Loewy Decomposition of Linear Third-Order PDE’s in the Plane

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ABSTRACT
Loewy’s decomposition of a linear ordinary differential operator as the product of largest completely reducible components is generalized to partial differential operators of order three in two variables. This is made possible by considering the problem in the ring of partial differential operators where both left intersections and right divisors of left ideals are not necessarily principal. Listings of possible decomposition types are given. Many of them are illustrated by worked out examples. Algorithmic questions and questions of uniqueness are discussed in the Summary.

1. INTRODUCTION

About one hundred years ago Loewy proved the fundamental result that any ordinary differential operator may be represented uniquely as the product of largest irreducible operators, i.e. operators that are the left intersection of irreducible operators of lower order [10]; see also Chapter 2 of the book [12]. This decomposition provides a detailed understanding of the structure of the solution space of the corresponding differential equation. Therefore it would be highly desirable to generalize it to partial differential operators as well. Amazingly this has never really been done since Loewy’s original work. In this article Loewy decompositions of third-order operators in two variables are considered in detail; the possible limitations are also discussed. In the subsequent section some basic notations from differential algebra are introduced; details may be found in the book by Kolchin [9] or the articles by Buium and Cassidy [2] or Quadrat [11]. The main part of the article is organized according to leading derivatives of the respective operators.

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2. BASIC DIFFERENTIAL ALGEBRA

In this section some basic terminology from differential algebra that is used throughout the article will be introduced. Rings of differential operators $D \equiv F[\partial_x, \partial_y]$ or $D \equiv Q(x, y)[\partial_x, \partial_y]$ are considered; $F$ is a universal differential field. $F$ or $Q(x, y)$ are called the base field. Let the left ideal $I$ be generated by elements $l_i \in D$, $i = 1, \ldots, p$. Then one writes $I = \langle l_1, \ldots, l_p \rangle$. Because right ideals are not considered, sometimes $I$ is simply called an ideal. As a rule, the $l_i$ are assumed to form a Janet basis in the term order grlex with $x > y$. If $z$ is some differential indeterminate, $l_i z = 0$, $i = 1, \ldots, p$, is the corresponding system of pde’s. The following shorthand notation is frequently used. If only the leading terms of the generators of an ideal are of interest the non-leading terms are omitted; it is denoted by $\langle \ldots \rangle_{LT}$.

Let $I \subseteq F[\partial_x, \partial_y]$ be an ideal and $H_I$ its Hilbert-Kolchin polynomial ([9], page 130; [2], page 602). The degree $deg(H_I)$ of $H_I$ is called the differential type of $I$. Its leading coefficient $lc(H_I)$ is called the typical differential dimension of $I$. The pair $(deg(H_I), lc(H_I))$ has been baptized the gauge of the ideal $I$ [7].

Let $I$ and $J$ be two ideals. Important constructions are the greatest common right divisor $Gcrd(I, J)$ or the sum ideal; and the least common left multiple $Lcm(I, J)$ or the left intersection. In [7] it has been shown how they are computed algorithmically. Finally the relative syzygy module $Syz(I, J)$ of $I$ and $J = (g_1, \ldots, g_n)$ is generated by

$$(h \equiv (h_1, \ldots, h_n) \in D^n)[h_1 g_1 + \ldots + h_n g_n \in I].$$

Define two ordinary differential operators by

\[
D_{x^n} \equiv d_x^n + a_1 d_x^{n-1} + \ldots + a_{m-1} d_x + a_m, \\
D_{y^n} \equiv d_y^n + b_1 d_y^{n-1} + \ldots + b_{n-1} d_y + b_n;
\]

$m$ and $n$ are natural numbers not less than 2. Several ideals generated by an operator of order three and one of these operators will occur later in this article. A special notation is introduced for them as shown in the table below. These ideals have an important meaning as divisors in the decompositions to be discussed in later chapters. Due to its close relation to the iteration scheme introduced by Laplace, see [6], vol. II, Chapter V, it is suggested to call them Laplace divisors. The pair of upper indices of the ideals in Table 1
denotes the gauge of the respective ideal.

<table>
<thead>
<tr>
<th>Notation</th>
<th>Leading term ideal</th>
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<tbody>
<tr>
<td>( J^{(1,1)} )</td>
<td>( \langle \partial_x y, \partial_x \rangle ) LT</td>
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<tr>
<td>( J^{(1,1)} )</td>
<td>( \langle \partial_x y, \partial_x \rangle ) LT</td>
</tr>
<tr>
<td>( J^{(1,2)} )</td>
<td>( \langle \partial_x y, \partial_x \rangle ) LT</td>
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<tr>
<td>( J^{(1,2)} )</td>
<td>( \langle \partial_x y, \partial_x \rangle ) LT</td>
</tr>
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Table 1. Some ideals of gauge (1, 1) and (1, 2) and order of derivatives not higher than 3. The ideals involving the parameter \( m \) or \( n \) will occur later on as Laplace divisors.

It turns out that the intersection ideals generated by two first-order operators are of fundamental importance for understanding the different decomposition types discussed later in this article. They are described in the following two theorems. The first theorem has been proved in [7].

**Theorem 2.1.** Let the ideals \( I_i = (\partial_x + a_i \partial_y + b_i) \) for \( i = 1, 2 \) with \( I_1 \neq I_2 \) be given. Both ideals have gauge (1, 1). There are three different cases for their intersection \( I_1 \cap I_2 \), all of gauge (1, 2).

1. **Separable case** \( a_1 \neq a_2 \). If \( \left( \frac{b_1 - b_2}{a_1 - a_2} \right) \) holds, \( I_1 \cap I_2 = (\partial_y \langle \partial_x \rangle ) \) and
   \[
   I_1 + I_2 = \left( \partial_x + \frac{a_1 b_2 - a_2 b_1}{a_1 - a_2}, \partial_y + \frac{b_1 - b_2}{a_1 - a_2} \right).
   
2. **Double root** \( a_1 = a_2 = a, b_1 \neq b_2 \). There holds \( I_1 \cap I_2 = (\partial_x \langle \partial_x \rangle ) \) and \( I_1 + I_2 = \langle \partial_y \rangle \).
3. **If the preceding two cases do not apply** there holds \( I_1 \cap I_2 = (\partial_x y, \partial_x) \) LT and \( I_1 + I_2 = \langle 1 \rangle \).

If a decomposition comprises an operator with leading derivative \( \partial_y \), the following result is required.

**Theorem 2.2.** Let the ideals \( I_1 = (\partial_x + a_1 \partial_y + b_1) \) and \( I_2 = (\partial_x + b_2) \) be given. There are two different cases for their intersection \( I_1 \cap I_2 \).

1. **If** \( b_1 - a_1 b_2 \neq b_2 \), there holds \( I_1 \cap I_2 = (\partial_y \langle \partial_x \rangle ) \) LT and
   \[
   I_1 + I_2 = (\partial_x + b_1 - a_1 b_2, \partial_y + b_2).
   
2. **If the preceding case does not apply** there holds \( I_1 \cap I_2 = (\partial_x y, \partial_x) \) LT and \( I_1 + I_2 = \langle 1 \rangle \).

The proof involves a slight modification of the proof of Theorem 2.1, it involves essentially Janet basis calculations. The special case that \( n \) first-order equations originate from the factorization of an operator of order \( n \) is treated next.

**Lemma 2.3.** Let \( L \) be a partial differential operator in \( x \) and \( y \) with leading term \( \partial_x \), and let \( l_i \equiv \partial_x + a_i \partial_y + b_i, i = 1, \ldots, n, a_i \neq a_j \) for \( i \neq j \), be right divisors of \( L \). Then the intersection ideal generated by the \( l_i \) is principal and is generated by \( L \).

