On a tropical dual Nullstellensatz

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Abstract

Since a tropical Nullstellensatz fails even for tropical univariate polynomials we study a conjecture on a tropical dual Nullstellensatz for tropical polynomial systems in terms of solvability of a tropical linear system with the Cayley matrix associated to the tropical polynomial system. The conjecture on a tropical effective dual Nullstellensatz is proved for tropical univariate polynomials.

Keywords: dual Nullstellensatz, solving tropical polynomial systems

Introduction

Let $T$ be a tropical semi-ring with operations $\oplus$, $\otimes$ (see e.g. [2], [3], [6], [12]). Typically $\oplus = \min$, $\otimes = +$. Examples of $T$ are $\mathbb{Z}$ and $\mathbb{Z}_\infty = \mathbb{Z} \cup \{\infty\}$. A tropical monomial has a form $Q = a \otimes X_1^{i_1} \otimes \cdots \otimes X_n^{i_n}$, $a \in T$. The tropical degree $\text{trdeg}(Q) := i_1 + \cdots + i_n$.

From the point of view of the classical algebra a tropical monomial is a linear function. A point $x = (x_1, \ldots, x_n) \in T^n$ (with some of $x_i \neq \infty$) is a tropical zero of a tropical polynomial $f = \bigoplus_i Q_i$ if the minimum $\min_i \{Q_i(x)\}$ is attained for at least two different tropical monomials $Q_i$.

We study the issue of a tropical Nullstellensatz. Its direct formulation fails even for tropical univariate polynomials: for example, two tropical polynomials $X \oplus 0$, $X \oplus 1$ have no common tropical zero, while the generated by them tropical ideal does not contain 1 or any other tropical monomial. That is why we consider a tropical "dual" Nullstellensatz.

One can treat the (customary) Hilbert’s Nullstellensatz as a reduction of solvability of a polynomial system to solvability of a suitable linear system. Namely, solvability of a polynomial system is equivalent to that the Cayley matrix $C$ associated to the system does not contain the vector $(1,0,\ldots,0)$ in the linear hull of its rows. In its turn it is equivalent to that the linear system $C \cdot (a_0, a_1, \ldots) = 0$ has a solution with $a_0 \neq 0$ (cf. Section 1).

The latter rephrasing of the Nullstellensatz we call the "dual" Nullstellensatz. It holds also for the infinite matrix $C$ (we call it the infinite "dual" Nullstellensatz) unlike the
customary Nullstellensatz, and it holds for a finite submatrix of $C$ with the size depending on $n$ and on the degrees of the polynomials in the system (we call it the effective ”dual” Nullstellensatz).

In Section 2 we formulate the conjecture on a tropical ”dual” Nullstellensatz. In Section 3 we give a rephrasing of the conjecture in terms of the combinatorial convex geometry. Finally, in Section 4 we prove the tropical effective ”dual” Nullstellensatz for univariate polynomials ($n = 1$).

Observe that the latter result in case of a system of two tropical polynomials $f_1, f_2$ follows from the approach of [11] which relies on the theorem due to Kapranov (see e. g. [11], [12]) applied to the (classical) resultant of a pair of (classical) polynomials. Since the theorem of Kapranov holds just for principal ideals, the resultant based approach fails for overdetermined systems in the tropical setting. We mention also that the problem of solvability of tropical polynomial systems is $NP$-complete even for tropical quadratic polynomials [12].

Solvability of tropical linear systems belongs to the complexity class $NP \cap \text{co-NP}$ [1], [5]. In [1], [5] two different algorithms for solving tropical linear systems were designed with the similar complexity bounds polynomial in $s, M$, where $s$ is the size of the tropical linear system (so, of its matrix) and $M$ majorates the absolute values of the finite (integer) coefficients of the system. We note that the algorithm from [5] possesses an extra feature that it has also a complexity bound polynomial in $\exp(s), \log M$. The open question is whether it runs in fact, within complexity polynomial in $s, \log M$ (which would provide a polynomial complexity for the problem of solvability of tropical linear systems)?

