Loewy Decomposition of Linear Third-Order PDE's in the Plane

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ABSTRACT

Loewy's decomposition of a linear ordinary differential operator as the product of largest completely reducible components is generalized to partial differential operators of order three in two variables. This is made possible by considering the problem in the ring of partial differential operators where both left intersections and right divisors of left ideals are not necessarily principal. Listings of possible decomposition types are given. Many of them are illustraded by worked out examples. Algorithmic questions and questions of uniqueness are discussed in the Summary.

1. INTRODUCTION

About one hundred years ago Loewy proved the fundamental result that any ordinary differential operator may be represented uniquely as the product of largest completely reducible operators, i.e. operators that are the left intersection of irreducible operators of lower order [10]; see also Chapter 2 of the book [12]. This decomposition provides a detailed understanding of the structure of the solution space of the corresponding differential equation. Therefore it would be highly desirable to generalize it to partial differential operators as well. Amazingly this has never really been done since Loewy's original work. In this article Loewy decompositions of third-order operators in two variables are considered in detail; the possible limitations are also discussed. In the subsequent section some basic notations from differential algebra are introduced; details may be found in the book by Kolchin [9] or the articles by Buium and Cassidy [2] or Quadrat [11]. The main part of the article is organized according to leading derivatives of the respective operators.

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2.BASIC DIFFERENTIAL ALGEBRA

In this section some basic terminology from differential algebra that is used throughout the article will be introduced. Rings of differential operators $\mathcal{D} \equiv \mathcal{F}[\partial_x, \partial_y]$ or $\mathcal{D} \equiv \mathbb{Q}(x,y)[\partial_x, \partial_y]$ are considered; $\mathcal{F}$ is a universal differential field. $\mathcal{F}$ or $\mathbb{Q}(x,y)$ are called the *base field*. Let the left ideal I be generated by elements $l_i \in \mathcal{D}$, $i = 1, \ldots, p$. Then one writes $I = \langle l_1, \ldots, l_p \rangle$. Because right ideals are not considered, sometimes I is simply called an ideal. As a rule, the l_i are assumed to form a Janet basis in the term order $grlex$ with $x > y$. If z is some differential indeterminate, $l_i z = 0$, $i = 1, \ldots, p$, is the corresponding system of pde's. The following shorthand notation is frequently used. If only the leading terms of the generators of an ideal are of interest the non-leading terms are omitted; it is denoted by $\langle \ldots \rangle_{LT}$.

Let $I \subseteq \mathcal{F}[\partial_x, \partial_y]$ be an ideal and H_I its Hilbert-Kolchin polynomial ([9], page 130; [2], page 602). The degree $deg(H_I)$ of H_I is called the *differential type* of I. Its leading coefficient $lc(H_I)$ is called the *typical differential dimension* of I. The pair $(deg(H_I), lc(H_I))$ has been baptized the *gauge* of the ideal I [7].

Let I and J be two ideals. Important constructions are the greatest common right divisor $Gcrd(I, J)$ or the sum ideal; and the least common left multiple $Lclm(I, J)$ or the left intersection. In [7] it has been shown how they are computed algorithmically. Finally the relative syzygy module $Syz(I, J)$ of I and $J = \langle g_1, \ldots, g_q \rangle$ is generated by

$$\{h \equiv (h_1, \ldots, h_q) \in \mathcal{D}^q | h_1 g_1 + \ldots + h_q g_q \in I\}.$$

Define two ordinary differential operators by

$$D_{x^m} \equiv d_{x^m} + a_1 d_{x^{m-1}} + \ldots + a_{m-1} d_x + a_m,$$
$$D_{y^n} \equiv d_{y^n} + b_1 d_{y^{n-1}} + \ldots + b_{n-1} d_y + b_n;$$

m and n are natural numbers not less than 2. Several ideals generated by an operator of order three and one of these operators will occur later in this article. A special notation is introduced for them as shown in the table below. These ideals have an important meaning as divisors in the decompositions to be discussed in later chapters. Due to its close relation to the iteration scheme introduced by Laplace, see [6], vol. II, Chapter V, it is suggested to call them *Laplace divisors*. The pair of upper indices of the ideals in Table 1 denotes the gauge of the respective ideal.

Notation	Leading term ideal
$\mathcal{J}_4^{(1,1)}(m)$	$\langle \partial_{xyy}, \partial_{x^m} \rangle_{LT}$
$\mathcal{J}_5^{(1,1)}(n)$	$\langle \partial_{xxy}, \partial_{y^n} \rangle_{LT}$
$\mathcal{J}_3^{(1,2)}(m)$	$\langle \partial_{xxy}, \partial_{x^m} \rangle_{LT}$
$\mathcal{J}_4^{(1,2)}(n)$	$\langle \partial_{xyy}, \partial_{y^n} \rangle_{LT}$
$\mathcal{J}_5^{(1,2)}$	$\langle \partial_{xxx}, \partial_{xxy} \rangle_{LT}$
$\mathcal{J}_6^{(1,2)}$	$\langle \partial_{xxy}, \partial_{xyy} \rangle_{LT}$

Table 1. Some ideals of gauge $(1,1)$ and $(1,2)$ and order of derivatives not higher than 3. The ideals involving the parameter m or n will occur later on as Laplace divisors.

It turns out that the intersection ideals generated by two first-order operators are of fundamental importance for understanding the different decomposition types discussed later in this article. They are described in the following two theorems. The first theorem has been proved in [7].

THEOREM 2.1. *Let the ideals $I_i = \langle \partial_x + a_i\partial_y + b_i \rangle$ for $i = 1,2$ with $I_1 \neq I_2$ be given. Both ideals have gauge $(1,1)$. There are three different cases for their intersection $I_1 \cap I_2$, all are of gauge $(1,2)$.*

i) Separable case $a_1 \neq a_2$. If $\left(\dfrac{b_1-b_2}{a_1-a_2}\right)_x = \left(\dfrac{a_1b_2-a_2b_1}{a_1-a_2}\right)_y$ there holds $I_1 \cap I_2 = \langle \partial_{xx} \rangle_{LT}$ and

$$I_1 + I_2 = \left\langle \partial_x + \frac{a_1b_2-a_2b_1}{a_1-a_2},\ \partial_y + \frac{b_1-b_2}{a_1-a_2} \right\rangle.$$

ii) Double root $a_1 = a_2 = a$, $b_1 \neq b_2$. There holds $I_1 \cap I_2 = \langle \partial_{xx} \rangle_{LT}$ and $I_1 + I_2 =< 1 \rangle$.

iii) If the preceding two cases do not apply there holds $I_1 \cap I_2 = \langle \partial_{xxx}, \partial_{xxy} \rangle_{LT}$ and $I_1 + I_2 = \langle 1 \rangle$.

If a decomposition comprises an operator with leading derivative ∂_y, the following result is required.

THEOREM 2.2. *Let the ideals $I_1 = \langle \partial_x + a_1\partial_y + b_1 \rangle$ and $I_2 = \langle \partial_y + b_2 \rangle$ be given. There are two different cases for their intersection $I_1 \cap I_2$.*

i) If $(b_1 - a_1b_2)_y = b_{2,x}$ there holds $I_1 \cap I_2 = \langle \partial_{xy} \rangle_{LT}$ and $I_1 + I_2 = \langle \partial_x + b_1 - a_1b_2, \partial_y + b_2 \rangle$.

ii) If the preceding case does not apply there holds $I_1 \cap I_2 = \langle \partial_{xxy}, \partial_{xyy} \rangle_{LT}$ and $I_1 + I_2 = \langle 1 \rangle$.

The proof involves a slight modification of the proof of Theorem 2.1, it involves essentially Janet basis calculations. The special case that n first-order equations originate from the factorization of an operator of order n is treated next.

LEMMA 2.3. *Let L be a partial differential operator in x and y with leading term ∂_{x^n}, and let $l_i \equiv \partial_x + a_i\partial_y + b_i$, $i = 1, \ldots, n$, $a_i \neq a_j$ for $i \neq j$, be n right divisors of L. Then the intersection ideal generated by the l_i is principal and is generated by L.*

Just like factoring an ordinary differential operator involves solving Riccati equations $y' + ay^2 + by + c = 0$, factoring pde's in the plane may require to solve equations of the form

$$z_x + az_y + bz^2 + cz + d = 0$$

where a, b, c, d are from the base field. For obvious reasons they are called *partial Riccati equations*. They have been discussed in detail in [7].

3. PDE'S WITH LEADING DERIVATIVE ∂_{xxx}

This case is also interesting for historical reasons because it was the first third order partial differential operator for which factorizations were considered [1]. Especially the operator discussed in Blumberg's thesis, see Example 3.7 below, attracted a lot of interest.