Just like factoring an ordinary differential operator involves solving Riccati equations \( y' + a y^2 + b y + c = 0 \), factoring pde’s in the plane may require to solve equations of the form
\[
z_x + a z_x + b z_x^2 + c z + d = 0
\]
where \( a, b, c, d \) are from the base field. For obvious reasons they are called **partial Riccati equations**. They have been discussed in detail in [7].

3. **PDE’S WITH LEADING DERIVATIVE \( \partial_{xxx} \)**

This case is also interesting for historical reasons because it was the first third order partial differential operator for which factorizations were considered [1]. Especially the operator discussed in Blumberg’s thesis, see Example 3.7 below, attracted a lot of interest.

**Proposition 3.1.** Let the third order linear partial differential operator
\[
D_{xxx} \equiv \partial_{xxx} + A_1 \partial_{xx} + A_2 \partial_{xxy} + A_3 \partial_{yyy}
+ A_4 \partial_{xx} + A_5 \partial_{xy} + A_6 \partial_{yy} + A_7 \partial_y + A_8 \partial_y + A_9
\]
be given with \( A_k \in \mathbb{Q}(x, y) \) for all \( k \). Its first order right factors \( \partial_x + a \partial_y + b \) are essentially determined by the roots \( a_1, a_2 \) and \( a_3 \) of \( a^3 - A_1 a^2 + A_2 a - A_3 = 0 \). The following alternatives may occur.

1. If \( a_1, a_2 \) and \( a_3 \) are three pairwise different rational solutions and the corresponding \( b \) are determined by (8), each \( a_1, b_1 \) satisfying (9) and (10) yields a factor \( 1 = \partial_x + a_1 \partial_y + b_1 \). If there are three factors, the operator is completely reducible and there holds \( D_{xxx} = L \partial_m (l_1, l_2, l_3) \). If there are two factors, their intersection may not be principal. If only a single pair satisfies conditions (9) and (10), or there is only a single rational solution which satisfies them, there is only a single factor.

2. If \( a_1 = a_2 \) is a twofold rational solution that does not satisfy (8), and \( a_3 \neq a_1 \) is a simple rational solution, a single right factor \( \partial_y + a_3 \partial_y + b_3 \) exists if \( a_3, b_3 \) as determined by (8) satisfy (9) and (10).

3. If \( a_1 = a_2 \) is a twofold rational solution satisfying (8), \( a_3 \neq a_1 \) is a simple rational solution, and \( a_3, b_3 \) as determined by (8) do not satisfy (9) and (10), the following subcases may occur.

a) There is a right factor of the form \( l(\Phi) \equiv \partial_y + a_1 \partial_y + R(x, y, \Phi(w)) \) whereas \( R(x, y, \Phi(w)) \) is the general solution of (9) satisfying (10); \( \omega(x, y) \) is the first integral of \( \frac{dy}{\Phi} = a_1(x, y) \). \( \Phi \) is an undetermined function; \( l(\Phi) \) leads to a second order right factor \( L \partial_m (l_1, l_2, \Phi) \) with \( \Phi \neq \Phi \) specializations of \( \Phi \).

b) There are two right factors \( l_i \equiv \partial_y + a_i \partial_y + r_i(x, y), i = 1, 2 \), if \( r(x, y) \) has rational solutions \( r_1(x, y) \) satisfying (10), or if it has a general rational solution as in case a), but constraint (10) singles out the special solutions \( r_1(x, y) \) and \( r_2(x, y) \); \( l_1 \) and \( l_2 \) generate a second order factor \( L \partial_m (l_1, l_2) \).

c) There is a single right factor \( l \equiv \partial_x + a_1 \partial_x + r(x, y) \) if \( r(x, y) \) has the single rational solution \( r(x, y) \) satisfying (10), or if it has two rational solutions or a general rational solution, but (10) singles out the single solution \( r(x, y) \).

4. If \( a_1 = a_2 \) is a twofold rational solution satisfying (8), \( a_3 \neq a_1 \) is a simple rational solution, and \( a_3 \) and \( b_3 \) as determined by (8) satisfy (9) and (10), the same distinctions occur as in the preceding case i), supplemented by an additional factor \( \partial_x + a_3 \partial_y + b_3 \).
v) If \( a_1 = a_2 = a_3 = \frac{1}{3} A_1 \) is a threefold rational solution, there holds
\[
A_1^2 A_4 - 3 A_1 A_5 + 9 A_6 = 0. \tag{2}
\]

The following subcases may occur.

a) If the coefficient of \( b \) in
\[
(A_{1,x} + \frac{1}{3} A_1 A_{1,y} + \frac{2}{3} A_1 A_{4} - A_5) b = \frac{1}{4} A_{1,x} + \frac{2}{3} A_1 A_{1,y} + \frac{2}{3} A_1 A_{1,y} - \frac{1}{3} A_1 A_{1,y} + \frac{1}{3} A_1 A_{1,y} + \frac{1}{3} A_1 A_{7} - A_8 = 0
\]
does not vanish, \( b \) may be determined uniquely from it. A factor exists if the constraint
\[
b_{xx} + \frac{1}{2} A_1 b_{xy} + \frac{1}{2} A_1^2 b_{yy} - 3 b_x + 4 A_2 b_y - A_1 b_y - (\frac{2}{3} A_1, x + \frac{2}{3} A_1 A_{1,y} + \frac{1}{3} A_1 A_4 - A_5) b_y + b_x - A_1 b^2 + A_2 b - A_3 = 0
\]
is satisfied. \tag{3}

b) If the coefficient of \( b \) in (3) vanishes, there holds
\[
A_{1,x} + \frac{1}{3} A_1 A_{1,y} + \frac{2}{3} A_1 A_{4} - A_5 = 0,
\]
\[
A_{1,xx} + \frac{2}{3} A_1 A_{1,xy} + \frac{2}{3} A_1^2 A_1 A_{yy} + A_4 A_{1,y} + \frac{1}{3} A_{1,x} A_{1,y} + \frac{2}{3} A_1 A_{1,y} + \frac{1}{3} A_1 A_4 - A_5 = 0
\]
and \( b \) is determined by
\[
b_{xx} + \frac{1}{2} A_1 b_{xy} + \frac{1}{2} A_1^2 b_{yy} - 3 b_x + 4 A_2 b_y - A_1 b_y + \frac{1}{3} (A_{1,x} + \frac{2}{3} A_1 A_{1,y} + A_4 A_{1,y} b_y + b_x - A_1 b^2 + A_2 b - A_3 = 0. \tag{4}
\]

Proof. Dividing the operator (1) by \( \partial_x + a \partial_y + b \), the requirement that this division be exact leads to the following set of equations between the coefficients.
\[
a^2 - A_1 a^2 + A_2 a - A_3 = 0, \tag{7}
\]
\[
(A_1 - 3 a) a_x + (3 a^2 - 3 A_1 a + 2 A_2) a_y - A_4 a^2 + A_5 a + (3 a^2 - 2 A_1 a + A_2) b = A_6, \tag{8}
\]
\[
(A_1 - 3 a) b_x + (3 a^2 - 3 A_1 a + 2 A_2) b_y - (A_1 - 3 a) b_y + (A_5 - 2 A_4 a - 2 A_1 a + 3 a A_2 - 3 a_x) b + a_{xx} + (A_1 - a) a_{xy} + (a^2 - A_1 a + A_2) a_{yy} - 2 a_x a_y + A_4 a_x - (A_1 - a) a_y - (A_4 a - A_5) a_y + A_7 a - A_8 = 0
\]
\tag{9}

\[
b_{xx} + (A_1 - a) b_{xy} + (a^2 - A_1 a + A_2) b_{yy} - (2 a_x + (A_1 - a) a_y + 2 A_4 a - A_5) b_y + (A_4 - 3 b) b_x + (3 a - 2 A_1) b b_y + b^3 - A_1 b^2 + A_2 b - A_3 = 0. \tag{10}
\]

The algebraic equation (7) determines \( a \). The following discussion is subdivided by the various cases.

