A tropical version of Hilbert polynomial (in dimension one)

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Abstract

For a tropical univariate polynomial f we define its tropical Hilbert function as the dimension of a tropical linear prevariety of solutions of the tropical Macaulay matrix of the polynomial up to a (growing) degree. We show that the tropical Hilbert function equals (for sufficiently large degrees) a sum of a linear function and a periodic function with an integer period. The leading coefficient of the linear function coincides with the tropical entropy of f. Also we establish sharp bounds on the tropical entropy.

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Introduction

One can find the basic concepts of tropical algebra in [9].

Consider a tropical univariate polynomial $f := \min_{0\le i\le n}\{iX + a_i\}$ where $a_i \in \mathbb{Z}$, $0 \le i \le n$. We call $z := (z_1, z_2, \ldots)$, $z_j \in \mathbb{R}$, $j \ge 1$ a *tropical recurrent sequence satisfying the vector* $\vec{a} := (a_0, \ldots, a_n)$ [4] if for any $j \ge 1$ the following tropical (linear) polynomial is satisfied:

$$\min_{0\le i\le n}\{z_{j+i} + a_i\}, \tag{1}$$

i. e. the minimum in (1) is attained at least for two different values among $0 \le i \le n$.

When one considers classical recurrent sequences $(x_1, x_2, \dots)$ satisfying relations $\sum_{0 \le i \le n} a_i x_{i+j} = 0$ similar to (1), the first n values $x_1, \dots, x_n$ determine the rest of the sequences uniquely. This is not the case for tropical recurrent sequences.

Denote by $D(k) \subset \mathbb{R}^s$ a tropical linear prevariety [9] of all the sequences $(z_1, \dots, z_k)$ satisfying (1) for $1 \le j \le k - n$. Therefore, $D(k)$ is a polyhedral complex [9]. The function $d(k) := d_{\vec{a}}(k) := \dim(D(k))$ we call *the tropical Hilbert function* of the tropical polynomial f (or equivalently, of the vector $\vec{a}$ of its coefficients). Obviously, $d(k) \le d(k+1) \le d(k) + 1$. It is observed in [4] that $d(k+t) \le d(k) + d(t)$. Therefore, due to Fekete's subadditivity lemma [11] there exists the limit

$$H := H_{\vec{a}} = \lim_{k \to \infty} d(k)/k \tag{2}$$

which is called [4] *the tropical entropy of the tropical polynomial f or of the vector $\vec{a}$*. Evidently, $0 \le H \le 1$.

In classical commutative algebra the Hilbert function of a polynomial $g = \sum_I g_I X^I \in F[X_1, \dots, X_m]$ is defined as the growth function of the quotient ring $F[X_1, \dots, X_m]/(g)$ in the filtration with respect to degree. For a given degree e this function coincides with the dimension of the space of solutions of a linear system

$$\sum_I g_I Y_{I+J} = 0 \tag{3}$$

for all vectors $J := (j_1, \dots, j_m) \in \mathbb{Z}^m$, $0 \le j_1, \dots, j_m$ such that for every vector $I = (i_1, \dots, i_m)$ from the support of g we have $i_1 + j_1 + \dots + i_m + j_m \le e$. Note that a linear system (3) forms the rows of Macaulay matrix.

Note that we define the tropical Hilbert function as the dimension of the space $D(k)$ of tropical recurrent sequences. We mention that multidimensional tropical recurrent sequences appear also as the solutions of the tropical Macaulay matrix [4] (generalizing tropical equations (1)). Macaulay matrix emerges in a tropical version of the weak Hilbert Nullstellensatz (see [2], [3], [6], [7], [8], [1]).

The main result of the paper (see Theorem 5.4 and Corollary 5.5) states that the tropical Hilbert function $d(k)$ is *quasi-linear*, i. e. coincides (for sufficiently big k) with a sum $Hk + r(k)$ of a linear function Hk (see (2)) and a periodic function $r(k)$ with an integer period.

Recall that in classical commutative algebra the Hilbert function of an ideal in $F[X_1, \dots, X_m]$ is a polynomial (for sufficiently large degrees). In its turn, the degree of this polynomial is less than m (in particular, in case $m = 1$ Hilbert polynomial is a constant). One can directly generalize the definition of a tropical Hilbert function to m-variate tropical polynomials based on the tropical

Macaulay matrix for m-variate polynomials. This function grows asymptotically as $H \cdot k^m$ where H is defined similary to (3) (cf. [4]). In case of dimension $m = 1$ which we study in the present paper, the tropical Hilbert function $d(k)$ coincides with the linear function Hk up to a periodic function (for sufficiently large k).

In [8] tropical ideals in the semiring of tropical polynomials are introduced, they have some features similar to classical ideals: in particular, it is proved that Hilbert function of a tropical homogeneous ideal is eventually a polynomial.

We mention that in [3], [4] it is proved that $H = 0$ iff each point $(i, a_i) \in \mathbb{R}^2$, $0 \le i \le n$ is a vertex of the Newton polygon. (The Newton polygon of f is the convex hull of the rays $\{(i, x \ge a_i)\}$, $0 \le i \le n$.)

It would be interesting to clarify, whether one can extend the results of the paper to vectors $(a_0, \dots, a_n)$ with $a_i \in \mathbb{R} \cup \{\infty\}$. Another problem is to improve the bound on the period in the function $r(k)$ and the bound on the minimal k starting with which the tropical Hilbert function coincides with $Hk + r(k)$ (cf. Corollary 5.5).

In section 1 we prove some auxiliary bounds on tropical recurrent sequences. In section 2 we describe a directed graph $G := G_{\vec{a}}$ and provide a recursive construction, which allows one to produce a tropical recursive sequence corresponding to a path in G. Vice versa, to each tropical recurrent sequence corresponds a path in G. Thus, all tropical recurrent sequences corresponding to a path T of a length k in G form a polyhedron $Q_T \subset \mathbb{R}^{k+n}$. We distinguish some edges of G and call them *augmenting*. Denote by $d(T)$ the number of augmenting edges in T, and by $n(T) \le n$ a certain integer depending just on the first vertex of T. The following theorem (see Theorem 3.5 in section 3 for more details) relates the tropical recurrent sequences with paths in G.

Theorem 0.1 *i)* $\dim(Q_T) = d(T) + n(T)$.

ii) The set $D(k+n)$ of all tropical recurrent sequences satisfying $\vec{a}$ of a length $k+n$ coincides with $\cup_T Q_T$, where T ranges over all paths in G of the length $k \ge 0$.

iii) The tropical Hilbert function $d_{\vec{a}}(k+n) = \max_T\{d(T) + n(T)\}$.

Theorem 0.1 allows one to express the tropical entropy explicitly in terms of G. For a path T in G by $l(T)$ denote its length.

Corollary 0.2 *(see Corollary 4.3 in section 4). The tropical entropy $H_{\vec{a}}$ equals the maximum of $d(T)/l(T)$ over all simple cycles T in G. In particular, $H_{\vec{a}}$ is a rational number.*

The main result of section 5 describes the behaviour of the tropical Hilbert function (one can find more details in Corollary 5.5). Its proof involves Theorem 0.1.

Theorem 0.3 *The tropical Hilbert function* $d_{\vec{a}}(k) = H_{\vec{a}} \cdot k + r(k)$ *for sufficiently big* k*, where* $r(k)$ *is a periodic function with an integer period. One can compute the function* $d_{\vec{a}}$*.*

In section 6 we study tropical recurrent sequences satisfying a *tropical boolean vector* $\vec{a} = (a_0, \dots, a_n)$ where $a_0 = a_n = 0$ and $a_i \in \{0, \infty\}, 0 \le i \le n$. We establish similar results to Theorem 0.1, Corollary 0.2, Theorem 0.3 for a tropical boolean vector $\vec{a}$.

Finally, in section 7 we prove the following result separating a positive tropical entropy from zero (see Theorem 7.1).

Theorem 0.4 *If* $H_{\vec{a}} > 0$ *then* $H_{\vec{a}} \ge 1/4$ *(and the bound is sharp).*

Also we show the sharp upper bound $H_{\vec{a}} \le 1 - 2/(n+1)$ in case when Newton polygon of $\vec{a}$ has a single bounded edge. We conjecture that the latter bound holds for an arbitrary vector $\vec{a}$.

1 Bounds on connected coordinates

Let $\vec{a} := (a_0, \dots, a_n) \in \mathbb{Z}^{n+1}$ be a vector, define its *amplitude* as

$$M := \max_{0 \le i \le n} \{a_i\} - \min_{0 \le i \le n} \{a_i\}. \tag{4}$$

Definition 1.1 *Consider a tropical recurrent sequence* $z := (z_0, z_1, \dots), z_j \in \mathbb{R}$ *satisfying vector* $\vec{a}$*. We call a coordinate* z_{j_0} *(or, more precisely,* j_0*) connected if there exists* $0 \le t_0 \le \min\{n, j_0\}$ *such that* $z_{j_0} + a_{t_0} = \min_{0 \le t \le n}\{z_{t+j_0-t_0} + a_t\}$*. In other words, one can't diminish the value of* z_{j_0} *without changing all other* z_j*,* $j \ne j_0$ *and keeping the property of being a tropical recurrent sequence satisfying* $\vec{a}$*. Otherwise, we call* z_{j_0} *disconnected. We say that connected coordinates* $j_0 < j_1$ *are neighbouring if any intermediate coordinate* $j_0 < j < j_1$ *is disconnected.*

Lemma 1.2 *Let* $\vec{a} \in \mathbb{Z}^{n+1}$ *be a vector with amplitude* M*, and let* z *be a tropical recurrent sequence satisfying* $\vec{a}$*. Let* $j_0 < j_1$ *be a pair of neighbouring connected coordinates. Then*
i) $j_1 - j_0 \le n$*;*
ii) $|z_{j_0} - z_{j_1}| \le 2M$*.*

Proof. To prove i) suppose the contrary. Then the minimum $\min_{0 \le t \le n}\{z_{j_0+t} + a_t\}$ is attained only for $t = 0$ which contradicts the assimption that z satisfies $\vec{a}$.

To prove ii) suppose the contrary. First, assume that $z_{j_1} \geq z_{j_0}$, hence $z_{j_1} - z_{j_0} > 2M$. There exists $0 \leq t_1 \leq n$ such that

$$z_{j_1} + a_{t_1} = \min_{0 \leq t \leq n} \{z_{j_1+t-t_1} + a_t\}. \tag{5}$$

If $j_1 - t_1 \leq j_0$ then $z_{j_0} + a_{j_0-j_1+t_1} < z_{j_1} - 2M + a_{j_0-j_1+t_1} \leq z_{j_1} - M + a_{t_1}$. and we get a contradiction with (5), thus $j_1 - t_1 > j_0$.

We claim that the minimum $\min_{0 \leq t \leq n}\{z_{j_0+t} + a_t\}$ is attained only for $t = 0$. Indeed, for any connected $j_2 \leq j_0 + n$ we have

$$z_{j_2} + a_{j_2-j_0} \geq z_{j_1} + a_{k_1} - a_{j_2-j_1+t_1} + a_{j_2-j_0} > z_{j_0} + a_0,$$

where the first inequality is due to (5), while the second inequality follows from $z_{j_1} - z_{j_0} > 2M$ and from (4). This proves the claim. We come to a contradiction with that z satisfies $\vec{a}$, which completes the proof of ii) in case $z_{j_1} \geq z_{j_0}$.

The case $z_{j_1} \leq z_{j_0}$ is handled in a similar way. The lemma is proved. □

Corollary 1.3 *Let z be a tropical recurrent sequence satisfying a vector $\vec{a} \in \mathbb{Z}^{n+1}$ of amplitude M, and let j be a connected coordinate. Then*

i) $z_s \leq z_j + 2M|s-j|$ for any connected coordinate s:

ii) $z_s \geq z_j - 2M \cdot \max\{|s-j|, n\}$ for any coordinate s:

iii) if $z_{s+n} > \min_{s_0 \leq t < s+n}\{z_t\} + 2Mn$ for some $s_0 \geq s$, $s_0 \geq 0$ then the coordinate $s+n$ is disconnected.

Proof. i) follows immediately from Lemma 1.2 ii).

ii) follows from i) when a coordinate s is connected, moreover, in this case

$$z_s \geq z_j - 2M|s-j|. \tag{6}$$

For a disconnected coordinate s one can assume w.l.o.g. that $s > j$. Take the maximal connected coordinate $s_0 < s$. Lemma 1.2 i) implies that $s - s_0 < n$. The minimum $\min_{0 \leq t \leq n}\{z_{t+s-n} + a_t\}$ is attained for some connected coordinate $t_0 + s - n \leq s_0$. Therefore, when $t_0 + s - n \geq j$, we obtain

$$z_s + a_n \geq z_{t_0+s-n} + a_{t_0} \geq z_j - 2M(t_0 + s - n - j) + a_{t_0}$$

due to (6) which proves ii) in this case.

When $k_0 + s - n < j$, we obtain

$$z_s + a_n \geq z_{t_0+s-n} + a_{t_0} \geq z_j - 2M(j - t_0 - s + n) + a_{t_0} \geq z_j - 2M(n-1) + a_{t_0}$$

again due to (6). This completes the proof of ii).

iii) follows from ii). □

Lemma 1.4 *Let a vector $\vec{a} \in \mathbb{Z}^n$ fulfill (4) and a sequence $(z_0, \dots, z_n) \in \mathbb{Z}^{n+1}$ satisfy $\vec{a}$. Denote $m_L := \min_{0 \le i < n}\{z_i\}$, $s_L := \min\{0 \le i < n \ : \ i = m_L\}$, $m_R := \min_{1 \le i \le n}\{z_i\}$, $s_R := \min\{1 \le i \le n \ : \ i = m_R\}$. Then $|m_L - m_R| \le M$, $s_L \le s_R$.*

Proof. It suffices to consider a case when either $m_L = z_0$ or $m_R = z_n$, (otherwise, $m_L = m_R$, $s_R = s_L$). If $m_L = z_0$ then $z_0 + a_0 \ge z_t + a_t = \min_{0 \le i \le n}\{z_i + a_i\}$ holds for a suitable $1 \le t \le n$. Hence $m_L = z_0 \ge z_t + a_t - a_0 \ge m_R - M$. If $m_R = z_n$, we obtain the inequality $m_R \ge m_L - M$ is a similar way. Otherwise, if $m_R = z_j$, $1 \le j < n$ then $m_L = z_0 \le z_j = m_R$. $\Box$

2 Construction of a graph of tropical recurrent sequences

We are producing by recursion a tropical recurrent sequence satisfying vector $\vec{a} = (a_0, \dots, a_n)$, and assume that a finite fragment (a prefix) of the sequence is already produced. Possibilities of choices of continuations of the fragment depend only on the last n entries (a suffix) of the fragment. That is why we consider only the last n entries and denote them by $(y_1, \dots, y_n) \in \mathbb{R}^n$. We will view $(y_1, \dots, y_n)$ also as the coordinates in $\mathbb{R}^n$. More precisely, if the minimum in $\min_{1 \le i \le n}\{y_i + a_{i-1}\}$ is attained

- once, then a contituation $y_{n+1} \in \mathbb{R}$ is determined uniquely;
- at least twice, then y_{n+1} ranges over an infinite interval bounded from below.

In this section we construct a directed finite graph $G := G_{\vec{a}}$ and for each vertex v of G a polyhedron $P_v \subset \mathbb{R}^n$. If $(y_1, \dots, y_n) \in P_v$, and y_{n+1} is a possible continuation (i.e. the sequence $(y_1, \dots, y_n, y_{n+1})$ satisfies vector $\vec{a}$) then $(y_2, \dots, y_n, y_{n+1}) \in P_w$ for a suitable vertex w of G such that (v, w) is an edge of G. The converse is also valid: if $(y_1, \dots, y_n) \in P_v$ and (v, w) is an edge of G then there exists a continuation y_{n+1} such that $(y_1, \dots, y_n, y_{n+1})$ satisfies vector $\vec{a}$ and $(y_2, \dots, y_n, y_{n+1}) \in P_w$. Observe that the construction of vertices of G (Definition 2.1) depends only on n, M, while the construction of its edges (Definitions 2.9, 2.11, 2.13) depends also on $\vec{a}$.

In section 3 we show that the tropical recurrent sequences satisfying $\vec{a}$ are encoded by paths in G (and vice versa).

2.1 Vertices of graph $G_{\vec{a}}$

Definition 2.1 *We define a vertex v of the graph G and a corresponding polyhedron $P := P_v \subset \mathbb{R}^n$. Each polyhedron P is determined by the following data. We fix a subset $\emptyset \ne B := B_v \subset \{1, \dots, n\}$, an element $s := s_v \in B$, for* *each pair* $1 \le r < l \le n,\ r, l \in B$ *integers* $m(r,l) := m_v(r,l)$ *and indicators* $e(r,l) := e_v(r,l) \in \{0,1\}$ *such that* P *is described by the following system of linear inequalities:*

$$y_s \le y_j,\ 1 \le j \le n, \tag{7}$$

$$m(s,q) \ge -nM,\ m(p,s) \le (n+s-p)M,\ 1 \le p < s < q \le n, \tag{8}$$

$$|m(r,l)| \le 2nM,\ 1 \le r < l \le n,\ p,q,r,l \in B, \tag{9}$$

$$y_r = y_l + m(r,l),\ 1 \le r < l \le n,\ r,l \in B\ if\ e(r,l) = 0, \tag{10}$$

$$y_r - 1 < y_l + m(r,l) < y_r,\ 1 \le r < l \le n,\ r,l \in B\ if\ e(r,l) = 1, \tag{11}$$

$$y_j - y_s > jM,\ 1 \le j \le n,\ j \notin B. \tag{12}$$

For definiteness, we take $s \in B$ *to be the minimal possible satisfying (7).*

Varying $B, s, m(r,l), e(r,l)$ *we obtain all vertices of the graph* G.

Coordinates y_r *of* $\mathbb{R}^n$ *for* $r \in B$ *we call bounded on* P, *while the coordinates* y_j *for* $j \notin B$ *we call unbounded.*

Remark 2.2 *i) The graph* G *consists of just the vertices* v *for which the polyhedron* P_v *is nonempty.*

ii) We define an equivalence relation on B *setting* $r, l \in B$ *to belong to the same equivalence class iff* $e(r,l) = 0$ *(i.e. the difference* $y_r - y_l$ *is an integer). When it is not an equivalence relation, the polyhedron* P *is empty.*

iii) Informally, the difference (repectively, the ceiling function of the difference) of each pair of bounded coordinates is given in (10) (respectively, in (11)), while for unbounded coordinates just lower bounds (12) via the minimal coordinate, which is always a bounded one, are given.

Remark 2.3 *Note that the inequlities (9) follow from the inequalities (7), (8), (10), (11). We keep (9) to have some apriori bound on* $m(k,l)$. *Indeed, assume that* $r < s < l,\ e(r,s) = e(s,l) = 1$ *(all other cases of orders between* r, s, l *and the values of* e *one can study in a similar manner). Then (11) imply that* $|y_r - y_l| < \max\{m(r,s)+1, -m(s,l)\}$ *taking into account that* $y_s \le y_r, y_l$. *Therefore,* $m(r,l) \le \max\{m(r,s)+1, -m(s,l)\}$.