PROPOSITION 3.1. *Let the third order linear partial differential operator*

$$D_{xxx} \equiv \partial_{xxx} + A_1\partial_{xxy} + A_2\partial_{xyy} + A_3\partial_{yyy} + A_4\partial_{xx} + A_5\partial_{xy} + A_6\partial_{yy} + A_7\partial_x + A_8\partial_y + A_9 \tag{1}$$

be given with $A_k \in \mathbb{Q}(x,y)$ for all k. Its first order right factors $\partial_x + a\partial_y + b$ are essentially determined by the roots a_1, a_2 and a_3 of $a^3 - A_1a^2 + A_2a - A_3 = 0$. The following alternatives may occur.

i) If a_1, a_2 and a_3 are three pairwise different rational solutions and the corresponding b_i are determined by (8), each pair a_i, b_i satisfying (9) and (10) yields a factor $l_i = \partial_x + a_i\partial_y + b_i$. If there are three factors, the operator is completely reducible and there holds $D_{xxx} = Lclm(l_1, l_2, l_3)$. If there are two factors, their intersection may not be principal. If only a single pair satisfies conditions (9) and (10), or there is only a single rational solution which satisfies them, there is only a single factor.

ii) If $a_1 = a_2$ is a twofold rational solution that does not satisfy (8), and $a_3 \neq a_1$ is a simple rational solution, a single right factor $\partial_x + a_3\partial_y + b_3$ exists if a_3, and b_3 as determined by (8) satisfy (9) and (10).

iii) If $a_1 = a_2$ is a twofold rational solution satisfying (8), $a_3 \neq a_1$ is a simple rational solution, and a_3 and b_3 as determined by (8) do not satisfy (9) and (10), the following subcases may occur.

a) There is a right factor of the form $l(\Phi) \equiv \partial_x + a_1\partial_y + R(x,y,\Phi(\phi))$ whereas $R(x,y,\Phi(\phi))$ is the general solution of (9) satisfying (10); $\phi(x,y)$ is the first integral of $\dfrac{dy}{dx} = a_1(x,y)$, Φ is an undetermined function; $l(\Phi)$ leads to a second order right factor $Lclm(l_1(\Phi_1), l_2(\Phi_2))$ with $\Phi_1 \neq \Phi_2$ specializations of Φ.

b) There are two right factors $l_i \equiv \partial_x + a_1\partial_y + r_i(x,y)$, $i = 1,2$, if (9) has rational solutions $r_i(x,y)$ satisfying (10), or if it has a general rational solution as in case a), but constraint (10) singles out the special solutions $r_1(x,y)$ and $r_2(x,y)$; l_1 and l_2 generate a second order factor $Lclm(l_1, l_2)$.

c) There is a single right factor $l \equiv \partial_x + a_1\partial_y + r(x,y)$ if (9) has the single rational solution $r(x,y)$ satisfying (10), or if it has two rational solutions or a general rational solution, but (10) singles out the single solution $r(x,y)$.

iv) If $a_1 = a_2$ is a twofold rational solution satisfying (8), $a_3 \neq a_1$ is a simple rational solution, and a_3 and b_3 as determined by (8) satisfy (9) and (10), the same distinctions occur as in the preceding case iii), supplemented by an additional factor $\partial_x + a_3\partial_y + b_3$.

v) If $a_1 = a_2 = a_3 = \frac{1}{3}A_1$ is a threefold rational solution, there holds

$$A_1^2 A_4 - 3A_1 A_5 + 9A_6 = 0. \qquad (2)$$

The following subcases may occur.

a) If the coefficient of b in

$$\begin{aligned}(A_{1,x} + \tfrac{1}{3}A_1 A_{1,y} + \tfrac{2}{3}A_1 A_4 - A_5)b = \\ \tfrac{1}{3}A_{1,xx} + \tfrac{2}{9}A_1 A_{1,xy} + \tfrac{2}{27}A_1^2 A_{1,yy} - \tfrac{2}{9}A_{1,x}A_{1,y} \\ +\tfrac{1}{3}A_4 A_{1,x} - \tfrac{1}{27}A_1 A_{1,y}^2 - \tfrac{1}{9}A_1 A_4 A_{1,y} \\ +\tfrac{1}{3}A_5 A_{1,y} + \tfrac{1}{3}A_1 A_7 - A_8 = 0\end{aligned} \qquad (3)$$

does not vanish, b may be determined uniquely from it. A factor exists if the constraint

$$\begin{aligned}b_{xx} + \tfrac{2}{3}A_1 b_{xy} + \tfrac{1}{9}A_1^2 b_{yy} - 3bb_x + A_4 b_x - A_1 bb_y \\ -(\tfrac{2}{3}A_{1,x} + \tfrac{2}{9}A_1 A_{1,y} + \tfrac{1}{3}A_1 A_4 - A_5)b_y \\ +b^3 - A_4 b^2 + A_7 b - A_9 = 0.\end{aligned} \qquad (4)$$

is satisfied.

b) If the coefficient of b in (3) vanishes, there holds

$$\begin{aligned}A_{1,x} + \tfrac{1}{3}A_1 A_{1,y} + \tfrac{2}{3}A_1 A_4 - A_5 = 0, \\ A_{1,xx} + \tfrac{2}{3}A_1 A_{1,xy} + \tfrac{2}{9}A_1^2 A_{1,yy} + A_4 A_{1,x} + \tfrac{1}{3}A_{1,x}A_{1,y} \\ +\tfrac{2}{9}A_1 A_{1,y}^2 + \tfrac{1}{3}A_1 A_4 A_{1,y} + \tfrac{1}{3}A_1 A_7 - 3A_8 = 0\end{aligned} \qquad (5)$$

and b is determined by

$$\begin{aligned}b_{xx} + \tfrac{2}{3}A_1 b_{xy} + \tfrac{1}{9}A_1^2 b_{yy} - 3bb_x + A_4 b_x - A_1 bb_y \\ +\tfrac{1}{3}(A_{1,x} + \tfrac{1}{3}A_1 A_{1,y} + A_1 A_4)b_y \\ +b^3 - A_4 b^2 + A_7 b - A_9 = 0.\end{aligned} \qquad (6)$$

PROOF. Dividing the operator (1) by $\partial_x + a\partial_y + b$, the requirement that this division be exact leads to the following set of equations between the coefficients.

$$a^3 - A_1 a^2 + A_2 a - A_3 = 0, \qquad (7)$$

$$\begin{aligned}(A_1 - 3a)a_x + (3a^2 - 3A_1 a + 2A_2)a_y \\ -A_4 a^2 + A_5 a + (3a^2 - 2A_1 a + A_2)b = A_6,\end{aligned} \qquad (8)$$

$$\begin{aligned}(A_1 - 3a)b_x + (3a^2 - 3A_1 a + 2A_2)b_y - (A_1 - 3a)b^2 \\ + (A_5 - 2A_4 a - 2A_1 a_y + 3aa_y - 3a_x)b \\ + a_{xx} + (A_1 - a)a_{xy} + (a^2 - A_1 a + A_2)a_{yy} \\ -2a_x a_y + A_4 a_x - (A_1 - a)a_y^2 - (A_4 a - A_5)a_y + A_7 a - A_8 = 0,\end{aligned} \qquad (9)$$

$$\begin{aligned}b_{xx} + (A_1 - a)b_{xy} + (a^2 - A_1 a + A_2)b_{yy} \\ - (2a_x + (A_1 - a)a_y + A_4 a - A_5)b_y \\ +(A_4 - 3b)b_x + (3a - 2A_1)bb_y + b^3 - A_4 b^2 + A_7 b - A_9 = 0.\end{aligned} \qquad (10)$$

The algebraic equation (7) determines a. The following discussion is subdivided by the various cases.

Case *i*). Assume at first that it has three simple roots a_1, a_2 and a_3. None of them may be rational, there may be a single rational solution, or all three roots are rational. For any of these roots the coefficient of b in (8) does not vanish because it is the derivative of (7) w.r.t. a. Therefore for each a_i, equation (8) determines the corresponding value b_i. For those values of a_i and b_i which satisfy the constraints (9) and (10), a right factor $l_i \equiv \partial_x + a_i\partial_y + b_i$ exists. If there are three right factors, by Lemma 2.3 D_{xxx} is completely reducible and there holds $D_{xxx} = Lclm(l_1, l_2, l_3)$.

Case *ii*). Assume now there is a twofold rational solution $a_1 = a_2 \neq a_3$ of equation (7). The double root $a_1 = \frac{1}{3}(A_1 - \sqrt{A_1^2 - 3A_2})$ is one of the roots of the equation $p \equiv 3a^2 - 2A_1 a + A_2 = 0$ of (7). The coefficient of b in (8) vanishes for $a = a_1$ because it is equal to p; (8) becomes a constraint for a_1. Assume that it is not obeyed. For the root a_3, the coefficient b_3 follows from (8). The existence of a right factor involving a_3 and b_3 depends on whether they satisfy (9) and (10). If this is true there is a single right factor $\partial_x + a_3\partial_y + b_3$.

Case *iii*). If the double root a_1 does satisfy the constraint following from (8), the corresponding value $b = b_1$ has to be determined from the partial Riccati equation (9) with $a = a_1$. Those values of a_1 and b_1 satisfying (10) have to be singled out. If a_3 and b_3 do not satisfy (9) and (10), the only possible factor is the one involving a_1 and b_1. A simple calculation shows that the terms of (9) involving derivatives of b simplify to $b_x + a_1 b_y$. According to Lemma 5.3 of [7], the general solution for b_1 may contain an undetermined function Φ of $\phi(x,y)$, the first integral of $\frac{dy}{dx} = a_1(x,y)$. The condition (10) may be satisfied without constraints for Φ, it may restrict it to a certain form, or it may completely eliminate the corresponding solution. Similar arguments apply if there is a single rational s! olution for b_1, or if there are two of them. Whenever two factors are obtained, by case *ii*) of Theorem 2.1 they generate a principal ideal.

Case *iv*). If again a_1 satisfies the constraint following from (8), and a_3 and b_3 satisfy (9) and (10) now, the same alternatives as in case *iii*) occur with an additional factor $\partial_x + a_3\partial_y + b_3$.

Case *v*). Finally assume there is a threefold solution $a_1 = a_2 = a_3 = \frac{1}{3}A_1$. Then $A_2 = \frac{1}{3}A_1^2$ and $A_3 = \frac{1}{27}A_1^3$. The coefficient of b in (8) vanishes again, it becomes the constraint (2). The coefficients of b_x, b_y and b^2 in (9) vanish.

If $A_{1,x} + \frac{1}{3}A_{1,y} + \frac{2}{3}A_1 A_4 - A_5 \neq 0$, b is determined by (3). In order for a factor to exist, in addition (4) must be satisfied. This is subcase *a*). In the exceptional case that the coefficient of b in (3) vanishes, it becomes a constraint and b has to be determined from (6). This is subcase *b*). □

In order to obtain a complete answer comprising all possible factorizations of the operator (1), second order right factors have to be taken into account as well. They are considered next.