In addition, the algorithm from [5] entails as a by-product the equivalence of solvability of a tropical linear system with the degeneration of its tropical rank and simultaneously with the degeneration of its Kapranov rank. The latter for systems with finite coefficients (say, from $\mathbb{Z}$) was shown in [3], also a part of this equivalence just for the tropical rank follows from [8].

Besides, we mention that in [7] the tropical (customary) Nullstellensatz was established for an introduced there a ”ghost” tropical semi-ring. In [10] the radical of a tropical ideal was explicitly described.

1 ”Dual” Nullstellensatz

Let $F_1, \ldots, F_s \in K[X_1, \ldots, X_n]$ be polynomials over an algebraically closed field $K$. Denote by $C := C(F_1, \ldots, F_s)$ the (infinite size) Cayley matrix over $K$ consisting of the coefficients of $F_1, \ldots, F_s$. The columns of $C$ correspond to all the monomials $X^I := X_1^{i_1} \cdots X_n^{i_n}$, $I = (i_1, \ldots, i_n)$, and the rows of $C$ correspond to all the polynomials of the form $X^I \cdot F_j, 1 \leq j \leq s$. Let the first column of $C$ correspond to the monomial $X^0 = 1$. For an integer $N$ denote by $C_N$ the (finite size) submatrix of $C$ formed by the rows $X^I \cdot F_j, 1 \leq j \leq s$ with the degrees $\deg X^I = i_1 + \cdots + i_n \leq N$ and the corresponding columns which contain a non-zero entry in at least one of these rows.
Nullstellensatz states that a polynomial system

\[ F_1 = \cdots = F_s = 0 \]  

(1)

has a solution in \( K^n \) iff for any \( N \) the linear hull of the rows of \( C_N \) does not contain the vector \((1,0,\ldots,0)\). An effective Nullstellensatz provides an upper bound on \( N \) for which the latter equivalence holds. The bound \( N < (\max_j \{ \deg(F_j) \})^{O(n)} \) close to optimal was obtained in [4], [9].

Thus, the effective Nullstellensatz is equivalent to the following. System (1) has a solution iff the linear system \( C_N \cdot (y_1, y_2, \ldots) = 0 \) has a solution with \( y_1 \neq 0 \) for an appropriate \( N \) depending on \( n \) and on \( \max_j \{ \deg(F_j) \} \). We call the latter statement the effective dual Nullstellensatz. The equivalence that (1) has a solution iff the linear system \( C_N \cdot (y_1, y_2, \ldots) = 0 \) has a solution with \( y_1 \neq 0 \) for any \( N \), we call the dual Nullstellensatz. Finally, the statement (also equivalent to Nullstellensatz) that (1) has a solution iff the infinite linear system \( C \cdot (y_1, y_2, \ldots) = 0 \) has a solution with \( y_1 \neq 0 \), we call the infinite dual Nullstellensatz. The latter infinite linear system makes sense because each row of \( C \) contains just a finite number of non-zero entries.

2 Conjecture on a tropical dual Nullstellensatz

Below we assume that the tropical semi-ring \( T = \mathbb{R}_\infty := \mathbb{R} \cup \{ \infty \} \), but for the sake of simplifying the exposition we study tropical zeroes defined over \( \mathbb{R} \) (although, one could also consider zeroes defined over \( \mathbb{R}_\infty \)). For each monomial \( Q_l = a_l \otimes X_1^{i_1,l} \otimes \cdots \otimes X_n^{i_n,l} \) of a tropical polynomial \( f = \bigoplus_l Q_l \) we plot the point \((i_1,l, \ldots, i_n,l, a_l) \in \mathbb{Z}^n \times \mathbb{R} \subset \mathbb{R}^{n+1} \). Then a point \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) is a tropical zero of \( f \) iff the linear function \((i_1, \ldots, i_n, a) \rightarrow a + i_1 \cdot x_1 + \cdots + i_n \cdot x_n \) attains its minimum at the plotted points at least twice.