Case i). Assume at first that it has three simple roots \( a_1, a_2 \) and \( a_3 \). None of them may be rational, there may be a single rational solution, or all three roots are rational. For any of these roots the coefficient of \( b \) in (8) does not vanish because it is the derivative of (7) w.r.t. \( a \). Therefore for each \( a_i \), equation (8) determines the corresponding value \( b_i \). For those values of \( a_i \) and \( b_i \) which satisfy the constraints (9) and (10), a right factor \( l_i \equiv \partial_x + a_i \partial_y + b_i \) exists. If there are three right factors, by Lemma 2.3 \( D_{x,y} \) is completely reducible and there holds \( D_{x,y} = Lcm(l_1, l_2, l_3) \).

Case ii). Assume now there is a twofold rational solution \( a_1 = a_2 \neq a_3 \) of equation (7). The double root \( a_1 = \frac{1}{2} (A_1 - \sqrt{A_1^2 - 3 A_2}) \) is one of the roots of the equation \( p \equiv 3 a^2 - 2 A_1 a + A_2 = 0 \) of (7). The coefficient of \( b \) in (8) vanishes for \( a = a_1 \) because it is equal to \( p \); (8) becomes a constraint for \( a_1 \). Assume that it is not obeyed. For the root \( a_3 \), the coefficient \( b_3 \) follows from (8). The existence of a right factor involving \( a_3 \) and \( b_3 \) depends on whether they satisfy (9) and (10). If this is true there is a single right factor \( \partial_x + a_3 \partial_y + b_3 \).

Case iii). If the double root \( a_1 \) does satisfy the constraint following from (8), the corresponding value \( b = b_1 \) has to be determined from the partial Riccati equation (9) with \( a = a_1 \). Those values of \( a_1 \) and \( b_1 \) satisfying (10) have to be singled out. If \( a_3 \) and \( b_3 \) do not satisfy (9) and (10), the only possible factor is the one involving \( a_1 \) and \( b_1 \). A simple calculation shows that the terms of (9) involving derivatives of \( b \) simplify to \( b_3 = a_1 b_1 \). According to Lemma 5.3 of [7], the general solution for \( b_1 \) may contain an undetermined function \( \Phi \) of \( \phi(x, y) \), the first integral of \( \frac{dx}{dy} = a_1(x, y) \). The condition (10) may be satisfied without constraints for \( \Phi \), it may restrict it to a certain form, or it may completely eliminate the corresponding solution. Similar arguments apply if there is a single rational solution for \( b_1 \), or if there are two of them. Whenever two factors are obtained, by case iii) of Theorem 2.1 they generate a principal ideal.

Case iv). If again \( a_1 \) satisfies the constraint following from (8), and \( a_3 \) and \( b_3 \) satisfy (9) and (10) now, the same alternatives as in case iii) occur with an additional factor \( \partial_x + a_3 \partial_y + b_3 \).

Case v). Finally assume there is a threefold solution \( a_1 = a_2 = a_3 = \frac{1}{3} A_1 \). Then \( A_2 = \frac{1}{3} A_1^2 \) and \( A_3 = \frac{1}{3} A_1^3 \). The coefficient of \( b \) in (8) vanishes again, it becomes the constraint (2). The coefficients of \( b_x, b_y \) and \( b^2 \) in (9) vanish.

If \( A_{1,x} + \frac{1}{3} A_1 A_1, y + \frac{1}{3} A_1 A_4 - A_5 \neq 0 \), \( b \) is determined by (3). In order for a factor to exist, in addition (4) must be satisfied. This is case subcase a). In the exceptional case that the coefficient of \( b \) in (3) vanishes, it becomes a constraint and \( b \) has to be determined from (6). This is case subcase b).

In order to obtain a complete answer concerning all possible factorizations of the operator (1), second order right factors have to be taken into account as well. They are considered next.

Proposition 3.2 Let the third order partial differential operator (1) be given with \( A_k \in C(x, y) \) for all \( k \). Its second order right factors
\[
\partial_x + a \partial_y + b \partial_y + c \partial_x + d \partial_y + f \tag{11}
\]
are determined by the roots \( a_1, a_2 \) and \( a_3 \) of
\[
a^3 - 2 A_1 a^2 + (A_1^2 + A_2) a - A_1 A_2 + A_3 = 0.
\]
The following alternatives may occur.

i) If \(a_1, a_2\) and \(a_3\) are three pairwise different rational solutions for each \(a\), the corresponding values of \(b, c, d\) and \(f\) follow from equations (20). Those values which satisfy constraints (22) yield a factor.

\[
(A_1 - a_1)[a_{1,x} + (A_1 - a_1)a_{1,y} + A_1a_1 - A_3] = b_{1,x} + (A_1 - a_1)b_{1,y} + A_4b_1 - A_6.
\]

The coefficient \(c\) has to be determined from the partial Riccati equation

\[
c_{xx} - (a - A_1)c_{xy} - c^2 + \frac{3}{2a - 2A_1}[(a - \frac{1}{2}A_1)x - (a - A_1)(a - \frac{1}{2}A_1)y + \frac{1}{2}(a - \frac{1}{2}A_1)A_4 - \frac{1}{2}A_5)c
\]

\[
+ \frac{1}{2a - 2A_1}a_{xx} - 2(a - A_1)a_{xy}
\]

\[
+ (a^2 - 2A_1a + A_1^2)a_{yy} - a_y(a - A_1)x
\]

\[
+ (a - A_1)(a - A_1)y - 2A_4a_y - A_4a_y + A_5y + A_7
\]

\[
+ A_5, a = 2A_4a_x - A_5x - A_4A_5 + A_8 + A_2^2a = 0.
\]

(12)

If it has a rational solution, \(d, f\) and \(c, d\) follow from the first equation of (21) and (22) respectively. The values satisfying the constraint

\[
c_{xx} - 2(a - A_1)c_{xy} + (a - A_1)^2c_{yy}
\]

\[
+ [(a - A_1)x - \frac{1}{2}(a^2 - A_1a - A_1^2)y + (a - A_1)(2A_4 - 3c)]c
\]

\[
+ [(a - A_1)(3c - A_4)y + A_4x + A_5^2 + A_7c]
\]

lead to a factor.

iii) If \(a_1 = a_2 = a_3 = \frac{3}{4}A_1\) is a threefold solution and \(b_1 = \frac{1}{4}A_1^3\), \(c\) is uniquely determined by

\[
(A_{1,x} + \frac{1}{2}A_{1,y} + \frac{3}{2}A_{1,y} - A_2)c = \frac{1}{2}A_{1,x} + \frac{1}{2}A_{1,x} + \frac{1}{2}A_{1,x}
\]

\[
\frac{3}{2}A_{1,x} + \frac{1}{2}A_{1,y} + \frac{3}{2}A_{1,y} - A_2c = \frac{1}{2}A_{1,x} + \frac{1}{2}A_{1,x} + \frac{1}{2}A_{1,x}
\]

\[
\frac{1}{2}A_{1,x} + \frac{3}{2}A_{1,y} - A_2c = \frac{1}{2}A_{1,x} + \frac{1}{2}A_{1,x} + \frac{1}{2}A_{1,x}
\]

\[
+ A_4A_5 - A_5x + A_8 = 0.
\]

(14)

if its coefficient does not vanish. The remaining coefficients follow from

\[
d - \frac{1}{2}A_1c + \frac{3}{2}A_{1,x} + \frac{3}{2}A_{1,y} + \frac{1}{2}A_4A_4 - A_2 = 0,
\]

\[
f + c_x + \frac{1}{2}A_1c_{xy} - c^2 + A_4c - A_7 = 0.
\]

(16)

In addition the two constraints

\[
A_1^2A_4 - 3A_1A_5 + 9A_6 = 0.
\]

(17)

\[
c_{xx} + \frac{3}{2}A_1c_{xy} + \frac{3}{2}A_1c_{yy} + (2A_4 - 3c)c_x
\]

\[
+ [(\frac{1}{2}A_{1,x} + \frac{1}{2}A_{1,y} + \frac{3}{2}A_{1,y} - A_1c)y]
\]

\[
+ c^3 - 2A_4c^2 + (\frac{1}{2}A_1A_4, y + A_{1,x} + A_5^2 + A_7)c
\]

\[
- \frac{1}{2}A_1A_4, y - A_4A_7 - A_7c_x + A_6 = 0
\]

must be satisfied. If the coefficient of \(c\) in (15) vanishes, it becomes a constraint and \(c\) has to be determined from (18).