PROPOSITION 3.2. *Let the third order partial differential operator (1) be given with $A_k \in \mathbb{Q}(x,y)$ for all k. Its second order right factors*

$$\partial_{xx} + a\partial_{xy} + b\partial_{yy} + c\partial_x + d\partial_y + f \qquad (11)$$

are determined by the roots a_1, a_2 and a_3 of

$$a^3 - 2A_1 a^2 + (A_1^2 + A_2)a - A_1 A_2 + A_3 = 0.$$

The following alternatives may occur.

i) If a_1, a_2 and a_3 are three pairwise different rational solutions, for each a_i the corresponding values of b, c, d and f follow from equations (20) … (22). Those values which satisfy constraints (23) yield a factor.

ii) If $a_1 = a_2 \neq a_3$ is a twofold solution and $b_1 = a_1^2 - A_1a_1 + A_2$, there must hold

$$(A_1 - a_1)[a_{1,x} + (A_1 - a_1)a_{1,y} + A_1a_1 - A_5] = b_{1,x} + (A_1 - a_1)b_{1,y} + A_4b_1 - A_6. \tag{12}$$

The coefficient c has to be determined from the partial Riccati equation

$$\begin{aligned} c_x - (a - A_1)c_y - c^2 + \frac{3}{3a - 2A_1}[(a - \tfrac{1}{3}A_1)_x \\ -(a - A_1)(a - \tfrac{1}{3}A_1)_y + \tfrac{4}{3}(a - \tfrac{1}{2}A_1)A_4 - \tfrac{1}{3}A_5]c \\ +\frac{1}{3a - 2A_1}[a_{xx} - 2(a - A_1)a_{xy} \\ +(a^2 - 2A_1a + A_1^2)a_{yy} - a_y(a - A_1)_x \\ +(a - A_1)(a_y^2 - A_{1,y}a_y - 2A_4a_y - A_{4,y}a + A_{5,y} + A_7) \\ +A_{4,x}a + 2A_4a_x - A_{5,x} - A_4A_5 + A_8 + A_4^2a = 0. \end{aligned} \tag{13}$$

If it has a rational solution, d and f follow from the first equation of (21) and (22) respectively. The values satisfying the constraint

$$\begin{aligned} c_{xx} - 2(a - A_1)c_{xy} + (a - A_1)^2c_{yy} \\ +[(a - A_1)_x - \tfrac{1}{2}(a^2 - A_1a - A_1^2)_y + (a - A_1)(2A_4 - 3c)]c_y \\ +[(a - A_1)(3c - A_4)_y + A_{4,x} + A_4^2 + A_7]c \\ +A_{7,x} - (a - A_1)A_{7,y} + A_4A_7 - A_9 = 0. \end{aligned} \tag{14}$$

lead to a factor.

iii) If $a_1 = a_2 = a_3 = \frac{2}{3}A_1$ is a threefold solution and $b_1 = \frac{1}{9}A_1^2$, c is uniquely determined by

$$\begin{aligned} (A_{1,x} + \tfrac{1}{3}A_1A_{1,y} + \tfrac{2}{3}A_1A_4 - A_5)c = \tfrac{2}{3}A_{1,xx} + \tfrac{4}{9}A_1A_{1,xy} \\ +\tfrac{2}{27}A_1^2A_{1,yy} + \tfrac{2}{9}A_{1,x}A_{1,y} + \tfrac{4}{3}A_4A_{1,x} + \tfrac{2}{27}A_1A_{1,y}^2 \\ +\tfrac{4}{9}A_1A_4A_{1,y} + \tfrac{2}{9}A_1^2A_{4,y} + \tfrac{2}{3}A_1(A_{4,x} + A_4^2) \\ -\tfrac{1}{3}A_1(A_{5,y} + A_7) - A_4A_5 - A_{5,x} + A_8 \end{aligned} \tag{15}$$

if its coefficient does not vanish. The remaining coefficients follow from

$$\begin{aligned} d - \tfrac{1}{3}A_1c + \tfrac{2}{3}A_{1,x} + \tfrac{2}{9}A_1A_{1,y} + \tfrac{2}{3}A_1A_4 - A_5 = 0, \\ f + c_x + \tfrac{1}{3}A_1c_y - c^2 + A_4c - A_7 = 0. \end{aligned} \tag{16}$$

In addition the two constraints

$$A_1^2A_4 - 3A_1A_5 + 9A_6 = 0, \tag{17}$$

$$\begin{aligned} c_{xx} + \tfrac{2}{3}A_1c_{xy} + \tfrac{1}{9}A_1^2c_{yy} + (2A_4 - 3c)c_x \\ +(\tfrac{1}{3}A_{1,x} + \tfrac{1}{9}A_1A_{1,y} + \tfrac{2}{3}A_1A_4 - A_1c)c_y \\ +c^3 - 2A_4c^2 + (\tfrac{1}{3}A_1A_{4,y} + A_{4,x} + A_4^2 + A_7)c \\ -\tfrac{1}{3}A_1A_{7,y} - A_4A_7 - A_{7,x} + A_9 = 0 \end{aligned} \tag{18}$$

must be satisfied. If the coefficient of c in (15) vanishes, it becomes a coonstraint and c has to be determined from (18).

PROOF. Dividing the operator (1) by (11), the requirement that this division be exact leads to the following set of equations for the coefficients.

$$a^3 - 2A_1a^2 + (A_1^2 + A_2)a - A_1A_2 + A_3 = 0, \tag{19}$$

$$b - a^2 + A_1a - A_2 = 0, \tag{20}$$

$$\begin{aligned} (A_1 - 2a)c + d + a_x + (A_1 - a)a_y + A_4a - A_5 = 0, \\ (a^2 - A_1a + A_2)c - (A_1 - a)d - b_x \\ -(A_1 - a)b_y - A_4b + A_6 = 0, \end{aligned} \tag{21}$$

$$f + c_x + (A_1 - a)c_y - c^2 + A_4c - A_7 = 0, \tag{22}$$

$$\begin{aligned} (A_1 - a)f - d_x - (A_1 - a)d_y - (A_4 - c)d + A_8 = 0, \\ f_x + (A_1 - a)f_y + (A_4 - c)f - A_9 = 0. \end{aligned} \tag{23}$$

The algebraic equation (19) determines always a. The following discussion is subdivided by the various cases of the above theorem.

Case *i)* Assume it has three simple roots $a_1 \neq a_2 \neq a_3$. The corresponding values b_i may be determined from (20). The coefficient determinant of c and d in (21) is $a^2 - \frac{4}{3}A_1a + \frac{1}{3}(A_1^2 + A_2)$, it is the derivative of (19) w.r.t. a that does not vanish for simple roots. Therefore for any simple root a_i, the corresponding values c_i and d_i may be determined from (21). Finally f_i follows from (22). These values have to satisfy the constraints (23) in order to determine a factor.

Case *ii)* Assume now there is a twofold rational solution $a_1 = a_2 \neq a_3$ of (19). The double root $a_1 = \frac{2}{3}A_1 + \frac{1}{3}\sqrt{A_1^2 - 3A_2}$ is one of the roots of the equation $3a^2 - 4A_1a + A_1^2 + A_2 = 0$ of (19). In order to exclude a triple root there must hold $A_1^2 - 3A_2 \neq 0$; b_1 follows from (20). Because the coefficient determinant of c and d in (21) vanishes for $a = a_1$, one has to proceed as follows. Eliminating d from the first equation (21) and substiuting it into the second one yields the constraint (12). Eliminating d from the first equation (21), f from (22) and substituting them into the first equation of (23), the partial Riccati equation (13) for c is obtained. If it has a rational solution c_1, the coefficient d_1 may be obtained from the first equation (21) and f_1 from (22). Substituting these values into the second equation of (23), the constraint (14) follows. It may exclude a factor, or it may constrain the undetermined elements in the solution of (13).

Case *iii)* Finally assume there is a threefold solution $a_1 = a_1 = a_3 = \frac{2}{3}A_1$ with the corresponding value $b_1 = \frac{1}{9}A_1^2$. Then $A_2 = \frac{1}{3}A_1^2$ and $A_3 = \frac{1}{27}A_1^3$. The coefficient determinant for c and d in (21) vanishes. By elimination equation (15) is obtained for c if $A_{1,x} + \frac{1}{3}A_1A_{1,y} + \frac{2}{3}A_1A_4 - A_5 \neq 0$, and (16) for the remaining coefficients d and e. In order for a factor to exist, the two constraints (17) and (18) must be satisfied. In the exceptional case that the coefficient of c in (15) vanishes, it becomes a constraint; c has to be determined from (18). □

The next corollary summarizes to what extent the factors of Propositions 3.1 and 3.2 may be obtained algorithmically.

COROLLARY 3.3. *Any first-order factor of (1) corresponding to a simple root of (7), or any second-order factor cor-*

responding to a simple root of (19), may be determined algorithmically. In order to determine any such factor corresponding to a twofold root of (7) or (19) requires to find rational solutions of a partial Riccati equation. Finally, a factor corresponding to a threefold root may be determined algorithmically if $A_{1,x} + \frac{1}{3}A_1A_{1,y} + \frac{2}{3}A_1A_4 \neq A_5$, otherwise a quasilinear second-oder pde has to be solved for which a solution algorithm is not known.

