Randomized Complexity Lower Bounds

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The complexity lower bound $\Omega(\log N)$ is proved for randomized computation trees (over reals with branching signs $\{\leq, >\}$) for recognizing an arrangement or a polyhedron with N faces. A similar lower bound is proved for randomized computation trees over any zero-characteristic field with branching signs $\{=, \neq\}$ for recognizing an arrangement. As consequences, this provides in particular, the randomized lower bound $\Omega(n^2)$ for the KNAPSACK problem (which was proved in case of the randomized computation trees over reals in [11]) and also the randomized lower bound $\Omega(n \log n)$ for the DISTINCTNESS problem (which is thereby the sharp bound). The technical core of the paper is a lower bound on the multiplicative complexity of a polynomial in terms of its singularities.

Introduction.

The complexity lower bounds for deterministic algebraic computation trees were obtained in [26], [2], [4], [29], [30], [22] where the topological methods were developed. In particular, these methods provide the lower bound $\Omega(\log N)$ for recognizing a union of planes (of different dimensions) with N faces, under a face we mean any nonempty intersection of several among these planes. As consequences we obtain the lower bound $\Omega(n \log n)$ for the DISTINCTNESS problem $\bigcup_{1\leq i<j\leq n} \{X_i = X_j\} \subset \mathbb{R}^n$, EQUALITY SET problem $\{(x_1, \dots, x_n,\ y_1, \dots, y_n) : (x_1, \dots, x_n) \text{ is a permutation of } (y_1, \dots, y_n)\} \subset \mathbb{R}^{2n}$ and the lower bound $\Omega(n^2)$ for the KNAPSACK problem $\bigcup_{I\subset\{1,\dots,n\}} \left\{\sum_{i\in I} x_i = 1\right\} \subset \mathbb{R}^n$. In [14], [15] a differential-geometric approach for recognizing polyhedra (to which the mentioned topological methods are not applicable) was proposed which gives the lower bound $\Omega(\log N/\log\log N)$ where N is the number of faces of the polyhedron.

The first results on the randomized computation trees (RCT) appeared in [24], [19], [9], [10] but for decade an open problem remained to obtain non-linear complexity lower bounds for recognizing natural problems by RCT. In [13] for the first time the nonlinear lower bound was obtained for somewhat weaker computational model of the randomized algebraic *decision* trees in which the testing polynomials in the branching nodes are of a fixed degree, rather than the *computation* trees in which the testing polynomials are computed along the path of the computation, so they could have in principle an exponential degree. The approach of [13] provides the lower bound $\Omega(\log N)$ for recognizing an arrangement, i.e. a union of hyperplanes, and for recognizing a polyhedron, where N is again the number of faces. In particular, this leads to the lower bound $\Omega(n \log n)$ for the DISTINCTNESS problem and $\Omega(n^2)$ for the KNAPSACK problem. For the EQUALITY SET problem a complexity lower bound on a randomized algebraic decision tree seems to be an open question.

But the method of [13] does not provide a lower bound for more interesting model of RCT. Only in [11] a method was developed which gives in particular, a lower bound $\Omega(n^2)$ for the KNAPSACK problem on RCT. This method relies on the obtained in [11] lower bound on the multiplicative border complexity of polynomials. The lower bound $\Omega(\log N)$ of [11] holds for arrangements or polyhedra which satisfy some special conditions which fail, for example, for the DISTINCTNESS problem.

In this paper we consider RCT over an arbitrary zero-characteristic field F with branching signs $\{=, \neq\}$ and also more customary RCT over reals with branching signs $\{\leq, >\}$. We remind (see e.g. [24], [19], [13]) that RCT $T = \{T_\alpha\}_\alpha$ is a collection of computation trees T_α which are chosen with the probabilities $p_\alpha \geq 0, \sum_\alpha p_\alpha = 1$ such that T gives for any input a correct output with a probability greater than $1 - \gamma$ for a certain $\gamma < 1/2$ which is called the error probability of RCT.

Let $H_1, \dots, H_m \subset F^n$ be a family of hyperplanes, denote by $S = H_1 \cup \dots \cup H_m$ the arrangement. Under k-face of S we mean any nonempty intersection $H_{i_1} \cap \dots \cap H_{i_{n-k}}$ of the dimension $\dim(H_{i_1} \cap \dots \cap H_{i_{n-k}}) = k$.

