

## FAST PARALLEL ALGORITHMS FOR SPARSE MULTIVARIATE POLYNOMIAL INTERPOLATION OVER FINITE FIELDS\*

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**Abstract.** The authors consider the problem of reconstructing (i.e., interpolating) a  $t$ -sparse multivariate polynomial given a *black box* which will produce the value of the polynomial for any value of the arguments. It is shown that, if the polynomial has coefficients in a finite field  $GF[q]$  and the black box can evaluate the polynomial in the field  $GF[q^{\lceil 2\log_q(nt)+3 \rceil}]$ , where  $n$  is the number of variables, then there is an algorithm to interpolate the polynomial in  $O(\log^3(nt))$  boolean parallel time and  $O(n^2 t^6 \log^2 nt)$  processors.

This algorithm yields the first efficient deterministic polynomial time algorithm (and moreover boolean NC-algorithm) for interpolating  $t$ -sparse polynomials over finite fields and should be contrasted with the fact that efficient interpolation using a black box that only evaluates the polynomial at points in  $GF[q]$  is not possible (cf. [M. Clausen, A. Dress, J. Grabmeier, and M. Karpinski, *Theoret. Comput. Sci.*, 1990, to appear]). This algorithm, together with the efficient deterministic interpolation algorithms for fields of characteristic 0 (cf. [D. Yu. Grigoriev and M. Karpinski, in *Proceedings of the 28th IEEE Symposium on the Foundations of Computer Science*, 1987, pp. 166–172], [M. Ben-Or and P. Tiwari, in *Proceedings of the 20th ACM Symposium on the Theory of Computing*, 1988, pp. 301–309]), yields for the first time the general deterministic sparse conversion algorithm working over arbitrary fields. (The reason for this is that every field of positive characteristic contains a primitive subfield of this characteristic, and so this method can be applied to the slight extension of this subfield.) The method of solution involves the polynomial enumeration techniques of [D. Yu. Grigoriev and M. Karpinski, *op. cit.*] combined with introducing a new general method of solving the problem of determining if a  $t$ -sparse polynomial is identical to zero by evaluating it in a *slight* extension of the coefficient field (i.e., an extension whose degree over this field is logarithmic in  $nt$ ).

**Key words.** sparse multivariate polynomials, finite fields, interpolation

**AMS(MOS) subject classifications.** 68C25, 12C05

**1. Introduction.** The polynomial interpolation algorithms play an important role in the design of efficient algorithms in algebra and their applications (cf. [G83], [G84], [K85], [BT88]). For the case of finite fields there were no deterministic polynomial time algorithms known (cf. [BT88]) for the sparse interpolation problem. The existing methods required large extension fields of order  $GF[q^n]$ ; so, for example, no effective procedures for finding primitive elements over an actual interpolation field were known without using randomization.

Here we remedy the situation by considering what we call a “slight” extension of fields, which is an extension whose degree over the coefficient field is logarithmic in  $nt$ ,  $GF[q^{\lceil c \log_q(nt) \rceil}]$ . The method of solution involves two major steps: (1) solving the zero identity problem of polynomials from  $GF[q]$  by evaluating in a slight extension  $GF[q^{\lceil 2\log_q(nt)+3 \rceil}]$ , and (2) using inductive enumeration of partial solutions for terms and coefficients over  $GF[q]$  by means of recursion on (1). We develop a general method involving Cauchy matrices to solve the zero-identity problem in Step 1, and combine this with the refined polynomial enumeration techniques of Grigoriev and Karpinski [GK87] to solve Step 2.

Because of the lower bound of  $\Omega(n^{\log t})$  (cf. [CDGK88]) for the interpolation over the same field  $GF[q]$  without an extension, our *slight* field extension is in a sense the smallest extension capable of carrying out the efficient interpolation.

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In what follows we shall use the basic notions of the theory of finite fields (cf. [LN86], [MS77]) and algorithms for computing in finite fields (cf. [L82]), and the basic models of parallel computation (cf. [C85], [G82]).