PROOF. Testing the conditions of case $i)$ of both Proposition 3.1 and Proposition 3.2 requires only differentiations and arithmetic in the base field. Case $ii)$, $iii)$ and $iv)$ of Proposition 3.1 dealing with a twofold root of (7) require to determine rational solutions of the partial Riccati equation (8) for the coefficient a. To this end a general first-order ode must be solved as discussed in Appendix B of [7]. The same argument applies to case $ii)$ of Proposition 3.2. In case $v)$ of Propopsition 3.1, b follows from the linear algebraic equation (3) if its coefficient does not vanish; this leads to the above condition. If it does vanish, b has to be determined from equation (6). Similar arguments apply for case $iii)$ of Proposition 3.2) and equation (18) for the coefficient c. □

After the possible factorizations of an operator (1) have been determined, a listing of its completely reducible components may be set up as follows.

THEOREM 3.4. *Any differential operator*

$$L \equiv \partial_{xxx} + A_1\partial_{xxy} + A_2\partial_{xyy} + A_3\partial_{yyy} + A_4\partial_{xx} + A_5\partial_{xy} + A_6\partial_{yy} + A_7\partial_x + A_8\partial_y + A_9 \quad (24)$$

decomposes into completely reducible components corresponding to one of the types $\mathcal{L}^{(k)}_{xxx}$, $k = 1, \ldots, 5$, defined as follows; L_{xxx}, L_{xx} and L_x are completely reducible operators with leading derivatives ∂_{xxx}, ∂_{xx} or ∂_x respectively; $\mathcal{J}_5^{(1,2)}$ is the ideal $\langle\partial_{xxx}, \partial_{xxy}\rangle_{LT}$ defined in Table 1. Upper indices distinguish different copies within a type definition.

$$\mathcal{L}^{(1)}_{xxx} : L_{xxx},\ \mathcal{L}^{(2)}_{xxx} : L_{xx}L_x,\ \mathcal{L}^{(3)}_{xxx} : L_xL_{xx},\ \mathcal{L}^{(4)}_{xxx} : L_x^{(1)}L_x^{(2)}L_x^{(3)},$$

$$\mathcal{L}^{(5)}_{xxx} : Syz\big(\langle L\rangle, \mathcal{J}_5^{(1,2)}\big)\mathcal{J}_5^{(1,2)}.$$

The decomposition $\mathcal{L}^{(1)}_{xxx}$ is completely reducible.

PROOF. It is based on Propositions 3.1 and 3.2. In the separable case $i)$ of Proposition 3.1 there may be three factors with a principal intersection, this yields type $\mathcal{L}^{(1)}_{xxx}$. There may be two factors which do not have a principal intersection. If they are divided out and the decompositions of the respective second-order factor are taken into account (see Section 4 of [7]), type $\mathcal{L}^{(5)}_{xxx}$ is obtained. If case $i)$ allows only a single factor, or if case $ii)$, or subcase $c)$ of case $iii)$, or case $v)$ applies, and again the decompositions of the corresponding second order left factor are taken into account, decomposition types $\mathcal{L}^{(2)}_{xxx}$ or $\mathcal{L}^{(4)}_{xxx}$ follow. For case $iii)$, the principality of the intersection of the right factors leads to type $\mathcal{L}^{(3)}_{xxx}$. In case $iv)$ the decompositions of case $iii)$ are extended by an additional first order factor. According to Lemma 2.3, combined with the factors already obtained they generate a principal intersection covered by $\mathcal{L}^{(1)}_{xxx}$. By definition, an irreducible operator (24) corresponds to a type $\mathcal{L}^{(1)}_{xxx}$ decomposition. □

COROLLARY 3.5. *Let l_x, l_{xx} and l_{xxx} denote irreducible operators with leading term ∂_x, ∂_{xx} or ∂_{xxx}. An additional upper index distinguishes different copies of the respective operator. Φ is an undetermined function of a single argument. The types $\mathcal{L}^{(i,j)}_{xx}$ defined in Table 2 are refinements of the types $\mathcal{L}^{(i)}_{xxx}$, $i = 1, \ldots, 5$ defined in Theorem 3.4.*

This is an immediate consequence of the proof given for the preceding theorem if the various factorization alternatives of Proposition 3.1 and Proposition 3.2 are not merged into completely reducible components.

EXAMPLE 3.6. Consider the operator

$$L \equiv \partial_{xxx} + \big(y+2\big)\partial_{xxy} + (y+1)\partial_{xyy} + \big(1 - \frac{1}{y}\big)\partial_{xx} + \big(y + 2 - \frac{1}{y}\big)\partial_{xy} + (y+1)\partial_{yy} - \frac{1}{y}\partial_x - \frac{1}{y}\partial_y.$$

Equation (7) reads $a^3 - (y+2)a^2 + (y+1)a = 0$ with the three roots $a_1 = 0$, $a_2 = 1$ and $a_3 = y+1$, i. e. case $i)$ of Proposition 3.1. The corresponding values of b are $b_1 = 1$, $b_2 = b_3 = 0$. It turns out that all pairs a_i, b_i satisfy conditions (9) and (10). Consequently there are three right factors $l_1 = \partial_x + 1$, $l_2 = \partial_x + \partial_y$ and $l_3 = \partial_x + (y+1)\partial_y$. There holds $L = Lclm(l_1, l_2, l_3)$, i. e. the decomposition type is $\mathcal{L}^{(1,2)}_{xxx}$. The general solution of $Lz = 0$ is

$$z = F(y)e^{-x} + G(x-y) + H[(y+1)e^{-x}]$$

where F, G and H are undetermined functions.

EXAMPLE 3.7. *(Blumberg 1912)* In his dissertation Blumberg [1] considered the third order operator

$$L \equiv \partial_{xxx} + x\partial_{xxy} + 2\partial_{xx} + 2(x+1)\partial_{xy} + \partial_x + (x+2)\partial_y \quad (25)$$

generating a principal ideal of gauge $(1,3)$. He gave its factorizations

$$L = \begin{cases} \big(\partial_{xx} + x\partial_{xy} + \partial_x + (x+2)\partial_y\big)(\partial_x + 1) \\ (\partial_{xx} + 2\partial_x + 1)(\partial_x + x\partial_y). \end{cases} \quad (26)$$

with two completely reducible second order left factors.

This result may be obtained by Proposition 3.1 as follows. Equation (7) is $a^3 - xa^2 = a^2(a-x) = 0$ with the double root $a_1 = 0$, and the simple root $a_2 = x$. The latter yields $b_2 = 0$. Because these values satisfy (9) and (10), the factor $l_2 = \partial_x + x\partial_y$ is obtained, i. e. case $iv)$ of Theorem 3.1 applies. For the double root $a_1 = 0$, from (9) the Riccati equation

$$b_x - b^2 + \Big(2 + \frac{2}{x}\Big)b - 1 - \frac{2}{x} = 0$$

is obtained with the general solution $b = 1 + \frac{1}{x} - \dfrac{1}{x + \Phi(y)}$. Substitution into (10) yields $\Phi = 0$, i.e. $b = 1$ and the factor $l_1 = \partial_x + 1$.

The second order left factor in the first line at the right hand side of (26) is absolutely irreducible, whereas the second order factor in the second line is the left intersection of two first order factors, i. e. (26) may be further decomposed into irreducibles as

$$L = \begin{cases} \big(\partial_{xx} + x\partial_{xy} + \partial_x + (x+2)\partial_y\big)(\partial_x + 1) \\ Lclm(\partial_x + 1, \partial_x + 1 - \frac{1}{x})(\partial_x + x\partial_y). \end{cases} \quad (27)$$

$\mathcal{L}^{(1)}_{xxx}$	$\mathcal{L}^{(1,1)}_{xxx}: l_{xxx},\ \ \mathcal{L}^{(1,2)}_{xxx}: Lclm(l^{(1)}_x, l^{(2)}_x, l^{(3)}_x),\ \ \mathcal{L}^{(1,3)}_{xxx}: Lclm(l_{xx}, l_x).$
$\mathcal{L}^{(2)}_{xxx}$	$\mathcal{L}^{(2,1)}_{xxx}: l_{xx}l_x,\ \ \mathcal{L}^{(2,2)}_{xxx}: Lclm(l^{(1)}_x, l^{(2)}_x)l^{(3)}_x,\ \ \mathcal{L}^{(2,3)}_{xxx}: Lclm(l^{(1)}_x(\Phi))l^{(2)}_x.$
$\mathcal{L}^{(3)}_{xxx}$	$\mathcal{L}^{(3,1)}_{xxx}: l_x l_{xx},\ \ \mathcal{L}^{(3,2)}_{xxx}: l^{(1)}_x Lclm(l^{(2)}_x, l^{(3)}_x),\ \ \mathcal{L}^{(3,3)}_{xxx}: l^{(1)}_x Lclm(l^{(2)}_x(\Phi)).$
$\mathcal{L}^{(4)}_{xxx}$	$\mathcal{L}^{(4,1)}_{xxx}: l^{(1)}_x l^{(2)}_x l^{(3)}_x.$
$\mathcal{L}^{(5)}_{xxx}$	$\mathcal{L}^{(5)}_{xxx}: Syz\big(\langle L\rangle, \mathcal{J}^{(1,2)}_5\big)Lclm\big(\langle(\partial_{xx}+\ldots)(\partial_x+\ldots)\rangle, \langle(\partial_{xy}+\ldots)(\partial_x+\ldots)\rangle\big).$

Table 2. The decomposition types of Corollary 3.5

The intersection ideal of l_1 and l_2 is not principal, by Theorem 2.1 it is

$$\begin{aligned}Lclm(l_1,l_2) = \langle L_1 &\equiv \partial_{xxx} - x^2\partial_{xyy}\\ &+3\partial_{xx} + (2x+3)\partial_{xy} - x^2\partial_{yy} + 2\partial_x + (2x+3)\partial_y,\\ L_2 &\equiv \partial_{xxy} + x\partial_{xyy} - \tfrac{1}{x}\partial_{xx} - \tfrac{1}{x}\partial_{xy} + x\partial_{yy}\\ &-\tfrac{1}{x}\partial_x - \big(1+\tfrac{1}{x}\big)\partial_y\rangle\end{aligned} \tag{28}$$

with gauge $(1,2)$, therefore the decomposition (26) is of type $\mathcal{L}^{(5)}_{xxx}$. Both generators L_1 and L_2 have the divisors l_1 and l_2. There holds $L \in Lclm(l_1,l_2)$, explicitly $L = L_1 + xL_2$.