Proof. Dividing the operator (1) by (11), the requirement that this division be exact leads to the following set of equations for the coefficients.

\[
a^3 - 2A_1a_3^2 + (A_1^2 + A_3)a - A_1A_2 + A_3 = 0,
\]

\[
b - a^2 + A_1a - A_2 = 0,
\]

\[
(A_1 - 2a)c + d + a_x + (A_1 - a)a_y + A_4a - A_5 = 0,
\]

\[
(a_3^2 - A_1a + A_2)c - (A_1 - a)d - b_x
\]

\[
- (A_1 - a)b_y - A_4b + A_6 = 0,
\]

\[
f + c_x + (A_1 - a)c_y - c^2 + A_4c - A_7 = 0,
\]

\[
(A_1 - a)f - dx - (A_1 - a)d_y - (A_4 - c)d + A_8 = 0,
\]

\[
f_x + (A_1 - a)f_y + (A_4 - c)f - A_9 = 0.
\]

(23)

The algebraic equation (19) determines always \(a\). The following discussion is subdivided by the various cases of the above theorem.

Case i) Assume it has three simple roots \(a_1 \neq a_2 \neq a_3\). The corresponding values \(b, c, d\) may be determined from (20). The coefficient determinant of \(c, d, f\) in (21) is \(a^2 - \frac{1}{4}A_1a + \frac{1}{3}(A_4^2 + A_2^2)\), it is the derivative of (19) w.r.t. \(a\) that does not vanish for simple roots. Therefore for any simple root \(a_1\), the corresponding values \(c, d, f\) may be determined from (21). Finally \(f\) follows from (22). These values have to satisfy the constraints (23) in order to determine a factor.

Case ii) Assume now there is a twofold rational solution \(a_1 = a_2 \neq a_3\) of (19). The double root \(a_1 = \frac{3}{4}A_1 + \frac{1}{2}\sqrt{A_4^2 - 3A_2^2}\) is one of the roots of the equation for (21) is \(A_1 - a = 0\). In order to exclude a triple root there must hold \(A_1^2 - 3A_2 \neq 0\); \(b_1\) follows from (20). Because the coefficient determinant of \(c, d, f\) in (21) vanishes for \(a = a_1\), \(b_1\) has to proceed as follows. Eliminating \(d\) from the first equation (21) and substituting it into the second one yields the constraint (12). Eliminating \(d\) from the first equation (21), \(f\) from (22) and substituting them into the first equation (23), the partial Riccati equation (13) for is obtained. If it has a rational solution \(c, d, f\) the coefficient \(d_1\) may be obtained from the first equation (21) and (13) follows. Substituting these values into the second equation of (23), the constraint (14) follows. It may exclude a factor, or it may constrain the undetermined elements in the solution of (13).

Case iii) Finally assume there is a threefold solution \(a_1 = a_2 = a_3 = \frac{3}{4}A_1\) with the corresponding value \(b_1 = \frac{1}{4}A_1^3\). Then \(A_2 = \frac{1}{2}A_1^2\) and \(A_3 = \frac{3}{2}A_1^2\). The coefficient determinant for \(c, d, f\) in (21) vanishes. By elimination equation (15) is obtained for \(c\) if \(A_{1,x} + \frac{1}{2}A_{1,y} + \frac{3}{2}A_{1,y} - A_1c \neq 0\). If \(c\) is obtained. If it has a rational solution \(c, d, f\) in (15) vanishes, it becomes a constraint: \(c\) has to be determined from (18).

The next corollary summarizes to what extent the factors of Propositions 3.1 and 3.2 may be obtained algorithmically.

**Corollary 3.3.** Any first-order factor of (1) corresponding to a simple root of (7), or any second-order factor co-
Proposition 3.2) and equation (18) for the coefficient $b$ above condition. If it does vanish, components may be set up as follows. After the possible factorizations of an operator (1) have been determined, a listing of its completely reducible components is obtained with the general solution $b$.

**Theorem 3.4.** Any differential operator

\[ L \equiv \partial_{xx} + A_1 \partial_{xy} + A_2 \partial_{yy} + A_3 \partial_{y}
\]

(24)

decomposes into completely reducible components corresponding to one of the types $L^{(i)}_{xx} k = 1, \ldots, 5$, defined as follows;

$L_{xx}, L_{xy}$ and $L_x$ are completely reducible operators with leading derivatives $\partial_{xx}, \partial_{xy}$ or $\partial_x$, respectively. $J^{(1,2)}_k$ is the ideal $(\partial_{xx}, \partial_{xy}) LT$ defined in Table 1. Upper indices distinguish different copies within a type definition.

$L^{(1)}_{xx} : L_{xx}, L^{(2)}_{xx} : L_{xy} L_x, L^{(3)}_{xx} : L_x L_{xy}, L^{(4)}_{xx} : L^{(1)}_x L^{(2)}_x (25), L^{(5)}_{xx} : Sy((L), J^{(1,2)}_5) J^{(1,2)}_5.$

The decomposition $L^{(1)}_{xx}$ is completely reducible.

**Proof.** It is based on Propositions 3.1 and 3.2. In the separable case $i$) of Proposition 3.1 there may be three factors with a principal intersection, this yields type $L^{(1)}_{xx}$. There may be two factors which do not have a principal intersection. If they are divided out and the decompositions of the respective second-order factors are taken into account (see Section 4 of [7]), type $L^{(5)}_{xx}$ is obtained. If case $i$) allows only a single factor, or if case $ii$), or subcase $c)$ of case $iii$), or case $v$) applies, and again the decompositions of the corresponding second order left factor are taken into account, decomposition types $L^{(2)}_{xx}$ or $L^{(3)}_{xx}$ follow. For case $iii$), the principality of the intersection of the right factors leads to type $L^{(3)}_{xx}$. In case $iv$) the decompositions of case $iii$) are extended by an additional first order factor. According to Lemma 2.3, combined with the factors already obtained they generate a principal intersection covered by $L^{(1)}_{xx}$. By definition, an irreducible operator (24) corresponds to a type $L^{(1)}_{xx}$ decomposition.

**Corollary 3.5.** Let $l_x, l_{xx}$ and $l_{xxx}$ denote irreducible operators with leading term $\partial_x, \partial_{xx}$ or $\partial_{xxx}$. An additional upper index distinguishes different copies of the respective operator. $\Phi$ is an undetermined function of a single argument. The types $L^{(i)}_{xx}$ defined in Table 2 are refinements of the types $L^{(i)}_{xx}, i = 1, \ldots, 5$ defined in Theorem 3.4.

This is an immediate consequence of the proof given for the preceding theorem if the various factorization alternatives of Proposition 3.1 and Proposition 3.2 are not merged into completely reducible components.

