

Entropy of tropical holonomic sequences

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Abstract

We introduce tropical holonomic sequences of a given order and calculate their entropy in case of the second order.

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Introduction

We consider a tropical analog of univariate holonomic sequences [2], [5]. Necessary concepts of tropical algebra one can find in [3].

Let $A_0(i), A_1(i), \dots, A_n(i) \in \mathbb{R}[i]$ be polynomials. We say that a *sequence* $w := \{w_i \in \mathbb{R} : i \geq 0\}$ *satisfies vector* (A_0, \dots, A_n) if the following tropical equation

$$\min\{w_i + A_0(i), w_{i+1} + A_1(i), \dots, w_{i+n} + A_n(i)\} \quad (1)$$

holds for any $i \geq 0$. According to tropical algebra [3] this means that the minimum in (1) is attained for at least two different j among $w_{i+j} + A_j(i)$, $0 \leq j \leq n$. In this case we say that w is a *tropical holonomic sequence of order* n .

For a classical holonomic sequence which satisfies equations $\sum_{0 \leq j \leq n} A_j(i)w_{j+i} = 0$ its element w_s is determined uniquely for s greater than $n - 1$ and greater than the roots of A_n .

This is not the case in the tropical setting. Therefore, in [1] we consider a set $W_N \subset \mathbb{R}^N$ consisting of sequences $\{w_i : 0 \leq i < N\}$ satisfying (1).

Then W_N is a tropical prevariety [3], so a union of a finite number of convex polyhedra. We define the *tropical entropy of vector* A_0, \dots, A_n as

$$H := H(A_0, \dots, A_n) := \limsup_{N \rightarrow \infty} \dim(W_N)/N \quad (2)$$

Evidently, $0 \leq H \leq 1$. Moreover, the proof of Proposition 5.1 [1] can be literally transferred to conclude that $H \leq 1 - 1/n$.

In [1] we studied the case of constant polynomials A_0, \dots, A_n . In this case the limit $\lim_{N \rightarrow \infty} \dim(W_N)/N (= H)$ does exist due to Fekete's subadditive lemma [4]. In addition, one can consider Newton's graph P consisting of the points (i, A_i) , $0 \leq i \leq n$ on the plane. It was proved in Corollary 5.7 [1] that the entropy H vanishes iff P is convex and (i, A_i) , $0 \leq i \leq n$ form its vertices. Moreover, Corollary 5.7 [1] states that if $H > 0$ then $H \geq 1/6$.

In the present paper in case of the second order $n = 2$ we prove the existence of the limit $\lim_{N \rightarrow \infty} \dim(W_N)/N (= H)$ and explicitly calculate the tropical entropy H (2). It appears that in this case the entropy is either 0, either $1/4$ or $1/3$.

1 Entropy of tropical holonomic sequences of the second order

Let $A(i), B(i), C(i) \in \mathbb{R}[i]$ and $\{u_i : i \geq 0\}$ be a tropical holonomic sequence (of the second order) satisfying vector (A, B, C) , i. e.

$$\min\{u_j + A(j), u_{j+1} + B(j), u_{j+2} + C(j)\} \quad (3)$$

for $j \geq 0$. Denote the entropy $H := H(A, B, C)$ (see (2)). We write for a polynomial $A > 0$ if $A(j) > 0$ for $j \gg 0$.

Theorem 1.1 *The limit $\lim_{N \rightarrow \infty} \dim(W_N)/N (= H)$ does exist and*

1) if

$$A(i) + C(i - 1) = B(i - 1) + B(i) \quad (4)$$

for all i then $H = 1/3$;

2) if

$$A(i) + C(i - 1) < B(i - 1) + B(i), \quad (5)$$

$$A(i + 1) - A(i) + C(i) - C(i - 1) = B(i + 1) - B(i - 1) \quad (6)$$

then $H = 1/4$;

3) otherwise $H = 0$.

In the next two sections we prove the Theorem.