Therefore, without changing the set of tropical zeroes of \( f \) one can replace the plotted points by their convex hull. Moreover, w.l.o.g. for any point \((b_1, \ldots, b_n, a) \in \mathbb{R}^{n+1} \) from this convex hull one can add the ray \( \{(b_1, \ldots, b_n, b) : b \geq a \} \). The resulting convex set \( P(f) \subset \mathbb{R}^{n+1} \) we call the (extended) Newton polyhedron of \( f \). Thus, w.l.o.g. one can modify \( f \) replacing it by a tropical polynomial whose plotted points are just the points of the form \((i_1, \ldots, i_n, a) \in (\mathbb{Z}^n \times \mathbb{R}) \cap P(f) \) with the minimal possible \( a \). Finally, so modified tropical polynomial has the same set of tropical zeroes as \( f \), and (in abuse of notations) we keep for it the same notation. We say that the modified tropical polynomial is in the convex form, and from now on we consider tropical polynomials only in the convex form. Observe that \( x \) is a tropical zero of \( f \) iff for the minimal \( b \in \mathbb{R} \) such that the hyperplane \( \{(z_1, \ldots, z_{n+1}) : x_1 \cdot z_1 + \cdots + x_n \cdot z_n + z_{n+1} = b \} \subset \mathbb{R}^{n+1} \) has a non-empty intersection with \( P(f) \), the hyperplane has at least two common points with \( P(f) \).

Similarly to the classical algebra to a system of tropical polynomials

\[ f_1, \ldots, f_s \]  

(2)

in \( n \) variables we associate the Cayley matrix \( C := C(f_1, \ldots, f_s) \) over \( \mathbb{R}_\infty \) consisting of the coefficients of (2). The columns of \( C \) correspond to the tropical monomials of the
form $X^\otimes I$, $I \in \mathbb{Z}^n$, and the rows of $C$ correspond to the tropical polynomials of the form $X^\otimes I \otimes f_j$, $I \in \mathbb{Z}^n$, $1 \leq j \leq s$. Note that unlike the classical algebra the tropical Cayley matrix is infinite in all 4 directions. Actually, one could consider the tropical Cayley matrix infinite in two directions, i.e. with multiindices $I = (i_1, \ldots, i_n)$, $i_j \geq 0$, $1 \leq j \leq n$ (similar to the classical algebra). This consideration would strengthen the conjectures below and would not change Theorem 4.1. However, in the tropical setting the Cayley matrix infinite in 4 directions looks more natural.

**Conjecture 1 on a tropical infinite dual Nullstellensatz.** System (2) has a tropical zero iff the matrix $C$ has a tropical zero.

The latter statement is obvious in the direction that if (2) has a zero then $C$ has a zero (the similar is true for two conjectures below as well).

Observe that being a particular case of tropical polynomials (of the tropical degree 1) matrix $C = (c_{i,j})$ (or in other words, a tropical linear system) has a tropical zero $(\ldots, y_I, \ldots)$ if for every row $i$ of $C$ (in the language of classical algebra) the minimum $\min_I \{c_{i,j} + y_I\}$ is attained at least for two different coordinates $I$. Note that a tropical zero of $C$ makes sense because every row of $C$ contains just a finite number of finite (so, from $\mathbb{R}$) entries.

Similarly to the classical algebra for an integer $N$ denote by $C_N$ a (finite) submatrix of $C$ formed by the rows $X^\otimes I \otimes f_j$, $I = (i_1, \ldots, i_n) \in \mathbb{Z}^n$, $1 \leq j \leq s$ with $|i_1| + \cdots + |i_n| \leq N$, and by the columns of $C$ which contain at least one finite entry at one of these rows.

**Conjecture 2 on a tropical dual Nullstellensatz.** System (2) has a tropical zero iff for any $N$ the matrix $C_N$ has a tropical zero.

**Conjecture 3 on a tropical effective dual Nullstellensatz.** There is a function $N$ on $n$ and on $\text{trdeg}(f_j)$, $1 \leq j \leq s$ such that (2) has a tropical zero iff the matrix $C_N$ has a tropical zero.

Clearly, Conjecture 3 implies Conjecture 2, which in its turn implies Conjecture 1.

### 3 Convex-geometric rephrasing of the tropical dual Nullstellensatz

In the present Section we give a rephrasing of Conjecture 1 (and similarly of Conjectures 2, 3) in terms of the convex geometry in $\mathbb{R}^{n+1}$. Thus, assume that Cayley matrix $C$ has a tropical zero $(\ldots, y_I, \ldots)$, $I \in \mathbb{Z}^n$.