**Example 3.6.** Consider the operator

\[ L \equiv \partial_{xxx} + (y + 2) \partial_{xy} + (y + 1) \partial_{yy} + (1 - \frac{1}{y}) \partial_{xx} + (y + 2 - \frac{1}{y}) \partial_{xy} + (y + 1) \partial_{yy} - \frac{1}{y} \partial_x - \frac{1}{y} \partial_y. \]

Equation (7) reads $a^3 - (y + 2)a^2 + (y + 1)a = 0$ with the three roots $a_1 = 0, a_2 = 1$ and $a_3 = y + 1$, i.e. case $i$) of Proposition 3.1. The corresponding values of $b$ are $b_1 = 1, b_2 = b_3 = 0$. It turns out that all pairs $a_i, b_i$ satisfy conditions (9) and (10). Consequently there are three right factors $l_1 = \partial_x + 1, l_2 = \partial_x + \partial_y$ and $l_3 = \partial_x + (y + 1) \partial_y$. This holds for $L = Lclm(l_1, l_2, l_3), i.e. \text{the decomposition type is } L^{(1,2)}_{xx}$. The general solution of $L = 0$ is

\[ z = F(y) e^{-y} + G(x - y) + H([y + 1] e^{-y}) \]

where $F, G$ and $H$ are undetermined functions.

**Example 3.7.** (Blumberg 1912) In his dissertation Blumberg [1] considered the third order operator

\[ L \equiv \partial_{xx} + x \partial_{xy} + 2 \partial_{xy} + 2(x + 1) \partial_{xy} + \partial_x + (x + 2) \partial_y \]

(25)

generating a principal ideal of gauge $(1, 3)$. He gave its factorizations

\[ L = \begin{cases} \partial_{xx} + x \partial_{xy} + \partial_x + (x + 2) \partial_y & (\partial_x + 1) \partial_y, \\
(\partial_x + 2 \partial_y + 1)(\partial_x + \partial_y), & (26) \end{cases} \]

with two completely reducible second order left factors. This result may be obtained by Proposition 3.1 as follows. Equation (7) is $a^3 - xa^2 = a^2(x - a) = 0$ with the double root $a_1 = 0$, and the simple root $a_2 = x$. The latter yields $b_2 = 0$. Because these values satisfy (9) and (10), the factor $l_2 = \partial_x + x \partial_y$ is obtained, i.e. case $iv$) of Theorem 3.1 applies. For the double root $a_1 = 0$, from (9) the Riccati equation

\[ b_y - b^2 + \frac{2}{x} b - 1 - \frac{2}{x} = 0 \]

is obtained with the general solution $b = 1 + \frac{1}{x} \frac{1}{x} + \frac{1}{x} + \Phi(y).$

Substitution into (10) yields $\Phi = 0$, i.e. $b = 1$ and the factor $l_1 = \partial_x + 1$. The second order left factor in the first line at the right hand side of (26) is absolutely irreducible, whereas the second order factor in the second line is the left intersection of two first order factors, i.e. (26) may be further decomposed into irreducibles as

\[ L = \begin{cases} \partial_{xx} + x \partial_{xy} + \partial_x + (x + 2) \partial_y & (\partial_x + 1) \\
Lclm(\partial_x + 1, \partial_x + 1 - \frac{1}{2})(\partial_x + x \partial_y), & (27) \end{cases} \]
The intersection ideal of \( l_1 \) and \( l_2 \) is not principal, by
Theorem 2.1 it is
\[
Lclm(l_1, l_2) = (L_1 \equiv \partial_{xxx} - x^2 \partial_{xyy} + 3 \partial_x + (2x + 3) \partial_y - x^2 \partial_{yy} + 2 \partial_x + (2x + 3) \partial_y, L_2 \equiv \partial_{xxx} + x \partial_{xyy} - \frac{1}{2} \partial_x - \frac{1}{2} \partial_y + x \partial_y - \frac{1}{2} \partial_x - (1 + \frac{1}{2}) \partial_y)
\]
with gauge (1, 2), therefore the decomposition (26) is of type \( L_{xxx}^{(5)} \). Both generators \( L_1 \) and \( L_2 \) have the divisors \( l_1 \) and \( l_2 \). There holds \( L \in Lclm(l_1, l_2) \) explicitly \( L = L_1 + x L_2 \).
As a consequence the system of equations \( L_1 \equiv 0 \) and \( L_2 \equiv 0 \) has the two solutions \( F(y, -\frac{1}{2} x^2 +) \) and \( G(y) e^{-x} \); they correspond to \( L_2 = 0 \) and \( L_1 = 0 \) respectively.
In order to obtain the remaining part of the solution of Blumberg’s equation \( Lz = 0 \) of gauge (1, 1) the relative syzygy modules
\[
Syz((L), (l_1, l_2)) = \{(1, x), (- \partial_y + \frac{1}{2} \partial_x - x \partial_y + 2 + \frac{1}{2})\}
\]
is constructed. Introducing the new differential indeterminates \( z_1 \) and \( z_2 \), the equations \( z_1 + x z_2 = 0 \) and \( z_2 x + (1 + \frac{1}{2}) z_2 = 0 \) are obtained with the solution \( z_1 = -H(y) e^{-x} \) and \( z_2 = H(y) \frac{1}{2} e^{-x} \). \( H \) an undetermined function of \( y \). The desired solution is a special solution of the inhomogeneous system \( L_1 z = -H(y) e^{-x} \) and \( L_2 z = H(y) \frac{1}{2} e^{-x} \). The solution of the corresponding homogeneous system is already known, therefore the general solution of \( Lz = 0 \) may be written in terms of integrations as
\[
z = F(y, -\frac{1}{2} x^2) + G(y) e^{-x} + \int \left( H(y) \frac{1}{2} e^{-x} \right) d x \bigg|_{y = y_0 - \frac{1}{2} x^2}
\]
where \( y = -\frac{1}{2} x^2 \).

Example 3.8. Consider the operator
\[
L \equiv \partial_{xxx} + (x + y - 1) \partial_{xyy} - (x + y - 1) \partial_{xxx} - (x + y - 1) \partial_{xy} - (x^2 + xy - x + 1) \partial_x - (x + y) \partial_y - (x^2 + xy + y) \partial_y - x y - 1.
\]
Equation (7) reads \( a^3 - (x + y + 1) a^2 - (x + y) a = 0 \) with roots \( a_1 = 0, a_2 = -1 \) and \( a_3 = x + y \), i.e. case 1) of Proposition 3.1 applies. Equation (8) yields the corresponding values
\[
b_1 = 1, \ b_2 = -x \text{ and } b_3 = -\frac{2x^2 + xy - y^2 - y + 2}{4x + 5y - 3}.
\]
Only \( a_1 \) and \( b_1 \) satisfy the constraints (2) and (4). Consequently there is a single first order factor \( l_1 \equiv \partial_x + 1 \). Pursuant to case i) of Proposition 3.2 a second order factor does not exist. Dividing \( L \) by \( l_1 \) yields the operator
\[
\partial_{x} + (x + 1) \partial_{xy} + x \partial_{yy} + (y + 1) \partial_x + (xy + 1) \partial_y + x + y = (\partial_x + x \partial_y + 1)(\partial_x + \partial_y + y)
\]
Consequently the operator \( L \) has a type \( L_{xxx}^{(5)} \) decomposition
\[
L = (\partial_x + x \partial_y + 1)(\partial_x + \partial_y + y)(\partial_x + 1).
\]
Example 3.9. Consider the operator
\[
L \equiv \partial_{xxx} + (x + y + 1) \partial_{xyy} + (x + y) \partial_{xxx} + (x^2 + xy + y^2 + 2y + 1) \partial_{xy} + (x + y) \partial_x + (x + y + y) \partial_y.
\]
Equation (7) reads \( a^3 (a - y - 1) = 0 \) with double root \( a_1 = 0 \) and single root \( a_3 = y + 1 \). It turns out that case iv) applies with \( a_1 = 0, b_1 = 1 \) and \( a_3 = y + 1, b_3 = 0 \). The corresponding first order factors yield the divisor as the principal intersection
\[
Lclm(\partial_x + 1, \partial_x + (y + 1) \partial_y) = \partial_x + (y + 1) \partial_y + \partial_x + (y + 1) \partial_y.
\]
Consequently \( L \) has the decomposition
\[
(\partial_x + x + y) Lclm(\partial_x + 1, \partial_x + (y + 1) \partial_y)
\]
of type \( L_{xxx}^{(5)} \).