Theorem 1. *Assume that for a certain constant $c_0 < 1$ any subarrangement $S_1 = H_{i_1} \cup \dots \cup H_{i_q}$ of S where $q > c_0 m$, has at least $N^{(0)}$ faces of all the dimensions. Then the depth of any RCT over F recognizing S, is greater than $\Omega(\log_2 N^{(0)} - 2n - \log_2 n)$.*

Corollary 1.1. *Any RCT over F solving the DISTINCTNESS problem, has the complexity greater than $\Omega(n \log n)$.*

The idea of the proof of the necessary in theorem 1 lower bound on $N^{(0)}$ one can find in [13]. Observe that the lower

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bound in the corollary is nearly sharp since it is possible to compute (deterministically) the discriminant $\prod_{1\le i<j\le n}(X_i - X_j)$ with the complexity $O(n\log^2 n)$ ([20], [27]). If to count only *nonscalar* multiplications/divisions (i.e. to consider the multiplicative complexity) then the lower bound from the corollary becomes sharp also due to [20], [27].

Corollary 1.2. *Any RCT over F solving the KNAPSACK problem, has the complexity greater than $\Omega(n^2)$.*

The proof of the necessary lower bound on $N^{(0)}$ one can find in [11].

Corollary 1.2 can be generalized to the complexity lower bound $\Omega(n^2\log j)$ for RCT solving the RESTRICTED INTEGER PROGRAMMING ([19]) $\bigcup_{a\in\{0,\dots,j-1\}^n} \subset F^n$ (obviously, it converts into the KNAPSACK problem when $j=2$).

In case of more customary RCT over reals $\mathbb{R}$ with the branching signs $\{\le,>\}$ we consider recognizing either an arrangement $S=\cup_{1\le i\le m}H_i\subset\mathbb{R}^n$ or a polyhedron $S^+=\cap_{1\le i\le m}H_i^+\subset\mathbb{R}^n$, where H_i^+ is a half-space bounded by the hyperplane $H_i, 1\le i\le m$. We say that $\Gamma=H_{i_1}\cap\cdots\cap H_{i_{n-k}}$ is k-face of S^+ if $\dim(\Gamma\cap S^+)=k$.

Theorem 2. *Let for some positive constants c, c_1 and $k\le(1-c_1)n$ an arrangement $\mathcal{S}=S=\cup_{1\le i\le m}H_i$ or a polyhedron $\mathcal{S}=S^+=\cap_{1\le i\le m}H_i^+$ have at least $\Omega(m^{c(n-k)})$ k-faces. Then for any RCT recognizing $\mathcal{S}$, its depth is greater than $\Omega(n\log m)$.*

Corollary 2.1. *Any RCT over reals solving the DISTINCTNESS problem, has the complexity greater than $\Omega(n\log n)$.*

Similar to the case of RCT over a zero-characteristic field (cf. corollary 1.1) the complexity bound is sharp since one can (deterministically) sort the input real numbers $x_1,\dots,x_n$ with the complexity $O(n\log n)$.

Corollary 2.2. *(see also [11]). Any RCT over reals solving the KNAPSACK problem, has the complexity greater than $\Omega(n^2)$.*

For the similar to the DISTINCTNESS problem SET DISJOINTNESS $\{(x_1,\dots,x_n,y_1,\dots,y_n): x_i\ne y_j\}\subset\mathbb{R}^{2n}$ (being a complement to an arrangement) one obtains (almost literally as in the corolla ries 1.1, 2.1) the lower bound $\Omega(n\log n)$ and the upper bound $O(n\log^2 n)$ (relying on the computin g of the resultant [20], [27]) on the randomized complexity.

In the next two sections we give sketches of the proofs of theorems 1,2.

The construction from [5] of RCT with the linear complexity $O(n)$ for the EQUALITY SET problem (which is the union of n-dimensional planes in $2n$-dimensional space, see above) shows that the consideration just of *hyperplanes* in theorems 1,2 is crucial, and the non-linear randomized complexity lower bounds cannot be directly extended to unions of planes of arbitrary dimensions.