Below we repeatedly make use of the following statements describing the set of solutions of a system of inequalities of the form (10), (11).

Lemma 2.4 *Let a system of inequalities*

$$x_r - x_l = m(r,l),\ 1 \le r < l \le n,\ e(r,l) = 0, \tag{13}$$

$$m(r,l) - 1 < x_r - x_l < m(r,l),\ 1 \le r < l \le n,\ e(r,l) = 1, \tag{14}$$

in variables $x_1, \dots, x_n$ be consistent, where $m(r,l) \in \mathbb{Z},\ e(r,l) \in \{0,1\}, 1 \le r < l \le n$. W.l.o.g. one can assume that $x_1 = 0$ replacing x_l by $x_l - x_1,\ 1 \le l \le n$. We agree that $m(l,l) = e(l,l) = 0$. Denote

$$\widehat{x_l} := x_l + m(1,l),\ 1 \le l \le n. \tag{15}$$

Clearly, $0 \le \widehat{x_l} < 1, 1 \le l \le n$. We claim that the system (13), (14) determines uniquely an order

$$0 = \widehat{x_1} \le \widehat{x_{\pi(2)}} \le \cdots \le \widehat{x_{\pi(n)}} < 1 \tag{16}$$

for a suitable permutation $\pi \in Sym(n-1)$. The set of solutions of the system (13), (14) is a polytope (open in its linear hull) isomorphic to the polytope (in $\mathbb{R}^n$ endowed with the coordinates $\widehat{x_1}, \dots, \widehat{x_n}$) given by the system (16). The isomorphism is assured by (15).

Vice versa, given $x_1, \dots, x_n$ and integers $m(1,l), 1 \le l \le n$ such that (16) is satisfied for (15), one can find uniquely integers $m(r,l) \in \mathbb{Z}, e(r,l) \in \{0,1\}, 1 \le r < l \le n$ for which (13), (14) hold. The integers $m(r,l), e(r,l), 1 \le r < l \le n$ are determined just by the integers $m(1,l), 1 \le l \le n$ and by the order (16).

In particular, (16) together with (15) allows one to find $1 \le s \le n$ such that $x_s \le x_i,\ 1 \le i \le n$ holds for any solution of (13), (14). Moreover, the set of such s is determined just by the integers $m(1,l), 1 \le l \le n$ and by the order (16).

Proof. First note that the partition of the set $\{1, \dots, n\}$ into classes such that $r < l$ belong to the same class iff $e(r,l) = 0$, provides an equivalence relation.

It holds $-1 < \widehat{x_l} - \widehat{x_r} < 1,\ 1 \le r \le l \le n$. If $e(r,l) = 0$ then (because of (15)) we have

$$\widehat{x_l} - \widehat{x_r} = -m(r,l) - m(1,l) + m(1,r) \in \mathbb{Z}, \tag{17}$$

hence $\widehat{x_l} = \widehat{x_r}$.

For $e(r,l) = 1$ we get $\widehat{x_l} \ne \widehat{x_r}$. On the other hand, due to (14), (15) it holds

$$\widehat{m(r,l)} := m(r,l) - m(1,l) + m(1,r) - 1 < \widehat{x_l} - \widehat{x_r} < \widehat{m(r,l)} + 1. \tag{18}$$

We deduce that either $\widehat{m(r,l)} = -1$ or $\widehat{m(r,l)} = 0$. In the former case it holds $\widehat{x_l} < \widehat{x_r}$, in the latter case it holds $\widehat{x_l} > \widehat{x_r}$. This assures a required permutation π satisfying (16). Obviously, π is independent from the particular values of $\widehat{x_l},\ 2 \le l \le n$, but rather depends only on the order between them.

For any tuple $\widehat{x_l},\ 2 \le l \le n$ satisfying (16), it holds $\widehat{x_l} = \widehat{x_r}$ when $e(r,l) = 0$ (see (17)), while it holds (18) when $e(r,l) = 1$. Therefore, $x_l,\ 1 \le l \le n$ defined by (15), satisfy (13), (14) (with $x_1 = 0$). $\square$

Corollary 2.5 *For any point $(x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$ satisfying the system (13), (14) (more precisely, its subsystem for $1 \le r < l < n$), the set of points $x_n \in \mathbb{R}$ such that the point $(x_1, \dots, x_n)$ satisfies (13), (14), consists of either*

i) a single point when $e(r,n) = 0$ for some $1 \le r < n$ or

ii) an open nonempty finite interval when $e(r,n) = 1$ for all $1 \le r < n$.

Note that the system (10), (11) is similar to the system (13), (14).

2.2 An edge of the graph G in case of a unique continuation of a prefix of a tropical recurrent sequence

Now we describe when G has an edge from a vertex v to a vertex w. Denote the coordinates of $\mathbb{R}^n$ such that the polyhedron under construction $P_w \subset \mathbb{R}^n$ by $x_1, \dots, x_n$. The polyhedron P_w relates to P_v informally as follows. For any point $(y_1, \dots, y_n) \in P_v$ there exists a point $(y_2, \dots, y_n, x_n) \in P_w$, and x_n fulfills the conditions described below in Definitions 2.9, 2.11, 2.13 (see Theorem 3.1 below). A value of x_n is either unique or varies in an open interval. Formally, in Definitions 2.6, 2.9, 2.11, 2.13 we describe linear inequalities determining P_w.

In the following definition we provide a part of equalities and inequalities of the forms (10), (11) describing P_w and complete the description in Definitions 2.9, 2.11, 2.13.

Definition 2.6 *Denote the coordinates of $\mathbb{R}^n$ for which $P_w \subset \mathbb{R}^n$ by $x_1, \dots, x_n$. First, we impose that a coordinate $x_{r-1}, 2 \le r \le n$ is bounded on P_w iff the coordinate y_r is bounded on P_v, in other words $B_w \supset (B \setminus \{1\}) - 1$ The status of boundness of the coordinate x_n is specified in Definitions 2.9, 2.11, 2.13, i.e. whether $n \in B_w$.*

The description of P_w contains inequalities (cf. (10), (11))

$$x_{r-1} = x_{l-1} + m(r,l),\ 2 \le r < l \le n,\ r,l \in B\ if\ e(r,l) = 0, \tag{19}$$

$$x_{r-1} - 1 < x_{l-1} + m(r,l) < x_{r-1},\ 2 \le r < l \le n,\ r,l \in B\ if\ e(r,l) = 1. \tag{20}$$

Thus, we put $m_w(r-1,l-1) := m(r,l),\ e_w(r-1,l-1) := e(r,l),\ 2 \le r < l \le n$. *In addition, the description of* P_w *contains inequalities (cf. (9))*

$$|m_w(r-1,l-1)| \le 2nM. \tag{21}$$

In Definitions 2.9, 2.11, 2.13 we impose equalities and inequalities of the forms (10), (11) which involve the coordinate x_n, and also inequalities of the forms (7), (12) for P_w.

Lemma 2.7 *For points* $(y_1,\dots,y_n) \in P_v$ *the minimum*

$$\min_{1\le r\le n}\{y_r + a_{r-1}\} \tag{22}$$

is attained on a suitable subset of the bounded coordinates $r \in B$ *that do not depend on a choice of point* $(y_1,\dots,y_n) \in P_v$.

Proof. Due to (12) the minimum in (22) is attained only on bounded coordinates y_r. Let two points $(y_1^{(1)},\dots,y_n^{(1)})$, $(y_1^{(2)},\dots,y_n^{(2)}) \in P_v$. Assume that for a pair of bounded coordinates y_r, y_t an inequality holds $y_r^{(1)} + a_{r-1} \le y_t^{(1)} + a_{t-1}$. Then $y_r^{(2)} + a_{r-1} \le y_t^{(2)} + a_{t-1}$ because of inequalities (10), (11), taking into the account that a_{r-1}, a_{t-1} are integers (cf. also Lemma 2.4). $\square$

Definition 2.8 *Denote by* $S := S_v (\subset B)$ *the set of* $r, 1 \le r \le n$ *on which the minimum in (22) is attained.*

In particular, all the elements from S belong to the same class (see Remark 2.2). First consider the case when S consists of a single element t.

Definition 2.9 *Let the set* $S = \{t\}$ *be a singleton. We define a unique edge in* G *outgoing from the vertex* v *(to a suitable vertex* w*) and describe a system of equations and inequalities defining a polyhedron* P_w. *Recall that the description of* P_w *contains inequalities (19), (20), (21). Declare the coordinate* x_n *to be bounded, this determines* $B_w := ((B \setminus \{1\}) - 1) \cup \{n\}$.

Apply Lemma 2.4 to the system (10), (11). We obtain a sequence of the form (16) between $\widehat{y_i},\ 1 \le i \le n,\ i \in B$. *Denote* $x_n := y_t + a_{t-1} - a_n$, *then* $\widehat{x_n} = x_n - a_{t-1} + a_n + m(1,t) = \widehat{y_t}$ *(see Lemma 2.4). Extend the obtained sequence by* $\widehat{x_n}$. *Again applying Lemma 2.4 to the extended sequence, we get a system of the form (13), (14) in the variables* $y_i,\ 1 \le i \le n,\ i \in B,\ x_n$. *Remove from this system equations and inequalities containing* y_1 *(provided that* $1 \in B$*) and replace* $y_i,\ 2 \le i \le n,\ i \in B$ *by* x_{i-1}, *respectively. Thus, we obtain a system which extends the system (19), (20) (and playing the role of (10), (11) for* P_w*)*

$$x_r = x_l + m_w(k,l),\ 1 \le r < l \le n,\ r,l \in B_w\ if\ e_w(r,l) = 0, \tag{23}$$

$$x_r - 1 < x_l + m_w(r,l) < x_r,\ 1 \le r < l \le n,\ r,l \in B_w\ if\ e_w(r,l) = 1. \quad (24)$$

Due to Lemma 2.4 $m_w(r,l), e_w(r,l)$ for $1 \le r < l < n$ coincide with the corresponding integers already constructed in Definition 2.6.

Applying Lemma 2.4 to the system (23), (24) we find the minimal possible $1 \le s_w \le n,\ s_w \in B_w$ such that $x_{s_w} \le x_i,\ i \in B_w$. The system (23), (24) together with the following inequalities (playing the role of (7), (8), (9), (12), respectively, for P_w):

$$x_{s_w} \le x_i,\ 1 \le i \le n, \quad (25)$$

$$m_w(s_w, q) \ge -nM,\ m_w(p, s_w) \le (n + s_w - p)M, \quad (26)$$

$$|m_w(r,l)| \le 2nM,\ 1 \le p < s_w < q \le n,\ 1 \le r < l \le n,\ p,q,r,l \in B_w, \quad (27)$$

$$x_j - x_{s_w} > jM,\ 1 \le j \le n, j \notin B_w \quad (28)$$

describe P_w.

Remark 2.10 *Only in case $B_v = \{1\}$ the system (23), (24) is void, in this case $B_w = \{n\},\ s_w = n$, and P_w is described by inequalities (28) with $s_w = n$ and (25) (being a consequence of (28)).*

2.3 Edges of G in case of non-uniqueness of continuations of a prefix of a tropical recurrent sequence

Now we study the case when the set S (see Definition 2.8) consists of more than one element. Take a minimal $t > 1$ such that $t \in S$. There can be several edges in the graph G outgoing from the vertex v.

Definition 2.11 *First define a single edge from the vertex v to a vertex w such that the coordinate x_n is unbounded in P_w, in other words we put $B_w := (B \setminus \{1\}) - 1$. Again recall that the description of P_w contains the inequalities (19), (20), (21).*

Applying Lemma 2.4 to the system (19), (20) one can find the minimal possible $1 \le s' < n,\ s' \in B_w$ such that $x_{s'} \le x_i,\ 1 \le i < n,\ i \in B_w$. We put $s_w := s'$ (cf. (7)). The description of P_w consists of the inequalities (19), (20), (21) together with the following inequalities (playing the role of (7), (8), (12), respectively, for P_w):

$$x_{s_w} \le x_i,\ 1 \le i \le n, \quad (29)$$

$$m(s_w,q) \geq -nM,\ m(p,s_w) \leq (n+s_w-p)M \tag{30}$$

for $1 \leq p < s_w < q \leq n,\ p,q \in B_w$,

$$x_n - x_{s'} > nM, \tag{31}$$

$$x_j - x_{s_w} > jM,\ 1 \leq j \leq n,\ j \notin B_w. \tag{32}$$

Remark 2.12 *We distinguish (31) among the latter inequalities of the form (32) (when* $j = n$*) for the sake of easier references below.*

The constructed vertex w is the unique one to which there is an edge in the graph G from the vertex v such that the coordinate x_n is unbounded. Still we assume that $|S| \geq 2, t \in S$ with a minimal possible $t > 1$. Now we construct vertices w with a bounded coordinate x_n to which there are edges from v.

Definition 2.13 *We declare the coordinate* x_n *to be bounded, i.e.* $B_w := ((B \setminus \{1\}) - 1) \cup \{n\}$. *Recall that the description of* P_w *already contains the equalities and inequalities (19), (20), (21). As in Definition 2.11 applying Lemma 2.4 to the system (19), (20) one can find the minimal possible* $1 \leq s' < n$ *such that* $x_{s'} \leq x_l,\ 1 \leq l < n,\ l \in B_w$.

We choose all possible integers $m_w(l,n),\ 0 \leq e_w(l,n) \leq 1,\ 1 \leq l < n,\ l \in B_w$ *for which it holds*

$$m_w(t-1,n) \leq a_n - a_{t-1} - e_w(t-1,n), \tag{33}$$

$$m_w(s',n) \geq -nM, \tag{34}$$

$$|m_w(l,n)| \leq 2nM,\ 1 \leq l < n,\ l \in B_w. \tag{35}$$

The description of P_w *contains inequalities (35) playing the role of (9) (a part of them are inequalities (21)) and the following inequalities playing the role of (10), (11) (a part of them are inequalities (19), (20)):*

$$x_r - x_l = m_w(r,l),\ 1 \leq r < l \leq n,\ r,l \in B_w,\ e_w(r,l) = 0, \tag{36}$$

$$m_w(r,l) - 1 < x_r - x_l < m_w(r,l),\ 1 \leq r < l \leq n,\ r,l \in B_w,\ e_w(r,l) = 1. \tag{37}$$

Applying Lemma 2.4 to the system (36), (37) one can find the minimal possible $1 \leq s_w \leq n,\ s_w \in B_w$ *such that* $x_{s_w} \leq x_l,\ 1 \leq l \leq n,\ l \in B_w$. *The description of* P_w *contains the following inequalities (playing the role of (7), (8), (12), respectively):*

$$x_{s_w} \leq x_i,\ 1 \leq i \leq n, \tag{38}$$

$$m_w(s_w, q) \geq -nM,\ m_w(p, s_w) \leq (n + s_w - p)M, \tag{39}$$

for $1 \leq p < s_w < q \leq n,\ p, q \in S_w$,

$$x_j - x_{s_w} > jM,\ 1 \leq j \leq n,\ j \notin B_w. \tag{40}$$

Thus, the description of P_w *consists of the inequalities (36) - (40) for all possible choices of integers* $m_w(l, n),\ 0 \leq e_w(l, n) \leq 1,\ 1 \leq l < n,\ l \in B_w$ *satisfying (33), (34), (35), provided that* P_w *is not empty.*

Remark 2.14 *In Definition 2.13 it holds either* $s_w = s'$ *or* $s_w = n$ *(the latter holds iff* $x_n < x_{s'}$*).*

This completes the description of all the edges outgoing from the vertex v in the graph G.

3 Description of tropical recurrent sequences via paths in the graph

3.1 Producing a short tropical recurrent sequence along an edge of the graph

In this subsection for any point $(y_1, \ldots, y_n) \in P_v$ we prove the following claim. If a sequence $(y_1, \ldots, y_n, x) \in \mathbb{R}^{n+1}$ satisfies the vector $\vec{a}$ then for exactly one of the edges (v, w) of the graph G it holds that $(y_2, \ldots, y_n, x) \in P_w$. Conversely, for every edge (v, w) of G constructed according to one of Definitions 2.9, 2.11, 2.13 there exists a point $(y_2, \ldots, y_n, x_n) \in P_w$ such that the point $(y_1, \ldots, y_n, x_n) \in \mathbb{R}^{n+1}$ satisfies the vector $\vec{a}$ (for more precise statements see Theorem 3.1).

We assume that a point $(y_1, \ldots, y_n, x_n) \in \mathbb{R}^{n+1}$ satisfies the vector $\vec{a}$. Denote $x_i := y_{i+1}, 1 \leq i \leq n-1$ (see Definition 2.6). According to Definition 2.6 a coordinate $y_{i+1}, 1 \leq i < n$ is bounded on P_v (i.e. $i + 1 \in B$) iff the coordinate x_i is bounded on P_w (i.e. $i \in B_w$). Then the bounded coordinates among $x_1, \ldots, x_{n-1}$ fulfill the inequalities (19), (20), (21) introduced in Definition 2.6.

Consider the case of a singleton $S = \{t\}$. Then $x_n = y_t + a_{t-1} - a_n$. We claim that $(y_2, \ldots, y_n, x_n) \in P_w$ where the edge (v, w) of G is constructed according to Definition 2.9. Recall that $B_w = ((B \setminus \{1\}) - 1) \cup \{n\}$ in this case. The inequalities (25), (23), (24) are fulfilled by the construction in Definition 2.9.

Now we verify (26). First assume that $1 \leq s_w < n$. Since $s \leq s_w + 1$ due to Lemma 1.4, the inequalities (26) for $1 \leq p < q < n$ follow from (8) taking into account that $y_s \leq y_{s_w+1} = x_{s_w}$. It holds $x_n = y_t + a_{t-1} - a_n \leq$

$x_{s_w} + a_{s_w} - a_n \leq x_{s_w} + M$, hence $m_w(s_w, n) \geq -M$ (taking into account the inequalities (23), (24)), which justifies (26) in case $1 \leq s_w < n$. Now assume that $s_w = n$. Lemma 1.4 implies that $x_n \geq y_s - M$, and (26) follows from (8).

Now we verify the inequalities (28). Recall that $j+1 \notin B, 1 \leq j < n$ iff $j \notin B_w$. The inequality $y_{j+1} - y_s > (j+1)M, j+1 \notin B$ (see (12)) implies that $x_j - x_{s_w} > jM$ since $x_{s_w} \leq y_s + M$ (due to Lemma 1.4). This justifies the inequalities (28). Thus, the point $(x_1, \dots, x_n)$ belongs to the polyhedron P_w for an edge (v, w) of G constructed according to Definition 2.9. This proves the claim in case $S = \{t\}$.