4. PDE’S WITH LEADING DERIVATIVE \( \partial_{xy} \)
If an equation does not contain a derivative \( \partial_{xx} \) but only \( \partial_{yy} \), permuting \( x \) and \( y \) leads to an equation of the form (1) such that the above theorem may be applied. If there is neither a term \( \partial_{xxx} \) or \( \partial_{yy} \), the general third order operator
\[
D_{xy} \equiv \partial_{xyy} + A_1 \partial_{xy} + A_2 \partial_{xxx} + A_3 \partial_{xy} + A_4 \partial_{yy} + A_5 \partial_x + A_6 \partial_y + A_7
\]
is obtained. Its possible decompositions are described in this subsection. Like in the previous case, first- and second-order factors are considered separately.

Proposition 4.1. Let the third order partial differential operator (30) be given with \( A_k \in Q(x, y) \) for all \( k \). The following first order right factors may occur.

i) If \( A_1 \neq 0, a_1 = 0, a_2 = A_1 \), and \( b \) is determined from \( a_x - (3a - 2A_1) a_y - A_2 a^2 + A_3 a - (2a - A_1) b = A_4 \), a first order right factor \( \partial_x + a \partial_y + b \), exists if \( a \) and \( b \) satisfy the constraints
\[
b_x = -3a - 2A_1) b_y - b^2 - (2a + 2A_2 a - A_1) b + a_{xy} - (a - A_1) a_{yy}
\]
\[
+ A_2 a_x - a_y^2 - (A_2 a - A_3) a_y + A_5 a = A_6,
\]
\( b_{xy} + (A_1 - a)b_y + A_2 b_x - (2b + a_y + A_2 a - A_3)b_y - A_2b^2 + A_3b = A_7. \)

ii) If \( a = A_1 = A_4 = 0 \), the factor \( \partial_x + b \) exists with \( b = A_{0,y} + A_2 A_4 - A_2 A_3 \) if \( A_{3,y} + A_2 A_3 - A_3 \neq 0 \) and \( b_y - b^2 + A_3 b = A_6. \)

iii) If \( a = A_1 = A_4 = 0 \) and the two constraints

\[
A_{0,y} + A_2 A_6 - A_7 = 0, \quad A_{3,y} + A_2 A_3 - A_5 = 0
\]

are satisfied, the following two subcases may occur.

a) There is a right factor of the form \( \partial_x + R(x, y, \Phi(y)) \), where \( R \) is the general rational solution of

\[
b_x - b^2 + A_3 b - A_6 = 0 \tag{31}
\]

involving an undetermined function \( \Phi(y) \).

b) There is a single factor, or there are two factors \( \partial_x + r_i(x, y) \) where \( r_i(x, y) \) are special rational solutions of (31).

iv) A factor \( \partial_x + A_2 \) exists if the following two constraints are satisfied.

\[
A_5 = A_2 A_3 + 2A_2 x + A_{2,y} A_1 - A_2^2 A_1, \\
A_7 = A_2 A_6 + A_2 A_3 A_4 - A_2 A_3^2 + A_2 A_3 x, \\
+ A_2 A_3 x + A_2 A_3 A_1 - 2A_2 x A_2 A_1.
\]

PROPOSITION 4.2. Let the third order partial differential operator (30) be given with \( A_1 \in \mathbb{Q}(x, y) \) for all \( k \). The following second order right factors may occur.

i) A factor with leading derivative \( \partial_x \) does not contain a derivative \( \partial_y \). A factor \( \partial_x + a_1 \partial_{x_3} + a_2 \partial_x + a_3 \partial_y + a_4 \) exists if the two constraints

\[
A_{1,y} + 2A_{1,y} A_2 - A_{2,y} A_1 - A_3, \quad A_4,y + 2A_{4,y} A_2 + 2A_4 A_2 - A_6 - A_7 = 0
\]

are satisfied. Then \( a_1 - a_1 = 0, a_2 = A_3 - A_1 A_2 - A_1 y, a_3 = A_4 \) and \( a_4 = A_6 - A_2 A_4 - A_4 A_2 \).

ii) A factor \( \partial_x + a_1 \partial_{y_3} + a_2 \partial_x + a_3 \partial_y + a_4 \) may exist if one of the four alternatives applies.

a) If the conditions \( A_1 = A_4 = 0, A_2 x + \frac{1}{2} A_2 A_3 - \frac{1}{2} A_3 = 0 \) and \( A_{2,y} A_1 + A_2 A_3 + A_4 x + A_3 x - A_7 = 0 \) are satisfied, the coefficients of the factor may be determined from \( a_1 = 0, a_2 = A_2, a_3 = a_3^2 + A_3 A_3 - A_6 = 0 \) and \( a_4 = A_2 A_3 + 2A_2 x \).

b) If the conditions \( A_1 = A_4 = 0 \) and \( A_2 x + \frac{1}{2} A_2 A_3 - \frac{1}{2} A_3 = 0 \) are satisfied, define \( P \equiv A_2 x + A_2 A_3 - A_5 \quad \text{and} \quad Q \equiv 2A_2 x + A_2 A_3 - A_5. \) Then \( a_1 = 0, \quad a_2 = A_2, \quad a_3 = \frac{Q}{P} + A_3 P - A_6 \quad \text{and} \quad a_4 = A_2 (a_3 - A_3) - A_2 x + A_3. \) The following condition must hold. \( a_3 A_3 - a_4^2 + A_3 A_3 - A_6 = 0. \)

c) If \( a_4 \neq 0 \) there is a factor with coefficients

\[
a_1 = A_1, \quad a_2 = A_2, \quad a_3 = A_3 + \frac{A_1 x}{A_1} - \frac{A_1}{A_1}, \quad a_4 = \left( \frac{A_1 x}{A_1} - \frac{A_1}{A_1} \right) A_2 - A_2 x + A_5
\]

if the following constraints are satisfied.

\[
a_{3,x} - a_3^2 + A_3 A_3 - A_6 = 0, \quad a_{4,x} - a_4 A_3 + A_3 A_4 - A_7 = 0.
\]

d) If \( A_1 \neq 0 \) there is a factor with coefficients \( a_1 = 0, \quad a_2 = A_2, \quad a_3 = \frac{A_1}{A_1} \quad \text{and} \quad a_4 = \frac{A_2 A_4}{A_1} - A_2 x - A_2 y A_1 + A_2 A_3 - A_5 \) if the following conditions are satisfied.

\[
a_{3,x} + a_3 A_3 - A_1 A_2 A_3 + A_1 A_4 + A_2 A_3 - A_4 = 0, \quad a_{4,x} + a_4 A_3 A_1 - A_1 A_2 A_4 + (A_3 - A_3) A_4 - A_4 = 0.
\]

The proof of the two preceding propositions are similar to those in the previous section and are therefore omitted.

In order to solve concrete problems it is important to know to what extent the factorizations described in this section may be determined algorithmically. The answer to this question is given in the following corollary.

COROLLARY 4.3. Any principal divisor of an operator (30) may be determined algorithmically. The same is true for a possible Laplace divisor of a given order.

PROOF. Testing the conditions (34) of case i) of Proposition 5.1 requires only differentiations and arithmetic in the base field. The same is true for case ii), subcase b). In subcase a) of case ii) the rational solutions of an ordinary Riccati equation have to be determined which is algorithmically possible. In [8] the algorithmic construction of Laplace divisors for fixed values of \( m \) or \( n \) has been shown.