In [3] deterministic computation trees with the branching signs $\{=,\ne\}$ over algebraically closed fields of *positive characteristics* were considered, and the complexity lower bound $\Omega(\log C)$ for recognizing an algebraic variety was established, where C is the degree of the Zeta-function of the variety. It is an open question to obtain non-linear complexity lower bounds for *randomized* computation trees over the fields of positive characteristics.

Let us also mention the paper [12] where a complexity lower bound was established for the randomized *analytic* decision trees (rather than for more customary algebraic ones) and also the paper [6] where a lower bound was ascertained for a randomized *parallel* computational model (rather than a sequential model considered in the quoted papers including the present one).

1 RCT over zero characteristic fields.

In this section we give a sketch of the proof of theorem 1 (the complete proof one can find in [7]).

Assume for the time being that the field $F=\bar{F}$ is algebraically closed. Denote by N_0 the number of 0-faces (in other words, vertices) of the arrangement $S=H_1\cup\cdots\cup H_m$.

Similar to [27], [17] consider the graph of the gradient map of a polynomial $0\not\equiv g\in F[X_1,\dots,X_n]$

$$G=\{(x=(x_1,\dots,x_n),\frac{\partial g}{\partial X_1}(x),\dots,\frac{\partial g}{\partial X_n}(x))\}\subset F^{2n}$$

The main technical tool in the proof of theorem 1 is the following lower bound on the degree $\deg G$ (defined as the degree of the projective closure of G [23], [25]).

Lemma 1.1. $\deg G\ge\frac{N_0}{2^{2n}}$

Denote by $C(g)$ the multiplicative complexity of g. The results from [27], [1] imply the inequality $\deg G\le 2^{3C(g)}$ which together with lemma 1.1 entail the following lower bound on the multiplicative complexity of g.

Proposition 1. *If a polynomial $0\not\equiv g\in F[X_1,\dots,X_n]$ vanishes on the arrangement S with N_0 vertices then $C(g)\ge\frac{1}{3}(\log_2 N_0-2n)$.*

We remark that if N_l denotes the number of l-faces of S then one obtains the similar lower bound $\frac{1}{3}(\log_2 N_l-2(n-l))$ by means of intersecting S with a $(n-l)$-dimensional plane.

Now let F be an arbitrary zero characteristic field. To complete the proof of theorem 1 observe that if RCT $T=\{T_\alpha\}_\alpha$ recognizes S with an error probability $\gamma<1/2$, then for every α CT T_α possesses the unique "thick" path (from the root to a leaf), along which all the testing polynomials $f_1,\dots,f_k\in F[X_1,\dots,X_n]$ have the branching sign $\ne$. One can prove that with a probability greater than $1-2\gamma>0$ the product $f_1\cdots f_k$ vanishes on at least $q>\frac{1-2\gamma}{1+2\gamma}m$ of hyperplanes among $H_1,\dots,H_m$. Taking into account that γ could be made as close to zero as desired at the expense of increasing the depth of RCT by a suitable constant factor [19], we apply proposition 1 and the remark just after it to the polynomial $f_1\cdots f_k$ (notice that the multiplicative complexity of the latter product does not exceed $2k-1$), and get a lower bound on k. Since the complexity of RCT under consideration is greater or equal to k, one completes the proof of theorem 1.

2 RCT over reals

In this section we give a sketch of the proof of theorem 2 (the complete proof one can find in [8]).

Again let F be a zero characteristic field and $\Gamma=H_{i_1}\cap\cdots\cap H_{i_{n-k}}$ be k-face of the arrangement $S=H_1\cap\cdots\cap H_m$. Fix arbitrary coordinates $Z_1,\dots,Z_k$ in Γ. Then treating $H_{i_1},\dots,H_{i_{n-k}}$ as the coordinate hyperplanes of the coordinates $Y_1,\dots,Y_{n-k}$, one gets the coordinates $Z_1,\dots,Z_k$, $Y_1,\dots,Y_{n-k}$ in F^n. The next construction of the leading terms of a polynomial is similar to [13], [11].