Now we study the case when $|S| \geq 2$ and the inequality (31) is true. We claim that in this case $(x_1, \dots, x_n) \in P_w$ where the edge (v, w) of G is constructed according to Definition 2.11. Recall that in this case $B_w = (B \setminus \{1\}) - 1$, and $2 \leq t \leq n$ is the minimal element of $S \setminus \{1\}$. The inequalities (29) are fulfilled according to the construction in Definition 2.11. The inequalities (30) follow from (8) taking into account that $s \leq s_w + 1$ due to Lemma 1.4 and that $y_s \leq y_{s_w+1} = x_{s_w}$ (cf. the similar argument in the case $|S| = 1$ above). The inequalities (32) for $1 \leq j < n$ are justified as above in case $|S| = 1$. The inequality (32) for $j = n$ coincides with (31). Thus, the point $(x_1, \dots, x_n)$ belongs to the polyhedron P_w for an edge (v, w) construced in Definition 2.11. This proves the claim when $|S| \geq 2$ and (31) holds.

Now we assume that $|S| \geq 2$ and (34) hold (in other words, (31) is not true). We claim that $(x_1, \dots, x_n)$ belongs to the polyhedron P_w where the edge (v, w) of G is constructed according to Definition 2.13. Recall that in this case we have $B_w = ((B \setminus \{1\}) - 1) \cup \{n\}$. For the minimal element $2 \leq t \leq n$ of $S \setminus \{1\}$ it holds $x_n \geq x_{t-1} + a_{t-1} - a_n$. The inequalities (36), (37) for suitable $m_w(r, n), e_w(r, n), 1 \leq r < n, r \in B_w$ are fulfilled according to the construction in Lemma 2.4 which we apply to $x_1, \dots, x_n$. Then the inequality $x_n \geq x_{t-1} + a_{t-1} - a_n$ implies (33).

We verify the inequalities (39) (for $m_w(r, l)$ constructed in Definition 2.13 invoking Lemma 2.4). Observe that it holds either $s_w = s'$ or $s_w = n$ (see Remark 2.14). First assume that $s_w = s'$. The equalities (39) for $q < n$ follow from the inequalities (8) taking into account that $s \leq s' + 1 = s_w + 1$ and that $y_s \leq y_{s'+1} = x_{s'} = x_{s_w}$ (cf. the similar argument in the consideration of the case $|S| = 1$ above). The inequality (39) for $q = n$ follows from (34) (taking into account the inequalities (36), (37)). Now assume that $s_w = n$. Lemma 1.4 implies that $x_n \geq y_s - M$, and therefore the inequalities (39) follow from (8) (cf. the similar argument in the consideration of the case $|S| = 1$ above). So, the inequalities (39) are justified.

The inequalities (40) we verify as in the consideration of the case $|S| = 1$ above. The inequalities (35) follow from the inequalities (39) (see the Remark 2.3). Thus, the point $(x_1, \dots, x_n)$ belongs to a polyhedron P_w for an appropriate edge (v, w) of G constructed in Definition 2.13. This completes the proof of the claim.

Conversely, assume that for a point $(x_1, \dots, x_n) \in P_v$ it holds $(x_1, \dots, x_n) := (y_2, \dots, y_n, x_n) \in P_w$ for an edge (v, w) of G constructed according to one of Definitions 2.9, 2.11, 2.13. First, we study the case

i) there exists $t \in S$, $2 \leq t \leq n$. If $S = \{t\}$ then $x_n = x_{t-1} + a_{t-1} - a_n$ (see Definition 2.9). Otherwise, if $|S| \geq 2$ then $x_n \geq x_{t-1} + a_{t-1} - a_n$ (see (31) in case of Definition 2.11 and (33) in case of Definition 2.13). Therefore, the point $(y_1, \dots, y_n, x_n) \in \mathbb{R}^{n+1}$ satisfies the vector $\vec{a}$.

Observe that if the edge (v, w) is constructed according to Definition 2.11 then the values of the coordinate x_n vary in an open infinite interval bounded from below (see (31)). If the edge (v, w) is constructed according to Definition 2.13, and the description of P_w contains an equality $x_l - x_n = m_w(l, n)$ of the form (36) for some $1 \leq l \leq n-1$ then the value of the coordinate x_n is unique. Otherwise, if $e_w(l, n) = 1, 1 \leq l < n, l \in B_w$, the values of the coordinate x_n vary in an open finite interval due to Corollary 2.5.

ii) Now assume that $S = \{1\}$, then the point $(x_1, \dots, x_n) \in P_w$ for an edge (v, w) constructed according to Definition 2.9. If $e(1, l_0) = 0$ for some $2 \leq l_0 \leq n, l_0 \in B$, i.e. the description of P_v contains an equality $y_1 = y_{l_0} + m(1, l_0)$ of the form (10), then the description of P_w contains the equality $x_n + a_n - a_0 = x_{l_0 - 1} + m(1, l_0)$ (see (23)). Hence in this case for any point $(y_2, \dots, y_n, x_n) \in P_w$ the point $(y_1, \dots, y_n, x_n) \in \mathbb{R}^{n+1}$ satisfies the vector $\vec{a}$ (in fact, $(y_2, \dots, y_n, x_n) \in P_w$ implies that $x_n = y_{l_0} + a_0 - a_n + m(1, l_0)$).

Otherwise, if $e(1, l) = 1, 2 \leq l \leq n, l \in B$ then the values of the coordinate x_n such that $(y_2, \dots, y_n, x_n) \in P_w$ vary in an open interval (in this case $e_w(r, n) = 1, 1 \leq r < n, r \in B_w$, see (24)). Observe that the latter interval is finite iff (24) is not void, i.e. $e(1, l) = 1$ for some $2 \leq l \leq n, l \in B$, in other words $B \setminus \{1\} \neq \emptyset$. If $B = \{1\}$ then this interval is infinite bounded from above (see (28) and Remark 2.10). In this case only the point $(y_1, \dots, y_n, x_n = y_n + a_0 - a_n)$ satisfies the vector $\vec{a}$.

Summarizing, we have proved the following theorem.

Theorem 3.1 *Let a point* $(y_1, \dots, y_n) \in P_v$.

If a point $(y_1, \dots, y_n, x_n) \in \mathbb{R}^{n+1}$ *satisfies the vector* $\vec{a}$ *then* $(y_2, \dots, y_n, x_n) \in P_w$ *holds for exactly one edge* (v, w) *of the graph* G *constructed according to Definitions 2.9, 2.11, 2.13.*

Conversely, let $(y_2, \dots, y_n, x_n) \in P_w$ *for an edge* (v, w) *of* G *constructed according to one of Definitions 2.9, 2.11, 2.13.*

i) In case when there exists $t \in S$, $2 \leq t \leq n$ *(see subsection 2.2) the point* $(y_1, \dots, y_n, x_n) \in \mathbb{R}^{n+1}$ *satisfies the vector* $\vec{a}$*. In case of an edge constructed according to*

- *Definition 2.9, the value of* x_n *is unique;*

• *Definition 2.11, the values of x_n vary in an open infinite interval bounded from below;*

• *Definition 2.13, the values of x_n depending on the edge (v, w), can be either unique or vary in an open finite interval.*

ii) If $S = \{1\}$ then only for the value $x_n = y_1 + a_0 - a_n$ the point $(y_1, \dots, y_n, x_n) \in \mathbb{R}^{n+1}$ satisfies the vector $\vec{a}$.

3.2 The polyhedron of tropical recurrent sequences produced along a path of the graph

We consider paths in the graph G and describe how they correspond to the tropical recurrent sequences satisfying the vector $\vec{a}$. Take an arbitrary vertex v_0 as the first vertex in a path and any sequence $y^{(0)} := (y_1^{(0)}, \dots, y_n^{(0)}) \in P_{v_0}$. As in subsection 2.2 consider a subset S. If $|S| = 1$ then there is a unique edge (v_0, w_0) in G outgoing from v_0. In this case one applies Definition 2.9 and obtains a unique $y_{n+1}^{(0)} := x_n^{(0)} \in \mathbb{R}$ such that $(y_2^{(0)}, \dots, y_{n+1}^{(0)}) \in P_{w_0}$ and $(y_1^{(0)}, \dots, y_{n+1}^{(0)})$ satisfies vector $\vec{a}$ (see Theorem 3.1).

Otherwise, if $|S| > 1$ then there are several edges outgoing from v_0. For each edge (v_0, v_1) one applies either Definition 2.11 or Definition 2.13, respectively, and produces $y_{n+1}^{(0)} := x_n^{(0)} \in \mathbb{R}$ such that $(y_2^{(0)}, \dots, y_{n+1}^{(0)}) \in P_{v_1}$ and $(y_1^{(0)}, \dots, y_{n+1}^{(0)})$ satisfies vector $\vec{a}$ (see Theorem 3.1). Recall (see Theorem 3.1 i)) that for certain edges (v_0, v_1) the value $y_{n+1}^{(0)}$ is unique, while for other edges $y_{n+1}^{(0)}$ runs over an open interval.

An edge (v_0, v_1) for which the value $y_{n+1}^{(0)}$ is unique we call *rigid*, otherwise if the values run over an open interval we call an edge *augmenting*. Due to Theorem 3.1 i) the property of an edge to be rigid or augmenting does not depend on a point $y^{(0)}$. Note that in case of S being a singleton, the edge is rigid, while an edge constructed according to Definition 2.11, is augmenting (cf. also Theorem 3.1 ii)).

So far, we have produced a short tropical recurrent sequence $(y_1^{(0)}, \dots, y_{n+1}^{(0)})$ corresponding to an edge of G. We treat this as a base of recursion. Suppose that we have produced by recursion a tropical recurrent sequence $(y_1^{(0)}, \dots, y_{n+k}^{(0)})$ satisfying the vector $\vec{a}$ corresponding to a path T of the length k in G (the length of a path is defined as the number of its edges). Let v be the last vertex of T. Then we apply to v and to the suffix $(y_{k+1}^{(0)}, \dots, y_{n+k}^{(0)})$ of the produced sequence one of Definitions 2.9, 2.11, 2.13 as above in the base of recursion, choosing an edge (v, w) of G and producing $y_{n+k+1}^{(0)}$. Thereby, we get a tropical recurrent sequence $(y_1^{(0)}, \dots, y_{n+k+1}^{(0)})$ satisfying the vector $\vec{a}$ and corresponding to the path T_w obtained by extending T by an edge (v, w). This completes the recursive step.

Summarizing, we have established in this subsection the following proposition.

Proposition 3.2 *For any path in the graph G any produced (by the described recursive process) sequence along this path is a tropical recurrent sequence satisfying vector $\vec{a}$.*

Denote by $Q_T \subset \mathbb{R}^{k+n}$ a polyhedron of all the tropical recurrent sequences which are produced along the path T as described above (see Proposition 3.2). Thus, any produced tropical recurrent sequence satisfies the vector $\vec{a}$. The polyhedron Q_T is presented by the systems of linear equations and linear inequalities produced in Definitions 2.9, 2.11, 2.13, respectively, applied to the edges of the path T (see Theorem 3.1). Observe that when $S \neq \{1\}$ Theorem 3.1 i) implies that for the inequalities describing Q_T just the inequalities describing P_v and P_w suffice, while when $S = \{1\}$ one has to add to the latter inequalities also the equality $x_n = y_1 + a_0 - a_n$ (see Theorem 3.1 ii)).

Observe that for a rigid edge (v, w) the polyhedron $Q_{T_w} \subset \mathbb{R}^{k+n+1}$ is homeomorphic to Q_T, and the homeomorphism is provided by the projection along the last coordinate. For an augmenting edge (v, w) the polyhedron Q_{T_w} is homeomorphic to the cylinder $Q_T \times \mathbb{R}$. In particular, in the latter case $\dim(Q_{T_w}) = \dim(Q_T) + 1$. Summarizing, we have established the following proposition.

Proposition 3.3 *Let T be a finite path of the graph G with an ending vertex v, and T_w be an extension of T by an edge (v, w). If the edge (v, w) is rigid then the polyhedron Q_{T_w} of all the finite tropical recurrent sequences produced along T_w (see Proposition 3.2) is homeomorphic to the polyhedron Q_T, while if (v, w) is augmenting then Q_{T_w} is homeomorphic to the cylinder $Q_T \times \mathbb{R}$.*

3.3 Completeness of the construction of tropical recurrent sequences

Now, conversely to Proposition 3.2, we claim that every tropical recurrent sequence $y := (y_1, y_2, \ldots)$ satisfying the vector $\vec{a}$ emerges along an appropriate path of the graph G (see subsection 3.2).

Proposition 3.4 *i) The union of the polyhedra P_v over all the vertices v of the graph G coincides with $\mathbb{R}^n$.*

ii) For any tropical recurrent sequence $y := (y_1, y_2, \ldots)$ satisfying the vector $\vec{a}$ and a vertex v of G such that $(y_1, \ldots, y_n) \in P_v$ there exists a unique path T of G starting with v such that y is produced along T as described in subsection 3.2 (see Proposition 3.2).

Proof. We prove ii) by recursion. For the base of recursion assume that $y_{s_0} := \min_{1\le j\le n}\{y_j\}$ holds for the minimal possible $1 \le s_0 \le n$. We construct a vertex v_0 of G such that $(y_1,\dots,y_n) \in P_{v_0}$ as follows. Put $B_{v_0} := \{1 \le j \le n \ :\ y_j - y_{s_0} \le jM\}$ (cf. (12)). Applying Lemma 2.4 to the set $\{y_l \ :\ l \in B_{v_0}\}$, one produces integers $m_{v_0}(r,l), e_{v_0}(r,l);\ 1 \le r < l \le n;\ r,l \in B_{v_0}$. Then the inequalities similar to (7) - (12) describe the polyhedron P_{v_0} such that $(y_1,\dots,y_n) \in P_{v_0}$. In particular, this proves i).

For the recursive step suppose that a path T of G of a length k is already constructed such that the sequence $(y_1,\dots,y_{n+k})$ is produced along T as in subsection 3.2 (see Proposition 3.2). Let v be the last vertex of T, then $y^{(k)} := (y_{k+1},\dots,y_{k+n}) \in P_v$. Apply Theorem 3.1 to $y^{(k)}$, this provides a unique edge (v,w) of G such that $(y_{k+2},\dots,y_{k+n+1}) \in P_w$, thus the sequence $(y_1,\dots,y_{n+k+1})$ is produced along the extended path T_w. This completes the proof (by recursion) of ii). $\Box$

Observe that one could choose, perhaps, another initial vertex v' of G such that $(y_1,\dots,y_n) \in P_{v'}$ (the latter inclusion is the only property of v' we require). In fact, one could declare (in an arbitrary way) any coordinate $y_j, 1 \le j \le n$ either bounded on $P_{v'}$ (i.e. $j \in B_{v'}$) or unbounded (i.e. $j \notin B_{v'}$) if it fulfills the inequalities either $jM < y_j - y_{s_0},\ \lfloor y_j - y_{s_0}\rfloor \le (n + s_0 - j)M$ when $1 \le j < s_0$ or $jM < y_j - y_{s_0} \le nM$ when $s_0 < j \le n$ (cf. (8), (12)). Observe that if $y_j - y_{s_0} \le jM$ then y_j should be bounded on $P_{v'}$, i.e. $j \in B_{v'}$ (cf. (12)), while if either $y_j - y_{s_0} \ge (n + s_0 - j)M$ when $1 \le j < s_0$ or $y_j - y_{s_0} > nM$ when $s_0 < j \le n$, then y_j should be unbounded on $P_{v'}$, i.e. $j \notin B_{v'}$ (cf. (8)).

After choosing an initial vertex v_0, the rest of a path T in G is constructed uniquely (see Theorem 3.1 and subsection 3.2). Therefore, each tropical recurrent sequence satisfying the vector $\vec{a}$ corresponds to just a finite number of paths in G as in subsection 3.2 (see Proposition 3.2). Moreover, this number does not exceed the number of vertices in G. Thus, the tropical prevariety of all the tropical recurrent sequences of a length $n+k$ satisfying the vector $\vec{a}$ has the same dimension as the union of polyhedra Q_T over all the paths T of the length k in G.

For a path T in the graph G denote by $d(T)$ the number of augmenting edges in T. By $n(T) \le n$ denote the number of (equivalence) classes of the coordinates in the first vertex of T (see Remark 2.2). Thus, we have established the following theorem taking into account Propositions 3.2, 3.3, 3.4.

Theorem 3.5 *For any vector $\vec{a} := (a_0,\dots,a_n) \in \mathbb{Z}^{n+1}$ the constructed finite directed graph $G := G_a$ satisfies the following properties. For a path T of a length k in G denote by $Q_T \subset \mathbb{R}^{k+n}$ the polyhedron of all the tropical recurrent sequences satisfying the vector $\vec{a}$ and being produced along the path T in G. Then* $\dim(Q_T) = d(T) + n(T)$. *Moreover, the union of polyhedra Q_T*

over all the paths T of the length k coincides with the tropical prevariety of all the tropical recurrent sequences of the length $k+n$ satisfying the vector $\vec{a}$.

4 Calculating the entropy via the graph of tropical recurrent sequences

In this section we study the tropical Hilbert function $d(k) := d_{\vec{a}}(k)$ (see the Introduction). Due to Theorem 3.5 $d(k)$ equals the maximum of $n(T) + d(T)$ over all the paths T of the length $k - n$ in the graph G.

We call a simple cycle in G *optimal* if the quotient of the number of augmenting edges in the cycle to the length of the cycle is the maximal among the simple cycles. This maximal quotient we denote by $\mathcal{H} := \mathcal{H}_{\vec{a}}$. Later we show that $\mathcal{H}$ equals the entropy $H := H_{\vec{a}}$ (Corollary 4.3). Clearly, $\mathcal{H}$ equals the maximum of the same quotient over all the cycles in G (not necessary, simple).

First, we prove a lower bound on the tropical Hilbert function $d(k)$.

Lemma 4.1 *$d(k) \geq \mathcal{H}(k-n)$.*

Proof. Take an optimal simple cycle U in G. Denote the length of U by L and the number of augmenting edges in U by m, then $\mathcal{H} = m/L$. Assign to each augmenting edge of U the number $1 - \mathcal{H}$ and to each rigid edge the number $-\mathcal{H}$. Then the sum of all these numbers equals 0. Due to the lemma about leaders [10] there exists a vertex u of U such that the sum of the assigned numbers along any subpath of U starting with u, is non-negative.

Consider a path T of a length $k - n$ starting with the vertex previous to u in U and following the cycle U (i. e. T can wind the cycle U several times). According to Theorem 3.5 $\dim(Q_T) \geq \mathcal{H}(k-n)$. $\square$

Denote by V the number of vertices in G. Now we proceed to an upper bound on the tropical Hilbert function.

Lemma 4.2 *$d(k) \leq \mathcal{H}k + (1-\mathcal{H})(V+n)$.*

Proof. Consider a path T of a length L in G. Take a vertex v_1 of G which occurs in T at least twice (provided that it does exist). Then the subpath of T between these two occurrings constitues a cycle of a length L_1. Remove this cycle from T, and continue removing cycles from the resulting paths, while it is possible. Let $L_2, L_3, \dots, L_q$ be the lengths of the consecutively removed cycles. Then

$$d(T) \leq \mathcal{H}(L_1 + \cdots + L_q) + (L - L_1 - \cdots - L_q) \leq \mathcal{H}(L_1 + \cdots + L_q) + V$$

(cf. Theorem 3.5). Therefore, $d(k) \leq \mathcal{H}(k-n) + (1-\mathcal{H})V + n$ taking into the account that $L - L_1 - \cdots - L_q \leq V$. $\Box$

Lemmata 4.1, 4.2 imply the following corollary (see (2)).