2 Case of a positive entropy

2.1 Case $A(i) + C(i - 1) = B(i - 1) + B(i)$

Under the assumption (4) we construct a family of holonomic sequences $\{u_i : i \geq 0\}$ satisfying vector (A, B, C) by induction on i . As an induction base put $u_0 = 0$.

Suppose that a sequence $\{u_i : i \leq 3j\}$ is already constructed for some $j \geq 0$. Then put

$$u_{3j+1} := u_{3j} + B(3j - 1) - C(3j - 1), \quad u_{3j+3} := u_{3j+1} + A(3j + 1) - C(3j + 1),$$

while u_{3j+2} define arbitrarily fulfilling the inequality $u_{3j+2} \geq u_{3j} + A(3j) - C(3j)$. Employing (4) one can verify by induction on j that the constructed sequence satisfies vector (A, B, C) . To this end, it suffices to show that $u_{3j+2} \geq u_{3j+1} + A(3j + 1) - B(3j + 1)$, $u_{3j+2} \geq u_{3j+3} + B(3j + 2) - A(3j + 2)$.

Therefore, denoting by $U_N := \{u_i : 0 \leq i < N\} \subset \mathbb{R}^N$ the tropical prevariety of sequences satisfying (3) for $0 \leq j < N - 2$, we get $\dim(U_N) \geq \lfloor N/3 \rfloor$ since the coordinates u_{3j+2} , $j \geq 0$ attain arbitrarily sufficiently large values. Hence,

$$\liminf_{N \rightarrow \infty} \dim(U_N)/N \geq 1/3 \tag{7}$$

To prove the opposite inequality $H \leq 1/3$ fix for the time being N , a sequence $(u_0, \dots, u_{N-1}) \in U_N$, and consider a graph $G := G_N$ with N vertices $\{0, \dots, N - 1\}$. For each $0 \leq j \leq N - 3$, if the minimum in (3) is attained on some two indices among $j, j + 1, j + 2$ then between these indices draw an edge in G (it is not excluded that there could be three edges between $j, j + 1, j + 2$). Observe that any such graph G determines a convex polyhedron in U_N whose dimension does not exceed the number c of connected components of G (moreover, one can show that this dimension equals c , although we don't make use of it here). Thus, our next goal is to bound c from above.

We call an interval of G a maximal (with respect to inclusion) subset (of at least two elements) of its vertices being an interval and belonging to the same connected component. Enumerate the connected components in an arbitrary way.

Lemma 2.1 *The following statements are valid under the assumption of either (4) or (5):*

- i) two intervals can't adjoin;*
- ii) the vertices between two neighbouring intervals belong to two alternating components s, t , respectively (or, perhaps, there is a single vertex inbetween, in this case we agree that $s = t$);*

iii) if neighbouring intervals belong to components p, q , respectively, then $\{p, q\} \subset \{s, t\}$ (or, in case $s = t$ we have $p = q \neq s$).

Proof. i) Suppose that $i, i + 1$ belong to the same connected component, while $i + 2, i + 3$ belong to a different connected component. Then (considering triples of vertices $i, i + 1, i + 2$ and $i + 1, i + 2, i + 3$) (3) implies that

$$(u_i + A(i) =)u_{i+1} + B(i) < u_{i+2} + C(i),$$

$$u_{i+1} + A(i + 1) > u_{i+2} + B(i + 1)(= u_{i+3} + C(i + 1))$$

which leads to a contradiction with (4) as well as with (5).

ii) If $i, i + 1$ belong to the same connected component p , while $i + 2$ belongs to a different component s then $i + 3$ belongs to p (considering triple $i, i + 1, i + 2$). Similarly, $i + 4$ belongs to s (provided that $i + 3$ does not belong to the next interval). Continuing this argument we establish ii) and in addition, iii) in case $p = q$. To establish iii) in case $p \neq q$ note that $s = q$ considering triple $j - 2, j - 1, j$ where j is the beginning of the next interval (belonging to q). \square

One can deduce statement 1) of the Theorem from Lemma 2.1. Indeed, for any new emerging component (by scanning G from the left to the right) its first element is located between two neighbouring intervals because of i), in addition, between two neighbouring intervals at most one new component can emerge due to ii), iii). This entails an upper bound $\lceil N/3 \rceil + 1$ on the number of connected components of G since the length of every interval is at least 2. Hence the entropy $H \leq 1/3$ and the limit $\lim_{N \rightarrow \infty} \dim(U_N)/N = 1/3$ taking into account (7).

