

Tropical Newton-Puiseux polynomials II

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Abstract

Tropical Newton-Puiseux polynomials defined as piece-wise linear functions with rational coefficients at the variables, play a role of tropical algebraic functions. We provide explicit formulas for tropical Newton-Puiseux polynomials being the tropical zeroes of a univariate tropical polynomial with parametric coefficients.

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Introduction

One can find basic concepts of tropical algebra in [2].

Given a tropical univariate polynomial

$$f = \min_{0 \leq k \leq n} \{x_k + kY\} \quad (1)$$

its tropical zero $y \in \mathbb{R}$ is such that the minimum in (1) is attained for at least two different values of $0 \leq k \leq n$. In this paper we treat the coefficients $X = (x_0, \dots, x_n)$ as parameters and find the zeroes of f as a function in x_0, \dots, x_n .

We show that f has exactly n such parametric zeroes g_1, \dots, g_n . Each g_k , $1 \leq k \leq n$ is a *tropical Newton-Puiseux polynomial*, i. e. a piece-wise linear function with rational coefficients at the variables. One can represent any tropical Newton-Puiseux polynomial in the form

$$\min_I \{a_I + (I, X)\} - \min_J \{b_J + (J, X)\}$$

of a difference (so, the tropical quotient) of two concave piece-wise linear functions, where $I, J \in \mathbb{Q}^{n+1}$; $a_I, b_J \in \mathbb{R}$, and (I, X) denotes the inner product (cf. [3]).

Tropical Newton-Puiseux polynomials play a role of algebraic functions in the tropical setting. Similar to Newton-Puiseux series in the classical algebra, $g_k(x_0, \dots, x_n)$ provides a tropical zero of f for any point $(x_0, \dots, x_n) \in \mathbb{R}^{n+1}$. We note that in the classical algebra one considers Newton-Puiseux series just in a single variable, while an advantage of the tropical algebra is that one considers tropical Newton-Puiseux polynomials in several variables.

Observe that if one considers a univariate tropical polynomial with its coefficients being tropical Newton-Puiseux polynomials then in its turn, the tropical zeroes of this tropical polynomial are again tropical Newton-Puiseux polynomials. Thus, one can view the semi-field of tropical Newton-Puiseux polynomials as a tropical algebraic closure of the semi-ring of tropical polynomials.

1 Tropical Newton-Puiseux polynomials as tropical zeroes

We say that a tropical Newton-Puiseux polynomial $g := g(X_0, \dots, X_n)$ is a (*tropical*) zero of f (1) if for any $(x_0, \dots, x_n) \in \mathbb{R}^{n+1}$ the value $y = g(x_0, \dots, x_n)$ is a tropical zero of the tropical polynomial f .

First, we describe the tropical Newton-Puiseux zeroes of f geometrically and show that there are exactly n of them. In the next section we provide for them the explicit formulas.

For a point $x := (x_0, \dots, x_n) \in \mathbb{R}^{n+1}$ its Newton polygon $N_x \subset \mathbb{R}^2$ is the convex hull of the vertical rays $\{(k, c) : c \geq x_k\}$, $0 \leq k \leq n$. Note that the slopes of the edges of N_x are just the tropical zeroes of f .

For a subset $S \subset \{1, \dots, n-1\}$ consider a convex polyhedron $P_S \subset \mathbb{R}^{n+1}$ (of dimension $n+1$) consisting of points $x = (x_0, \dots, x_n)$ such that its Newton polygon N_x has the vertices $(0, x_0), (n, x_n), \{(s, x_s) : s \in S\}$. Thus, $\{P_S : S \subset \{1, \dots, n-1\}\}$ constitute a partition of \mathbb{R}^{n+1} into 2^{n-1} polyhedra.

Take the (open) polyhedron $P := P_{\{1, \dots, n-1\}}$ consisting of points x such that the Newton polygon N_x has $n+1$ vertices. Then there are exactly n continuous piece-wise linear functions g_1, \dots, g_n on P being tropical zeroes of f (1). Namely, $g_k(x_0, \dots, x_n) = x_{k-1} - x_k$, $1 \leq k \leq n$.

Observe that each g_k , $1 \leq k \leq n$ has a unique (continuous) continuation on every polyhedron P_S . Namely, take the unique pair $0 \leq i \leq k-1$, $k \leq j \leq n$ such that $i, j \in S \cup \{0, n\}$, and there are no $s \in S \cup \{0, n\}$ satisfying inequalities $i < s < j$.

Lemma 1.1 *The unique continuation of g_k on P_S coincides with $\frac{x_i - x_j}{j - i}$.*

Proof. For any point (x_0, \dots, x_n) which belongs to both boundaries of P and of P_S holds $x_s - x_{s+1} = x_{k-1} - x_k$, $i \leq s < j$, hence $x_{k-1} - x_k = \frac{x_i - x_j}{j-i}$. \square

Note that $\frac{x_i - x_j}{j-i}$ is the slope of the edge with the end-points (i, x_i) , (j, x_j) . Thus, we have shown that there are exactly n tropical Newton-Puiseux polynomials on \mathbb{R}^{n+1} being tropical zeroes of f (1).

2 Explicit formulas for tropical zeroes

Theorem 2.1 *A tropical polynomial $f = \min_{0 \leq k \leq n} \{x_k + kY\}$ with parametric coefficients (x_0, \dots, x_n) has exactly n tropical zeroes g_1, \dots, g_n being tropical Newton-Puiseux polynomials in (x_0, \dots, x_n) . For each $0 \leq k \leq n$ one can represent g_k as follows. For every $0 \leq p < k$ consider a tropical Newton-Puiseux polynomial*

$$t_p := \max_{k \leq q \leq n} \left\{ \frac{x_p - x_q}{q - p} \right\}.$$

Then $g_k = \min_{0 \leq p < k} \{t_p\}$.

Proof. Fix for the time being a polyhedron P_S and follow the notations from Lemma 1.1. For any point $x := (x_0, \dots, x_n) \in P_S$ its Newton polygon N_x has an edge with the end-points (i, x_i) , (j, x_j) . Therefore, for every $0 \leq p < k$ the following inequality for the slopes holds:

$$\frac{x_p - x_j}{j - p} \geq \frac{x_i - x_j}{j - i}.$$

Hence $t_p \geq \frac{x_i - x_j}{j - i}$.

On the other hand, $t_i = \frac{x_i - x_j}{j - i}$ since for every $k \leq q \leq n$ the following inequality for the slopes holds:

$$\frac{x_i - x_q}{q - i} \leq \frac{x_i - x_j}{j - i}.$$

Thus, g_k coincides with $\frac{x_i - x_j}{j - i}$ on P_S which completes the proof due to Lemma 1.1. \square

Remark 2.2 *In the formula for g_k in the theorem a tropical Newton-Puiseux polynomial t_p is involved in which a left-end point (p, x_p) of the intervals is fixed. In a dual way one can define $r_q := \min_{0 \leq p < k} \left\{ \frac{x_p - x_q}{q - p} \right\}$ by fixing a right end-point (q, x_q) . Then, similarly to the theorem we get $g_k = \max_{k \leq q \leq n} \{r_q\}$.*

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