For any $I \in \mathbb{Z}^n$ consider the shift $P(f_j) + (I, 0) \subset \mathbb{R}^{n+1}$, $1 \leq j \leq s$ of the Newton polyhedron. We say that a set $U \subset \mathbb{R}^{n+1}$ lies above (with respect to the last coordinate) a set $V \subset \mathbb{R}^{n+1}$ if for any pair of points $(w_1, \ldots, w_n, u) \in U$, $(w_1, \ldots, w_n, v) \in V$ we have $u \geq v$.

**Proposition 3.1** The following statement is equivalent to Conjecture 1.
For $I \in \mathbb{Z}^n$, $1 \leq j \leq s$ take the minimal $a \in \mathbb{R}$ such that the polyhedron $P(f_j) + (I,a)$ lies above the set $Y := \{(J,-y_J) : J \in \mathbb{Z}^n\}$. Assume that for any $I \in \mathbb{Z}^n$, $1 \leq j \leq s$ the polyhedron and $Y$ have at least two common points. Then there exists a hyperplane $H \subset \mathbb{R}^{n+1}$ defined by a linear equation $b_1 \cdot z_1 + \cdots + b_n \cdot z_n + z_{n+1} = 0$ such that for each $1 \leq j \leq s$ for the minimal $b \in \mathbb{R}$ with the property that the polyhedron $P(f_j)$ lies above the hyperplane $H - (0,b)$, the intersection of $P(f_j)$ and $H - (0,b)$ has at least two points.

For an equivalent statement to Conjecture 2 one has for any $I = (i_1, \ldots, i_n)$ such that $|i_1| + \cdots + |i_n| \leq N$. Respectively, for Conjecture 3 one has to take $N$ as a suitable function in $n$ and in $\text{trdeg}(f_j)$, $1 \leq j \leq s$.

4 Tropical effective dual Nullstellensatz for univariate polynomials

Now let $n = 1$. In this case for a pair of tropical polynomials $f_1$, $f_2$ a tropical effective dual Nullstellensatz follows from [11] with the bound $N \leq \text{trdeg}(f_1) + \text{trdeg}(f_2)$, but since this approach relies on the (classical) resultant of a pair of (classical) polynomials being liftings of $f_1$, $f_2$, respectively, the approach fails for overdetermined tropical systems ($s \geq 3$).

**Theorem 4.1** A tropical effective dual Nullstellensatz for univariate tropical polynomials $f_1, \ldots, f_s$ holds with $N \leq 4 \cdot (\text{trdeg}(f_1) + \cdots + \text{trdeg}(f_s))$.

**Proof.** Fix $1 \leq j \leq s$ for the time being. For the convex polyhedron $P := P(f_j) \subset \mathbb{R}^2$ and $i \in \mathbb{Z}$ take the minimal $a_i \in \mathbb{R}$ such that the shifted polygon $P_i := P(f_j) + (i,a_i)$ lies above the set $Y = \{(l,-y_l) : l \in \mathbb{Z}\}$ (see Proposition 3.1). By the assumption for any $i \in \mathbb{Z}$ there exist at least two points $(l_1,u_1)$, $(l_2,u_2) \in P_i \cap Y$, $l_1 < l_2$. Points from the latter intersection we call extremal points of $P_i$.

**Lemma 4.2** The function $i \mapsto a_i$ is convex.

**Proof of Lemma 4.2.** Suppose the contrary and let $2 \cdot a_i > a_{i-1} + a_{i+1}$ for a certain $i$. Let $(l_1,u_1)$, $(l_2,u_2) \in P_i \cap Y$. Denote by

$$S = \{(w,v) : v-w \cdot (a_i-a_{i-1}) \leq u_1-l_1 \cdot (a_i-a_{i-1}), v+w \cdot (a_i-a_{i+1}) \geq u_1+l_1 \cdot (a_i-a_{i+1})\} \subset \mathbb{R}^2$$

the sector with the vertex at the point $(l_1,u_1)$ between two rays $R_+ = (l_1,u_1) + \{(\lambda \cdot (-1, a_i-a_{i+1}) : \lambda \geq 0\}$ and $R_- = (l_1,u_1) + \{(\lambda \cdot (1,a_i-a_{i-1}) : \lambda \geq 0\}$. We claim that $P_i \subset S$.