As a consequence the system of equations $L_1 z = 0$ and $L_2 z = 0$ has the two solutions $F(y - \frac{1}{2}x^2)$ and $G(y)e^{-x}$; they correspond to $l_2 z = 0$ and $l_1 z = 0$ respectively.

In order to obtain the remaining part of the solution of Blumberg's equation $Lz = 0$ of gauge $(1,1)$ the relative syzygies module

$$\begin{aligned}Syz(\langle L\rangle, \langle L_1, L_2\rangle)&\\ &= \langle (1,x), \big(-\partial_y + \tfrac{1}{x}, \partial_x - x\partial_y + 2 + \tfrac{1}{x}\big)\rangle\\ &= \langle (1,x), \big(0, \partial_x + 1 + \tfrac{1}{x}\big)\rangle\end{aligned}$$

is constructed. Introducing the new differential indeterminates z_1 and z_2, the equations $z_1 + xz_2 = 0$ and $z_{2,x} + (1 + \frac{1}{x})z_2 = 0$ are obtained with the solution $z_1 = -H(y)e^{-x}$ and $z_2 = H(y)\frac{1}{x}e^{-x}$, H an undetermined function of y. The desired solution is a special solution of the inhomogeneous system $L_1 z = -H(y)e^{-x}$, $L_2 z = H(y)\frac{1}{x}e^{-x}$. The solution of the corresponding homogeneous system is already known, therefore the general solution of $Lz = 0$ may be written in terms of integrations as

$$z = F(y - \frac{1}{2}x^2) + G(y)e^{-x} + \int H\big(\bar{y} + \frac{1}{2}x^2\big)e^{-x}dx\Big|_{\bar{y}=y-\frac{1}{2}x^2} \tag{29}$$

where $\bar{y} = y - \frac{1}{2}x^2$.

EXAMPLE 3.8. Consider the operator

$$\begin{aligned}L \equiv \partial_{xxx} &+ (x+y-1)\partial_{xxy} - (x+y)\partial_{xyy} - (x-y-1)\partial_{xx}\\ &- (x-y-1)\partial_{xx} - (x^2+xy-x+1)\partial_{xy} - (x+y)\partial_{yy}\\ &- (xy+x-y+1)\partial_x - (x^2+xy+y)\partial_y - xy - 1.\end{aligned}$$

Equation (7) reads $a^3 - (x+y+1)a^2 - (x+y)a = 0$ with roots $a_1 = 0$, $a_2 = -1$ and $a_3 = x+y$, i. e. case $i)$ of Proposition 3.1 applies. Equation (8) yields the corresponding values

$$b_1 = 1,\quad b_2 = -x \quad \text{and} \quad b_3 = -\frac{2x^2+xy-x-y^2-y+2}{4x+5y-3}.$$

Only a_1 and b_1 satisfy the constraints (2) and (4). Consequently there is a single first order factor $l_1 \equiv \partial_x + 1$. Pursuant to case $i)$ of Proposition 3.2 a second order factor does not exist. Dividing L by l_1 yields the operator

$$\begin{aligned}\partial_{xx} + (x+1)\partial_{xy} + x\partial_{yy} + (y+1)\partial_x + (xy+1)\partial_y + x + y&\\ = (\partial_x + x\partial_y + 1)(\partial_x + \partial_y + y)&\end{aligned}$$

Consequently the operator L has a type $\mathcal{L}^{(4,1)}_{xxx}$ decomposition

$$L = (\partial_x + x\partial_y + 1)(\partial_x + \partial_y + y)(\partial_x + 1).$$

EXAMPLE 3.9. Consider the operator

$$\begin{aligned}L \equiv \partial_{xxx} + (y+1)\partial_{xxy} + (x+y+1)\partial_{xx} + (xy+x+y^2+2y+1)\partial_{xy}&\\ + (x+y)\partial_x + (xy+x+y^2+y)\partial_y.&\end{aligned}$$

Equation (7) reads $a^2(a-y-1) = 0$ with double root $a_1 = a_2 = 0$ and single root $a_3 = y+1$. It turns out that case $iv)$ applies with $a_1 = 0$, $b_1 = 1$ and $a_3 = y+1$, $b_3 = 0$. The corresponding first order factors yield the divisor as the principal intersection

$$Lclm(\partial_x + 1, \partial_x + (y+1)\partial_y) = \partial_{xx} + (y+1)\partial_{xy} + \partial_x + (y+1)\partial_y.$$

Consequently L has the decomposition

$$\big(\partial_x + x + y\big)Lclm\big(\partial_x + 1, \partial_x + (y+1)\partial_y\big)$$

of type $\mathcal{L}^{(3,2)}_{xxx}$.

4. PDE'S WITH LEADING DERIVATIVE ∂_{xxy}

If an equation does not contain a derivative ∂_{xxx} but only ∂_{yyy}, permuting x and y leads to an equation of the form (1) such that the above theorem may be applied. If there is neither a term ∂_{xxx} or ∂_{yyy}, the general third order operator

$$\begin{aligned}D_{xxy} \equiv \partial_{xxy} + A_1\partial_{xyy} + A_2\partial_{xx} + A_3\partial_{xy} + A_4\partial_{yy}&\\ + A_5\partial_x + A_6\partial_y + A_7&\end{aligned} \tag{30}$$

is obtained. Its possible decompositions are described in this subsection. Like in the previous case, first- and second-order factors are considered separately.

PROPOSITION 4.1. *Let the third order partial differential operator (30) be given with $A_k \in \mathbb{Q}(x,y)$ for all k. The following first order right factors may occur.*

i) If $A_1 \neq 0$, $a_1 = 0$, $a_2 = A_1$, and b is determined from

$$a_x - (3a - 2A_1)a_y - A_2a^2 + A_3a - (2a - A_1)b = A_4,$$

a first order right factor $\partial_x + a_i\partial_y + b_i$ exists if a_i and b_i satisfy the constraints

$$\begin{aligned}b_x - (3a-2A_1)b_y - b^2 - (2a_y + 2A_2a - A_3)b + a_{xy} - (a - A_1)a_{yy}&\\ + A_2a_x - a_y^2 - (A_2a - A_3)a_y + A_5a = A_6,&\end{aligned}$$

$$b_{xy} + (A_1 - a)b_{yy} + A_2 b_x - (2b + a_y + A_2 a - A_3)b_y \\ -A_2 b^2 + A_5 b = A_7.$$

ii) *If* $a = A_1 = A_4 = 0$, *the factor* $\partial_x + b$ *exists with* $b = \dfrac{A_{6,y} + A_2 A_6 - A_7}{A_{3,y} + A_2 A_3 - A_5}$ *if* $A_{3,y} + A_2 A_3 - A_5 \neq 0$ *and* $b_x - b^2 + A_3 b = A_6$.

iii) *If* $a = A_1 = A_4 = 0$ *and the two constraints*

$$A_{6,y} + A_2 A_6 - A_7 = 0,\ A_{3,y} + A_2 A_3 - A_5 = 0$$

are satisfied, the following two subcases may occur.

a) *There is a right factor of the form* $\partial_x + R(x, y, \Phi(y))$, *where* R *is the general rational solution of*

$$b_x - b^2 + A_3 b - A_6 = 0 \tag{31}$$

involving an undetermined function $\Phi(y)$.

b) *There is a single factor, or there are two factors* $\partial_x + r_i(x, y)$ *where* $r_i(x, y)$ *are special rational solutions of (31).*

iv) *A factor* $\partial_y + A_2$ *exists if the following two constraints are satisfied.*

$$A_5 = A_2 A_3 + 2A_{2,x} + A_{2,y} A_1 - A_2^2 A_1,$$
$$A_7 = A_2 A_6 + A_{2,y} A_4 - A_4 A_2^2 + A_3 A_{2,x} \\ + A_{2,xx} + A_{2,xy} A_1 - 2A_{2,x} A_2 A_1.$$

PROPOSITION 4.2. *Let the third order partial differential operator (30) be given with* $A_k \in \mathbb{Q}(x, y)$ *for all* k. *The following second order right factors may occur.*

i) *A factor with leading derivative* ∂_{xx} *does not contain a derivative* ∂_{yy}. *A factor* $\partial_{xx} + a_1\partial_{xy} + a_2\partial_x + a_3\partial_y + a_4$ *exists if the two constraints*

$$A_{1,yy} + 2A_{1,y} A_2 + A_{2,y} A_1 - A_{3,y} \\ + A_2^2 A_1 - A_2 A_3 + A_5 = 0,$$
$$A_{4,yy} + 2A_{4,y} A_2 + A_4 A_{2,y} - A_{6,y} \\ + A_2^2 A_4 - A_2 A_6 + A_7 = 0$$

are satisfied. Then $a_1 - A_1 = 0$, $a_2 = A_3 - A_1 A_2 - A_{1,y}$, $a_3 = A_4$ *and* $a_4 = A_6 - A_{5,y} - A_2 A_4$.