Corollary 4.3 *$\mathcal{H} = H$.*

Remark 4.4 *The entropy H is a rational number.*

5 Quasi-linearity of the tropical Hilbert function

Lemma 5.1 *Any path T of a length $k-n$ greater than $V^2(V+n)+V$ in the graph G such that $n(T)+d(T) = d(k)$, contains a vertex from an optimal simple cycle.*

Proof. First consider the case when $H = 0$. Then any simple cycle in G is optimal, and the statement of the lemma is true even with a better bound $k - n > V$. Thus, from now on in the proof of the lemma we assume that $H > 0$.

Recall that according to Theorem 3.5 it holds that $\dim(Q_T) = n(T)+d(T)$. Slightly modifying the construction from the proof of Lemma 4.2, take the first repetition of some vertex v in T (provided that it is possible). Then the subpath of T between these two occurrences of v constitutes a simple cycle of a length L_1 in T. Remove this simple cycle from T and continue removing simple cycles from the resulting paths in a similar way, while it is possible. Denote by $L_2, L_3, \dots, L_q$ the lengths of the consecutively removed simple cycles. Denote by B the denominator of H (cf. Remark 4.4), obviously $B \leq V$ (see section 4).

Assume the contrary to the claim of the lemma. Then

$$d(T) \leq H(L_1+\cdots+L_q) - q/B + (k-n-L_1-\cdots-L_q) \leq H(k-n) - q/B + V.$$

The first inequality follows from the statement that the amount of augmenting edges in the cycle with the length L_i, $1 \leq i \leq q$ is not greater than $H \cdot L_i - \frac{1}{B}$. Making use of Lemma 4.1 we obtain an inequality $q/B \leq V + n$, hence $q \leq V(V+n)$. The path T consists of q simple cycles and a path without cycles. Each simple cycle has length not more than V as well as the path without cycles. Therefore, $k - n \leq V^2(V+n) + V$ since $L_1, \dots, L_q \leq V$. $\Box$

Denote by R the least common multiple of the lengths of all the optimal simple cycles.

Lemma 5.2 *For any $k > (V^2+1)(V+n)$ we have $d(k+R) \geq d(k) + HR$.*

Proof. Take a path T of the length $k-n$ in G such that $n(T)+d(T) = d(k)$ (cf. Theorem 3.5). Due to Lemma 5.1 T contains a vertex v which belongs to an optimal simple cycle C of a length c. Glue in the path T at the vertex v the number R/c of copies of the cycle C, the resulting path of the length $k-n+R$ denote by T_1. In other words, in T_1 one follows first T till the vertex v, then there are R/c windings of the cycle C (finishing at v), finally after that one again follows path T (starting at v). Clearly, $d(T_1) = d(T) + (R/c)Hc$. $\Box$

Lemma 5.3 *If for some* $k > (V^2 + 1)(V + n)$ *we have*

$$d(k + iR) = d(k) + HiR,\ 0 \le i \le V((1 - H)V + n + 1)$$

then $d(k + jr) = d(k) + HjR$ *for any* $j \ge 0$.

Proof. Due to Lemma 5.2 it holds $d(k+jR) \ge d(k)+HjR$. Suppose that

$$d(k + jR) > d(k) + HjR \tag{41}$$

for some $j > V((1-H)V + n + 1)$, and take the minimal such j. There exists a path T of the length $k + jR - n$ in G for which $n(T) + d(T) = d(k + jR)$. For $0 \le i \le V((1-H)V+n+1)$ denote by T_i the beginning of T of the length $k + iR - n$. One can represent the path $T = T_i\overline{T_i}$ as a concatenation of two paths.

There exists a subsequence $0 \le i_0 < i_1 < \cdots < i_{(1-H)V+n+1} \le V((1 - H)V + n + 1)$ such that each path T_{i_l}, $0 \le l \le (1 - H)V + n + 1$ ends with the same vertex v of G. Assume that there exists $0 \le l \le (1 - H)V + n$ for which it holds that

$$d(T_{i_{l+1}}) \le d(T_{i_l}) + H(i_{l+1} - i_l)R. \tag{42}$$

Then we consider a concatenation $\overline{T} := T_{i_l}\overline{T_{i_{l+1}}}$ being a path of the length $k + jR - n - (i_{l+1} - i_l)R$ in G. We obtain

$$d(\overline{T}) = d(T) + d(T_{i_l}) - d(T_{i_{l+1}}) > d(k) + HjR - H(i_{l+1} - i_l)R$$

due to (41), (42), and we get a contradiction with the choice of the minimal j (see (41)).

Thus, for every $0 \le l \le (1 - H)V + n$ we have

$$d(T_{i_{l+1}}) \ge d(T_{i_l}) + H(i_{l+1} - i_l)R + 1.$$

Summing up these inequalities for $0 \le l \le (1 - H)V + n$ we conclude that

$$d(T_{i_{(1-H)V+n+1}}) - d(T_{i_0}) \ge H(i_{(1-H)V+n+1} - i_0)R + (1 - H)V + n + 1$$

which contradicts to Lemmata 4.1, 4.2. $\Box$

Note that $V < (O(Mn))^n$ (see Definition 2.1) and $R < \exp(V)$. Lemmata 4.1, 4.2, 5.2, 5.3 entail the following theorem.

Theorem 5.4 *For $k > (V^2+1)(V+n)+V((1-H)V+n+1)^2$ the tropical Hilbert function $d_{\vec{a}}(k)$ of the integer vector $\vec{a} = (a_0, \dots, a_n)$ with an amplitude M (4) fulfils the following equality:*

$$d_{\vec{a}}(k+R) = d_{\vec{a}}(k) + HR.$$

for some integer $R < \exp((O(Mn))^n)$ where $H := H_{\vec{a}}$ is the tropical entropy of the vector $\vec{a}$.

We call a function (from the natural numbers to themselves) *quasi-linear* if it is a sum of a linear function and a periodic function with an integer period.

Corollary 5.5 *The tropical Hilbert function*

$$d(k) = Hk + r(k)$$

is quasi-linear for $k > (Mn)^{O(n)}$ where $r(k)$ is a periodic function with an integer period less than $\exp((O(Mn))^n)$.

Now we illustrate the constructions in sections 2 - 5 for three vectors $\vec{a} \in \mathbb{Z}^3$, thus $n = 2$ (we use notations from sections 2 - 5). In each example we construct a graph $G := G_{\vec{a}}$ whose vertices v are in a bijective correspondence with polygons $P_v \subset \mathbb{R}^2$. Denote by (y, x) coordinates in $\mathbb{R}^2$.

Example 5.6 *First, we consider a vector $\vec{a} := (0, 0, 0)$. Therefore, $M = 0$.*

The graph G contains three vertices v, we list their corresponding polygons $P_v \subset \mathbb{R}^2$:

$$P_{v_\infty} = \{x - y > 0\},\ P_{v_0} = \{x - y = 0\},\ P_{v_{-\infty}} = \{x - y < 0\}.$$

Note that $B_{v_\infty} = S_{v_\infty} = \{1\}$, $B_{v_0} = S_{v_0} = \{1, 2\}$, $B_{v_{-\infty}} = S_{v_{-\infty}} = \{2\}$ (see Definitions 2.1, 2.8).

The edges of G are the following:

$$(v_\infty, v_{-\infty}), (v_0, v_0), (v_0, v_\infty), (v_{-\infty}, v_0).$$

The edges $(v_\infty, v_{-\infty}), (v_{-\infty}, v_0)$ are constructed according to Definition 2.9, the edge (v_0, v_0) is constructed according to Definition 2.13, and the edge (v_0, v_∞) is constructed according to Definition 2.11. The only augmenting edge is (v_0, v_∞).

An optimal cycle is $v_{-\infty}, v_0, v_\infty$. It contains a single augmenting edge, hence the entropy $H = 1/3$ (see Corollary 4.3, also [4]).

Consider a path

$$T = \underbrace{(v_{-\infty}, v_0, v_\infty) \cdots (v_{-\infty}, v_0, v_\infty)}_{p}$$

of the length $3p-1$ in G. For any reals $z_0 < z_1, \dots, z_{p+1}$ take the point $(u_1, \dots, u_{3p+1})$ such that $u_{3l-1} = u_{3l} = z_0, 1 \le l \le p, u_{3j-2} = z_j, 1 \le j \le p+1$. Then $(u_1, \dots, u_{3p+1})$ belongs to the polyhedron $Q_T \subset \mathbb{R}^{3p+1}$ (see Theorem 3.5). Vice versa, one can check that any point of Q_T has this form. Therefore, $\dim Q_T = p+2$. One can verify that for every $1 \le q \le 3p-1$ the maximum $\max_{T'}\{\dim Q_{T'}\}$, where T' ranges over all paths in G of the length q, is attained at the prefix of T of the length q. Hence, the tropical Hilbert function $d(k) = \lfloor (k-1)/3 \rfloor + 2, k \ge 2$ (cf. Theorem 3.5, Corollary 5.5).

Example 5.7 *Let a vector $\vec{a} := (1,0,1)$, therefore $M = 1$. If to follow the bounds (8) formally, then the graph $G := G_{\vec{a}}$ should have 13 vertices. We simplify the construction of G imposing stronger bounds than (8), namely, $|m(1,2)| \le 1$. One can verify that in case of the chosen $\vec{a}$ this simplification still computes the entropy $H := H_{\vec{a}}$ and the tropical Hilbert function $d := d_{\vec{a}}$.*

The graph G has 7 vertices v, we list their corresponding polygons $P_v \subset \mathbb{R}^2$:

$$P_{v_\infty} = \{x-y > 1\}, P_{v_1} = \{x-y = 1\}, P_{v_{0.5}} = \{0 < x-y < 1\}, P_{v_0} = \{x-y = 0\},$$

$$P_{v_{-0.5}} = \{-1 < x-y < 0\}, P_{v_{-1}} = \{x-y = -1\}, P_{v_{-\infty}} = \{x-y < -1\}$$

(see Definition 2.1).

Note that the set $B_{v_\infty} = \{1\}$, $B_{v_{-\infty}} = \{2\}$, for all other vertices v of G the set $B_v = \{1,2\}$ (see Definition 2.1), the set $S_{v_\infty} = \{1\}$, $S_{v_1} = \{1,2\}$, for all other vertices v of G the set $S_v = \{2\}$ (see Definition 2.8).

The graph G has 12 following edges:

$$e_1 := (v_\infty, v_{-\infty}),\ e_2 := (v_1, v_{-1}), e_3 := (v_1, v_{-0.5}),\ e_4 := (v_1, v_0),$$

$$e_5 := (v_1, v_{0.5}),\ e_6 := (v_1, v_1), e_7 := (v_1, v_\infty),\ e_8 := (v_{0.5}, v_{-1}),$$

$$e_9 := (v_0, v_{-1}),\ e_{10} := (v_{-0.5}, v_{-1}),\ e_{11} := (v_{-1}, v_{-1}),\ e_{12} := (v_{-\infty}, v_{-1}).$$

Observe that the edges $e_1, e_8, e_9, e_{10}, e_{11}, e_{12}$ (so, the edges not outgoing from the vertex v_1) are constructed according to Definition 2.9, the edge e_7 is constructed according to Definition 2.11, the edges e_2, e_3, e_4, e_5, e_6 are constructed according to Definition 2.13. Just the edges e_3, e_5, e_7 are augmenting (cf. Proposition 3.3).

The graph G has the source v_1 (i.e. the vertex without incoming edges), and the sink v_{-1} (i.e. the vertex without outgoing edges). Since no cycle in G contains an augmenting edge, the entropy $H = 0$ due to Corollary 4.3. This also follows from [4].

Every path in G starts with several loops in v_1 (perhaps, empty), ends with several loops in v_{-1} (perhaps, empty), and has at most two edges inbetween. Consider a path

$$T_0 = \underbrace{v_1 \dots v_1}_{p} v_\infty v_{-\infty} \underbrace{v_{-1} \dots v_{-1}}_{q}.$$

Then the polyhedron $Q_{T_0} \subset \mathbb{R}^{p+q+3}$ (see Theorem 3.5) is described as follows. For arbitrary $z_0, z_1 > z_0 + p + 1 \in \mathbb{R}$ the point

$$(z_0, z_0 + 1, \dots, z_0 + p, z_1, z_0 + p, z_0 + p - 1, \dots, z_0 + p - q - 1)$$

belongs to Q_{T_0}. Vice versa, these points exhaust Q_{T_0}. Therefore, $\dim Q_{T_0} = 2$, it holds $n(T_0) = d(T_0) = 1$ (cf. Theorem 3.5). One can verify that any path T in G provides a polyhedron Q_T of dimension at most 2. Indeed, if T starts with the vertex v_1 then $n(T) = 1, d(T) \leq 1$ because any path contains at most one augmenting edge. Otherwise, if T starts with with a vertex different from v_1 then $n(T) \leq 2, d(T) = 0$. Thus, the Hilbert function $d(k) = 2, k \geq 2$ (cf. Theorem 3.5, Corollary 5.5).

Example 5.8 *Now consider a vector $\vec{a} := (0, 1, 0)$. Thus, $M = 1$. Again as in Example 5.7 we simplify the construction of the graph G. It contains the same 7 vertices as in Example 5.7.*

It holds $B_{v_\infty} = \{1\}, B_{v_{-\infty}} = \{2\}$, for all other vertices v of G it holds $B_v = \{1, 2\}$. It holds $S_{v_{-1}} = \{1, 2\}, S_{v_{-\infty}} = \{2\}$, for all other vertices v of G it holds $S_v = \{1\}$.

The graph G has the following edges:

$$e_1 := (v_\infty, v_{-\infty}),\ e_2 := (v_1, v_{-1}),\ e_3 := (v_{0.5}, v_{-0.5}),\ e_4 := (v_0, v_0),$$

$$e_5 := (v_{-0.5}, v_{0.5}),\ e_6 := (v_{-1}, v_1),\ e_7 := (v_{-1}, v_\infty),\ e_8 := (v_{-\infty}, v_1).$$

The edge e_6 is constructed according to Definition 2.13, the edge e_7 is constructed according to Definition 2.11, all other edges are constructed according to Definition 2.9. The only augmenting edge is e_7.

The unique simple cycle of G which contains the augmenting edge is $(v_{-\infty}, v_1, v_{-1}, v_\infty)$. Therefore, the entropy $H = 1/4$ (see also [4]).

Consider a path

$$T := \underbrace{(v_{-\infty}, v_1, v_{-1}, v_\infty), \dots, (v_{-\infty}, v_1, v_{-1}, v_\infty)}_{p}$$

of the length $4p - 1$. Then for any reals $z_0 + 1 < z_1, \dots, z_{p+1}$ take the point $(u_1, \dots, u_{4p+1})$ such that $u_{4l+1} = z_{l+1}, 0 \leq l \leq p$, $u_{4j+2} = u_{4j+4} = u_{4j+3} - 1 = z_0, 0 \leq j < p$. The point $(u_1, \dots, u_{4p+1})$ belongs to the polyhedron Q_T (see Theorem 3.5). Vice versa, one can check that any point of Q_T has the described form, hence $\dim Q_T = p + 2$.

One can verify that for every $1 \leq q \leq 4p - 1$ the maximum in $\max_{T'}\{\dim Q_{T'}\}$, where T' ranges over all paths of the length q of G, is attained at the prefix of the length q of T. Thus, the tropical Hilbert function $d(k) = \lfloor (k-1)/4 \rfloor + 2$ (cf. Theorem 3.5, Corollary 5.5).

Remark 5.9 *In case when the tropical entropy $H = H_{\vec{a}} = 0$ Lemma 4.2 implies that $d(k) = const$ for sufficiently large k, taking into account that $d(k)$ is a non-decreasing function. Recall (see [4]) that Newton polygon $\mathcal{N}(\vec{a}) \subset \mathbb{R}^2$ for a vector $\vec{a} = (a_0, \ldots, a_n)$ is defined as the convex hull of the rays $\{(i, y) : y \geq a_i\}$ for $0 \leq i \leq n$. We say that the vector $\vec{a}$ is regular [4] if each point (i, a_i) with $a_i < \infty$ is a vertex of $\mathcal{N}(\vec{a})$, and the indices i for which $a_i < \infty$ constitute an arithmetic progression. It was proved in [4, Corollary 5.7] that $H_{\vec{a}} = 0$ iff $\vec{a}$ is regular. For regular $\vec{a}$ in case when each (i, a_i), $0 \leq i \leq n$ is a vertex of $\mathcal{N}(\vec{a})$ one can deduce from [3, Corollary 4.9] that $d(k) = k$ for $k \leq n$ and $d(k) = n$ for $k \geq n$.*

6 Tropical boolean vectors

As we already mentioned it would be interesting to extend the results of the paper to arbitrary vectors $\vec{a}$ involving infinite coordinates. The first step to implementing this idea can be considered as the construction of an appropriate graph $G_{\vec{a}}$ (cf. section 2) for the case when $\vec{a}$ is a tropical boolean vector (see the Introduction). In this case, the construction looks simpler and contains less technical details comparing to the case considered in the previous sections 2, 3.

6.1 Construction of a graph for tropical boolean vectors

We call a vector $\vec{a} = (a_0, \ldots, a_n)$ *tropical boolean vector* if for all $0 \leq i \leq n$ it holds either $a_i = 0$ or $a_i = \infty$, and $a_0 = a_n = 0$.

Below we construct a directed graph $G := G_{\vec{a}}$. First we define the vertices of G.

Definition 6.1 *Every vertex v of G corresponds to an (open in its linear hull) nonempty polyhedron $P := P_v \subset \mathbb{R}^n$ with the condition that for each pair of coordinates $y_r, y_t, 1 \leq r, t \leq n$ a system of equations and strict inequalities defining P contains either $y_r = y_t$ or $y_r < y_t$.*

These linear restrictions set the order on the coordinates $y_1, \ldots, y_n$. The polyhedra $\{P_v\}_v$ constitute a partition of $\mathbb{R}^n$. Now we define the edges of G.

Definition 6.2 *There is an edge (v, w) in G iff there exist vectors $(y_0, \ldots, y_{n-1}) \in P_v, (y_1, \ldots, y_n) \in P_w$ such that the sequence $(y_0, \ldots, y_n) \in \mathbb{R}^{n+1}$ satisfies the vector $\vec{a}$.*

Similar to subsection 2.2 for a vertex v of G define $S := S_v$ as a set of $0 \leq t \leq n-1$ such that $y_t = a_t + y_t = \min_{0 \leq j \leq n-1}\{a_j + y_j\}$. In other words,

$t \in S$ iff $a_t = 0$ and $y_t \leq y_j$ for each $0 \leq j \leq n-1$ such that $a_j = 0$. The definition of S does not depend on a choice of a point $(y_0, \dots, y_{n-1}) \in P_v$ (cf. Lemma 2.7). The following theorem is similar to Theorem 3.1.

Theorem 6.3 *Let v be a vertex of the graph $G := G_{\vec{a}}$ (see Definitions 6.1, 6.2) and $(y_0, \dots, y_{n-1}) \in P_v$.*

If a point $(z_0, \dots, z_{n-1}) \in P_v$ and a sequence $(z_0, \dots, z_n) \in \mathbb{R}^{n+1}$ satisfies the vector $\vec{a}$ then $(z_1, \dots, z_n) \in P_w$ for some edge (v, w) of G.