**2. Interpolation problem over finite fields.** We consider the problem of interpolation for multivariate polynomials given by *black boxes* (special cases of it are the explicit interpolations of polynomials given by straight-line programs (cf. [K85]), or polynomials given by determinants (cf. [L79], [GK87])). In this setting we are given a polynomial  $f$  in  $GF[q]$  as a black box that allows us to evaluate  $f$  in extensions of  $GF[q]$  and information about its sparsity  $t$  (the bound on the number of its nonzero coefficients). Given this, we must determine an extension  $GF[q^s]$  of  $GF[q]$ ,  $s$  as small as possible, and an efficient polynomial time interpolation algorithm working over  $GF[q^s]$  to determine all coefficients of  $f$  in  $GF[q]$ .

We say that the *black box* interpolation problem (over a finite field extension  $GF[q^s]$ ) is in  $NC^k$  (cf. [C85]), if there exists a class of uniform  $(ntq)^{O(1)}$ -size and  $O(\log^k(ntq))$ -depth boolean circuits with oracle nodes  $S$  (returning values of a black box over the field extension  $GF[q^s]$ ) computing for an arbitrary  $n$ -variate polynomial  $f \in GF[q][x_1, \dots, x_n]$  all the nonzero coefficients and monomial vectors of  $f$ , with the oracle  $S_f^s$ , defined by  $S_f^s(x_1, \dots, x_n, y)$  if and only if  $f(x_1, \dots, x_n) = y$  over  $GF[q^s]$ . If the *lifting* of a *black box* (given explicitly by a straight-line program, determinant, boolean circuit, etc.) from  $GF[q]$  to the extension  $GF[q^s]$ , and the computation of  $f(x_1, \dots, x_n)$  over  $GF[q^s]$  by a black box, are both in boolean  $NC$  (in  $P$ ), then the explicit interpolation problem lies also in boolean  $NC$  (in  $P$ ).

We note that the interpolation problem over finite fields deals not only with the interpolation of polynomials but with arbitrary functions in their  $t$ -sparse ring sum expansion representation (RSE) ([W87]).

We shall develop an interpolation algorithm (for polynomials over  $GF[q]$ ) for the *slight* extension of a field of order  $s = \lceil 2 \log(nt) + 3 \rceil$ . This allows us for the first time to efficiently find the generators in  $GF[q^s]$ , as the size of this field is polynomial in the size of the input polynomial under interpolation. Our *slight* field extension is in a sense the best possible, as the efficient interpolation over the same field (i.e., for  $s = 1$ ) is not possible. In [CDGK88] the tight lower and upper bounds  $\Theta(n^{\log t})$  have been established for the number of steps needed to determine identity to zero of polynomials  $f \in GF[2][x_1, \dots, x_n]$ .

**3. The algorithm.** We now formulate the Interpolation Theorem and the underlying Interpolation Algorithm over Finite Fields.

THE INTERPOLATION THEOREM. *Given any  $t$ -sparse polynomial  $f \in GF[q][x_1, \dots, x_n]$ . For an arbitrary  $q$ , there exists a deterministic parallel algorithm ( $NC^3$ ) for interpolating  $f$  over a slight field extension  $GF[q^{\lceil 2 \log_q(nt) + 3 \rceil}]$  working in  $O(\log^3(ntq))$  parallel boolean time and  $O(n^2 t^6 \log^2(ntq) + q^{2.5} \log^2 q)$  processors. For a fixed field the algorithm works in  $O(\log^3(nt))$  parallel boolean time and  $O(n^2 t^6 \log^2 nt)$  processors.*

SPARSE INTERPOLATION ALGORITHM OVER FINITE FIELDS

**Input:** A black-box oracle allowing one to evaluate a  $t$ -sparse polynomial  $f \in GF[q^s][x_1, \dots, x_n]$  for  $s = 1, \dots$ . (A  $t$ -sparse polynomial is a polynomial with at most  $t$  nonzero coefficients.)