ii) *A factor* $\partial_{xy} + a_1\partial_{yy} + a_2\partial_x + a_3\partial_y + a_4$ *may exists if one of the four alternatives applies.*

a) *If the conditions* $A_1 = A_4 = 0$, $A_{2,x} + \frac{1}{2}A_2 A_3 - \frac{1}{2}A_5 = 0$ *and* $A_{2,xx} + A_{2,x} A_3 + \frac{1}{2}A_2 A_6 - \frac{1}{2}A_7 = 0$ *are satisfied, the coefficients of the factor may be determined from* $a_1 = 0$, $a_2 = A_2$, $a_{3,x} - a_3^2 + A_3 a_3 - A_6 = 0$ *and* $a_4 = A_2 a_3 + 2A_{2,x}$.

b) *If the conditions* $A_1 = A_4 = 0$ *and* $A_{2,x} + \frac{1}{2}A_2 A_3 - \frac{1}{2}A_5 \neq 0$ *are satisfied, define* $P \equiv A_{2,x} + A_2 A_3 - A_5$ *and* $Q \equiv 2A_{2,x} + A_2 A_3 - A_5$. *Then* $a_1 = 0$, $a_2 = A_2$, $a_3 = \dfrac{1}{Q}(P_x + A_3 P - A_2 A_6 + A_7)$ *and* $a_4 = A_2(a_3 - A_3) - A_{2,x} + A_5$. *The following condition must hold.* $a_{3,x} - a_3^2 + A_3 a_3 - A_6 = 0$.

c) *If* $A_1 \neq 0$ *there is a factor with coefficients* $a_1 = A_1$, $a_2 = A_2$,

$$a_3 = A_3 + \frac{A_{1,x}}{A_1} - \frac{A_4}{A_1},\ a_4 = \Big(\frac{A_{1,x}}{A_1} - \frac{A_4}{A_1}\Big)A_2 - A_{2,x} + A_5$$

if the following constraints are satisfied.

$$a_{3,x} - a_3^2 + A_3 a_3 - A_6 = 0,\quad a_{4,x} - a_3 a_4 + A_3 a_4 - A_7 = 0.$$

d) *If* $A_1 \neq 0$ *there is a factor with coefficients* $a_1 = 0$, $a_2 = A_2$, $a_3 = \dfrac{A_4}{A_1}$ *and*

$$a_4 = \frac{A_2 A_4}{A_1} - A_{2,x} - A_{2,y} A_1 + A_2^2 A_1 - A_2 A_3 + A_5$$

if the following conditions are satisfied

$$a_{3,x} + a_{3,y} A_1 - A_1 A_2 a_3 + A_3 a_3 + A_1 a_4 - A_6 = 0,$$
$$a_{4,x} + a_{4,y} A_1 - A_1 A_2 a_4 + (A_3 - a_3)a_4 - A_7 = 0.$$

The proof of the two preceding propositions are similar to those in the previous section and are therefore omitted.

In order to solve concrete problems it is important to know to what extent the factorizations described in this section may be determined algorithmically. The answer to this question is given in the following corollary.

COROLLARY 4.3. *Any principal divisor of an operator (30) may be determined algorithmically. The same is true for a possible Laplace divisor of a given order.*

PROOF. Testing the conditions (34) of case *i)* of Proposition 5.1 requires only differentiations and arithmetic in the base field. The same is true for case *ii)*, subcase *b)*. In subcase *a)* of case *ii)* the rational solutions of an ordinary Riccati equation have to be determined which is algorithmically possible. In [8] the algorithmic construction of Laplace divisors for fixed values of m or n has been shown. □

THEOREM 4.4. *Any differential operator*

$$D_{xxy} \equiv \partial_{xxy} + A_1\partial_{xyy} \\ + A_2\partial_{xx} + A_3\partial_{xy} + A_4\partial_{yy} + A_5\partial_x + A_6\partial_y + A_7. \tag{32}$$

decomposes uniquely into largest completely reducible components corresponding to one of the types $\mathcal{L}_{xxy}^{(k)}$, $k = 1, \ldots, 11$, *defined as follows;* L_{xxy}, L_{xy}, L_{xx}, L_x *and* L_y *are completely reducible operators with leading terms* ∂_{xxy}, ∂_{xy}, ∂_{xx}, ∂_x *or* ∂_y *respectively.*

$$\mathcal{L}_{xxy}^{(1)} : L_{xxy},\quad \mathcal{L}_{xxy}^{(2)} : L_{xy}L_x,\quad \mathcal{L}_{xxy}^{(3)} : L_{xx}L_y,$$
$$\mathcal{L}_{xxy}^{(4)} : L_x L_{xy},\quad \mathcal{L}_{xxy}^{(5)} : L_y L_{xx},$$
$$\mathcal{L}_{xxy}^{(6)} : L_x^{(1)} L_x^{(2)} L_y,\quad \mathcal{L}_{xxy}^{(7)} : L_x^{(1)} L_y L_x^{(2)},\quad \mathcal{L}_{xxy}^{(8)} : L_y L_x^{(1)} L_x^{(2)},$$
$$\mathcal{L}_{xxy}^{(9)} : Syz\big(\langle\partial_{xxy}\rangle, \mathcal{J}_4^{(1,1)}(m)\big)\mathcal{J}_4^{(1,1)}(m),$$
$$\mathcal{L}_{xxy}^{(10)} : Syz(\langle L\rangle, \mathcal{J}_4^{(1,2)}(n))\mathcal{J}_4^{(1,2)}(n),$$
$$\mathcal{L}_{xxy}^{(11)} : Syz\big(\langle L\rangle, \mathcal{J}_6^{(1,2)}\big)\mathcal{J}_6^{(1,2)}.$$

If the completely reducible components are split into irreducible ones, the following refined decomposition scheme is obtained.

COROLLARY 4.5. *Let* l_x, l_y, l_{xx}, l_{xy} *and* l_{xyy} *denote irreducible operators with leading derivatives as determined by the respective subindex. An additional upper index distinguishes different copies of the respective operator;* Φ *is an undetermined function of a single argument. The types* $\mathcal{L}_{xyy}^{(i,j)}$ *defined in Table 3 are refinements of the types* $\mathcal{L}_{xyy}^{(i)}$, $i = 1, \ldots, 10$ *of Theorem 5.4.*

This is an immediate consequence of the proof given for the preceding theorem if the various factorization alternatives of Proposition 3.1 and Proposition 3.2 are not merged into completely reducible components.

$\mathcal{L}_{xxy}^{(1)}$	$\mathcal{L}_{xxy}^{(1,1)}: l_{xyy}$, $\mathcal{L}_{xxy}^{(1,2)}: Lclm(l_x, l_y^{(1)}, l_y^{(2)})$, $\mathcal{L}_{xxy}^{(1,3)}: Lclm(l_{yy}, l_x)$, $\mathcal{L}_{xxy}^{(1,4)}: Lclm(l_{xy}, l_y)$, $\mathcal{L}_{xxy}^{(1,5)}: Lclm(\mathcal{J}_4^{(1,1)}(m), \mathcal{J}_4^{(1,2)}(n))$.
$\mathcal{L}_{xxy}^{(2)}$	$\mathcal{L}_{xxy}^{(2,1)}: l_{yy}l_x$, $\mathcal{L}_{xxy}^{(2,2)}: Lclm(l_y^{(1)}, l_y^{(2)})l_x$, $\mathcal{L}_{xxy}^{(2,3)}: Lclm(l_y(\Phi))l_x$.
$\mathcal{L}_{xxy}^{(3)}$	$\mathcal{L}_{xxy}^{(3,1)}: l_{xy}l_y$, $\mathcal{L}_{xxy}^{(3,2)}: Lclm(l_x, l_y^{(1)})l_y^{(2)}$, $\mathcal{L}_{xxy}^{(3,3)}: Lclm(\mathcal{J}_2^{(1,1)}(m), \mathcal{J}_3^{(1,1)}(n))l_y$.
$\mathcal{L}_{xxy}^{(4)}$	$\mathcal{L}_{xxy}^{(4,1)}: l_x l_{yy}$, $\mathcal{L}_{xxy}^{(4,2)}: l_x Lclm(l_y^{(1)}, l_y^{(2)})$, $\mathcal{L}_{xxy}^{(4,3)}: l_x Lclm\big(l_y(\Phi)\big)$.
$\mathcal{L}_{xxy}^{(5)}$	$\mathcal{L}_{xxy}^{(5,1)}: l_y l_{xy}$, $\mathcal{L}_{xxy}^{(5,2)}: l_y^{(1)} Lclm(l_x, l_y^{(1)})$.
$\mathcal{L}_{xxy}^{(6,7,8)}$	$\mathcal{L}_{xxy}^{(6,1)}: l_x l_y^{(1)} l_y^{(2)}$. $\mathcal{L}_{xxy}^{(7,1)}: l_y^{(1)} l_y^{(2)} l_x$. $\mathcal{L}_{xxy}^{(8,1)}: l_y^{(1)} l_x l_y^{(2)}$.
$\mathcal{L}_{xxy}^{(9)}$	$\mathcal{L}_{xxy}^{(9,1)}: Syz(\langle D_{xyy}\rangle, \langle D_{xyy}, D_{x^m}\rangle)\langle D_{xyy}, (\partial_{x^{m-1}} + \ldots)(\partial_x + A_2)\rangle$,
$\mathcal{L}_{xxy}^{(10)}$	$\mathcal{L}_{xxy}^{(10,1)}: Syz(\langle D_{xyy}\rangle, \langle D_{xyy}, D_{y^n}\rangle)\langle D_{xyy}, (\partial_{y^{n-1}} + \ldots)(\partial_y + A_2)\rangle$.
$\mathcal{L}_{xxy}^{(11)}$	$\mathcal{L}_{xxy}^{(11,1)}: Syz\big(\langle \partial_{xyy}\rangle, \mathcal{J}_6^{(1,2)}\big) Lclm((\partial_{xy} + \ldots)(\partial_x + \ldots), (\partial_{xy} + \ldots)(\partial_y + \ldots))$.