For any polynomial $f(Z_1,\dots,Z_k,Y_1,\dots,Y_{n-k})\in F[Z_1,\dots,Z_k,Y_1,\dots,Y_{n-k}]$ following [13], [11] define its leading term

$$\alpha Z_1^{m_1'}\cdots Z_k^{m_k'}Y_1^{m_1}\cdots Y_{n-k}^{m_{n-k}}$$

$0 \neq \alpha \in F$ (with respect to the coordinate system $Z_1, \dots, Z_k$, $Y_1, \dots, Y_{n-k}$) as follows. First take the minimal integer m_{n-k} such that $Y_{n-k}^{m_{n-k}}$ occurs in the terms of $f = f^{(0)}$. Consider the polynomial

$$0 \not\equiv f^{(1)} = \left(\frac{f}{Y_{n-k}^{m_{n-k}}}\right)(Z_1, \dots, Z_k, Y_1, \dots, Y_{n-k-1}, 0) \in F[Z_1, \dots, Z_k, Y_1, \dots, Y_{n-k-1}]$$

which could be viewed as a polynomial on the hyperplane $H_{i_{n-k}}$. Observe that m_{n-k} depends only on $H_{i_{n-k}}$ and not on $Z_1, \dots, Z_k, Y_1, \dots, Y_{n-k-1}$, since a linear transformation of the coordinates $Z_1, \dots, Z_k$, $Y_1, \dots$, Y_{n-k-1} changes the coefficients (being the polynomials from $F[Z_1, \dots, Z_k, Y_1, \dots, Y_{n-k-1}]$) of the expansion of f in the variable Y_{n-k}, and a coefficient vanishes identically if and only if it vanishes identically after the transformation. Then $f^{(1)}$ is the coefficient of the expansion of f at the power $Y_{n-k}^{m_{n-k}}$.

Second, take the minimal integer m_{n-k-1} such that $Y_{n-k-1}^{m_{n-k-1}}$ occurs in the terms of $f^{(1)}$. In other words, $Y_{n-k-1}^{m_{n-k-1}}$ is the minimal power of Y_{n-k-1} occurring in the terms of f in which occurs the power $Y_{n-k}^{m_{n-k}}$. Therefore, m_{n-k}, m_{n-k-1} depend only on the hyperplanes H_{n-k}, H_{n-k-1} and not on $Z_1, \dots, Z_k$, $Y_1, \dots, Y_{n-k-2}$, since (as above) a linear transformation of the coordinates $Z_1, \dots, Z_k$, $Y_1, \dots, Y_{n-k-2}$ changes the coefficients (being the polynomials from $F[Z_1, \dots, Z_k, Y_1, \dots, Y_{n-k-2}]$) of the expansion of f in the variables Y_{n-k}, Y_{n-k-1} and a coefficient vanishes identically if and only if it vanishes identically after the transformation. Denote by $0 \not\equiv f^{(2)} \in F[Z_1, \dots, Z_k$, $Y_1, \dots, Y_{n-k-2}]$ the coefficient of the expansion of f at the monomial $Y_{n-k-1}^{m_{n-k-1}}\ Y_{n-k}^{m_{n-k}}$. Obviously

$$f^{(2)} = \left(\frac{f^{(1)}}{Y_{n-k-1}^{m_{n-k-1}}}\right)(Z_1, \dots, Z_k, Y_1, \dots, Y_{n-k-2}, 0)$$

One could view $f^{(2)}$ as a polynomial on the $(n-2)$-dimensional plane $H_{i_{n-k}} \cap H_{i_{n-k-1}}$.

Continuing in the similar way, we obtain consecutively the (non-negative) integers $m_{n-k}, m_{n-k-1}, \dots, m_1$ and the polynomials

$$0 \not\equiv f^{(l)} \in F[Z_1, \dots, Z_k,\ Y_1, \dots, Y_{n-k-l}]$$

$1 \leq l \leq n-k$, by induction on l. Herewith, $Y_{n-k-l+1}^{m_{n-k-l+1}}$ is the minimal power of $Y_{n-k-l+1}$ occurring in the terms of f, in which occurs the monomial $Y_{n-k-l+2}^{m_{n-k-1+2}} \cdots Y_{n-k}^{m_{n-k}}$ for each $1 \leq l \leq n-k$. Notice that $m_{n-k}, \dots, m_{n-k-l}$ depend only on the hyperplanes $H_{i_{n-k}}, \dots, H_{i_{n-k-l}}$ and not on $Z_1, \dots, Z_k$, $Y_1, \dots, Y_{n-k-l-1}$. Then $f^{(l)}$ is the coefficient of the expansion of f at the monomial $Y_{n-k-l+1}^{m_{n-k-l+1}} \cdots Y_{n-k}^{m_{n-k}}$ and