Conversely, let $(y_1, \dots, y_n) \in P_w$ for an edge (v, w) of G, and the sequence $(y_0, \dots, y_n) \in \mathbb{R}^{n+1}$ satisfy the vector $\vec{a}$. If $t \in S$ for some $0 \leq t \leq n-1$ then $y_n \geq y_t$.

i) Let $t \in S$ for some $1 \leq t \leq n-1$ and $y_r = y_n$ for some $1 \leq r \leq n-1$. Assume that a point $(z_0, \dots, z_{n-1}) \in P_v$. If a point $(z_1, \dots, z_{n-1}, z) \in P_w$ then $z = z_r$. The point $(z_1, \dots, z_{n-1}, z_r) \in P_w$, and the sequence $(z_0, \dots, z_{n-1}, z_r) \in \mathbb{R}^{n+1}$ satisfies the vector $\vec{a}$.

ii) Let $t \in S$ for some $1 \leq t \leq n-1$. Assume that $y_{r_1} < y_n$ for some $1 \leq r_1 \leq n-1$ and for every $1 \leq r \leq n-1$ neither $y_{r_1} < y_r \leq y_n$ nor $y_n \leq y_r$ holds. Then for any point $(z_0, \dots, z_{n-1}) \in P_v$ if a point $(z_1, \dots, z_{n-1}, z) \in P_w$ then $z_{r_1} < z$ and for every $1 \leq r \leq n-1$ neither $z_{r_1} < z_r \leq z_n$ nor $z_n \leq z_r$ holds. For any $z_{r_1} < z_n \in \mathbb{R}$ the point $(z_1, \dots, z_n) \in P_w$ and the sequence $(z_0, \dots, z_n) \in \mathbb{R}^{n+1}$ satisfies the vector $\vec{a}$.

iii) Let $t \in S$ for some $1 \leq t \leq n-1$. Assume that $y_{r_1} < y_n < y_{r_2}$ for some $1 \leq r_1, r_2 \leq n-1$, and for every $1 \leq r \leq n-1$ neither $y_{r_1} < y_r \leq y_n$ nor $y_n \leq y_r < y_{r_2}$ holds. Then for any point $(z_0, \dots, z_{n-1}) \in P_v$ if a point $(z_1, \dots, z_{n-1}, z) \in P_w$ then $z_{r_1} < z < z_{r_2}$ and for every $1 \leq r \leq n-1$ neither $z_{r_1} < z_r \leq z_n$ nor $z_n \leq z_r < z_{r_2}$ holds. For any $z_n \in \mathbb{R}, z_{r_1} < z_n < z_{r_2}$ the point $(z_1, \dots, z_n) \in P_w$ and the sequence $(z_0, \dots, z_n) \in \mathbb{R}^{n+1}$ satisfies the vector $\vec{a}$.

iv) Let $S = \{0\}$. Then $y_n = y_0$. For any point $(z_0, \dots, z_{n-1}) \in P_v$ the point $(z_1, \dots, z_{n-1}, z_0) \in P_w$ and the sequence $(z_0, \dots, z_{n-1}, z_0) \in \mathbb{R}^{n+1}$ satisfies the vector $\vec{a}$.

Proof. An informal idea of the proof is to transfer inequalities on the differences between the coordinates y to the corresponding inequalities on the coordinates z, and back.

Let $(z_0, \dots, z_{n-1}) \in P_v$ and a sequence $(z_0, \dots, z_{n-1}, z_n) \in \mathbb{R}^{n+1}$ satisfy the vector $\vec{a}$. Assume that $t \in S$ for some $1 \leq t \leq n-1$, then

$$z_t = a_t + z_t = \min_{0 \leq j \leq n} \{a_j + z_j\}. \tag{43}$$

First, consider the case when $t \in S$ for some $1 \leq t \leq n-1$ and $z_n = z_r$ for some $1 \leq r \leq n$ (cf. i)). Then the sequence $(y_0, \dots, y_{n-1}, y_n = y_r) \in \mathbb{R}^{n+1}$ also satisfies the vector $\vec{a}$. Indeed, (43) implies that $y_t = a_t + y_t = \min_{0 \le j \le n}\{a_j + y_j\}$. Therefore, due to Definition 6.2 there exists an edge (v, w) of G such that $(y_1, \dots, y_{n-1}, y_n = y_r) \in P_w$. Hence $(z_1, \dots, z_{n-1}, z_n = z_r) \in P_w$ as well. This proves the first statement of the theorem in the case under consideration.

Now consider the case when $t \in S$ for some $1 \le t \le n-1$ and $z_n > z_r$ for each $1 \le r \le n-1$ (cf. ii)). Then for any $y > \max_{1 \le j \le n-1}\{y_j\}$ the sequence $(y_0, \dots, y_{n-1}, y) \in \mathbb{R}^{n+1}$ satisfies the vector $\vec{a}$. Indeed, (43) implies that $y_t = a_t + y_t = \min\{\min_{0 \le j \le n-1}\{a_j + y_j\}, y\}$. Due to Definition 6.2 there exists an edge (v, w) (independent of y) of G such that $(y_1, \dots, y_{n-1}, y) \in P_w$. Hence $(z_1, \dots, z_{n-1}, z_n) \in P_w$ as well. This proves the first statement of the theorem in the case under consideration.

The case when $t \in S$ for some $1 \le t \le n-1$ and $z_{r_1} < z_n < z_{r_2}$ for some $1 \le r_1, r_2 \le n-1$ such that for each $1 \le r \le n-1$ neither $z_{r_1} < z_r \le z_n$ nor $z_n \le z_r < z_{r_2}$ holds (cf. iii)) can be studied in a similar manner as the previous case.

Finally, consider the case $S = \{0\}$ (cf. iv)). Then $z_0 < z_l$ for each $1 \le l \le n-1$ for which $a_l = 0$. Therefore, $z_n = z_0$. Hence the sequence $(y_0, \dots, y_{n-1}, y_0) \in \mathbb{R}^{n+1}$ satisfies the vector $\vec{a}$. Due to Definition 6.2 there exists an edge (v, w) of G such that $(y_1, \dots, y_{n-1}, y_0) \in P_w$. Therefore $(z_1, \dots, z_{n-1}, z_0) \in P_w$ as well. This proves the first statement of the theorem.

One can directly verify the second statement of the theorem. $\Box$

Corollary 6.4 *The edges of the graph G (see Definition 6.2) do not depend on choices of points $(y_0, \dots, y_{n-1}) \in P_v$.*

Remark 6.5 *Let an edge (v, w) fulfill the assumptions of one of the items Theorem 6.3 i), ii), iii) and a point $(z_0, \dots, z_{n-1}) \in P_v$. Then for any $z \in \mathbb{R}$ such that $(z_1, \dots, z_{n-1}, z) \in P_w$ the sequence $(z_0, \dots, z_{n-1}, z) \in \mathbb{R}^{n+1}$ satisfies the vector $\vec{a}$. In contrast, in case of Theorem 6.3 iv) only for the value $z = z_0$ it holds that the sequence $(z_0, \dots, z_{n-1}, z)$ satisfies the vector $\vec{a}$ (cf. Theorem 3.1).*

6.2 The polyhedron of tropical recurrent sequences produced along a path of the graph

Consider an arbitrary path T of a length k with vertices $v_0, \dots, v_k$ in the graph $G_{\vec{a}}$. Similar to subsection 2.2 we describe a recursive process producing along T tropical recurrent sequences satisfying the vector $\vec{a}$. For the first vertex v_0 take any vector $(y_1, \dots, y_n) \in P_{v_0}$. Assume by recursion that a tropical recurrent sequence $(y_1, \dots, y_{k+n})$ is already produced along T. Then $(y_{k+1}, \dots, y_{k+n}) \in P_{v_k}$. Take an edge (v_k, w) of G and denote by T_w the extension of T by (v_k, w). We choose $y_{k+n+1} \in \mathbb{R}$ such that $(y_{k+2}, \dots, y_{k+n+1}) \in P_w$ and the sequence $(y_{k+1}, \dots, y_{k+n+1}) \in \mathbb{R}^{n+1}$ satisfies the vector $\vec{a}$. Thus, the tropical recurrent sequence $(y_1, \dots, y_{k+n+1})$ is produced along T_w. Theorem 6.3 justifies that a required y_{k+n+1} exists and moreover, Theorem 6.3 describes all possible y_{k+n+1}. This completes the description of the recursive process.

Denote by $Q_T \subset \mathbb{R}^{k+n}$ the set of all the tropical recurrent sequences produced along T by the described recursive process. One can define Q_T by imposing linear inequalities for each edge of T. Say, for an edge $(v_i, v_{i+1}), 0 \leq i \leq k-1$ we impose that the point $(y_{i+1}, \dots, y_{i+n+1})$ belongs to P_{v_i}, the point $(y_{i+2}, \dots, y_{i+n+2})$ belongs to $P_{v_{i+1}}$. This suffices for edges (v_i, v_{i+1}) fulfilling the items Theorem 6.3 i), ii), iii). In case of Theorem 6.3 iv) one has to impose an extra condition that $y_{i+1} = y_{i+n+2}$, i.e. the sequence $(y_{i+1}, \dots, y_{i+n+2}) \in \mathbb{R}^{n+1}$ satisfies the vector $\vec{a}$. Thus, Q_T is (an open in its linear hull) polyhedron.

If an edge (v_i, v_{i+1}) fulfills one of the items Theorem 6.3 i), iv) we call the edge *rigid*, otherwise, if the edge fulfills one of the items Theorem 6.3 ii), iii) we call the edge *augmenting*. Similar to subsection 2.2 when the edge (v_k, w) is rigid the value of y_{k+n+1} is unique, while when the edge is augmenting the values of y_{k+n+1} vary in an open interval. Therefore, when the edge (v_k, w) is rigid the polyhedron Q_{T_w} is homeomophic to Q_T, while when the edge is augmenting the polyhedron Q_{T_w} is homeomophic to $Q_T \times \mathbb{R}$.

Conversely, Theorem 6.3 implies that any tropical recurrent sequence satisfying the vector $\vec{a}$ emerges along a suitable path of G in the described above recursive process. Thus, the tropical prevariety of all tropical recurrent sequences of a length $k+n$ satisfying the vector $\vec{a}$ coincides with the union of polyhedra Q_T over all the paths of the length k in G.

For a path T in the graph G denote by $d(T)$ the number of augmenting edges in T. By $n(T) \leq n$ denote the number of the pairwise distinct coordinates in $(y_1, \dots, y_n) \in P_{v_0}$ for the first vertex v_0 of T. We summarize the proved above in the following theorem which is analogous to the Theorem 3.5 for the case when $\vec{a}$ is a tropical boolean vector.

Theorem 6.6 *For any tropical boolean vector $\vec{a} := (a_0, \dots, a_n)$ (i. e. $a_0 = a_n = 0$ and each a_i, $0 \leq i \leq n$ equals either 0 or ∞) a finite directed graph $G := G_{\vec{a}}$ is constructed with the following properties. For an arbitrary path T of a length k in G denote by $Q_T \subset \mathbb{R}^{k+n}$ the polyhedron of all the tropical recurrent sequences satisfying the vector $\vec{a}$ and corresponding (as described above in this subsection) to the path T in G. Then $\dim(Q_T) = d(T) + n(T)$. Moreover, the union of polyhedra Q_T over all the paths T of the length k coincides with the tropical prevariety of all the tropical recurrent sequences of the length $k+n$ satisfying the vector $\vec{a}$.*

Now let us notice that all the arguments presented in sections 4 and 5 for the graph constructed in section 2 are also true in the case of tropical boolean vectors. Indeed, both definitions of $n(T)$ and $d(T)$ and thereby, Hilbert function $d_{\vec{a}}(s)$ coincide, respectively, with the definitions for the case when $\vec{a}$

has a finite amplitude. Moreover, an analogue of Theorem 3.5 holds in the tropical boolean case (Theorem 6.6). As all the statements from sections 4 and 5 (except of Theorem 5.4 and Corollary 5.5) depend only on $d_{\vec{a}}(s)$ and on Theorem 3.5, we can formulate the following corollaries.

Corollary 6.7 *Lemmata 4.1, 4.2, 5.1, 5.2, 5.3 and Corollary 4.3 hold when $\vec{a}$ is a tropical boolean vector.*

Proof. Follows from the proofs of the mentioned statements. □

Corollary 6.8 *Theorem 5.4 and Corollary 5.5 hold when $\vec{a}$ is a tropical boolean vector putting in the bounds $M = 1$.*

Proof. From subsection 6.1 it follows that the number of vertices V in G is less than the amount of orders on an n-element set, hence it is less than n^n. Thus, we can put $M = 1$ in the bounds. The remaining part of the proof is literally as in the proofs of the mentioned statements. □

Let us note that for any tropical boolean vector $\vec{a}$ every vertex of $G_{\vec{a}}$ could be presented as a sequence of numbers from 0 to n which reflects the order between the coordinates of the corresponding polyhedron. For example polyhedron $P_v \subset \mathbb{R}^3$ defined by a system of equalities and inequalities $\{y_1 < y_2,\ y_1 < y_3,\ y_2 = y_3\}$ could be presented as sequence $\{0, 1, 1\}$.

Example 6.9 *Now we illustrate constructions in this section for vector $\vec{a} = (0, 0, \infty, 0)$.*

It is not hard to see that there are 13 different orderings on 3 coordinates. We list corresponding polyhedra (see Definition 6.1) $P_v \subset \mathbb{R}^3$:

$$\begin{gathered}
P_{v_{0,0,0}} = \{y_1 = y_2 = y_3\},\ P_{v_{1,0,0}} = \{y_1 > y_2,\ y_2 = y_3\},\\
P_{v_{0,0,1}} = \{y_1 = y_2,\ y_2 < y_3\},\ P_{v_{0,1,0}} = \{y_1 < y_2,\ y_1 = y_3\},\\
P_{v_{1,0,1}} = \{y_1 > y_2,\ y_1 = y_2\},\ P_{v_{1,1,0}} = \{y_1 = y_2,\ y_2 > y_3\},\\
P_{v_{0,1,1}} = \{y_1 < y_2,\ y_2 = y_3\},\ P_{v_{0,2,1}} = \{y_1 < y_2,\ y_1 < y_3\},\\
P_{v_{1,2,0}} = \{y_1 > y_3,\ y_2 > y_1\},\ P_{v_{2,1,0}} = \{y_1 > y_2,\ y_2 > y_3\},\\
P_{v_{2,0,1}} = \{y_1 > y_3,\ y_2 < y_3\},\ P_{v_{0,1,2}} = \{y_1 < y_2,\ y_2 < y_3\},\\
P_{v_{1,0,2}} = \{y_1 > y_2,\ y_1 < y_3\}.
\end{gathered}$$

The graph $G_{\vec{a}}$ has the following 13 edges (see Definition 6.2):

$$\begin{gathered}
e_1 := (v_{0,0,0}, v_{0,0,1}),\ e_2 = (v_{1,0,0}, v_{0,0,0}),\ e_3 = (v_{0,0,1}, v_{0,1,0}),\\
e_4 := (v_{0,0,1}, v_{0,1,1}),\ e_5 = (v_{0,0,1}, v_{0,2,1}),\ e_6 = (v_{0,0,1}, v_{0,1,2}),\\
e_7 := (v_{0,1,0}, v_{1,0,0}),\ e_8 = (v_{1,0,1}, v_{0,1,0}),\ e_9 = (v_{0,1,1}, v_{1,1,0}),\\
e_{10} := (v_{0,2,1}, v_{2,1,0}),\ e_{11} = (v_{2,0,1}, v_{0,1,0}),\ e_{12} = (v_{0,1,2}, v_{1,2,0}),\\
e_{13} := (v_{1,0,2}, v_{0,1,0}).
\end{gathered}$$

Moreover, just the edges e_1, e_5 and e_6 are augmenting (see Theorem 6.3).

There is only one cycle in this graph:

$$(v_{0,0,0}, v_{0,0,1}, v_{0,1,0}, v_{1,0,0}).$$

As only one of these edges is augmenting we obtain that $H_{\vec{a}} = \frac{1}{4}$ (cf. Corollary 6.7).

Now we give the detailed description of the tropical Hilbert function $d(k)$ in this case (cf. Corollary 6.8):

- *if $k = 1, 2, 3$ then $d(k) = 3$. The maximum of the dimension $\dim(Q_T)$ over paths of a given length in $G_{\vec{a}}$ (see Theorem 6.6) is attained at the path*
$$T := e_{11} e_7 e_2 e_1;$$
- *if $k = 4p$, where $p \geq 1$ then $d(k) = 3 + p$. The maximum is attained at the path*
$$T := e_{11} \underbrace{(v_{0,0,0}, v_{0,0,1}, v_{0,1,0}, v_{1,0,0}) \cdots (v_{0,0,0}, v_{0,0,1}, v_{0,1,0}, v_{1,0,0})}_{p-1} e_7 e_2 e_1;$$
- *if $k = 4p + 1$, where $p \geq 1$ then $d(k) = 3 + (p + 1)$. The maximum is attained at the path*
$$T := e_{11} \underbrace{(v_{0,0,0}, v_{0,0,1}, v_{0,1,0}, v_{1,0,0}) \cdots (v_{0,0,0}, v_{0,0,1}, v_{0,1,0}, v_{1,0,0})}_{p-1} e_7 e_2 e_1 e_5;$$
- *if $k = 4p + 2$, where $p \geq 1$ then $d(k) = 3 + (p + 1)$. The maximum is attained at the path*
$$T := e_{11} \underbrace{(v_{0,0,0}, v_{0,0,1}, v_{0,1,0}, v_{1,0,0}) \cdots (v_{0,0,0}, v_{0,0,1}, v_{0,1,0}, v_{1,0,0})}_{p-1} e_7 e_2 e_1 e_6 e_{12};$$
- *if $k = 4p + 3$, where $p \geq 1$ then $d(k) = 3 + p$. The maximum is attained at the path*
$$T := e_{11} \underbrace{(v_{0,0,0}, v_{0,0,1}, v_{0,1,0}, v_{1,0,0}) \cdots (v_{0,0,0}, v_{0,0,1}, v_{0,1,0}, v_{1,0,0})}_{p} e_7 e_2.$$

7 Sharp bounds on the tropical entropy

7.1 Sharp lower bound on the positive entropy

In this section our main goal is to prove that if for a vector $\vec{a} = (a_0, \ldots, a_n) \in \mathbb{Z}^{n+1}$ its tropical entropy $H(\vec{a}) > 0$ then $H(\vec{a}) \geq \frac{1}{4}$. Together with the example

[4, Example 5.5] demonstrating that $H(0,\,1,\,0) = 1/4$ (cf. also Example 5.8) we will conclude that this bound is sharp. This result is the answer to the hypothesis that was formulated in [4, Remark 5.6] (for the criterion of positivity of the tropical entropy see [4, Corollary 5.7], cf. also Remark 5.9).