Indeed, consider a left adjacent to $(l_1,u_1)$ point $(l_1-1,u_+)$ in $\partial P_i$ on the boundary of $P_i$ (provided that such a point does exist). If $u_+ < u_1 + l_1 \cdot (a_i-a_{i+1})$ (in other words, the point $(l_1-1,u_+)$ lies strictly below the ray $R_+$, cf. the description of $S$) then the point $(l_1-1,u_+) + (1,a_{i+1} - a_i) \in P_{i+1}$ lies strictly below $Y$, the achieved contradiction implies that $(l_1-1,u_+) \in S$. In a similar way a right adjacent to $(l_1,u_1)$ point $(l_1+1,u_-) \in \partial P_i$ (provided that it does exist) belongs to $S$, which justifies the claim.
By the same token the parallel shift \( S + (l_2, u_2) - (l_1, u_1) \) of the sector \( S \) (with its vertex at the point \((l_2, u_2)\)) also contains \( P_i \). This contradicts to the convexity of \( P_i \) and completes the proof of Lemma 4.2. ■

Denote by \( E := E(f_j) \subset \mathbb{R}^2 \) the polygon with the vertices at the extremal points of \( P_i \) for all \( i \in \mathbb{Z} \). Each edge of \( E \) connects an adjacent (with respect to the first coordinate) pair of extremal points. Below we enumerate the (finite) edges of the polygon \( P \) from the left to the right. Denote by \((1, b_r)\) the vector parallel to the \( r \)-th edge of \( P \).

**Lemma 4.3** Let \((l_1, u_1), \ldots, (l_t, u_t) \in P_i, l_1 < \cdots < l_t \) be all the extremal points of \( P_i \). Let the point \((l_t, u_t) - (i, a_i) \in P \) lie in the \( r \)-th (finite) edge of \( P \) (when the latter point belongs both to the \( r \)-th and to the \((r+1)\)-th edges we agree that the point lies in the \( r \)-th edge). Then \( a_{i+1} - a_i \geq b_r \).

For any extremal point \((k, v)\) of \( P_{i+1} \) the point \((k, v) - (i+1, a_{i+1}) \in P \) lying in the \( q \)-th edge of \( P \) either satisfies an inequality \( q \geq r \) or \((k, v) - (i+1, a_{i+1}) \) is the common vertex of the \((r-1)\)-th and \( r \)-th edges of \( P \) (in the latter case \((k, v)\) is the leftmost extremal point of \( P_{i+1} \)). There exists an extremal point \((k, v)\) for which either \( q = r \) and \((k, v) - (i+1, a_{i+1})\) not being the common vertex of the \( r \)-th and \((r+1)\)-th edges of the polygon \( P \) or \((k, v) - (i+1, a_{i+1})\) is the common vertex of the \((r-1)\)-th and \( r \)-th edges of \( P \) iff \( a_{i+1} - a_i = b_r \). Moreover, when \( a_{i+1} - a_i = b_r \) any extremal point \((l_m, u_m) \) of \( P_i \) with \((l_m, u_m) - (i, a_i) \) lying in the \( r \)-th edge of \( P \) is also an extremal point of \( P_{i+1} \).

**Proof of Lemma 4.3.** Consider the point \((l_t, u_t) - (1, b_r) \in P_i \). Then the point \(((l_t, u_t) - (1, b_r)) + (1, a_{i+1} - a_i) \in P_{i+1} \) should lie above the extremal point \((l_t, u_t)\), this entails the inequality \( a_{i+1} - a_i \geq b_r \).