2.2 Case $A(i) + C(i - 1) < B(i - 1) + B(i)$,

$$A(i + 1) - A(i) + C(i) - C(i - 1) = B(i + 1) - B(i - 1)$$

Now we proceed to the proof of statement 2) of the Theorem. First similar to section 2.1 construct by induction a family of holonomic sequences $V_N := \{v_i : 0 \leq i < N\} \subset \mathbb{R}^N$ satisfying (3) such that $\dim(V_N) \geq N/4 - \text{const}$. Let $4j_0$ be greater than all the roots of polynomial $B(i - 1) + B(i) - A(i) - C(i - 1)$. Obviously, any sequence v_0, \dots, v_i satisfying (3) one can continue to v_0, \dots, v_i, v_{i+1} also satisfying (3). Construct an arbitrary sequence v_0, \dots, v_{4j_0} satisfying (3). Then family U_{4j_0+1} consists of this single sequence. Consider the latter as a base of induction.

Suppose that a family $V_{4j+1} = \{v_i : 0 \leq i \leq 4j\}$ is already constructed for some $j \geq j_0$. Put

$$v_{4j+1} := v_{4j} + B(4j - 1) - C(4j - 1),$$

$$\begin{aligned} v_{4j+2} &:= v_{4j} + A(4j) - C(4j), \\ v_{4j+4} &:= v_{4j+2} + A(4j+2) - C(4j+2). \end{aligned}$$

Then we take v_{4j+3} in an arbitrary way satisfying inequality $v_{4j+3} \geq v_{4j+2} + B(4j+1) - C(4j+1)$. One can verify that the constructed family $V_{4j+5} := \{v_i : 0 \leq i \leq 4j+4\}$ satisfies (3). To this end it suffices to show that

$$u_{4j} + A(4j) < u_{4j+1} + B(4j), \quad u_{4j+1} + A(4j+1) = u_{4j+2} + B(4j+1),$$

$v_{4j+3} \geq v_{4j+2} + A(4j+2) - B(4j+2)$, $v_{4j+3} \geq v_{4j+4} + B(4j+3) - A(4j+3)$ employing (5), (6). Similar to section 2.1 conclude that $\dim(V_{4j+1}) = j - j_0$, thus

$$\liminf_{N \rightarrow \infty} \dim(V_N)/N \geq 1/4 \tag{8}$$

Now we prove the opposite inequality $H \leq 1/4$ in statement 2) of the Theorem. It goes analogously to the proof of the upper bound on the entropy in section 2.1 considering graph G and applying Lemma 2.1 with a difference that now the length of every interval is at least 3 (with a possible finite number of intervals of length 2 lying in $\{0, \dots, 4j_0\}$). Indeed, if $i, i+1$ was an interval then

$$u_i + B(i-1) = u_{i+1} + A(i-1), \quad u_i + A(i) = u_{i+1} + B(i),$$

and we get a contradiction with (5). Thus, $H \leq 1/4$ and (8) implies the existence of the limit $\lim_{N \rightarrow \infty} \dim(V_N)/N = H = 1/4$. The statement 2) of the Theorem is proved.

3 Case of zero entropy

Now we proceed to a proof of statement 3) of the Theorem.

3.1 Case $A(i) + C(i-1) > B(i) + B(i-1)$

First we assume that

$$A(i) + C(i-1) > B(i) + B(i-1) \tag{9}$$

(cf. (4), (5)). Denote by $W_N := \{w_i : 0 \leq i < N\} \subset \mathbb{R}^N$ the set of all sequences satisfying (3). We show that $\dim(W_N)$ is bounded from above by a constant independent from N .