Table 3. The decomposition types of Corollary 4.5

EXAMPLE 4.6. Consider the operator

$$L \equiv \partial_{xxy} + \partial_{xyy} + y\partial_{xx} + (x - y - 1)\partial_{xy}$$
$$-\partial_{yy} + (xy + x + 1)\partial_x - (x - y)\partial_y + xy + 1.$$

By case $i)$ of Proposition 4.1 the two factors $l_1 \equiv \partial_x - 1$ and $l_2 \equiv \partial_x + \partial_y - y$ exist. They have a principal left intersection $\partial_{xx} + \partial_{xy} - (y+1) - \partial_y + y$; consequently by Theorem 4.4, L has the type $\mathcal{L}_{xxy}^{(5)}$ decomposition

$$L = (\partial_y + x)Lclm(\partial_x - 1, \partial_x + \partial_y - y).$$

The two first-order right factors contribute $F(y)\exp(-x)$ and $G(x-y)\exp(\frac{1}{2}x^2 - xy)$ to the solution.

5. PDE'S WITH LEADING DERIVATIVE ∂_{xyy}

If an equation contains a single mixed derivative of order three it may be assumed to be ∂_{xyy}. The corresponding operator is

$$D_{xyy} \equiv \partial_{xyy} + A_1\partial_{xx} + A_2\partial_{xy} + A_3\partial_{yy} + A_4\partial_x + A_5\partial_y + A_6. \tag{33}$$

Its possible decompositions are discussed in this subsection. First- and second-order factors are considered one after another. Proofs are again omitted.

PROPOSITION 5.1. *Let the third order partial differential operator (33) be given with $A_k \in \mathbb{Q}(x,y)$ for all k. The following first order right factors may occur.*

i) A factor with leading derivative ∂_x exists if the two constraints

$$A_5 = 2A_{3,y} + A_2A_3,$$
$$A_6 = A_{3,yy} + A_{3,y}A_2 + A_{3,x}A_1 + A_3(A_4 - A_1A_3) \tag{34}$$

are satisfied; then the factor is $\partial_x + A_3$.

ii) For a factor $\partial_y + c$ to exist there must hold $A_1 = 0$. Define

$$P \equiv A_{4,x} + A_3A_4 - A_6 \quad \text{and} \quad Q = A_{2,x} + A_2A_3 - A_5. \tag{35}$$

The following two subcases may occur.

a) If $P = 0$ and $Q = 0$, a factor $\partial_y + c$ exists if c is a solution of $c_y - c^2 + A_2c - A_4 = 0$.

b) If $Q \neq 0$ and $P_yQ - P(Q_y + P) + A_2PQ - A_4Q^2 = 0$ there is the factor $\partial_y + \dfrac{A_{4,x} + A_3A_4 - A_6}{A_{2,x} + A_2A_3 - A_5}$.

A complete listing of all possible Loewy decompositions has to include the divisors of order 2; they are considered next.

PROPOSITION 5.2. *Let the third order partial differential operator (33) be given with $A_k \in \mathbb{Q}(x,y)$ for all k. The following second order right factors may occur.*

i) A factor $\partial_{yy} + a_1\partial_x + a_2\partial_y + a_3$ exists if there holds

$$A_5 = A_{2,x} + A_2A_3,$$
$$A_6 = A_{4,x} - A_{3,x}A_1 - 2A_3A_{1,x} - A_{1,xx} + A_3(A_4 - A_1A_3).$$

Then $a_1 = A_1$, $a_2 = A_2$ and $a_3 = A_4 - A_1A_3 - A_{1,x}$.

ii) A factor with leading derivative ∂_{xy} does not contain a term with ∂_{yy}. A factor $\partial_{xy} + a_1\partial_x + a_2\partial_y + a_3$ exists if one of the following two subcases applies.

a) There holds $A_1 = 0$, $A_5 - 2A_{3,y} - A_2A_3 = 0$ and $A_6 + A_{3,yy} + A_{2,y}A_3 - A_{5,y} - A_3A_4 = 0$. The coefficient a_1 may be determined from the Riccati equation

$$a_{1,y} - a_1^2 + A_2a_1 - A_4 = 0. \tag{36}$$

The remaining coefficients are $a_2 = A_3$ and $a_3 = A_3a_1 + A_{3,y}$.

b) There holds $A_1 = 0$ and $A_5 \neq 2A_{3,y} + A_2A_3$. Then $a_2 = A_3$, $a_3 = A_3a_1 - A_{3,y} - A_2A_3 + A_5$,

$$a_1 = A_2 + \frac{A_{3,yy} + A_{2,y}A_3 - A_{5,y} + A_6 - A_3A_4}{2A_{3,y} + A_2A_3 - A_5}.$$

Substituting a_1 into $a_{1,y} - a_1^2 + A_2a_1 - A_4 = 0$ yields a constraint for the coefficients $A_1, \ldots, A_5$.

Similar as for operators (30), the decompositions of (33) may be determined algorithmically to a large extent as is shown next.

COROLLARY 5.3. *Any principal divisor of an operator (33) may be determined algorithmically. The same is true for a possible Laplace divisor of a given order.*

$\mathcal{L}_{xyy}^{(1)}$	$\mathcal{L}_{xyy}^{(1,1)}: l_{xyy},\ \ \mathcal{L}_{xyy}^{(1,2)}: Lclm(l_x, l_y^{(1)}, l_y^{(2)}),\ \ \mathcal{L}_{xyy}^{(1,3)}: Lclm(l_{yy}, l_x),$ $\mathcal{L}_{xyy}^{(1,4)}: Lclm(l_{xy}, l_y),\ \ \mathcal{L}_{xyy}^{(1,5)}: Lclm(\mathcal{J}_4^{(1,1)}(m), \mathcal{J}_4^{(1,2)}(n)).$
$\mathcal{L}_{xyy}^{(2)}$	$\mathcal{L}_{xyy}^{(2,1)}: l_{yy}l_x,\ \ \mathcal{L}_{xyy}^{(2,2)}: Lclm(l_y^{(1)}, l_y^{(2)})l_x,\ \ \mathcal{L}_{xyy}^{(2,3)}: Lclm(l_y(\Phi))l_x.$
$\mathcal{L}_{xyy}^{(3)}$	$\mathcal{L}_{xyy}^{(3,1)}: l_{xy}l_y,\ \ \mathcal{L}_{xyy}^{(3,2)}: Lclm(l_x, l_y^{(1)})l_y^{(2)},\ \ \mathcal{L}_{xyy}^{(3,3)}: Lclm(\mathcal{J}_2^{(1,1)}(m), \mathcal{J}_3^{(1,1)}(n))l_y.$
$\mathcal{L}_{xyy}^{(4)}$	$\mathcal{L}_{xyy}^{(4,1)}: l_x l_{yy},\ \ \mathcal{L}_{xyy}^{(4,2)}: l_x Lclm(l_y^{(1)}, l_y^{(2)}),\ \ \mathcal{L}_{xyy}^{(4,3)}: l_x Lclm\big(l_y(\Phi)\big).$
$\mathcal{L}_{xyy}^{(5)}$	$\mathcal{L}_{xyy}^{(5,1)}: l_y l_{xy},\ \ \mathcal{L}_{xyy}^{(5,2)}: l_y^{(1)} Lclm(l_x, l_y^{(1)}).$
$\mathcal{L}_{xyy}^{(6,7,8)}$	$\mathcal{L}_{xyy}^{(6,1)}: l_x l_y^{(1)} l_y^{(2)}.\ \ \mathcal{L}_{xyy}^{(7,1)}: l_y^{(1)} l_y^{(2)} l_x.\ \ \mathcal{L}_{xyy}^{(8,1)}: l_y^{(1)} l_x l_y^{(2)}.$
$\mathcal{L}_{xyy}^{(9)}$	$\mathcal{L}_{xyy}^{(9,1)}: Syz(\langle D_{xyy}\rangle, \langle D_{xyy}, D_{x^m}\rangle)\langle D_{xyy}, (\partial_{x^{m-1}} + \ldots)(\partial_x + A_2)\rangle,$
$\mathcal{L}_{xyy}^{(10)}$	$\mathcal{L}_{xyy}^{(10,1)}: Syz(\langle D_{xyy}\rangle, \langle D_{xyy}, D_{y^n}\rangle)\langle D_{xyy}, (\partial_{y^{n-1}} + \ldots)(\partial_y + A_2)\rangle.$
$\mathcal{L}_{xyy}^{(11)}$	$\mathcal{L}_{xyy}^{(11,1)}: Syz\big(\langle \partial_{xyy}\rangle, \mathcal{J}_6^{(1,2)}\big) Lclm((\partial_{xy} + \ldots)(\partial_x + \ldots), (\partial_{xy} + \ldots)(\partial_y + \ldots)).$

Table 4. The decomposition types of Corollary 5.5

The proof is almost identical to that of Corollary 4.3 and is therefore omitted.

A full listing of maximal completely reducible components may be obtained from the preceding results. Any operator (33) decomposes uniquely according to one of the following decomposition schemes.