$$f^{(l+1)} = \left(\frac{f^{(l)}}{Y_{n-k-l}^{m_{n-k-l}}}\right)(Z_1, \dots, Z_k, Y_1, \dots, Y_{n-k-l-1}, 0)$$

Thus, $f^{(l)}$ depends only on $H_{i_{n-k}}, \dots, H_{i_{n-k-l}}$ and not on $Z_1, \dots, Z_k, Y_1, \dots, Y_{n-k-l-1}$. One could view $f^{(l)}$ as a polynomial on the $(n-l)$ dimensional plane $H_{i_{n-k}} \cap \cdots \cap H_{i_{n-k-l+1}}$. Continuing, we define also $m'_k, \dots, m'_1$.

Finally, the leading term $lm(f) = \alpha Z_1^{m'_1} \cdots Z_k^{m'_k} Y_1^{m_1} \cdots Y_{n-k}^{m_{n-k}}$ is the minimal term of f in the lexicographical ordering with respect to the ordering $Z_1 > \cdots > Z_k > Y_1 > \cdots > Y_{n-k}$. The leading term $lm(f^{(l)}) = \alpha Z_1^{m'_1} \cdots Z_k^{m'_k}$ $Y_1^{m_1} \cdots Y_{n-k-l}^{m_{n-k-l}}$, we refer to this equality as the maintenance property (see also [13], [11]).

From now on the construction and the definitions differ from the ones in [13], [11].

For any polynomial $g \in F[X_1, \dots, X_n]$ one can rewrite it in the coordinates $\overline{g}(Z_1, \dots, Z_k, Y_1, \dots, Y_{n-k})$ and expand $\overline{g} = g_s + g_{s+1} + \cdots + g_{s_1}$, where $g_j \in F[Z_1, \dots, Z_k$, $Y_1, \dots, Y_{n-k}]$, $s \leq j \leq s_1$ is homogeneous with respect to the variables $Y_1, \dots, Y_{n-k}$ of degree j and $g_s = g_s^{(0)} \not\equiv 0$. Consider the leading term $lm(g_s) = \alpha Z_1^{m'_1} \cdots Z_k^{m'_k} Y_1^{m_1} \cdots Y_{n-k}^{m_{n-k}}$ and denote by $\mathrm{Var}^{(\Gamma)}(g) = \mathrm{Var}^{(H_{i_1}, \dots, H_{i_{n-k}})}(g)$ the number of positive (in other words, nonzero) integers among $m_{n-k}, \dots, m_1$, note that $s = m_1 + \cdots + m_{n-k}$. As we have shown above $\mathrm{Var}^{(H_{i_1}, \dots, H_{i_{n-k}})}(g)$ is independent from the coordinates $Z_1, \dots, Z_k$ of Γ. Obviously, $\mathrm{Var}^{(H_{i_1}, \dots, H_{i_{n-k}})}(g)$ coincides with the number of $1 \leq l \leq n-k$ such that $Y_{n-k-l} | g_s^{(l)}$, the latter condition is equivalent to that the variety $\{g_s^{(l)} = 0\} \cap H_{i_{n-k}} \cap \cdots \cap H_{i_{n-k-l+1}}$ contains the plane $H_{i_{n-k}} \cap \cdots \cap H_{i_{n-k-l+1}} \cap H_{i_{n-k-l}}$ (being a hyperplane in $H_{i_{n-k}} \cap \cdots \cap H_{i_{n-k-l+1}}$).

It is convenient (see also [13], [11]) to reformulate the introduced concepts by means of infinitesimals in case of a real closed field F (see e.g. [18]). We say that an element ε transcendental over F is an infinitesimal (relative to F) if $0 < \varepsilon < a$ for any element $0 < a \in F$. This uniquely induces the order on the field $F(\varepsilon)$ of rational functions and further on the real closure $\widetilde{F(\varepsilon)}$ (see [18]).