Lemma 3.1 *Let $w := \{w_i : 0 \leq i \leq 4j_0\} \in W_{4j_0+1}$ (cf. section 2.1). Then w has at most one-dimensional continuation in any W_N for $N > 4j_0+1$.*

Proof. Let $\{w_i : 0 \leq i \leq N\} \in W_{N+1}$ be a continuation of w . denote by $s \geq 4j_0$ the maximal integer such that for every $4j_0 \leq t < s$ it holds $w_t + A(t) = w_{t+1} + B(t)$. Suppose w.l.o.g. that $s < N$. We claim that $\{w_i : 0 \leq i \leq s + 1\} \in W_{s+2}$ has a unique continuation in any W_N for $N > s + 1$.

We consider two cases. In the first one

$$w_s + A(s) > w_{s+1} + B(s) \quad (10)$$

Then $w_{s+2} + C(s) = w_{s+1} + B(s)$ because of (3). This together with (9) implies that $w_{s+1} + A(s+1) > w_{s+2} + B(s+1)$. Continuing this arguing by induction on i we obtain that $w_{i+2} + C(i) = w_{i+1} + B(i)$ for any $i \geq s$. This proves the claim in case (10).

In the second case

$$w_s + A(s) < w_{s+1} + B(s) \quad (11)$$

Then we have

$$w_{s+2} + C(s) = w_s + A(s) \quad (12)$$

Therefore, $w_{s+1} + A(s+1) > w_{s+2} + B(s+2)$ since otherwise, summing up the opposite inequality with (11), (12), we get a contradiction with (9). Hence, we arrive to already considered in the first case inequality (10) which proves the claim.

Thus, all the continuations of w in any W_N for $N > 4j_0$ have the following form. For some $4j_0 \leq s < N$ for each $4j_0 \leq t < s$ it holds $w_t + A(t) = w_{t+1} + B(t)$. After that, w_{s+1} can take an arbitrary value not less than $w_{s-1} + A(s-1) - C(s-1)$ (when $s > 4j_0$ or when $s = 4j_0$ and $w_{4j_0-1} + A(4j_0-1) = w_{4j_0} + B(4j_0-1)$). Subsequently, a continuation of $\{w_i : 0 \leq i \leq s+1\}$ in any W_N for $N > s$ is unique. Lemma is proved. \square

Lemma 3.1 implies that the entropy $H = 0$ in case (9).

3.2 Case $A(i) + C(i-1) < B(i) + B(i-1)$,

$$A(i+1) - A(i) + C(i) - C(i-1) > B(i+1) - B(i-1)$$

Now we assume (5) and

$$A(i+1) - A(i) + C(i) - C(i-1) > B(i+1) - B(i-1) \quad (13)$$

Lemma 3.2 *Under assumptions (5), (13) let a sequence $\{w_i : 0 \leq i < N\} \in W_N$. If $w_{k+2} + C(k) = w_{k+1} + B(k)$ (cf. (3)) for a sufficiently large k (namely, $k > 4j_0$, cf. section 2.1) then*

$$w_{k+2j+2} + C(k+2j) = w_{k+2j+1} + B(k+2j), \quad (14)$$

$$w_{k+2j+3} + C(k+2j+1) = w_{k+2j+1} + A(k+2j+1) \quad (15)$$

for any $j \geq 0$.