THEOREM 5.4. *Any differential operator*

$$L \equiv \partial_{xyy} + A_1\partial_{xx} + A_2\partial_{xy} + A_3\partial_{yy} + A_4\partial_x + A_5 \qquad (37)$$

decomposes uniquely into largest completely reducible components corresponding to one of the types $\mathcal{L}_{xyy}^{(k)}$, $k = 1, \ldots, 11$, defined as follows; L_{xyy}, L_{xy}, L_{xx}, L_x and L_y are completely reducible operators with leading derivatives ∂_{xyy}, ∂_{xy}, ∂_{xx}, ∂_x or ∂_y respectively.

$$\mathcal{L}_{xyy}^{(1)}: L_{xyy},\quad \mathcal{L}_{xyy}^{(2)}: L_{yy}L_x,\quad \mathcal{L}_{xyy}^{(3)}: L_{xy}L_y,$$
$$\mathcal{L}_{xyy}^{(4)}: L_x L_{yy},\quad \mathcal{L}_{xyy}^{(5)}: L_y L_{xy},$$
$$\mathcal{L}_{xyy}^{(6)}: L_x L_y^{(1)} L_y^{(2)},\quad \mathcal{L}_{xyy}^{(7)}: L_y^{(1)} L_x L_y^{(2)},\quad \mathcal{L}_{xyy}^{(8)}: L_y^{(1)} L_y^{(2)} L_x,$$
$$\mathcal{L}_{xyy}^{(9)}: Syz\big(\langle \partial_{xyy}\rangle, \mathcal{J}_4^{(1,1)}(m)\big)\mathcal{J}_4^{(1,1)}(m)$$
$$\mathcal{L}_{xyy}^{(10)}: Syz\big(\langle L\rangle, \mathcal{J}_4^{(1,2)}(n)\big)\mathcal{J}_4^{(1,2)}(n).$$
$$\mathcal{L}_{xyy}^{(11)}: Syz\big(\langle \partial_{xyy}\rangle, \mathcal{J}_6^{(1,2)}\big)\mathcal{J}_6^{(1,2)}.$$

The decomposition into irreducible components leads to the following refined decomposition scheme.

COROLLARY 5.5. *Let l_x, l_y, l_{xx}, l_{xy} and l_{xyy} denote irreducible operators with leading derivatives as determined by the respective subindex. An additional upper index distinguishes different copies of the respective operator; Φ is an undetermined function of a single argument. The types $\mathcal{L}_{xyy}^{(i,j)}$ defined in Table 4 are refinements of the types $\mathcal{L}_{xyy}^{(i)}$, $i = 1, \ldots, 10$ defined in Theorem 5.4.*

This is an immediate consequence of the proof given for the preceding theorem if the various factorization alternatives of Proposition 3.1 and Proposition 3.2 are not merged into completely reducible components.

EXAMPLE 5.6. Consider the operator

$$L \equiv \partial_{xyy} + 2\big(x + \tfrac{1}{y}\big)\partial_{xy} + y\partial_{yy}$$
$$+x\big(x + \tfrac{2}{y}\big)\partial_x + 2(xy + 2)\partial_y + x^2 + 4x + \tfrac{2}{y}.$$

Its coefficients satisfy conditions (34) of Proposition 5.1; consequently there is a factor $\partial_y + x$. Furthermore there holds $P = 0$ and $Q = 0$ for P and Q defined by (35). The Riccati equation for c is

$$c_y - c^2 + 2\big(x + \tfrac{1}{y}\big)c - x\big(x + \tfrac{2}{y}\big) = 0$$

with the rational solutions $c = x$ and $c = x + \frac{1}{y}$; they yield the factors $\partial_y + x$ and $\partial_y + x + \frac{1}{y}$ with the type $\mathcal{L}_{xyy}^{(1,2)}$ decomposition $L = Lclm(\partial_x + y, \partial_y + x, \partial_y + x + \frac{1}{y})$. The general solution of $Lz = 0$ is obtained by three integrations as

$$z = \big[F(x) + G(x)\frac{1}{y} + H(y)\big]\exp{(-xy)}.$$

F, G and H are undetermined functions.

EXAMPLE 5.7. Consider the operator

$$L \equiv \partial_{xyy} + \Big(\frac{1}{x} - 2\Big)\partial_{xy} - \partial_{yy} - \frac{1}{x}\partial_x - \Big(\frac{1}{x^2} + \frac{1}{x} - 2\Big)\partial_y + \frac{2}{x^2} + \frac{2}{x}.$$

There holds $A_1 = 0$ and $P = Q = 0$, i. e. case *ii*), subcase *a*) of Proposition 5.1 applies. The equation

$$c_y - c^2 + \Big(\tfrac{1}{x} - 2\Big)c + \tfrac{1}{x} = 0$$

has the general solution $c = \dfrac{y\exp{(\frac{1}{2}y^2 + y)} - C}{\exp{(\frac{1}{2}y^2 + y)} + C}$ where C is a constant. For $C = 0$ and $C \to \infty$ the two rational solutions $c = y$ and $c = -1$ are obtained. They yield the factors $\partial_y + y$ and $\partial_y - 1$ and the type $\mathcal{L}_{xyy}^{(4,2)}$ decomposition

$$L = (\partial_x - x)Lclm(\partial_y + y, \partial_y + 1) =$$
$$(\partial_x - x)\Big(\partial_{yy} + \frac{y^2 - 2}{y + 1}\partial_y - \frac{y^2 + y - 1}{y + 1}\Big).$$

The general solution of $Lz = 0$ is

$$z = F(x)\exp{\big(-\tfrac{1}{2}y^2\big)} + G(x)\exp{(-y)} + \exp{\big(\tfrac{1}{2}x^2 - y\big)}$$
$$\times \int H(y)e^y\frac{dy}{y + 1} - \exp{\big(\frac{1}{2}x^2 - \frac{1}{2}y^2\big)}\int H(y)\exp{\big(\frac{1}{2}y^2\big)}\frac{dy}{y + 1}$$

F, G and H are undetermined functions.

EXAMPLE 5.8. Consider the operator

$$L \equiv \partial_{xyy} + \big(x + 1 + \tfrac{1}{y}\big)\partial_{xy} + y\partial_{yy}$$
$$+\big(x + \frac{1}{y} - \frac{1}{y^2}\big)\partial_x + (xy + y + 3)\partial_y + xy + x + 2.$$

By case i) and case ii), subcase b) of Proposition 5.1 the type $\mathcal{L}_{xyy}^{(5,2)}$ decomposition

$$L = (\partial_y + 1)Lclm\big(\partial_x + y, \partial_y + x + \tfrac{1}{y}\big)$$
$$= (\partial_y + 1)\big(\partial_{xy} + (x + \frac{1}{y})\partial_x + y\partial_y + xy + 2\big)$$

is obtained. The two arguments of the $Lclm$ yield the solutions $z_1 = F(y)\exp(-xy)$ and $z_2 = G(x)\frac{1}{y}\exp(-xy)$; F and G are undetermined functions. L may be factorized as $L = (\partial_y + 1)(\partial_y + x + \frac{1}{y})(\partial_x + y)$ from which the third solution

$$z_3 = \exp\big(-(x+1)y\big)\frac{1}{y}\int H(x)\frac{xy - y - 1}{(x-1)^2}\exp(xy)dx$$

follows; H is again an undetermined function. Finally the general solution of $Lz = 0$ is $z = z_1 + z_2 + z_3$.

EXAMPLE 5.9. Consider the operator

$$L \equiv \partial_{xyy} - 2x\partial_{xy} + y\partial_{yy} + x^2\partial_x - 2(xy-1)\partial_y + x(xy-2).$$

By case i) and case ii), subcase b) of Proposition 5.1 the factorizations

$$L = \begin{cases} (\partial_{yy} - 2x\partial_y + x^2)(l_1 \equiv \partial_x + y) \\ (\partial_{xy} - x\partial_x + y\partial_y - xy + 3)(l_2 \equiv \partial_y - x). \end{cases}$$

are obtained. The intersection of l_1 and l_2 is

$$Lclm(l_1, l_2) =$$
$$\langle \partial_{xyy} - 2x\partial_{xy} + y\partial_{yy} + x^2\partial_x - 2(xy-1)\partial_y + x(xy-2),$$
$$\partial_{xxy} - x\partial_{xx} + 2y\partial_{xy} - 2(xy+1)\partial_x + y^2\partial_y - y(xy+2)\rangle.$$

It is not principal, i.e. L has decomposition type $\mathcal{L}_{xyy}^{(11)}$. The two equations $l_1 z = 0$ and $l_2 z = 0$ yield the solutions $z_1 = F(y)\exp(-xy)$ and $z_2 = G(x)\exp(xy)$. A third solution may be obtained from the factorization $\partial_{yy} - 2x\partial_y + x^2 = (\partial_y - x)(\partial_y - x)$. It yields

$$z_3 = y\exp(-xy)\int xH(x)\exp(2xy)dx.$$

The general solution of $Lz = 0$ is $z = z_1 + z_2 + z_3$.

6. SUMMARY AND CONCLUSION

In this article for the first time Loewy decompositions for a full class of partial differential equations of differential type different from zero are considered. It became clear that Theorems 2.1 and 2.2 are fundamental for a full understanding of how the various decompositions arise. Furthermore it has been shown by numerous examples that obtaining such a decomposition is essentially synonymous with finding its solutions in closed form. However, two basic questions remain to be discussed.

In the first place this is the uniqueness of a decomposition as it is the case for ordinary operators. To this end, one probably has to make the problem more specific by the requirement that the admitted divisors have differential type 1, as it is true for the given operators.

Secondly there is the question to what extent these decompositions may be obtained algorithmically. For ordinary operators it is sufficient to determine rational solutions of ordinary Riccati equations. It has been shown that for decomposing linear pde's of the plane in general rational solutions of partial Riccati equations are required. This problem boils down to finding rational first integrals of a general quasilinear first-order ode; for this problem see [4] and [3]. Up to this point the discussion concerns principal divisors. Deciding the non-existence of a Laplace divisor requires an upper bound for its order. It may not be possible to find such a bound, i.e. this problem may not be decidable. Finally there remain the exceptional cases mentioned at the end of Corollary 3.3.

Obviously there are many possible extensions of the results presented in this article. To mention just a few, there should be a close connection between the type of solution and the decomposition type as it is true for ordinary differential equation, Section 2.1 of [12]. For applications decompositions in more than two independent variables would by highly desirable. Dealing with more than a single dependent variables would involve modules over the respective rings of differential operators.

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