For convenience, we use the following assumptions in this section:

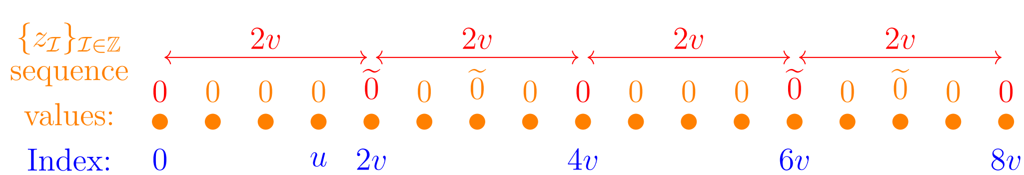
- we consider tropical sequences $\{z_{\mathcal{I}}\}_{\mathcal{I}\in\mathbb{Z}}$ infinite in both directions. To obtain finite tropical sequence it is enough to consider only $\mathcal{I} \in \mathbb{N}$;
- for most of the considered cases we attach diagrams with simple examples. To save place on these diagrams the $\geq$ sign is replaced by sign $\sim$ above the symbol.

Theorem 7.1 *If a vector $\vec{a}$ is not regular then $H(\vec{a}) \geq \frac{1}{4}$.*

Proof. Consider Newton polygon $\mathcal{N}(\vec{a})$ of the vector $\vec{a}$ (see Remark 5.9). It has several bounded edges and two unbounded edges. First, assume that there is a bounded edge of $\mathcal{N}(\vec{a})$ such that there are at least three points of $\vec{a}$, i. e. of the form (i, a_i) (in this case we follow the proof of [4, Theorem 5.5]). Making a suitable affine transformation of the plane one can suppose w.l.o.g. that this edge lies on the abscissas axis and $(0,0)$ is its left endpoint (the transformation, perhaps, converts the tropical polynomial f, see (1), into a tropical Laurent polynomial, the proof still goes through for the latter). Consider the points of $\vec{a}$ located on this edge: $E_0 := \{(e,0) : a_e = 0\}$, then $|E_0| \geq 3$ by our assumption. One can assume w.l.o.g. that the greatest common divisor $GCD(E_0)$ of the differences $e_1 - e_2$ of all the pairs of the elements $e_1,\, e_2 \in E_0$ of E_0 equals 1. Otherwise, one can consider separately all $GCD(E_0)$ arithmetic progressions with the difference $GCD(E_0)$.

Pick any three elements of E_0 not all with the same parity, say $0, 2v, u$ w.l.o.g. where $v \geq 1$ and u being odd. Consider the following tropical recurrent sequence $\{z_{\mathcal{I}}\}_{\mathcal{I}\in\mathbb{Z}}$ satisfying a :

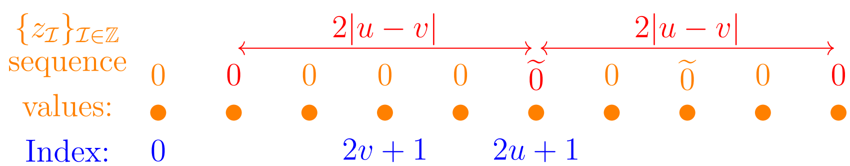
- $z_{2l+1} = 0$, for $0 \leq l \in \mathbb{Z}$;
- $z_{2(2qv+r)} = 0$, for $q \in \mathbb{Z}$ and $0 \leq r < v$;
- $z_{2((2q+1)v+r)} \geq 0$, for $q \in \mathbb{Z}$ and $0 \leq r < v$.



Taking finite fragments $(z_1, \dots, z_N)$ with growing N we conclude that $H(\vec{a}) \geq \frac{1}{2+1+1} = \frac{1}{4}$.

In the other case we have 0, $2v+1$ and $2u+1 \in E_0$ and thus we have the following sequence $\{z_\mathcal{I}\}_{\mathcal{I}\in\mathbb{Z}}$:

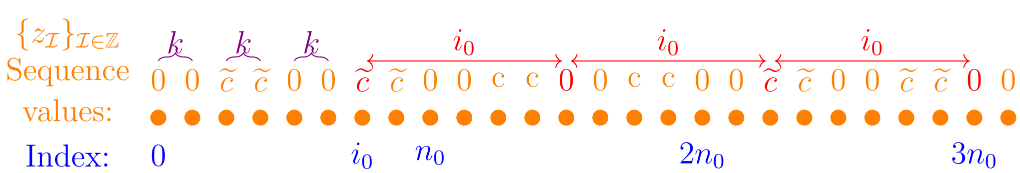
- $z_{2l} = 0$, for $0 \leq l \in \mathbb{Z}$;
- $z_{2(2q(u-v)+r)+1} = 0$, for $q \in \mathbb{Z}$ and $0 \leq r < u-v$;
- $z_{2((2q+1)(u-v)+r)+1} \geq 0$, for $q \in \mathbb{Z}$ and $0 \leq r < u-v$.



Now we assume that no edge of $\mathcal{N}(\vec{a})$ contains a point of $\vec{a}$ other than two vertices of this edge. We take an edge of $\mathcal{N}(\vec{a})$ with the biggest difference of indices of its vertices. Due to a suitable affine transformation we suppose w.l.o.g. that these vertices are $(0,0)$ and $(n_0,0)$. There exists $i \in J$ such that n_0 does not divide i, since $\vec{a}$ is not regular. Among such i we pick i_0 for which $c := a_{i_0}$ is minimal. Then $c > 0$. Denote $k = GCD(n_0, i_0)$. When $\frac{n_0}{k}$ is even we consider the sequence $\{z_\mathcal{I}\}_{\mathcal{I}\in\mathbb{Z}}$:

- $z_{qn_0-2ji_0+i} = 0$, when $0 \leq 2j \leq \frac{n_0}{k}$;
- $z_{2qn_0-(2j+1)i_0+i} = c$, when $0 < 2j+1 < \frac{n_0}{k}$;
- $z_{(2q+1)n_0-(2j+1)i_0+i} \geq c$, when $0 < 2j+1 < \frac{n_0}{k}$,

for $q \in \mathbb{Z}, 0 \leq i < k$.



This sequence satisfies $\vec{a}$ and taking finite fragments $(z_1, \dots, z_N)$ with growing N we conclude that $H(\vec{a}) \geq \frac{1}{2+1+1} = \frac{1}{4}$. Thus further we suppose that $\frac{n_0}{k}$ is odd.

We denote the first (respectively, the last) index of a by $\mathcal{B}$ (respectively, by $\mathcal{E}$). Thus, the projection of $\mathcal{N}(\vec{a})$ is the interval from $\mathcal{B}$ to $\mathcal{E}$ on the abscissas axis. Before we prove the statement of the theorem in general case let us prove the following lemma.

Lemma 7.2 *If there exists* $i_1 \neq i_0$ *such that* $n_0 \nmid i_1$ *and* $a_{i_1} = a_{i_0}$ *then* $H(\vec{a}) \geq \frac{1}{4}$.

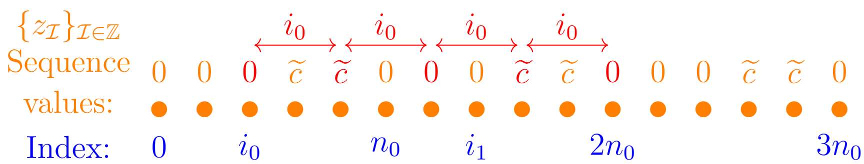
Proof.

L.I Let $n_0 | (i_1 - i_0)$.

Then we consider a sequence $\{z_\mathcal{I}\}_{\mathcal{I}\in\mathbb{Z}}$ such that:

- $z_{qn_0 - 2ji_0 + i} = 0$ when $0 \leq 2j < \frac{n_0}{k}$;
- $z_{qn_0 - (2j+1)i_0 + i} \geq c$ when $0 < 2j + 1 < \frac{n_0}{k}$

for $q \in \mathbb{Z}$, $0 \leq i < k$.



This sequence satisfies a.

Indeed,

- For $m = qn_0 - 2ji_0 + i$ we have $\min_{\mathcal{B}\leq v\leq\mathcal{E}}\{a_v + z_{v+m}\} = 0$, the minimum is attained at indices m and $m + n_0$.
- For $m = qn_0 - (2j+1)i_0 + i$ we have $\min_{\mathcal{B}\leq v\leq\mathcal{E}}\{a_v + z_{v+m}\} = c$, the minimum is attained at indices $m + i_0$ and $m + i_1$.

Taking finite fragments $(z_1, \dots, z_N)$ with growing N we conclude that $H(\vec{a}) \geq \frac{1}{2}$ for even $\frac{n_0}{k}$ and $H(\vec{a}) \geq \frac{\frac{\frac{n_0}{k}-1}{2}\cdot k}{n_0} \geq \frac{1}{3}$.

L.II Let $n_0 \nmid (i_1 - i_0)$.

L.II.1 Assume that $\mathbf{k = 1}$ (since we consider the case where $\frac{n_0}{k}$ is odd, thus n_0 is odd).

First, consider sequence $\{w_\mathcal{I}\}_{\mathcal{I}\in\mathbb{Z}}$ such that:

- $w_{qn_0 - 2ji_0} = 0$ when $0 \leq 2j \leq n_0$;
- $w_{2qn_0 - (2j+1)i_0} = c$ when $0 < 2j + 1 < n_0$;
- $w_{(2q+1)n_0 - (2j+1)i_0} \geq c$ when $0 < 2j + 1 < n_0$

for $q \in \mathbb{Z}$.

This sequence satisfies a.

Indeed,

- For $m = qn_0 - 2ji_0$ we have $\min_{\mathcal{B}\le v\le\mathcal{E}}\{a_v + w_{v+m}\} = 0$, the minimum is attained at indices m and $m+n_0$.

- For $m = 2qn_0 - (2j+1)i_0$ we have $\min_{\mathcal{B}\le v\le\mathcal{E}}\{a_v + w_{v+m}\} = c$, the minimum is attained at indices $m+i_0$ and m.

- For $m = (2q+1)n_0 - (2j+1)i_0$ we have $\min_{\mathcal{B}\le v\le\mathcal{E}}\{a_v + w_{v+m}\} = c$, the minimum is attained at indices $m+i_0$ and $m+n_0$.

Now, we claim that there exists $0 < 2l+1 < n_0$ such that $w_{qn_0-(2l+1)i_0+i_1} = 0$ for all $q \in \mathbb{Z}$. Note, that if we found $2l'+1$ such that $w_{qn_0-(2l'+1)i_0+i'_1} = 0$ for all $q \in \mathbb{Z}$ for some $i'_1 \equiv i_1$, then $w_{qn_0-(2l'+1)i_0+i_1} = 0$ for all $q \in \mathbb{Z}$.

Recall that $GCD(i_0, n_0) = 1$ and $n_0 \nmid i_1$, therefore there exists m_{i_1} such that $m_{i_1}i_0 \equiv i_1\ (mod\ \ n_0)$ and $0 < m_{i_1} < n_0$.

- If m_{i_1} is odd then the required $2l+1$ equals $n_0 - 2$. Indeed, $qn_0 - (n_0 - 2)i_0 + m_{i_1}i_0 = qn_0 - (n_0 - 2 - m_{i_1})i_0$. $0 \le n_0 - 2 - m_{i_1} < n_0 - 2$ and $(n_0 - 1 - m_{i_1})$ is even, thus $w_{qn_0-(n_0-2-m_{i_1})i_0} = 0$.

- If m_{i_1} is even then the required $2l+1$ equals $m_{i_1}-1$. Indeed, $qn_0-(m_{i_1} - 1)i_0 + m_{i_1}i_0 = qn_0 + i_0 = (q + i_0)n_0 - (n_0 - 1)i_0$. $0 < n_0 - 1 < n_0$ and $n_0 - 1$ is even, thus $w_{(q+i_0)n_0-(n_0-1)i_0} = 0$.

Now consider a sequence $\{z_{\mathcal{I}}\}_{\mathcal{I}\in\mathbb{Z}}$ such that:

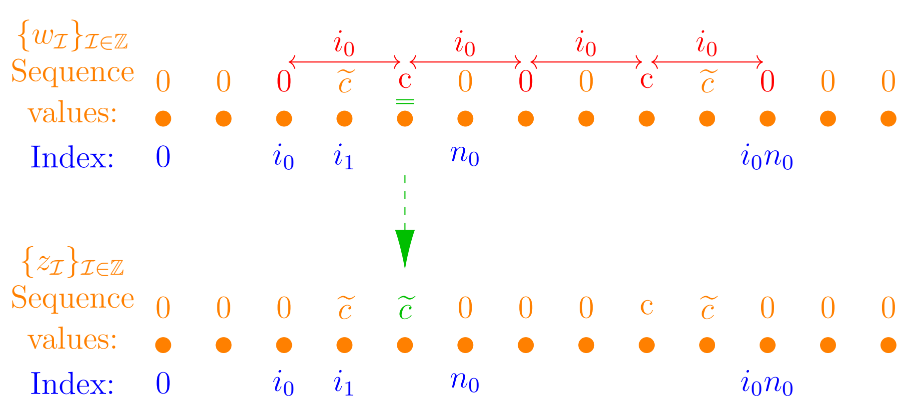
- $z_{qn_0-2ji_0} = 0$ when $0 \le 2j \le n_0$;

- $z_{2qn_0-(2j+1)i_0} = c$ when $0 < 2j+1 < n_0$ and $l \ne j$;

- $z_{2qn_0-(2l+1)i_0} \ge 0$;

- $z_{(2q+1)n_0-(2j+1)i_0} \ge c$ when $0 < 2j+1 < n_0$

for $q \in \mathbb{Z}$.



This sequence satisfies a. Indeed,

- For $m = qn_0 - 2ji_0$ we have $\min_{\mathcal{B}\le v\le\mathcal{E}}\{a_v + z_{v+m}\} = 0$, the minimum is attained at indices m and $m + n_0$.
- For $m = 2qn_0 - (2j+1)i_0$ and $j \neq l$ we have $\min_{\mathcal{B}\le v\le\mathcal{E}}\{a_v + z_{v+m}\} = c$, the minimum is attained at indices $m + i_0$ and m.
- For $m = 2qn_0-(2l+1)i_0$ we have $\min_{\mathcal{B}\le v\le\mathcal{E}}\{a_v+z_{v+m}\} = c$, the minimum is attained at indices $m + i_0$ and $m + i_1$.
- For $m = (2q+1)n_0 - (2j+1)i_0$ we have $\min_{\mathcal{B}\le v\le\mathcal{E}}\{a_v + z_{v+m}\} = c$, the minimum is attained at indices $m + i_0$ and $m + n_0$.

Taking finite fragments $(z_1, \ldots, z_N)$ with growing N we conclude that $H(\vec{a}) \ge \frac{\frac{n-1}{2}+1}{2n} \ge \frac{1}{4}$.

L.II.2 Now assume that $\mathbf{k > 1}$.

Define $k_1 := GCD(i_1, n_0)$. W.l.o.g. we can consider that $k_1 \ge k$ (otherwise we can swap i_0 and i_1).

We will find k different indices $l_1, \ldots l_k$ such that $w_{qn_0+l_j+i_1} = 0$ for all $q \in \mathbb{Z}$ and for all $1 \le j \le k$ and $l_{j_1} \not\equiv l_{j_2}$ for all $j_1 \neq j_2$. Note, that if $i'_1 \equiv i_1\,(mod\ n_0)$ and $w_{qn_0+l_j+i'_1} = 0$ for all $q \in \mathbb{Z}$ and for all $1 \le j \le k$ then it is true for i_1. Thus, we can assume that $0 \le i_1 < n_0$. We can represent i_1 as $s \cdot k + r$, where $0 \le r < k$. We study two different cases:

L.II.2.1 $r = 0$.

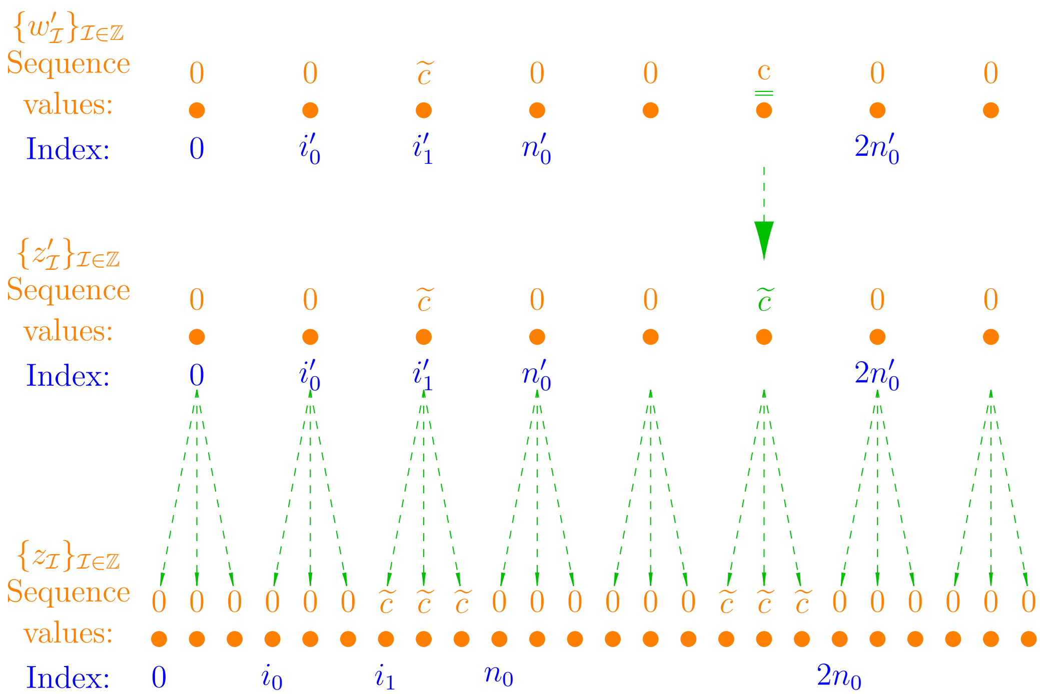
Denote $n' := \frac{n_0}{k}$, $i'_0 = \frac{i_0}{k}$ and $i'_1 := \frac{i_1}{k}$.Consider $a' = (a_j)_{j\equiv 0\,(mod\ k)}$. Sequence $\{w'_{\mathcal{I}}\}_{\mathcal{I}\in\mathbb{Z}}$ is defined as follows:

- $w'_{q'\frac{n_0}{k}-2j\frac{i_0}{k}} = 0$, where $0 \le 2j < \frac{n_0}{k}$;
- $w'_{2q'\frac{n_0}{k}-(2j+1)\frac{i_0}{k}} = c$, where $0 < 2j+1 < \frac{n_0}{k}$;
- $w'_{(2q'+1)\frac{n_0}{k}-(2j+1)\frac{i_0}{k}} \ge c$, where $0 < 2j+1 < \frac{n_0}{k}$

for $q' \in \mathbb{Z}$.

Similar to the previous case ($k = 1$) we can consider sequence $\{z'_{\mathcal{I}}\}_{\mathcal{I}\in\mathbb{Z}}$ that provides the bound $H(a') \ge \frac{1}{4}$ for n'_0, i'_0, i'_1 and a'. Now take sequence $\{z_{\mathcal{I}}\}_{\mathcal{I}\in\mathbb{Z}}$ as follows:

- $z_{i\cdot k+r} = z'_i$, for $0 \le i \in \mathbb{Z}$ and $0 \le r < k$.