One could make the order in $\widetilde{F(\varepsilon)}$ clearer by embedding it in the larger real closed field $F((\varepsilon^{1/\infty}))$ of Puiseux series (cf. e.g. [16]). A nonzero Puiseux series has the form $b = \sum_{i \geq i_0} \beta_i \varepsilon^{i/\delta}$, where $-\infty < i_0 < \infty$ is an integer, $\beta_i \in F$ for every integer i; $\beta_{i_0} \neq 0$ and the denominator of the rational exponents $\delta \geq 1$ is an integer. The order on $F((\varepsilon^{1/\infty}))$ is defined as follows: $sgn(b) = sgn(\beta_{i_0})$. When $i_0 \geq 1$, then b is called an infinitesimal, when $i_0 \leq -1$, then b is called infinitely large. For any not infinitely large b we define its standard part $st(b) = st_\varepsilon(b) \in F$ as follows: when $i_0 = 0$, then $st(b) = \beta_{i_0}$, when $i_0 \geq 1$, then $st(b) = 0$. In the natural way we extend the standard part to the vectors from $(F((\varepsilon^{1/\infty})))^n$ and further to subsets in this space.

Now let $\varepsilon_1 > \varepsilon_2 \cdots > \varepsilon_{n+1} > 0$ be infinitesimals, where ε_1 is an infinitesimal relative to $\mathbb{R}$; then ε_{i+1} is an infinitesimal relative to $\widetilde{\mathbb{R}(\varepsilon_1, \dots, \varepsilon_i)}$ for all $0 \leq i \leq n$. Denote the real closed field $\mathbb{R}_i = \widetilde{\mathbb{R}(\varepsilon_1, \dots, \varepsilon_i)}$, in particular, $\mathbb{R}_0 = \mathbb{R}$. For an element $b \in \mathbb{R}_{n+1}$ for brevity denote the standard part $st_i(b) = st_{\varepsilon_{i+1}}(st_{\varepsilon_{i+2}} \cdots (st_{\varepsilon_{n+1}}(b) \cdots)) \in \mathbb{R}_i$ (provided that it is definable).

Also we will use the Tarski's transfer principle [28]. Namely, for two real closed fields $F_1 \subset F_2$ a closed (so, without free variables) formula in the language of the first-order theory of F_1 is true over F_1 if and only if this formula is true over F_2.

An application of Tarski's transfer principle is the concept of the completion. Let $F_1 \subset F_2$ be real closed fields and Ψ be a formula (with quantifiers and, perhaps, with n free variables) of the language of the first-order theory of the field F_1. Then Ψ determines a semialgebraic set $V \subset F_1^n$. The completion $V^{(F_2)} \subset F_2^n$ is a semialgebraic set determined by the same formula Ψ (obviously, $V \subset V^{(F_2)}$).

One could easily see that for any point $(z_1, \dots, z_k) \in \mathbb{R}_k^k$ and a polynomial $g \in \mathbb{R}[X_1, \dots, X_n]$ such that

$g_s^{(n-k)}(z_1,\ldots,z_k) \neq 0$ (we utilize the introduced above notations) the following equality for the signs

$$\sigma_1^{m_1}\ldots\sigma_{n-k}^{m_{n-k}}\ sgn(g_s^{(n-k)}(z_1,\ldots,z_k)) = sgn(\overline{g}(z_1,\ldots,z_k,\sigma_1\varepsilon_{k+1}\varepsilon_{n+1},\ldots,\sigma_{n-k}\varepsilon_n\varepsilon_{n+1})) \quad (1)$$

holds for any $\sigma_1,\ldots,\sigma_{n-k}\in\{-1,1\}$. For any $1\leq i\leq n-k$ such that $m_i=0$ (1) holds also for $\sigma_i=0$, agreeing that $0^0=1$. Moreover, the following polynomial identity holds:

$$g_s^{(n-k)}(Z_1,\ldots,Z_k) = st_k\left(\frac{\overline{g}(Z_1,\ldots,Z_k,\varepsilon_{k+1}\varepsilon_{n+1},\ldots,\varepsilon_n\varepsilon_{n+1})}{\varepsilon_{k+1}^{m_1}\cdots\varepsilon_n^{m_{n-k}}\varepsilon_{n+1}^{s}}\right)$$