Let \((k, v)\) be an extremal point of \( P_{i+1} \) with \((k, v) - (i+1, a_{i+1}) \) lying in the \( q \)-th edge of \( P \). The point \((k, v) - (1, a_{i+1} - a_i) \) lies in the \( q \)-th edge of the polygon \( P_i \). If \( q < r \) and the point \((k, v) - (i+1, a_{i+1}) \) is not the common vertex of the \((r-1)\)-th and \( r \)-th edges of \( P \) then its shift \((k, v) = ((k, v) - (1, a_{i+1} - a_i)) + (1, a_{i+1} - a_i) \) lies strictly inside the polygon \( P_i \), and therefore \((k, v)\) can not be an extremal point. The achieved contradiction implies that either \( q \geq r \) or \((k, v) - (i+1, a_{i+1}) \) is the vertex of \((r-1)\)-th and \( r \)-th edges of \( P \).

When \( a_{i+1} - a_i > b_r \) a similar argument shows that either \( q > r \) or \((k, v) - (i+1, a_{i+1}) \) is the common vertex of the \( r \)-th and \((r+1)\)-th edges of \( P \). Finally, when \( a_{i+1} - a_i = b_r \), for any extremal point \((l_m, u_m) \) of \( P_i \) with \((l_m, u_m) - (i, a_i) \) lying in the \( r \)-th edge of \( P \) take the point \((l_m, u_m) - (1, a_{i+1} - a_i) \in P_i \), then the point \((l_m, u_m) = ((l_m, u_m) - (1, a_{i+1} - a_i)) + (1, a_{i+1} - a_i) \in P_{i+1} \) is also an extremal point of \( P_{i+1} \). ■

**Remark 4.4** Lemma 4.3 is formulated for the shifts passing from the polygon \( P_i \) to \( P_{i+1} \) (so, from the left to the right). By the same token a similar statement holds while passing from \( P_{i+1} \) to \( P_i \) (so, from the right to the left).

**Lemma 4.5** The polygon \( E \) is convex.
Proof of Lemma 4.5. Denote by \( E_i \) the polygon with the set of vertices being the union of the extremal points of \( P_0, \ldots, P_i \) and with the edges connecting the adjacent vertices. In particular, the leftmost and the rightmost vertices of \( E_i \) are both incident to single edges. We prove by induction on \( i \) that the polygon \( E_i \) is convex. At the inductive step we consider the polygon \( P_{i+1} \) (so, we move from the left to the right). By the same token one can alternatively consider the polygon \( P_{i-1} \) (so, move from the right to the left). This would entail Lemma 4.5.

We say that a polygon with the vertices \( \ldots, w_{i-1}, w_i, w_{i+1}, \ldots \) is convex at the vertex \( w_i \) if there is a line passing through \( w_i \) such that both points \( w_{i-1}, w_{i+1} \) lie above this line.

Let \((l_1', u_1'), (l_2', u_2'), \ldots, l_i' < l_{i+1}' < \cdots \) be all the extremal points of \( P_{i+1} \) (if they exist) being not extremal points of \( P_i \). Lemma 4.3 implies that the point \((l_i', u_i') - (i + 1, a_{i+1})\) either lies in the \( q \)-th edge of \( P \) with \( q > r \) or it is the common vertex of the \( r \)-th and \((r + 1)\)-th edges of \( P \) (we keep the notations from Lemma 4.3).

The inductive hypothesis and Lemma 4.3 entail that the polygon \( E_{i+1} \) is convex at all the vertices of \( E_i \), perhaps, with the exception of the rightmost extremal point \((l_i, u_i)\) of \( E_i \) (and simultaneously of \( P_i \)). The point \((l_i, u_i)\) lies in the \( r \)-th edge of the polygon \( P_i \), and both polygons \( P_i, P_{i+1} \) lie above the line \( L \) spanned by this edge (due to Lemma 4.3), whence \( E_{i+1} \) is convex at its vertex \((l_i, u_i)\) as well.

Since the extremal points \((l_1', u_1'), (l_2', u_2'), \ldots \) are located on the convex polygon \( P_{i+1} \) we get that \( E_{i+1} \) is convex at its vertices \((l_2', u_2'), \ldots \). Thus, it remains to verify that \( E_{i+1} \) is convex at its vertex \((l_i', u'_i)\).