This provides us the bound $H(\vec{a}) \geq \frac{1}{4}$.

L.II.2.2 $r \neq 0$

Note, that in this case $k_1 > k$ and thus $s \neq 0$ and $s+1 \neq \frac{n_0}{k}$. Here we have three different cases:

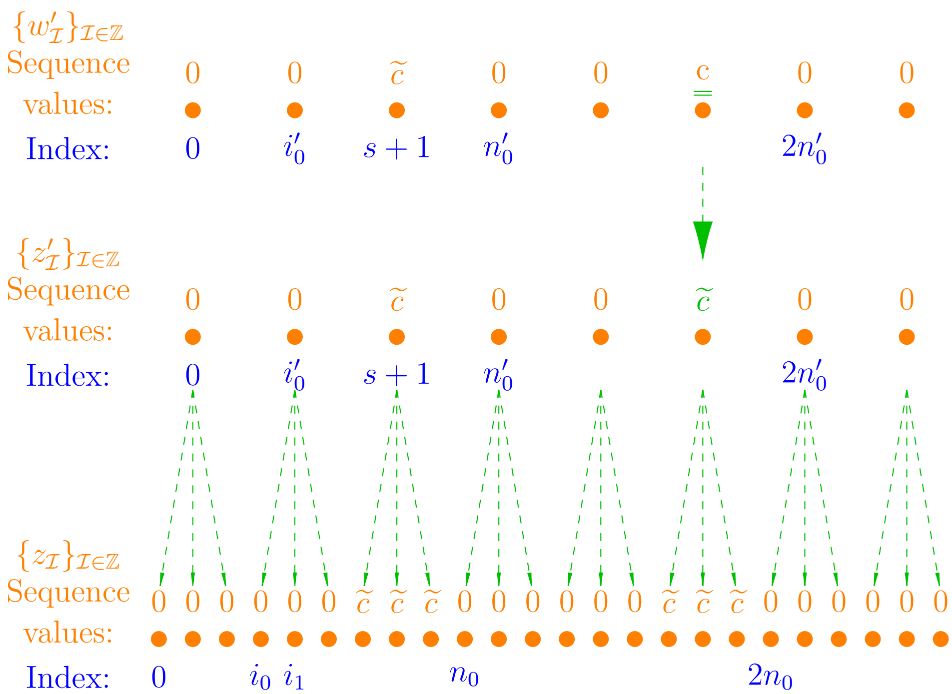
L.II.2.2a $s \equiv \frac{i_0}{k}\ (mod\ \frac{n_0}{k})$.

From the proof for $k = 1$ we know that there exists $0 < 2l+1 < \frac{n_0}{k}$ such that $w'_{q'\frac{n_0}{k}-(2l+1)\frac{i_0}{k}+(s+1)} = 0$ for all $q' \in \mathbb{Z}$.

Consider $\{z_{\mathcal{I}}\}_{\mathcal{I}\in Z}$ as follows:

- $z_{qn_0-2ji_0+i}$, where $0 \leq 2j < n_0$ and $0 \leq i < k$;
- $z_{2qn_0-(2j+1)i_0+i} = c$, where $0 < 2j+1 < n_0$, $j \neq l$ and $0 \leq i < k$;
- $z_{2qn_0-(2l+1)i_0+i} \geq c$, where $0 \leq i < k$;
- $z_{(2q+1)n_0-(2l+1)i_0+i}$, where $0 < 2j+1 < n_0$ and $0 \leq i < k$

for $q \in \mathbb{Z}$.



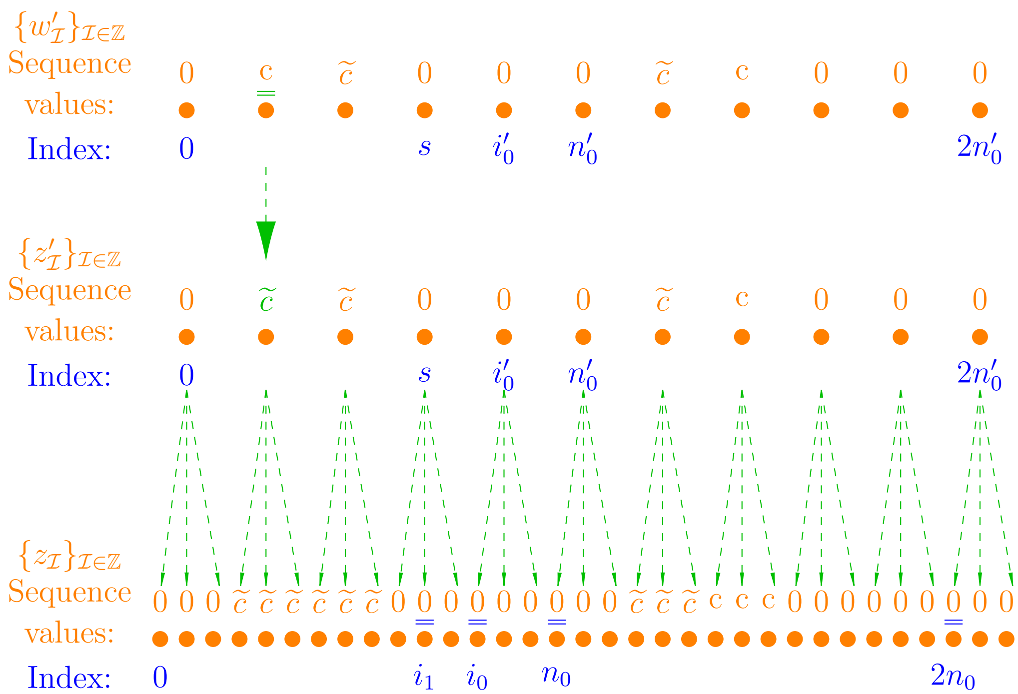
We claim that $\{z_{\mathcal{I}}\}_{\mathcal{I}\in\mathbb{Z}}$ satisfies a. Indeed,

- For $m = qn_0 - 2ji_0 + i$ we have $\min_{\mathcal{B}\le v\le\mathcal{E}}\{a_v + z_{v+m}\} = 0$, the minimum is attained at indices m and $m + n_0$.
- For $m = 2qn_0 - (2j+1)i_0 + i$, $j \neq l$ we have $\min_{\mathcal{B}\le v\le\mathcal{E}}\{a_v + z_{v+m}\} = c$, the minimum is attained at indices m and $m + i_0$.
- For $m = (2q+1)n_0 - (2j+1)i_0 + i$, $j \neq l$ we have $\min_{\mathcal{B}\le v\le\mathcal{E}}\{a_v + z_{v+m}\} = c$, the minimum is attained at indices $m + n_0$ and $m + i_0$.
- For $m = qn_0 - (2l+1)i_0 + i$ we have $\min_{\mathcal{B}\le v\le\mathcal{E}}\{a_v + z_{v+m}\} = c$, the minimum is attained at indices $m + i_1$ and $m + i_0$.

Taking finite fragments $(z_1, \ldots, z_N)$ with growing N we conclude that $H(\vec{a}) \ge \frac{\frac{\frac{n}{k}-1}{2}\cdot k + k}{2n} \ge \frac{1}{4}$

L.II.2.2b $s+1 \equiv \frac{i_0}{k}\ (mod\ \frac{n_0}{k})$.

This case is the same as the previous one except that we need to find $0 < 2l+1 < \frac{n_0}{k}$ such that $w'_{q'\frac{n_0}{k}-(2l+1)\frac{i_0}{k}+s} = 0$ for all $q' \in \mathbb{Z}$.



Taking finite fragments $(z_1, \dots, z_N)$ with growing N we conclude that $H(\vec{a}) \geq \frac{\frac{\frac{n_0}{k}-1}{2}\cdot k + k}{2n_0} \geq \frac{1}{4}$

L.II.2.2c $s,\ s+1 \not\equiv \frac{i_0}{k}\ (mod\ \frac{n_0}{k})$.

From the proof for $k=1$ we know that there exist $0 < 2l+1,\ 2l'+1 < \frac{n_0}{k}$ such that $w'_{q'\frac{n_0}{k}-(2l+1)\frac{i_0}{k}+s} = 0$ and $w'_{q'\frac{n_0}{k}-(2l'+1)\frac{i_0}{k}+(s+1)} = 0$ for all $0 \leq q' \in \mathbb{Z}$.

Consider $\{z_{\mathcal{I}}\}_{\mathcal{I}\in\mathbb{Z}}$ as follows:

- $z_{qn_0-2ji_0+i}$, where $0 \leq 2j < n_0$ and $0 \leq i < k$;
- $z_{2qn_0-(2j+1)i_0+i} = c$, where $0 < 2j+1 < n_0$, $j \neq l,\ l'$ and $0 \leq i < k$;
- $z_{2qn_0-(2l+1)i_0+i} \geq c$, where $0 \leq i < k-r$;
- $z_{2qn_0-(2l+1)i_0+i} = c$, where $k-r \leq i < k$;
- $z_{2qn_0-(2l'+1)i_0+i} = c$, where $0 \leq i < k-r$;
- $z_{2qn_0-(2l'+1)i_0+i} \geq c$, where $k-r \leq i < k$;
- $z_{2qn_0-(2l+1)i_0+i} \geq c$, where $0 \leq i < k$;
- $z_{(2q+1)n_0-(2l+1)i_0+i}$, where $0 < 2j+1 < n_0$ and $0 \leq i < k$

for $q \in \mathbb{Z}$. We claim that $\{z_{\mathcal{I}}\}_{\mathcal{I}\in\mathbb{Z}}$ satisfies a.

Indeed,

- For $m = qn_0 - 2ji_0 + i$ we have $\min_{\mathcal{B} \le v \le \mathcal{E}}\{a_v + z_{v+m}\} = 0$, the minimum is attained at indices m and $m + n_0$.

- For $m = 2qn_0 - (2j+1)i_0 + i$, $j \neq l,\ l'$ we have $\min_{\mathcal{B} \le v \le \mathcal{E}}\{a_v + z_{v+m}\} = c$, the minimum is attained at indices m and $m + i_0$.

- For $m = 2qn_0 - (2l+1)i_0 + i$, $k - r \le i < k$ we have $\min_{\mathcal{B} \le v \le \mathcal{E}}\{a_v + z_{v+m}\} = c$, the minimum is attained at indices m and $m + i_0$.

- For $m = 2qn_0 - (2l'+1)i_0 + i$, we have $0 \le i < k - r$ $\min_{\mathcal{B} \le v \le \mathcal{E}}\{a_v + z_{v+m}\} = c$, the minimum is attained at indices m and $m + i_0$.

- For $m = (2q+1)n_0 - (2j+1)i_0 + i$, $j \neq l, l'$ we have $\min_{\mathcal{B} \le v \le \mathcal{E}}\{a_v + z_{v+m}\} = c$, the minimum is attained at indices $m + n_0$ and $m + i_0$.

- For $m = (2q+1)n_0 - (2l+1)i_0 + i$, $k - r \le i < k$ we have $\min_{\mathcal{B} \le v \le \mathcal{E}}\{a_v + z_{v+m}\} = c$, the minimum is attained at indices $m + n_0$ and $m + i_0$.

- For $m = (2q+1)n_0 - (2l'+1)i_0 + i$, $0 \le i < k - r$ we have $\min_{\mathcal{B} \le v \le \mathcal{E}}\{a_v + z_{v+m}\} = c$, the minimum is attained at indices $m + n_0$ and $m + i_0$.

- For $m = qn_0 - (2l+1)i_0 + i$, $0 \le i < k - r$ we have $\min_{\mathcal{B} \le v \le \mathcal{E}}\{a_v + z_{v+m}\} = c$, the minimum is attained at indices $m + i_1$ and $m + i_0$.

- For $m = qn_0 - (2l'+1)i_0 + i$, $k - r \le i < k$ we have $\min_{\mathcal{B} \le v \le \mathcal{E}}\{a_v + z_{v+m}\} = c$, the minimum is attained at indices $m + i_1$ and $m + i_0$.

Taking finite fragments $(z_1, \ldots, z_N)$ with growing N we conclude that $H(\vec{a}) \ge \frac{\frac{\frac{n_0}{k} - 1}{2} \cdot k + k}{2n_0} \ge \frac{1}{4}$. $\square$

Now we are returning to the proof of the theorem. Assume that there is no such $n \nmid i_i$, $i_0 \neq i_1$ that $a_{i_1} = a_{i_0}$. As in the proof of Lemma 7.2 we will consider two different cases.

T.1 $k = 1$.

Define the following sequence $\{w_{\mathcal{I}}\}_{\mathcal{I} \in \mathbb{Z}}$:

- $w_{qn_0 - 2ji_0} = 0$ when $0 \le 2j < n_0$;

- $w_{2qn_0 - (2j+1)i_0} = c$ when $0 < 2j + 1 < n_0$;

- $w_{(2q+1)n_0 - (2j+1)i_0} \ge c$ when $0 < 2j + 1 < n_0$

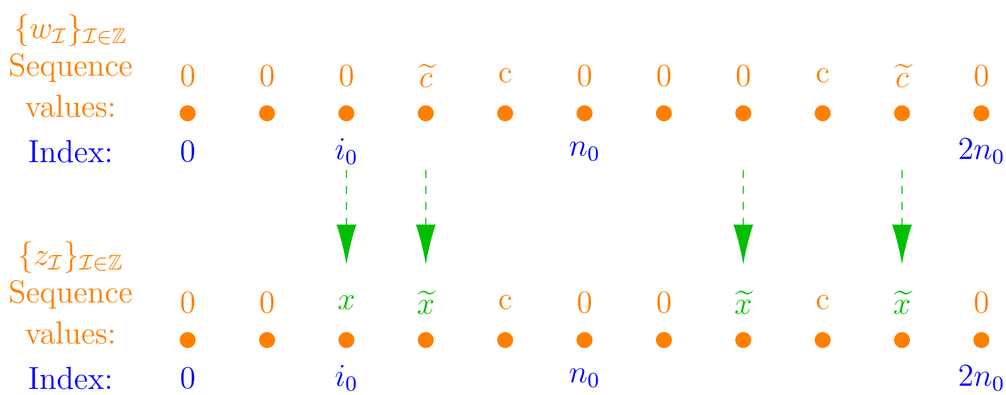
for $q \in \mathbb{Z}$.

We define $\mathcal{L}_0 := \{\mathcal{B} \le v \le \mathcal{E}, n_0 \nmid v, v \neq 0, n_0$, such that $w_{qn_0 - (n_0 - 1)i_0 + v} = 0$ for all $q \in \mathbb{Z}\}$. Set $x := \min\ \{a_v \mid v \in \mathcal{L}_0\}$. Also define i_x by the equation $a_{i_x} = x$. If such i_x is not unique then we choose any i_x with such property.

T.1.1 First assume that $x \leq 2c$.
In this case we define a sequence $\{z_{\mathcal{I}}\}_{\mathcal{I}\in\mathbb{Z}}$ as follows:

- $z_{qn_0-2ji_0} = 0$ when $0 \leq 2j < n_0$, $2j \neq n_0 - 1$;
- $z_{2qn_0-(2j+1)i_0} = c$ when $0 < 2j+1 < n_0$;
- $z_{(2q+1)n_0-(2j+1)i_0} \geq x$ when $0 < 2j+1 < n_0$
- $z_{2qn_0-(n_0-1)i_0} = x$;
- $z_{(2q+1)n_0-(n_0-1)i_0} \geq x$

for $q \in \mathbb{Z}$.



We claim that $\{z_{\mathcal{I}}\}_{\mathcal{I}\in\mathbb{Z}}$ satisfies a.
Indeed,

- For $m = qn_0 - 2ji_0$, $2j \neq n_0 - 1$ we have $\min_{\mathcal{B}\leq v\leq\mathcal{E}}\{a_v + z_{v+m}\} = 0$, the minimum is attained at indices m and $m + n_0$.
- For $m = 2qn_0 - (2j+1)i_0$, $2j \neq n_0 - 1$ we have $\min_{\mathcal{B}\leq v\leq\mathcal{E}}\{a_v + z_{v+m}\} = c$, the minimum is attained at indices m and $m + i_0$.
- For $m = (2q+1)n_0 - (2j+1)i_0$ we have $\min_{\mathcal{B}\leq v\leq\mathcal{E}}\{a_v + z_{v+m}\} = c$, the minimum is attained at indices $m + n_0$ and $m + i_0$.
- For $m = 2qn_0 - (n-1)i_0$, we have $\min_{\mathcal{B}\leq v\leq\mathcal{E}}\{a_v + z_{v+m}\} = x$, the minimum is attained at indices m and $m + i_x$ (because $a_v + z_{v+m}$ is at least x if $v \in \mathcal{L}_0$ and $a_v + z_{v+m} \geq min\{c+c, c+x\} \geq x$ if $v \notin \mathcal{L}_0$).
- For $m = (2q+1)n_0 - (n_0-1)i_0$ we have $\min_{\mathcal{B}\leq v\leq\mathcal{E}}\{a_v + z_{v+m}\} = x$, the minimum is attained at indices $m + n_0$ and $m + i_x$ (because $a_v + z_{v+m}$ is at least x if $v \in \mathcal{L}_0$ and $a_v + z_{v+m} \geq min\{c+c, c+x\} \geq x$ if $v \notin \mathcal{L}_0$).

Taking finite fragments $(z_1, \dots, z_N)$ with growing N we conclude that $H(\vec{a}) \geq \frac{\frac{n_0-1}{2}+1}{2n_0} \geq \frac{1}{4}$.

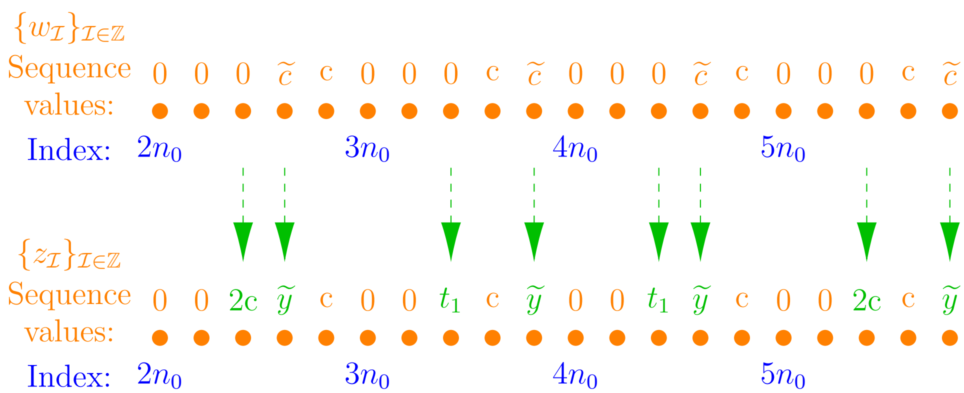
T.1.2 Now we assume that $x > 2c$.