**Output:** All  $(\mathbf{k}, f_{\mathbf{k}})$  such that  $f = \sum f_{\mathbf{k}} x^{\mathbf{k}}$  where  $0 \neq f_{\mathbf{k}} \in GF[q]$  and  $x^{\mathbf{k}} = x_1^{k_1} \dots x_n^{k_n}$ .

We begin by first describing a Subalgorithm.

SUBALGORITHM (IDENTITY-TO-ZERO TEST):

**Input:** Same as above.

**Output:** Yes, if  $f \equiv 0$ ; No, if  $f \neq 0$ .

**Step 1:** Choose  $s$  so that  $q^s - 1 > 4nq(n-1)\binom{2}{2}$ . So let  $s = \lceil 2 \log_q (nt) + 3 \rceil$ .

**Step 2:** Construct the field  $GF[q^s]$  by looking over all polynomials of degree  $s$  with coefficients in  $GF[q]$  and testing irreducibility with the help of the Berlekamp algorithm [B70]. We find an irreducible  $\phi \in GF[q][z]$ , and then  $GF[q^s]$  is isomorphic to  $GF[q][z]/(\phi)$ . We find an  $\omega$  that is generator of the cyclic group  $GF[q^s]^*$  in the following way. Factor  $q^s - 1 = \prod p_i^{n_i}$ ,  $p_i$  prime. For any  $a \in GF[q^s]$ , calculate  $a^{(q^s-1)/p_i}$  for each  $i$ . We do this using the binary expansion of the exponent and by techniques from [L82]. An element is a generator of the cyclic group if and only if all these powers are distinct from 1.

**Step 3:** Denote  $N = \lceil (q^s - 1) / 4nq \rceil$ . Use the sieve of Eratosthenes to find a prime  $p$  with  $2N < p \leq 4N$ . Such a prime exists by Bertrand's postulate (cf. [HW78]).

**Step 4:** Now construct an  $N \times N$  Cauchy matrix  $C$  (cf. [C], [PS64], [MS77]) over the field  $GF[p]$ ,  $y_i = x_i = i$ ,  $1 \leq i \leq N$  by  $C = [1/(x_i + y_j)] = [1/(i + j)]$ . We have

$$\det C = \frac{\prod_{1 \leq i < j \leq n} (x_j - x_i)(y_j - y_i)}{\prod_{1 \leq i, j \leq n} (x_i + y_j)}.$$

For any of its minors  $\neq 0$ , a similar formula holds. Therefore any minor of any size is nonsingular. Compute, using the Euclidean algorithm  $c_{ij} \in \mathbb{Z}$ , such that  $c_{ij} \equiv 1/(i + j) \pmod{p}$  and  $0 \leq c_{ij} < p \leq 4N$ .

**Step 5:** Denote by  $\bar{C} = [\bar{c}_{ij}]$  an arbitrary submatrix of  $C$  of size  $N \times n$ .

**Step 6:** Pick out in parallel any row  $\bar{c}_i = (\bar{c}_{ij})$ ,  $1 \leq j \leq n$ , of the matrix  $\bar{C}$  and, for each  $l$ ,  $0 \leq l < t$ , plug  $\omega l \bar{c}_{ij}$  for each  $x_j$  in the black-box (with  $s = \lceil 2 \log_q (nt) + 3 \rceil$ ) for the polynomial  $f = \sum f_{\mathbf{k}} x^{\mathbf{k}} = \sum f_{\mathbf{k}} x_1^{k_1} \cdots x_n^{k_n}$ , where  $\mathbf{k} = (k_1, \dots, k_n)$  and the number of  $\mathbf{k}$ 's is less than  $t$ ,  $0 \leq k_j < q - 1$ ,  $f_{\mathbf{k}} \in GF[q]$ .