Denote the vector \( w := (l_2', u_2') - (l_i', u_i') \). The points \( p := (l_i', u_i') - (1, a_{i+1} - a_i), (l_2', u_2') - (1, a_{i+1} - a_i) \in P_i \). Therefore, the point \( p \) lies in a sector \( S_0 \) with the vertex \((l_i, u_i)\) formed by the rays \((l_i, u_i) + \{\lambda \cdot (1, b_r) : \lambda \geq 0\} \subset L \) and \((l_i, u_i) + \{\lambda \cdot w : \lambda \geq 0\} \). Now consider a sector \( S_1 \subset S_0 \) parallel to \( S_0 \) with the vertex \( p \) formed by the rays \( p + \{\lambda \cdot (1, b_r) : \lambda \geq 0\} \) and \( p + \{\lambda \cdot w : \lambda \geq 0\} \). The point \((l_i', u_i') = p + (1, a_{i+1} - a_i)\) is located in \( S_1 \) due to Lemma 4.3 and taking into the account that the point \((l_i', u_i')\) is extremal in \( P_{i+1} \) and thereby, can not lie strictly inside \( P_i \). Hence the polygon \( E_{i+1} \) is convex at its vertex \((l_i', u_i')\).

Remark 4.6 The latter statement that \( E_{i+1} \) is convex at its vertex \((l_i', u_i')\) becomes obvious when the point \((l_i, u_i)\) is an extremal point of \( P_{i+1} \), this is equivalent to the equality \( a_{i+1} - a_i = b_r \) due to Lemma 4.3. In case when \( a_{i+1} - a_i > b_r \) the polygon \( P_{i+1} \) has no common extremal points with \( E_i \).

Corollary 4.7 Any edge \( e = ((l, u), (l', u')) \) of the convex polygon \( E \) is one of the following three types:

1) either \((l, u), (l', u') \in P_i \) for a certain \( i \in \mathbb{Z} \) where the point \((l, u)\) lies in the \( r \)-th edge of \( P_i \), the point \((l', u')\) lies in the \( r' \)-th edge of \( P_i \) for some \( r < r' \), except the case when \((l, u)\) is the common vertex of the \((r-1)\)-th and \( r \)-th edges of \( P_i \) and \((l', u')\) lies in the \( r \)-th edge of \( P_i \) (in the latter case \( e \) is parallel to the \( r \)-th edge of \( P_i \), cf. 3) below);

2) either the point \((l, u)\) lies in the \( r \)-th edge of \( P_i \) for a certain \( i \in \mathbb{Z} \), the point \((l', u')\) lies in the \( r' \)-th edge of \( P_{i+1} \) for some \( r, r' \), and the point \((l', u') - (1, a_{i+1} - a_i)\) lies in the \( r' \)-th edge of \( P_i \), moreover either \( r < r' \) or \((l', u') - (1, a_{i+1} - a_i)\) is the common vertex of
the $r$-th and $(r+1)$-th edges of $P_i$. Case 2) occurs when $a_{i+1} - a_i > b_r$ (see Lemma 4.3 and Remark 4.6):

3) or $e$ is parallel to an edge of $P$.

The edges $e$ of $E$ of types 1), 2) we call intermediate and the edges of types 3) we call $r$-principal when $e$ is parallel to the $r$-th edge of $P$. For an edge of the type either 1) or 2) we define its projection (to the first coordinate) as the interval $(l - i, l' - i)$ for the type 1) and as $(l - i, l' - i - 1)$ for the type 2).

Lemma 4.8 The polygon $E$ lies above $Y$.

Proof of Lemma 4.8. Consider a point $(m, -y_m) \in Y$. If $(m, -y_m)$ is a vertex of $E$ or $m$ is a projection of a point strictly inside an edge of $E$ of a type either 1) or 3) then the claim of Lemma 4.8 is obvious.

Else if $m$ is a projection of a point strictly inside an edge $e$ of the type 2) (we keep the notations of 2) of Corollary 4.7) then the point $(m, -y_m)$ lies below the interval $((l, u), (l' - 1, u' - a_{i+1} + a_i))$ with its endpoints on the polygon $P_i$, and it lies also below the interval $((l + 1, u + a_{i+1} - a_i), (l', u'))$ with its endpoints on the polygon $P_{i+1}$. Hence the point $(m, -y_m)$ lies below the edge $((l, u), (l', u'))$ of $E$. ■

Corollary 4.9 i) For a pair of adjacent intermediate edges of $E$ their projections are also adjacent (in the same order).

ii) For each $r$ all $r$-principal edges of $E$ (if they exist) constitute an interval in $E$ (parallel to the $r$-th edge of $P$). We call it $r$-interval. Among these intervals there are either two intervals infinite in one of directions or one interval infinite in both directions.