THEOREM 4.4. Any differential operator

\[
D_{x,y} \equiv D_{x,y} + A_1 \partial_{x,y}
\]

+ \( A_2 \partial_x + A_3 \partial_y + A_4 \partial_y + A_5 \partial_x + A_6 \partial_y + A_7 \)

decomposes uniquely into largest completely reducible components corresponding to one of the types \( L(x,y) \), \( k = 1, \ldots, 11 \), defined as follows: \( L_{x,y}, L_{x,y}, L_{x,z}, L_{z}, L_{y} \) are completely reducible operators with leading terms \( \partial_{x,y}, \partial_{y,z}, \partial_{z}, \partial_{x} \) or \( \partial_{y} \) respectively.

\[
L_{x,y}^{(1)} : L_{x,y}, \quad L_{x,y}^{(2)} : L_{x,y} L_{x}, \quad L_{x,y}^{(3)} : L_{x,y} L_{y}, \\
L_{x,y}^{(4)} : L_{x,y}, L_{x,y}, \quad L_{x,y}^{(5)} : L_{x,y} L_{y}, \\
L_{x,y}^{(6)} : L_{x}^{(1)} L_{x}^{(2)} L_{x}, \quad L_{x,y}^{(7)} : L_{x}^{(1)} L_{y} L_{x}^{(2)}, \quad L_{x,y}^{(8)} : L_{y} L_{x}^{(1)} L_{x}^{(2)},
\]

\[
L_{x,y}^{(9)} : \text{Syz}((\partial_{x,y}), \mathfrak{J}_{(1)}^{(1)}(m)) \mathfrak{J}_{(1)}^{(1)}(m), \\
L_{x,y}^{(10)} : \text{Syz}((L), \mathfrak{J}_{(1)}^{(2)}(n)) \mathfrak{J}_{(1)}^{(2)}(n), \\
L_{x,y}^{(11)} : \text{Syz}((L), \mathfrak{J}_{(2)}^{(2)}(n)) \mathfrak{J}_{(2)}^{(2)}(n).
\]

If the completely reducible components are split into irreducible ones, the following refined decomposition scheme is obtained.

COROLLARY 4.5. Let \( l_{x}, l_{y}, l_{x,z}, l_{x,y} \) and \( l_{x,y} \) denote irreducible operators with leading derivatives as determined by the respective subindex. An additional upper index distinguishes different copies of the respective operator; \( \Phi \) is an undetermined function of a single argument. The types \( L_{x,y}^{(1)} \) defined in Table 3 are refinements of the types \( L_{x,y}^{(1)} \), \( i = 1, \ldots, 10 \) of Theorem 5.4.

This is an immediate consequence of the theorem given for the preceding theorem if the various factorization alternatives of Proposition 3.1 and Proposition 3.2 are not merged into completely reducible components.
The remaining coefficients are $a_2 = A_3$ and $a_3 = A_4$. The following second order right factors may occur.

- A factor with leading derivative $\partial_x$ does not contain a term with $\partial_y^k$. A factor $\partial_x a_1 \partial_x + a_2 \partial_y + a_3$ exists if one of the following two subcases applies.
  
  a) There holds $a_1 = 0$, $A_2 = 2A_3 - A_2A_3 = 0$ and $A_6 + A_3A_2 = A_2A_3 - A_2A_3 = 0$. The coefficient $a_1$ may be determined from the Riccati equation
  \[ a_1, y - a_1^2 + A_2a_1 - A_4 = 0. \]  
  b) If $Q \neq 0$ and $P_3Q - (P_3Q + P)A_2PQ - A_2Q^2 = 0$ there is the factor $\partial_y + A_4 + A_3A_2 - A_6$.

A complete listing of all possible Loewy decompositions has to include the divisors of order 2; they are considered next.

**Proposition 5.2.** Let the third order partial differential operator (33) be given with $A_3 \in \mathbb{Q}(x, y)$ for all $k$. The following second order right factors may occur.

- A factor $\partial_y a_1 \partial_x + a_2 \partial_y + a_3$ exists if there holds $A_3 = A_2A_3 = 0$.
  
  a) There holds $A_1 = 0$, $A_2 - 2A_3 - A_2A_3 = 0$ and $A_6 - A_3A_2 = A_2A_3 - A_2A_3 = 0$. The coefficient $a_1$ may be determined from the Riccati equation
  \[ a_1, y - a_1^2 + A_2a_1 - A_4 = 0. \]  
  b) If $Q \neq 0$ and $P_3Q - (P_3Q + P)A_2PQ - A_2Q^2 = 0$ there is the factor $\partial_y + A_4 + A_3A_2 - A_6$.

A complete listing of all possible Loewy decompositions has to include the divisors of order 2; they are considered next.
The proof is almost identical to that of Corollary 4.3 and is therefore omitted.

A full listing of maximal completely reducible components may be obtained from the preceding results. Any operator (33) decomposes uniquely according to one of the following decomposition schemes.

**Theorem 5.4.** Any differential operator

\[ L \equiv \partial_{xy} + A_1 \partial_x + A_2 \partial_{xy} + A_3 \partial_y + A_4 \partial_t + A_5 \]

(37) decomposes uniquely into the largest completely reducible components corresponding to one of the types \( \mathcal{L}^{(i)} \), \( i = 1, \ldots, 11 \), defined as follows; \( \mathcal{L}^{(1)} \), \( \mathcal{L}^{(2)} \), \( \mathcal{L}^{(4)} \), \( \mathcal{L}^{(5)} \), \( \mathcal{L}^{(6)} \), \( \mathcal{L}^{(7)} \), \( \mathcal{L}^{(8)} \), \( \mathcal{L}^{(9)} \), \( \mathcal{L}^{(10)} \), \( \mathcal{L}^{(11)} \), with leading derivatives \( \partial_{xy}, \partial_{xy}, \partial_{xy}, \partial_{xy}, \partial_{xy} \), respectively.

<table>
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<tr>
<th>Type</th>
<th>Description</th>
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<tbody>
<tr>
<td>( \mathcal{L}^{(1)} )</td>
<td>( L_{xy} + L_{xy} )</td>
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<tr>
<td>( \mathcal{L}^{(2)} )</td>
<td>( L_{xy} + L_{xy} )</td>
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<tr>
<td>( \mathcal{L}^{(4)} )</td>
<td>( L_{xy} + L_{xy} )</td>
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<tr>
<td>( \mathcal{L}^{(5)} )</td>
<td>( L_{xy} + L_{xy} )</td>
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<td>( \mathcal{L}^{(6)} )</td>
<td>( L_{xy} + L_{xy} )</td>
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<td>( \mathcal{L}^{(7)} )</td>
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<td>( \mathcal{L}^{(8)} )</td>
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<td>( \mathcal{L}^{(9)} )</td>
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<td>( \mathcal{L}^{(10)} )</td>
<td>( L_{xy} + L_{xy} )</td>
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<td>( \mathcal{L}^{(11)} )</td>
<td>( L_{xy} + L_{xy} )</td>
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The decomposition into reducible components leads to the following refined decomposition scheme.

**Corollary 5.5.** Let \( L, L_1, L_2, L_3 \) \( \Phi \) denote reducible operators with leading derivatives as determined by the respective subindex. An additional upper index distinguishes different copies of the respective operator; \( \Phi \) is an undetermined function of a single argument. The types \( \mathcal{L}^{(i,j)} \) defined in Table 4 are refinements of the types \( \mathcal{L}^{(i,j)} \), \( i = 1, 2, \ldots, 11 \), defined in Theorem 5.4.

This is an immediate consequence of the proof given for the preceding theorem if the various factorization alternatives of Proposition 3.1 and Proposition 3.2 are not merged into completely reducible components.