Now let F be an algebraically closed field of zero characteristic. Take a certain $0<\eta\leq 1$ (it will be specified later). We call k-face $\Gamma = H_{i_1}\cap\cdots\cap H_{i_{n-k}}$ of the arrangement S *strongly singular* (with respect to a polynomial $g\in F[X_1,\ldots,X_n]$) if $\mathrm{Var}^{(H_{i_1},\ldots,H_{i_{n-k}})}(g)\geq\eta(n-k)$. Denote by N the number of strongly singular k-faces of S with respect to g (since g will be fixed for the time being, in the sequel we omit mentioning of g in this context).

The following lower bound on the degree of the graph G of the gradient map of g (see section 1) strengthens lemma 1.1, being the main technical tool in the proof of theorem 2.

Lemma 2.1 $\deg G\geq\Omega(N/(m^{(1-\eta)(n-k)}2^{4n}))$

Similar to proposition 1 from section 1 this lemma implies the following proposition.

Proposition 2. *Let a polynomial $g\in F[X_1,\ldots,X_n]$ have N strongly singular k-faces in an arrangement $H_1\cup\cdots\cup H_m\subset F^n$. Then the multiplicative complexity $C(g)\geq 1/3(\log N-(n-k)(1-\eta)\log m-4n-$ const).*

For a family of polynomials $f_1,\ldots,f_t\in\mathbb{R}[X_1,\ldots,X_n]$ we define $\mathrm{Var}^{(\Gamma)}(f_1,\ldots,f_t)$ to be the number of the variables among $Y_1,\ldots,Y_{n-k}$ which occur in at least one of the leading terms $lm(f_{1,s_1}),\ldots,lm(f_{t,s_t})$, where $H_{i_1},\ldots,H_{i_{n-k}}$ are the coordinate hyperplanes of the coordinates $Y_1,\ldots,Y_{n-k}$, respectively; $\overline{f}_j(Z_1,\ldots,Z_k,Y_1,\ldots,Y_{n-k}) = f_j(X_1,\ldots,X_n)$ and $\overline{f}_j = f_{j,s_j}+f_{j,s_j+1}+\cdots$, herewith $f_{j,l}$ is homogeneous with respect to the variables $Y_1,\ldots,Y_{n-k}$ of degree l and $f_{j,s_j}\not\equiv 0$, $1\leq j\leq t$. Because the expansion into the homogeneous components $\overline{f}_1\cdots\overline{f}_t=(f_{1,s_1}\cdots f_{t,s_t})+\cdots$ starts with $f_{1,s_1}\cdots f_{t,s_t}$, we have $lm(f_{1,s_1}\cdots f_{t,s_t}) = lm(f_{1,s_1})\cdots lm(f_{t,s_t})$ and hence $\mathrm{Var}^{(H_{i_1},\ldots,H_{i_{n-k}})}(f_1\cdots f_t) = \mathrm{Var}^{(\Gamma)}(f_1\cdots f_t) = \mathrm{Var}^{(\Gamma)}(f_1,\cdots,f_t)$.

For any CT T_1 we denote by $\mathrm{Var}^{(\Gamma)}(T_1) = \mathrm{Var}^{(H_{i_1},\ldots,H_{i_{n-k}})}(T_1)$ the maximum of the $\mathrm{Var}^{(\Gamma)}(f_1\cdots f_t)$ taken over all the paths of T_1, whose $f_1,\ldots,f_t$ are testing polynomials along the path.

The proof of the following "local" (i.e. concerning a single face) lemma relies on the relation (1) and is similar to lemma 1 [13], [11], but differs from it due to the different definition of the leading term lm.

Lemma 2.2. *Let $T=\{T_\alpha\}$ be an RCT recognizing*

a) an arrangement $S=\cup_{1\leq i\leq m}H_i$ such that $\Gamma=H_{i_1}\cap\cdots\cap H_{i_{n-k}}$ is k-face of S, or

b) a polyhedron $S^+=\cap_{1\leq i\leq m}H_i^+$ such that $\Gamma=\cap_{1\leq j\leq n-k}H_{i_j}$ is k-face of S^+

with error probability $\gamma<\frac{1}{2}$. Then $Var^{(H_{i_1},\ldots,H_{i_{n-k}})}(T_\alpha)\geq(1-2\gamma)^2(n-k)$ forx a fraction of $\frac{1-2\gamma}{2-2\gamma}$ of all T_α's.