Let $e_- := ((l_-, u_-), (l'_-, u'_-))$ be an edge of $E$ adjacent to the $r$-interval from the left (provided that the $r$-interval is not infinite to the left). Then $e_-$ is intermediate. Assume for definiteness that $(l'_-, u'_-) \in P_i$ for a certain $r$, while either $(l_-, u_-) \in P_i$ in case of the type 1) (see Corollary 4.7) or $(l_-, u_-) \in P_{i-1}$ in case of the type 2). Then the point $(l'_-, u'_-)$ lies in the $r$-th edge of $P_i$.

Similarly, let an edge $e_+ := ((l_+, u_+), (l'_+, u'_+))$ be an edge adjacent to the $r$-interval from the right (provided that the $r$-interval is not infinite to the right). Then $e_+$ is intermediate. Assume that $(l_+, u_+) \in P_i$ for a certain $r$ and either $(l'_+, u'_+) \in P_i$ in case of the type 1) or $(l'_+, u'_+) \in P_{i+1}$ in case of the type 2). Then the point $(l'_+, u'_+)$ either lies in the $r$-th edge of $P_i$ or $(l_+, u_+)$ is the common vertex of the $(r-1)$-th and $r$-th edges of $P_i$.

iii) Denote by $(k_1, d_1), (k_2, d_2)$ the endpoints of the $r$-th edge of $P$. Then for any pair of adjacent extremal points $(k'_1, d'_1), (k'_2, d'_2)$ in the $r$-interval of $E$ we have $k'_2 - k'_1 \leq k_2 - k_1$.

Finally, we complete the proof of Theorem 4.1. So far, we studied the convex polygon $E(f_j)$ for a fixed $1 \leq j \leq s$ (see Lemma 4.5). Now we consider the intersection $E := \bigcap_{1 \leq j \leq s} E(f_j)$. Every edge $e$ of the convex polygon $E$ is some subinterval of either an intermediate edge of $E(f_j)$ or an $r$-interval for certain $1 \leq j \leq s$ and $r$. The total sum of the lengths of the projections of the edges being subintervals of intermediate edges of $E(f_j)$, $1 \leq j \leq s$ does not exceed $3 \cdot \sum_{1 \leq j \leq s} \text{trdeg } f_j$ (due to i), ii) of Corollary 4.9.
Observe that if $\epsilon$ is a subinterval of an $r$-interval of $E(f_j)$ for a certain $1 \leq j \leq s$ and not all the points of $\epsilon$ belong to all polygons $E(f_{j_1})$, $1 \leq j_1 \leq s$ (the latter is equivalent to that any strictly inside point of $\epsilon$ does not belong to all $E(f_{j_1})$, $1 \leq j_1 \leq s$) then $\epsilon$ can not contain extremal points strictly inside itself according to Lemma 4.8. Hence the total sum of the lengths of the projections of all the edges of $\mathcal{E}$ being subintervals of some $r$-intervals of $E(f_j)$, $1 \leq j \leq s$ does not exceed $\sum_{1 \leq j \leq s} \trdeg f_j$ by virtue of iii) of Corollary 4.9.

Thus, a truncation $\mathcal{E}_N$ of $\mathcal{E}$ with the length of the projection to the first coordinate equal $N$, where $N \geq 4 \cdot \sum_{1 \leq j \leq s} \trdeg f_j$, contains an edge which is a common subinterval of $r_j$-intervals for appropriate $r_j$ of all $E(f_j)$, $1 \leq j \leq s$. Taking into the account the Proposition 3.1 we conclude with Theorem 4.1.

It would be interesting to improve the factor 4 in Theorem 4.1.

We observe that one of the difficulties towards generalizing the proof of Theorem 4.1 to the multidimensional case $n \geq 2$ is that a direct multidimensional generalization of Lemma 4.5 does not always hold.

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References