We now pause to justify that if  $f \neq 0$ , then for some  $\bar{c}_i l$  as above  $f(\omega^{l \bar{c}_i}) \neq 0$ , where  $\omega l \bar{c}_{ij}$  has been substituted for  $x_j$ . We first show that for a suitable vector  $\bar{c}_i$ ,  $1 \leq i \leq N$ , after substituting  $\omega^{l \bar{c}_{ij}}$  for  $x_j$ , any two monomials  $x^{\mathbf{k}}$ ,  $x^{\mathbf{k}'}$  would give different elements of  $GF[q]$ . Suppose that for some pair  $\mathbf{k}$ ,  $\mathbf{k}'$  and  $\bar{c}_i$  we have  $\omega^{l \bar{c}_i \cdot \mathbf{k}} = \omega^{l \bar{c}_i \cdot \mathbf{k}'}$ . This means that  $\sum k_j \bar{c}_{ij} \equiv \sum k'_j \bar{c}_{ij} \pmod{q^s - 1}$  and so  $\sum (k_j - k'_j) \bar{c}_{ij} \equiv 0 \pmod{q^s - 1}$ . Since  $|k_j - k'_j| \leq q - 1$ ,  $\bar{c}_{ij} < 4N$ , we have  $|\sum_{1 \leq j \leq n} (k_j - k'_j) \bar{c}_{ij}| < (q - 1)n4N < (q^s - 1)$ ; therefore  $\sum (k_j - k'_j) \bar{c}_{ij} = 0$ . For any pair of monomials  $x^{\mathbf{k}}$ ,  $x^{\mathbf{k}'}$ , we consider all the "bad" vectors  $\bar{c}_i$ ,  $1 \leq i \leq N$ , i.e., those  $\bar{c}_i$  for which  $\sum_{1 \leq j \leq n} (k_j - k'_j) \bar{c}_{ij} = 0$ . There cannot be more than  $(n - 1)$  "bad" vectors for this pair, since if there exist such  $n$  vectors  $\bar{c}_{i_1}, \dots, \bar{c}_{i_n}$ , the corresponding  $n \times n$  submatrix of  $\bar{C}$  would have determinant zero. As there are at most  $\binom{2}{2}$  pairs of monomials, there is a vector  $\bar{c}_{i_0}$ ,  $1 \leq i_0 \leq N$ , that is not "bad" for any pair of monomials  $\mathbf{k}$ ,  $\mathbf{k}'$ , since  $\binom{2}{2}(n - 1) < N$ .

Let  $\bar{c}_{i_0}$  be some vector such that distinct monomials  $x^{\mathbf{k}}$ ,  $x^{\mathbf{k}'}$  yield distinct elements of  $GF[q^s]$  after substituting  $\omega^{l \bar{c}_{i_0}}$ . We now show that  $f(\omega l \bar{c}_{i_0}) \neq 0$  for some  $0 \leq l < t$ . If  $f(\omega l \bar{c}_{i_0}) = 0$  for all  $l$ ,  $0 \leq l < t$ , then  $XV = 0$ , where  $X = (f_{\mathbf{k}})_{\mathbf{k}}$  and  $V = (\omega l \bar{c}_{i_0} \cdot \mathbf{k})$  is the  $t \times t$  matrix whose rows are indexed by  $l$ ,  $0 \leq l < t$ , and columns are indexed by the  $\mathbf{k}$  that appears as an exponent in  $f$ .

Note that  $\det(V)^2 = \prod_{\mathbf{k} \neq \mathbf{k}'} (\omega^{\sum \bar{c}_{i_0}^{k_j} j} - \omega^{\sum \bar{c}_{i_0}^{k'_j} j}) \neq 0$  (it is a Vandermonde matrix), so we have a contradiction. Therefore the identity-to-zero subalgorithm is correct.

We now continue with the main algorithm. Assume  $n = 2^m$  for simplicity of notation. Define  $S_{\alpha, \beta} = \{(k_1, \dots, k_{2^{\alpha-1}}): x_{\beta 2^{\alpha-1} + 1}^{k_1} \cdots x_{\beta 2^{\alpha-1} + 2^{\alpha-1}}^{k_{2^{\alpha-1}}}$  occurs as a subterm in some nonzero term of  $f\}$ , where  $1 \leq \alpha \leq m + 1$  and  $0 \leq \beta < 2^{m+1-\alpha}$ . We produce  $S_{\alpha, \beta}$  recursively for  $\alpha = 1, \dots, m + 1$ .

