A_{s+1,t} & x_{s+t+1}E \\ 0 & -A_{s,t+1} \end{vmatrix}$$

where E is the identity matrix with side $r_{s+1,t+1} = \binom{s+t}{s}$.

Conditions 2) and 4) are verified directly using the induction assumption. Conditions 1) and 3) follow from the identity $\binom{s+t-1}{s} + \binom{s+t-1}{s-1} = \binom{s+t}{s}$. We verify condition 5):

$$A_{s,t+1} \cdot A_{s+1,t} = \begin{vmatrix} A_{s,t} & x_{s+t}E \\ 0 & -A_{s-1,t+1} \end{vmatrix} \cdot \begin{vmatrix} A_{s+1,t-1} & x_{s+t}E \\ 0 & -A_{s,t} \end{vmatrix} = 0$$

by the induction assumption. This matrix equality is meaningful for $s,t \geqslant 2$. It can also be given a meaning for $s=1$ or $t=1$, and condition 5) can be verified directly for such values.

From condition 5) we obtain

$$\mathrm{rg}\, A_{s+1,t+1} = \mathrm{rg} \begin{vmatrix} A_{s,t+1} & x_{s+t+1}E \\ E & 0 \end{vmatrix} \begin{vmatrix} A_{s+1,t} & x_{s+t+1}E \\ 0 & -A_{s,t+1} \end{vmatrix} = r_{s+1,t+1}.$$

It remains to evaluate $\mathrm{Rg}\, A_{s+1,t+1}$. For the sake of definiteness, we assume that $s \leq t$. Let $BA_{s+1,t+1}C = 0$, where B, C are F-matrices, and let $p = \mathrm{rg}\, B$, $q = \mathrm{rg}\, C$. We denote by b_i (c_j) the i-th column (j-th row) of the matrix B (C). For each n ($1 \leq n \leq s+t+1$), let $t_{n,1}, \ldots, t_{n,r}$ ($s_{n,1}, \ldots, s_{n,r}$) be the indices of the rows (columns) in which the variable x_n ($r = r_{s+1,t+1}$) appears. Then by condition 3) we have

$$\mathrm{rg}(b_{t_{n,1}}, \ldots, b_{t_{n,r}}) + \mathrm{rg}(c_{s_{n,1}}, \ldots, c_{s_{n,r}}) \leq r \tag{n}$$

Assume the $b_{i_1}, \ldots, b_{i_p}$ are linearly independent and let the $c_{j_1}, \ldots, c_{j_q}$ also be linearly independent. We write p_n for the cardinality of the set $\{i_1, \ldots, i_p\} \cap \{t_{n,1}, \ldots, t_{n,r}\}$ and q_n for the cardinality of $\{j_1, \ldots, j_q\} \cap \{s_{n,1}, \ldots, s_{n,r}\}$. Then by condition 4) $\sum_{1 \leq n \leq s+t+1} p_n = (t+1)p$, $\sum_{1 \leq n \leq s+t+1} q_n = (s+1)q$. On the other hand, the inequality (n) implies that $p_n + q_n \leq r$. Therefore, $(1+s)(p+q) \leq (1+s)p + (1+t)q \leq r(s+t+1)$, i.e., $(p+q) \leq \binom{s+t+1}{s+1} = v_{s+1,t+1}$. Hence by Lemma 2.1, $\mathrm{Rg}\, A_{s+1,t+1} \geqslant v_{s+1,t+1}$. Condition 6) is verified. Finally, we put $A_n = A_{n,n}$. The theorem is proved.

Remark 2.5. The above construction of the matrices $A_{s,t}$ is similar to the construction of the mapping cone for complexes [4, Chap. 2]. That is, let $P = F[x_1, x_2, \ldots]$ be the polynomial ring in infinitely many variables. Consider the following sequence of finite complexes consisting of free finitely generated P-modules:

$$\begin{array}{ll}
C_0: & \cdots 0 \to P^1 \to 0 \cdots \\
C_1: & \cdots 0 \to P^1 \xrightarrow{A_{1,1}} P^1 \to 0 \cdots \\
C_2: & \cdots 0 \to P^1 \xrightarrow{A_{2,1}} P^2 \xrightarrow{A_{1,2}} P^1 \to 0 \cdots \\
\vdots & \\
C_\ell: & \cdots 0 \to P^1 \xrightarrow{A_{\ell,1}} P^\ell \cdots P^{\binom{\ell}{i}} \xrightarrow{A_{\ell-i,i+1}} P^{\binom{\ell}{i+1}} \cdots P^\ell \xrightarrow{A_{1,\ell}} P^1 \to 0 \cdots \\
\vdots &
\end{array}$$

Then the complex $C_{\ell+1}$ $(\ell \geqslant 0)$ is the cone of the chain map $\varphi_\ell : C_\ell \longrightarrow C_\ell$ given by componentwise multiplication of the modules by the variable $x_{\ell+1}$. Condition 5) in the statement of Theorem 2 just means that C_ℓ $(\ell \geqslant 0)$ is a complex. In this connection it would be interesting to give a "homological proof" of the estimate for $\mathrm{Rg}\, A_{s,t}$. This would probably shed light on the properties of Rg for arbitrary matrices.

In this section we have studied the behavior of Rg for square-free matrices. I do not know any analogous answers for matrices of arbitrary form; e.g., is $\sup_{\mathrm{rg} A = r} \mathrm{Rg}_K A$ bounded for $r \geqslant 2$ and K a regular ring of dimension greater than two? I conjecture that we always have the inequality $\mathrm{Rg}_K A \leqslant c_K \cdot \mathrm{rg} A$, where c_K depends only on the ring K. The analog of Corollary 2.2 (additivity of Rg) for an arbitrary regular ring is false. We give a counterexample. Put $K = \mathbb{Z}[\sqrt{5}]$, $A = \begin{vmatrix} \sqrt{5}-1 & 2 \\ 2 & \sqrt{5}+1 \end{vmatrix}$ and let $A_n = \begin{vmatrix} A & & 0 \\ & A & \\ 0 & & \ddots A \end{vmatrix}$ be the matrix containing n copies of A along the diagonal. Then $\mathrm{rg}\, A_n = n$, and therefore since K is Dedekind, $\mathrm{Rg}_K A_n \leqslant n+1$, by Corollary 1.3. On the other hand, $\mathrm{Rg}_K A = 2$. At the same time, I believe that additivity of Rg holds for polynomial rings.

When the ring K is not commutative, I know a reasonable definition of rg only in the case when K is a division ring (using the Dieudonne determinant [11]). In this case the equality $\mathrm{Rg}_K A = \mathrm{rg}\, A$ is satisfied for every matrix A over the division ring K.

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