Proof goes by induction on j . Suppose that (14) holds for some $j \geq 0$ (the base is for $j = 0$). Then $w_{k+2j+2} + B(k+2j+1) > w_{k+2j+1} + A(k+2j+1)$ because of (5), therefore, (15) is true (invoking (3)). Summing up (14), (15) and employing (13) for $i := k+2j$, we obtain that

$$w_{k+2j+3} + B(k+2j+2) < w_{k+2j+2} + A(k+2j+2).$$

Hence, (14) is valid for $j := j+1$, this proves the inductive hypothesis and lemma. \square

Consider graph $G := G_N$ constructed in section 2.1 corresponding to the sequence $\{w_i : 0 \leq i < N\}$. Lemma 3.2 entails that G contains at most one interval of the form $k, k+1, \dots$ where $k > 4j_0$. Moreover, if there is such an interval then it is of the form $k, k+1, \dots, N$, so ends at N . Therefore, G_N can contain at most one new connected component in comparison with G_{4j_0} . Indeed, if $m > 4j_0$ is the first element of a new component then $m-2, m-1$ belong to the same component (due to (3)), hence $m-2, m-1$ is the end of an interval which contradicts to Lemma 3.2. Note that G_N can contain even less components than G_{4j_0} does.

From this follows that the number of components of G is bounded from above by a constant independent from N . Hence (cf. section 2.1), $\dim(W_N)$ is bounded from above by a constant independent from N , thus the entropy $H = 0$ under assumptions (5), (13).

3.3 Case $A(i) + C(i-1) < B(i) + B(i-1)$,

$$A(i+1) - A(i) + C(i) - C(i-1) < B(i+1) - B(i-1)$$

Lemma 3.3 *Let (5) and*

$$A(i+1) - A(i) + C(i) - C(i-1) < B(i+1) - B(i-1) \quad (16)$$

be fulfilled. Assume that a sequence $\{w_i : 0 \leq i < N\} \in W_N$ and for some $k > 4j_0$ it holds

$$w_{k+2} + C(k) = w_{k+1} + B(k). \quad (17)$$

Then for any $j \geq 1$ we have

$$w_{k+j+2} + C(k+j) = w_{k+j} + A(k+j). \quad (18)$$

Proof goes by induction on j . Suppose that (18) is already established for all $1 \leq j \leq s$ for some $s \geq 0$ (thus, one can treat the void case $s = 0$ as a base of induction).

Summing up inequalities (16) for $i := k + 2t$, $0 \leq t \leq r$ we obtain that

$$B(k) + \sum_{1 \leq m \leq 2r} (-1)^m A(k+m) < B(k+2r) + \sum_{0 \leq m \leq 2r-1} (-1)^m C(k+m) \quad (19)$$

First, let s be odd. Summing up equations (18) multiplied by signs $(-1)^j$ for $1 \leq j \leq s$, with (19) for $r := (s+1)/2$ and also with (17) we get inequality

$$w_{k+s+2} + B(k+s+1) > w_{k+s+1} + A(k+s+1). \quad (20)$$

Hence, (3) entails that

$$w_{k+s+3} + C(k+s+1) = w_{k+s+1} + A(k+s+1)$$

which proves (18) for $j := s+1$.

Now let s be even. Applying (19) for $k := k+1$, $r := s/2$ we obtain inequality

$$B(k+1) + \sum_{1 \leq m \leq s} (-1)^m A(k+m+1) < B(k+s+1) + \sum_{0 \leq m \leq s-1} (-1)^m C(k+m+1).$$

Summing up the latter inequality with (5) for $i := k+1$, we establish inequality

$$-B(k) + \sum_{0 \leq m \leq s} (-1)^m A(k+m+1) < B(k+s+1) + \sum_{0 \leq m \leq s} (-1)^m C(k+m) \quad (21)$$

Similar to the case of odd s summing up equations (18) multiplied by signs $(-1)^j$ for $1 \leq j \leq s$, with (21) and with (17), we again get inequality (20). Then again (3) entails (18) for $j := s+1$ which completes the proof of the inductive hypothesis and Lemma. \square

As in section 3.2 Lemma 3.3 implies that graph $G := G_N$ can contain at most one interval of the form $k, k+1, \dots$ where $k > 4j_0$. Also as in section 3.2 we conclude that the entropy $H = 0$. This completes the proof of the Theorem. \square

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