The following "global" (i.e, concerning the set of all faces) lemma is similar to lemma 2 from [13], [11], but its proof is considerably simpler.

Lemma 2.3. *Let $\mathcal{S}=S$ or $\mathcal{S}=S^+$ satisfy the conditions of the theorem 2. Assume that CT T' for some constant $\eta>1-c$, satisfies the inequality $Var^{(\Gamma)}(T')\geq\eta(n-k)$ for at least $M\geq\Omega(m^{c(n-k)})$ of k-faces Γ of $\mathcal{S}$. Then the depth t of T' is greater than $\Omega(n\log m)$.*

Proof of lemma 2.3: To each k-face Γ of $\mathcal{S}$ satisfying the inequality $\mathrm{Var}^{(\Gamma)}(T')\geq\eta(n-k)$, we correspond a path in T' with the testing polynomials $f_1,\ldots,f_{t_0}\in\mathbb{R}[X_1,\ldots,X_n]$, $t_0\leq t$ such that $\mathrm{Var}^{(\Gamma)}(f_1\cdots f_{t_0})\geq\mathrm{Var}^{(\Gamma)}(T')$ (in other words, Γ is strongly singular k-face for $f_1\cdots f_{t_0}$, see section 1). Denote $f=f_1\cdots f_{t_0}$.

Assume that $3^t\leq O(m^{(\eta-1+c)(n-k)/2})$, otherwise we are done. Then there exists a path of T' (let us keep the notation $f_1,\ldots,f_{t_0}$ for the testing polynomials along this path) which corresponds to at least $N=\Omega(m^{(c-\eta+1)(n-k)/2})$ of strongly singular k-faces Γ for f (because there are most 3^t paths in T'). Proposition 2 implies that the multiplicative complexity $C(f)\geq\frac{1}{3}((\eta-1+c)(n-k)\log m-4n-\mathrm{const})$. Obviously $C(f)\leq t+t_0-1\leq 2t-1$ (cf. the proof of theorem 1 in section 1). Hence $t\geq\Omega(n\log m)$ that proves lemma 2.3.

Finally we show how to deduce the theorem 2 from lemmas 2.2 and 2.3. Consider RCT $\{T_\alpha\}$ recognizing $\mathcal{S}$ with error probability $\gamma<\frac{1}{2}$. Lemma 2.2 and counting imply the existence of T_{α_0} such that the inequality $\mathrm{Var}^{(\Gamma)}(T_{\alpha_0})\geq(1-2\gamma)^2(n-k)$ is true for $M=\frac{1-2\gamma}{2(1-\gamma)}\Omega(m^{c(n-k)})$ of k-faces Γ of $\mathcal{S}$. Apply lemma 2.3 to CT $T'=T_{\alpha_0}$ with $\eta=(1-2\gamma)^2$. Since the error probability γ could be made a positive constant as close to zero as desired at the expense of increasing by a constant factor the depth of RCT [19], take γ such that $\eta>1-c$. Then lemma 2.3 entails that $t\geq\Omega(n\log m)$, which proves theorem 2.

3 Deterministic computation trees

Treating a deterministic computation tree (CT) as a particular case of RCT one can release the restriction on s ubarrangements in theorem 1 and obtain the following result.

Corollary 1.3 *If a CT (over a zero characteristic field) recognizes an arrangement with N faces (of all the dimensions) then its depth exceeds $\Omega(\log N)$.*

For CT over reals in a similar way one can release the restriction on the number of faces in theorem 2.

Corollary 2.3 *If a CT (over reals) recognizes either an arrangement or a polyhedron $\mathcal{S}$ with N faces (of all the d imensions) then its depth exceeds $\Omega(\log N)$.*

In case of an arrangement one could deduce corollary 2.3 from [2], in case of a polyhedron the corolla ry strengthens the result from [15].

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