**Basis Step:** Let  $\alpha = 1$ . Let  $\{a_1, a_2, \dots\}$  be an enumeration of  $GF[q]$ . In parallel for each  $a \in GF[q]$ , substitute  $a$  for  $x_{\beta+1}$  in  $f$ . Find a vector  $u_l \in (GF[q])^q$  such that  $u_l \cdot (a_i^l) = (0, \dots, 1, \dots, 0)$  where all entries of this latter vector are 0 except for a 1 in the  $l$ th place. We then have  $u_l \cdot (f(x_1, \dots, x_\beta, a_1, x_{\beta+2}, \dots, x_n), \dots, f(x_1, \dots, x_\beta, a_q, x_{\beta+2}, \dots, x_n)) = P_l$  where  $f = \sum_l x_{\beta+1}^l P_l$  and  $P_l \in GF[q][x_1, \dots, x_\beta, x_{\beta+2}, \dots, x_n]$ . We see that  $P_l$  may be evaluated at any point  $(b_1, \dots, b_{\beta-1}, b_{\beta+1}, \dots, b_n)$  by evaluating  $f$  at the  $q$  points  $(b_1, \dots, b_{\beta-1}, a_i, b_{\beta+1}, \dots, b_n)$ ,  $i = 1, \dots, q$  and using this last formula, where  $u_l$  has been found by inverting the matrix  $(a_i^l)$  and extracting the  $l$ th row. This gives a black box for  $P_l$ . The identity-to-zero subalgorithm now allows us to determine which  $P_l$ 's are not identically zero, and so to determine  $S_{1,\beta}$ .

**Recursion Step:** Assume that we have produced  $S_{\alpha,\beta}$  for all  $\beta$ ,  $0 \leq \beta < 2^{m+1-\alpha}$ . We now produce  $S_{\alpha+1,\beta}$  for fixed  $\beta$ ,  $0 \leq \beta < 2^{m-\alpha}$ . For each element from the set  $S_{\alpha,2\beta}$  and for each element from the set  $S_{\alpha,2\beta+1}$ , consider the corresponding product  $x_{\beta 2^\alpha+1}^k, \dots, x_{\beta 2^\alpha+2^\alpha}^k$ . For all such products (observe that the number of them is at most  $t^2$ , since  $|S_{\alpha,2\beta}|, |S_{\alpha,2\beta+1}| \leq t$ ), we can find (in parallel) a vector  $v \in \mathbb{N}^{2^\alpha}$  as in Step 6 such that  $v = (v_1, \dots, v_{2^\alpha})$ ,  $0 \leq v_i < 4N_1$ , where  $s_1$  is chosen such that  $(\lceil q^{s_1} - 1 \rceil) / 4nq = N_1 > (n-1) \binom{t^2}{2}$  and for any two products  $x_{\beta 2^\alpha+1}^{k_1} \dots x_{\beta 2^\alpha+2^\alpha}^{k_2}$  and  $x_{\beta 2^\alpha+1}^{k'_1} \dots x_{\beta 2^\alpha+2^\alpha}^{k'_2}$ ,  $q^{s_1} - 1 \nmid (\sum k_i v_i - \sum k'_i v_i)$ . Let  $\omega_1 \in GF[q^{s_1}]$  be a generator of the cyclic group  $GF[q^{s_1}]^*$ . For any  $0 \leq l < t^2$ , we replace  $x_{\beta 2^\alpha+j}$  with  $\omega_1^{v_j l}$ . Consider the  $t^2 \times t^2$  matrix  $B = (\omega_1^{(\sum_j k_j v_j) l}) = (b_k, l)$ . Note that  $\det(B)^2 = \prod_{k \neq k'} (\omega_1^{(\sum_j k_j v_j)} - \omega_1^{(\sum_j k'_j v_j)}) \neq 0$ , since  $q^{s_1} - 1 \nmid (\sum_j k_j v_j - \sum_j k'_j v_j)$ . Calculate vectors  $u_j \in (GF[q^{s_1}])^{t^2}$  such that  $u_j B = (0, \dots, 0, 1, 0, \dots, 0)$  where this latter vector has 1 in the  $i$ th position and zeros everywhere else. We then have  $u_i \cdot Y = \bar{P}_i$  where  $f = \sum_k x^k \bar{P}_k$ , where  $x^k = x_{\beta 2^\alpha+1}^{k_1} \dots x_{\beta 2^\alpha+2^\alpha}^{k_2}$  and  $\bar{P}_k \in GF[q][x_1, \dots, x_{\beta 2^\alpha}, x_{(\beta+1)2^\alpha+1}, \dots, x_n]$ , and  $Y$  is the  $1 \times t^2$  vector whose  $l$ th entry is  $f(x_1, \dots, x_{\beta 2^\alpha}, \omega_1^{v_1 l}, \dots, \omega_1^{v_{2^\alpha} l}, x_{(\beta+1)2^\alpha+1}, \dots, x_n)$ . Using this last formula with black box evaluations of  $f$  gives us the new black boxes for the  $\bar{P}_i$  as before. The identity-to-zero subalgorithm now allows us to determine which  $\bar{P}_i$  are not identically zero and thus to determine  $S_{\alpha+1,\beta}$ . Notice that when  $\alpha = m+1$  we have determined all the terms of  $f$  in the form of  $(k, f_k)$  such that  $f = \sum_k f_k x^k$ ,  $0 \neq f_k \in F[q]$  and  $x^k = x_1^{k_1}, \dots, x_n^{k_n}$ .  $\square$