Denote $\min_{v \neq i_0,\ v \nmid n_0}\{a_v\}$ by s. Note, that $s > c$. Indeed, otherwise we can use lemma 7.2 and get the required bound. Denote $\min_{v \neq 0, n_0,\ v | n_0}\{a_v\}$ by d. Note, that $d > 0$. Finally, set $y := \min\{s + c, x, 2c + d\}$.

Define a sequence $\{z_{\mathcal{I}}\}_{\mathcal{I} \in \mathbb{Z}}$ as follows:

- $z_{qn_0 - 2ji_0} = 0$ when $0 \leq 2j < n_0$, $2j \neq n_0 - 1$;
- $z_{2qn_0-(2j+1)i_0} = c$ when $0 < 2j + 1 < n_0$;
- $z_{(2q+1)n_0-(2j+1)i_0} \geq y$ when $0 < 2j + 1 < n_0$
- $z_{4qn_0-(n_0-1)i_0} = 2c$;
- $z_{(4q+1)n_0-(n_0-1)i_0} = t_q$, is a free variable, $t_q \in [2c, y]$;
- $z_{(4q+2)n_0-(n_0-1)i_0} = t_q$, is a free variable, $t_q \in [2c, y]$;
- $z_{(4q+3)n_0-(n_0-1)i_0} = 2c$

for $q \in \mathbb{Z}$.



We claim that $\{z_{\mathcal{I}}\}_{\mathcal{I}\in\mathcal{Z}}$ satisfies a.

Indeed,

- For $m = qn_0 - 2ji_0$, we have $2j \neq n_0 - 1$ $\min_{\mathcal{B} \leq v \leq \mathcal{E}}\{a_v + z_{v+m}\} = 0$, the minimum is attained at indices m and $m + n_0$.
- For $m = 2qn_0 - (2j+1)i_0$, $2j \neq n_0 - 1$ we have $\min_{\mathcal{B} \leq v \leq \mathcal{E}}\{a_v + z_{v+m}\} = c$, the minimum is attained at indices m and $m + i_0$.

- For $m = (2q+1)n_0 - (2j+1)i_0$ we have $\min_{\mathcal{B}\leq v\leq\mathcal{E}}\{a_v + z_{v+m}\} = c$, the minimum is attained at indices $m + n_0$ and $m + i_0$.

- For $m = 4qn_0 - (n_0 - 1)i_0$, we have $\min_{\mathcal{B}\leq v\leq\mathcal{E}}\{a_v + z_{v+m}\} = 2c$, the minimum is attained at indices m and $m + i_0$ (because $a_v + z_{v+m}$ is at least $x > 2c$ if $v \in \mathcal{L}_0$, and $a_v + z_{v+m} \geq min\{s + c, 0 + t_q, d + 2c\} \geq 2c$ if $v \notin \mathcal{L}_0$).

- For $m = (4q+1)n_0 - (n_0 - 1)i_0$ we have $\min_{\mathcal{B}\leq v\leq\mathcal{E}}\{a_v + z_{v+m}\} = t_q$, the minimum is attained at indices m and $m+n_0$ (because a_v+z_{v+m} is at least $x > y \geq t_q$ if $v \in \mathcal{L}_0$, and $a_v + z_{v+m} \geq min\{c + y, s + c, d + 2c\} \geq y \geq t_q$ if $v \notin \mathcal{L}_0$).

- For $m = (4q+2)n_0 - (n_0 - 1)i_0$, we have $\min_{\mathcal{B}\leq v\leq\mathcal{E}}\{a_v + z_{v+m}\} = 2c$, the minimum is attained at indices $m + n_0$ and $m + i_0$ (because $a_v + z_{v+m}$ is at least $x > 2c$ if $v \in \mathcal{L}_0$, and $a_v + z_{v+m} \geq min\{s+c, 0+t_q, d+2c\} \geq 2c$ if $v \notin \mathcal{L}_0$).

- For $m = (4q+3)n_0 - (n_0 - 1)i_0$, we have $\min_{\mathcal{B}\leq v\leq\mathcal{E}}\{a_v + z_{v+m}\} = 2c$, the minimum is attained at indices m and $m + n_0$ (because $a_v + z_{v+m}$ is at least $x > 2c$ if $v \in \mathcal{L}_0$, and $a_v+z_{v+m} \geq min\{c+y, s+c, 0+t_q, d+2c\} \geq 2c$ if $v \notin \mathcal{L}_0$).

Taking finite fragments $(z_1, \dots, z_N)$ with growing N we conclude that $H(\vec{a}) \geq \frac{n_0-1+1}{4n_0} = \frac{1}{4}$.

T.2 $k > 1$.
Consider the following sequence $\{z'_{\mathcal{I}}\}_{\mathcal{I}\in\mathbb{Z}}$:

- $z'_{qn_0-2ji_0+i} = 0$ when $0 \leq 2j \leq n_0$;

- $z'_{2qn_0-(2j+1)i_0+i} = c$ when $0 < 2j+1 < n_0$;

- $z'_{(2q+1)n_0-(2j+1)i_0+i} \geq c$ when $0 < 2j+1 < n_0$

for $q \in \mathbb{Z}$ and $0 \leq i < k$.

For $0 \leq i < k$ define $mathcalL_{0,r} := \{\mathcal{B} \leq v \leq \mathcal{E},\ n_0 \nmid v,\ v \neq 0, n_0$ such that $z'_{qn_0-(n_0-1)i_0+r+v} = 0$ for all $q \in \mathbb{Z}$ and such that $qn_0-(n_0-1)i_0+r+v \neq q'n_0-(n_0-1)i_0+r'$ for any q' and for any $0 \leq r' < k\}$. Set $x_r := \min\{a_v \mid v \in \mathcal{L}_{0,r}\}$. Define $i_{x,r}$ by the equation $a_{i_{x,r}} = x_r$.

Denote $\min_{v\neq i_0,\ v\nmid n_0}\{a_v\}$ by s. Note, that $s > c$. Indeed, otherwise we can use lemma 7.2 and get the required bound. Denote $\min_{v\neq 0,n_0,\ v|n_0}\{a_v\}$ by d. Note, that $d > 0$. For $0 \leq r < k$ set $y_r := min\{s+c, x_r, 2c+d\}$. Finally, define $M := \max_{0\leq r<k}\{x_r, y_r\}$.

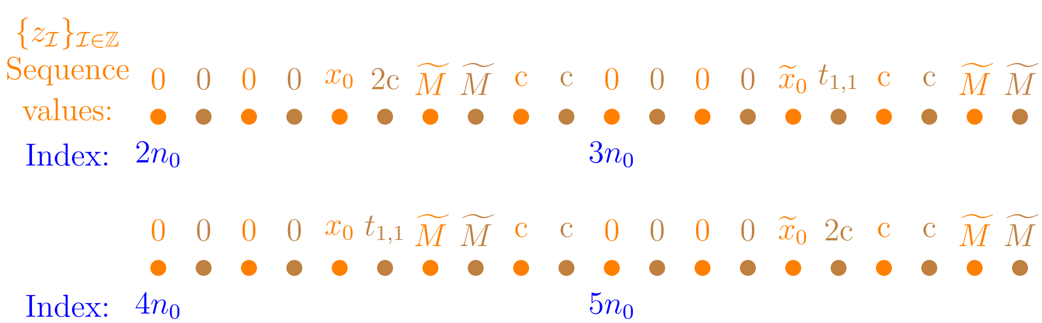
Define a sequence $\{z_{\mathcal{I}}\}_{\mathcal{I}\in\mathcal{Z}}$ as follows:

- $z_{qn_0-2ji_0+r} = 0$ when $0 \le 2j \le n_0$, $2j \ne (n_0-1)$, where $0 \le r < k$;
- $z_{2qn_0-(2j+1)i_0+r} = c$ when $0 < 2j+1 < n_0$, where $0 \le r < k$;
- $z_{(2q+1)n_0-(2j+1)i_0+r} \ge M$ when $0 < 2j+1 < n_0$, where $0 \le r < k$;

  For $0 \le r < k$ set:

  1. if $x_r \le 2c$ then:
     - $z_{2qn_0-(n_0-1)i_0+r} = x_r$;
     - $z_{(2q+1)n_0-(n_0-1)i_0+r} \ge M$;
  2. if $x_r > 2c$ then:
     - $z_{4qn_0-(n_0-1)i_0+r} = 2c$;
     - $z_{(4q+1)n_0-(n_0-1)i_0+r} = t_{q,r}$, is a free variable, $t_{q,r} \in [2c, y_r]$;
     - $z_{(4q+2)n_0-(n_0-1)i_0+r} = t_{q,r}$, is a free variable, $t_{q,r} \in [2c, y_r]$;
     - $z_{(4q+3)n_0-(n_0-1)i_0+r} = 2c$

for $q \in \mathbb{Z}$.



We claim that this sequence satisfies a. It is sufficient to check that a subsequence $\{z_{qn_0-(n-1)i_0+r'}\}_{q\in\mathbb{Z}}$ does not change the minima in the subsequence $\{z_{qn_0-(n-1)i_0+r}\}_{0\le q\in\mathbb{Z}}$ with $r \ne r'$ in the definition of satisfiability of the vector $\vec{a}$ (see (1)). The latter is true because $z_{qn_0-(n_0-1)i_0+r'} \ge c$ and thus $z_{qn_0-(n_0-1)i_0+r'}+a_v \ge c+s \ge x_r$ (if $x_r \le 2c$) and $z_{qn_0-(n_0-1)i_0+r'}+a_v \ge c+s \ge y_r$ (if $x_r > 2c$).

Taking finite fragments $(z_1, \dots, z_N)$ with growing N we conclude that in the worst case $H(\vec{a}) \ge \frac{(\frac{n_0}{k}-1)k+k}{4n_0} = \frac{1}{4}$. $\square$

7.2 Sharp upper bound on the tropical entropy in case of a single bounded edge of Newton polygon

The last theorem is an upper bound on $H(\vec{a})$ in case of a single bounded edge of Newton polygon $\mathcal{N}(\vec{a})$. We conjecture that this bound holds for an arbitrary vector $\vec{a}$. We mention that in [4] a weaker upper bound $1-1/n$ was established for an arbitrary vector $\vec{a}$. Together with the result $H(\vec{a}) = 1 - 2/(n+1)$ for a vector $a = (a_0, \dots, a_n)$ with $a_0 = \dots = a_n = 0$ [4, Example 5.2] it demonstrates the sharpness of the obtained upper bound. The full proof will be provided in the future.

Theorem 7.3 *If Newton polygon for $\vec{a}$ has only one bounded edge then $H(\vec{a}) \leq 1 - \frac{2}{n+1}$.*

Proof. For convenience we make a suitable affine transformation such that $a_0 = a_n = 0$.

Consider the polyhedral complex $D(s)$. It is a union of a finite number of polyhedra such that each of these polyhedra Q satisfies the following conditions. For every $0 \leq j \leq s-n$ there exists a pair $0 \leq i_1 < i_2 \leq n$ such that

$$z_{j+i_1} + a_{i_1} = z_j + a_{i_2} = \min_{0 \leq p \leq n} \{z_{p+j} + a_p\} \tag{44}$$

for any $(z_1, \dots, z_s) \in Q$.

For every Q we consider the following *restriction graph* $RG(Q)$:

- vertices are the indices of coordinates from 1 to s;
- there is an edge between vertices i and j if there is a linear condition of the form $y_i + \gamma = y_j$ which is true for all $(y_1, \dots, y_s) \in Q$.

Let us notice that $RG(Q)$ is the union of connected components where each component is the complete subgraph. Moreover, the dimension of Q equals the number of components of $RG(Q)$ (cf. [5]).

Let us fix some Q from the finite union above. For arbitrary $(t_1, \dots, t_s) \in Q$ we construct the following sequence by recursion:

- The first element of the sequence equals the least index i_0 such that $t_{i_0} = \min_{1 \leq f \leq s} t_f$;
- Let i_v be the last current constructed element of the sequence. If $i_v + n > s$ then we terminate the process and declare i_v to be the last constructed element of the sequence.

- If $i_v + n \leq s$ then we consider $\min_{0\leq p\leq n}\{t_{i_v+p} + a_p\}$. According to the definition of a tropical sequence and the definition of Q there exist $0 \leq p_1 < p_2$ such that $\min_{0\leq p\leq n}\{z_{i_v+p} + a_p\} = z_{i_v+p_1} + a_{p_1} = z_{i_v+p_2} + a_{p_2}$ for all $(z_1, \ldots, z_s) \in Q$. If $p_1 > 0$ then we set $i_{v+1} = i_v + p_1$ and $i_{v+2} = i_v + p_2$. Otherwise, we just set $i_{v+1} = i_v + p_2$.

  Note that there can be more than two indices where $\min_{0\leq p\leq n}\{z_{i_v+p} + a_p\}$ is attained for all $(z_1, \ldots, z_s) \in Q$. We pick some pair $p_1 < p_2$.

We will call this sequence an *equality row* for $(t_1, \ldots, t_s)$. Now we claim two important statements:

- $$i_0 < n + 1 \tag{45}$$

  Indeed, suppose the contrary. Then consider $\min_{0\leq p\leq n}\{t_{i_0-n+p} + a_p\}$. As $t_{i_0} = \min_{1\leq f\leq s}\{t_f\}$ and $a_n = 0$ then this minimum equals $\min_{1\leq f\leq s}\{t_f\}$ and there exist $p_1 < p_2 \leq n$ such that $t_{i_0-n+p_1} + a_{p_1} = t_{i_0-n+p_2} + a_{p_2} = \min_{1\leq f\leq s}\{t_f\}$. As $a_p \geq 0$ then we obtain that $a_{p_2} = a_{p_1} = 0$ and $t_{i_0-n+p_1} = t_{i_0-n+p_2} = t_{i_0}$. However, $i_0 - n + p_1 < i_0$ and we get a contradiction with that i_0 is the least index such that $t_{i_0} = \min_{1\leq f\leq s}\{t_f\}$.

- $$t_{i_v} = t_{i_0}, \tag{46}$$

  for all i_v in the equality row.

  We prove this by recursion. For i_0 the statement is already true. Suppose we have proved this statement for i_v and we consider $\min_{0\leq p\leq n}\{t_{i_v+p} + a_p\}$ then either $t_{i_v} + a_0 = t_{i_{v+1}} + a_{p_2}$ equals this minimum or $t_{i_{v+1}} + a_{p_1} = t_{i_{v+2}} + a_{p_2}$. However, this minimum is less or equal to $t_{i_v} + a_0 = t_{i_v} = \min_{1\leq f\leq s}\{t_f\}$. Recalling the fact that $a_p \geq 0$ for $0 \leq p \leq n$ we obtain that $a_{p_1} = a_{p_2} = 0$ and either $t_{i_{v+1}} = t_{i_v} = t_{i_0}$ or $t_{i_{v+2}} = t_{i_{v+1}} = t_{i_v} = t_{i_0}$.

Let us fix $(t_1, \ldots, t_s) \in Q$ and its equality row $\{i_0, \ldots, i_E\}$. Consider another arbitrary point $(t'_1, \ldots, t'_s) \in Q$. We prove the following lemma:

Lemma 7.4 *If for some v it is true that $t'_{i_v} = \min_{0\leq f\leq s}\{t'_f\}$ then for all $w \geq v$ it is true that $t'_{i_w} = t'_{i_v}$.*

Proof of lemma. Indeed, during the recursive construction of the equality row i_v for $(t_1, \ldots, t_s)$ there could appear one of the following three possibilities:

- $v = 0$. Then the processes of construction of equality row for $(t_1, \ldots, t_s)$ and for $(t'_1, \ldots, t'_s)$ completely coincide.

- We considered $\min_{0\le p\le n}\{t_{i_{v-1}+p}+a_p\}$ which is equal to $t_{i_{v-1}+p_1}+a_{p_1}=t_{i_{v-1}+p_2}+a_{p_2}$ for some $p_1<p_2$ and $i_v=i_{v-1}+p_2$. Then the processes of construction of equality row for $(t_1,\ldots,t_s)$ and for $(t'_1,\ldots,t'_s)$ completely coincide starting from the next step.

- We considered $\min_{0\le p\le n}\{t_{i_{v-1}+p}+a_p\}$ which is equal to $t_{i_{v-1}+p_1}+a_{p_1}=t_{i_{v-1}+p_2}+a_{p_2}$ for some $p_1<p_2$ and $i_v=i_{v-1}+p_1$. We recall that these equalities are true for arbitrary $(z_1,\ldots,z_s)\in Q$ and so they are true for $(t'_1,\ldots,t'_s)$. Thus $t'_{i_{v-1}+p_2}$ also equals $\min_{1\le f\le s}\{t'_f\}$ and we come to the previous case.

Now we define Q_b as

$\{(y_1,\ldots,y_s)\in Q$ and b is the least index such that $y_b=\min_{1\le f\le s}\{t_f\}\}$

According to the statement 45 $Q=\bigcup_{b=1}^{n}Q_b$. Next we prove the crucial lemma.

Lemma 7.5 *The number of connected components in the $RG(Q_b)$ is not greater than $s+4-\frac{2s}{n+1}$.*

Proof of lemma. According to the definition of Q_b and according to lemma 7.4 for every $i_v>b$ from the equality row, (b,i_v) is an edge in $RG(Q_b)$. We partition $[b,s]$ into disjoint intervals each of length $n+1$ starting from b. Now we produce the following sequence $\{G'_r\}_{r=0}^{[\frac{s-q}{n+1}]}$ of subgraphs by recursion on an interval number:

- G'_0 is just $RG(Q_b)$ without edges;

- Suppose we have produced G'_r and now we are considering $(r+1)$-th interval of length $(n+1)$. The interval contains at least one element i_v from the equality row. If there are at least two elements from the equality row then for each i_v from this interval we add an edge (b,i_v) to the graph G'_r and obtain G'_{r+1}.

  Otherwise, we consider $(r+1)$-th interval:

  $$[b+(n+1)\cdot r;\ \ b+(n+1)\cdot r+n].$$

  Consider $\min_{0\le p\le n}\{y_{b+(n+1)\cdot r+p}+a_p\}$. According to the definition of the tropical sequence there exist $p_1<p_2$ such that this minimum equals $y_{b+(n+1)\cdot r+p_1}+a_{p_1}=y_{b+(n+1)\cdot r+p_2}+a_{p_2}$ for all $(y_1,\ldots,y_s)\in Q_b$. Thus there is an edge from $RG(Q_b)$ whose vertices have indices from the $(r+1)$-th interval and at least one of them does not lie in the equality row. We call this edge a non-equality edge. Then we set G'_{r+1} as G'_r with one added edge (b,i_v) and one added non-equality edge.

We claim that for every r the number of components in G'_r is at least by two less than G'_{r-1}. It follows from the fact that at each step all edges have at least one end-point which does not belong to the transitive closure of previous subgraph.

Thus we obtain that the number of components is less than $s - 2 \cdot [\frac{s-b}{n+1}] \leq s + 2 - 2\frac{s-b}{n+1} \leq s + 4 - 2\frac{s}{n+1}$.

Now we note that $\dim Q = \max_{1 \leq b \leq n}\{\dim Q_b\}$ and therefore, according to lemma 7.5 we obtain that $\dim Q \leq s + 4 - \frac{2s}{n+1}$. Tending to the limit on s we obtain the required statement of the theorem. $\square$

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