**Example 5.6.** Consider the operator

\[ L \equiv \partial_{xy} + 2(x + \frac{1}{2}) \partial_x + y \partial_y + x^2 + 4x + \frac{2}{9} \]

Its coefficients satisfy conditions (34) of Proposition 5.1; consequently there is a factor \( \partial_x + y \). Furthermore there holds \( P = 0 \) and \( Q = 0 \) for \( P \) and \( Q \) defined by (35). The Riccati equation for \( c \) is

\[ cy - c^2 + 2(x + \frac{1}{9})c - x^2 + \frac{2}{9} = 0 \]

with the rational solutions \( c = x \) and \( c = x + \frac{1}{9} \); they yield the factors \( \partial_x + y \) and \( \partial_x + y + \frac{1}{9} \) with the type \( \mathcal{L}^{(1,2)} \) decomposition \( L = \text{Lclm}(\partial_x + y, \partial_x + y, \partial_x + y + \frac{1}{9}) \). The general solution of \( Lz = 0 \) is obtained by three integrations as

\[ z = \left[ F(x) + G(x) \frac{1}{y} + H(y) \right] 
\exp(-xy). \]

\( F, G \) and \( H \) are undetermined functions.

**Example 5.7.** Consider the operator

\[ L \equiv \partial_{xy} + \left( \frac{1}{2} \right) \partial_x - \partial_y - \frac{1}{x} \partial_x - \frac{1}{x^2} \partial_x + \frac{2}{x} \]

There holds \( A_1 = 0 \) and \( P = Q = 0 \), i.e. case ii), subcase a) of Proposition 5.1 applies. The equation

\[ cy - c^2 + \left( \frac{1}{2} - 2 \right) c + \frac{1}{2} = 0 \]

has the general solution \( c = \frac{y \exp\left(\frac{1}{2}y^2 + y\right) - C}{\exp\left(\frac{1}{2}y^2 + y\right) + C} \) for \( C \) constant. For \( C = 0 \) and \( \Phi \rightarrow \infty \) the two rational solutions \( c = y \) and \( c = -1 \) are obtained. They yield the factors \( \partial_x + y \) and \( \partial_x - 1 \) and the type \( \mathcal{L}^{(4,2)} \) decomposition

\[ L = (\partial_x - x) \text{Lclm}(\partial_x + y, \partial_x + 1) \]

The general solution of \( Lz = 0 \) is

\[ z = F(x) \exp\left(-\frac{1}{2}y^2\right) + G(x) \exp(-y) + \exp\left(\frac{1}{2}x^2 - y\right) \times \int H(y) \exp\left(\frac{1}{2}y^2\right) \frac{dy}{y+1} - \exp\left(\frac{1}{2}x^2 - \frac{1}{2}y^2\right) \int H(y) \exp\left(\frac{1}{2}y^2\right) \frac{dy}{y+1} \]

\( F, G \) and \( H \) are undetermined functions.
EXAMPLE 5.8. Consider the operator
\[ L \equiv \partial_{x y y} + (x + 1 + \frac{1}{y}) \partial_{x y} + y \partial_{y y} + (x + 1 - \frac{1}{y}) \partial_{x} + (x y + y + 3) \partial_{y} + x y + x + 2. \]
By case i) and case ii), subcase b) of Proposition 5.1 the type \( L_{x y y} \) decomposition
\[ L = (\partial_{y} + 1)L_{cl}m(\partial_{x} + y, \partial_{y} + x + \frac{1}{y}) \]
follows; \( H \) solution \( L_{cl}m \) is undetermined functions. The intersection of \( \partial F \) is \( \partial_{y} + 1 \) \( (\partial_{x} + (x + \frac{1}{y}) \partial_{y} + y \partial_{y} + x y + 2) \) obtained. The two arguments of the \( L_{cl}m \) yield the solutions \( z_{1} = F(y) \exp(-x y) \) and \( z_{2} = G(x) \frac{1}{y} \exp(-x y); F \) and \( G \) are undetermined functions. \( L_{cl}m \) may be factorized as\( L = (\partial_{y} + 1)(\partial_{x} + x + \frac{1}{y})(\partial_{x} + y) \) from which the third solution
\[ z_{3} = \exp \left( - (x + 1) y \right) \frac{1}{y} \int H(x) \frac{y}{x y - y - \frac{1}{x - 1}} \exp(x y) d x \]
follows; \( H \) is again an undetermined function. Finally the general solution of \( L z = 0 \) is \( z = z_{1} + z_{2} + z_{3} \).

EXAMPLE 5.9. Consider the operator
\[ L \equiv \partial_{x y y} - 2x \partial_{x y} + y \partial_{y y} + x^{2} \partial_{x} - 2(x y - 1) \partial_{y} + x(x y - 2). \]
By case i) and case ii), subcase b) of Proposition 5.1 the factorizations
\[ L = \left\{ \begin{array}{l}
(\partial_{y y} - 2x \partial_{x} + x^{2})(l_{1} \equiv \partial_{x} + y) \\
(\partial_{x y} - x \partial_{x} + y \partial_{y} - x y + 3)(l_{2} \equiv \partial_{y} - x). 
\end{array} \right. \]
are obtained. The intersection of \( l_{1} \) and \( l_{2} \) is
\[ L_{cl}m(l_{1}, l_{2}) = \langle \partial_{y y} - 2x \partial_{x} + x^{2}, \partial_{x} - 2(x y - 1) \partial_{y} + x(x y - 2), \partial_{x y} - x \partial_{x} + 2y \partial_{y} - 2(x y + 1) \partial_{x} + y^{2} \partial_{y} - y(x y + 2) \rangle. \]
It is not principal, i.e. \( L \) has decomposition type \( \phi(11) \). The two equations \( l_{1} z = 0 \) and \( l_{2} z = 0 \) yield the solutions \( z_{1} = F(y) \exp(-x y) \) and \( z_{2} = G(x) \exp(x y) \). A third solution may be obtained from the factorization \( \partial_{y} - 2x \partial_{x} + x^{2} = (\partial_{y} - x)(\partial_{y} - x) \). It yields
\[ z_{3} = y \exp(-x y) \int x H(x) \exp(2x y) d x. \]
The general solution of \( L z = 0 \) is \( z = z_{1} + z_{2} + z_{3} \).

6. SUMMARY AND CONCLUSION

In this article for the first time Loewy decompositions for a full class of partial differential equations of differential type different from zero are considered. It became clear that Theorems 2.1 and 2.2 are fundamental for a full understanding of how the various decompositions arise. Furthermore it has been shown by numerous examples that obtaining such a decomposition is essentially synonymous with finding its solutions in closed form. However, two basic questions remain to be discussed.

In the first place this is the uniqueness of a decomposition as it is the case for ordinary operators. To this end, one probably has to make the problem more specific by the requirement that the admitted divisors have differential type 1, as it is true for the given operators.

Secondly there is the question to what extent these decompositions may be obtained algorithmically. For ordinary operators it is sufficient to determine rational solutions of ordinary Riccati equations. It has been shown that for decomposing linear pde’s of the plane in general rational solutions of partial Riccati equations are required. This problem boils down to finding rational first integrals of a general quasi-linear first-order ode; for this problem see [4] and [3]. Up to this point the discussion concerns principal divisors. Deciding the non-existence of a Laplace divisor requires an upper bound for its order. It may not be possible to find such a bound, i.e. this problem may not be decidable. Finally there remain the exceptional cases mentioned at the end of Corollary 3.3.

Obviously there are many possible extensions of the results presented in this article. To mention just a few, there should be a close connection between the type of solution and the decomposition type as it is true for ordinary differential equation, Section 2.1 of [12]. For applications decompositions in more than two independent variables would be highly desirable. Dealing with more than one dependent variables would involve modules over the respective rings of differential operators.

7. REFERENCES