**4. Analysis of the Algorithm.** Let  $N = (\lceil q^{s-1} \rceil / 4nq)$ . Note that  $N < nt^2q$ . The parallel time of our algorithm is  $O(\log^3 N)$ . This is because the identity-to-zero test takes  $O(\log^2 N)$  parallel time, the recursive step calls this test and uses matrix inversion, which requires  $O(\log^2 N)$  parallel time [M86], and the recursion depth is  $O(\log n)$ . Steps 1-5 take  $O(N \log^2(Nnq))$  processors. Step 6 takes  $O(Nnt \log^2(Nnq))$  processors. Therefore the total cost (in processors) of the identity-to-zero subalgorithm is  $O(Nnt \log^2(Nnq))$ .

We now proceed to analyze the complexity of the rest of the algorithm. In the basic step, we must invert the  $q \times q$  matrix  $(a_i^l)$  over  $GF[q]$ . This requires  $O(q^{2.5} \log^2 q)$  processors by [M86]. In applying Steps 1-6 to test whether  $P_l$  is identically zero, we refer  $q$  times to substituting  $\omega^{c_{ij}}$  in a black box and calling the identity-to-zero test. Thus we need  $Nntq \log^2 Nnq$  processors. In the recursion step, we calculate  $N_1 t^2$  sums  $\sum_j k_j v_j$  of length  $n$  and compute  $\omega_1^{\sum_j k_j v_j}$  in the field  $GF[q^{s_1}]$ . This takes  $N_1 t^2 n \log^2 N_1$  processors. Notice that  $N_1 < nt^4q$ . Inverting the  $t^2 \times t^2$  matrix  $B$  over  $GF[q^{s_1}]$  requires  $t^5 \log^2 N_1$  processors [M86]. Therefore the total number of processors would be  $O(t^6 n^2 q \log^2(tnq) + q^{2.5} \log^2 q)$ . For a fixed field, the algorithm works in  $O(\log^3 nt)$  time and  $O(n^2 t^6 \log^2 nt)$  processors.

**5. Further research.** Our parallel algorithm enjoys very good parallel time bound. Concerning the number of processors, would it be possible to improve on the number of processors of the interpolation algorithm?

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