Tropical recurrent sequences

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Abstract

Tropical recurrent sequences are introduced satisfying a given vector (being a tropical counterpart of classical linear recurrent sequences). We consider the case when Newton polygon of the vector has a single (bounded) edge. In this case there are periodic tropical recurrent sequences which are similar to classical linear recurrent sequences. A question is studied when there exists a non-periodic tropical recurrent sequence satisfying a given vector, and partial answers are provided to this question. Also an algorithm is designed which tests existence of non-periodic tropical recurrent sequences satisfying a given vector with integer coordinates. Finally, we introduce a tropical entropy of a vector, provide some bounds on it and extend this concept to tropical multi-variable recurrent sequences.

keywords: tropical recurrent sequence, periodic sequence, tropical entropy

Introduction

A classical (linear) recurrent sequence $\{z_l\}_{l\in\mathbb{Z}}$ (e. g. Fibonacci numbers) satisfies conditions $\sum_{0\le i\le n} a_i z_{i+k}=0$, $k\in\mathbb{Z}$, $a_0\neq 0$, $a_n\neq 0$. It well known that the linear space of all such sequences has dimension n and can be explicitly described via the roots of polynomial $\sum_{0\le i\le n} a_i x^i$ and the derivatives in case of multiple roots.

We study *tropical recurrent sequences* satisfying similar tropical linear polynomials $\min_{0\le i\le n}\{a_i+y_{i+k}\}$, $k\in\mathbb{Z}$ where as it is adopted in tropical algebra [4] we assume that the minimum is attained for at least two different indices $0\le i_1(k)<i_2(k)\le n$ for each $k\in\mathbb{Z}$. We say that in this

case a tropical recurrent sequence $y = \{y_i \in \mathbb{R} \cup \{\infty\}\}_{i \in \mathbb{Z}}$ *satisfies vector* $a := (a_0, \dots, a_n) \in (\mathbb{R} \cup \{\infty\})^{n+1}, a_0 < \infty$, $a_n < \infty$. One can treat a tropical recurrent sequence as a solution of an (infinite) tropical Macauley matrix whose rows are obtained from vector a by all possible shifts. We mention that Macauley matrix plays a key role in the tropical Nullstellensatz [2], [3].

Throughout the paper (except for sections 5, 6) we impose the requirement of *minimality* of tropical recurrent sequences: for any $j \in \mathbb{Z}$ one can not diminish y_j keeping all the rest y_i, $i \neq j$ without violation of being a tropical recurrent sequence satisfying a.

A description of tropical recurrent sequences satisfying a given vector a is more complicated than its classical counterpart, and we don't provide a complete answer.

A crucial feature of a is its Newton polygon $P(a)$ on the plane which is the convex hull of the vertical rays $\{(i, b \geq a_i), 0 \leq i \leq n\}$. For each (bounded) edge of $P(a)$ with a slope s a tropical recurrent sequence $\{y_j\}_{j \in \mathbb{Z}}$ satisfies a where points (j, y_j), $j \in \mathbb{Z}$ are located on a line with the slope $-s$. There can be more general periodic tropical recurrent sequences, so for some period $d \in \mathbb{Z}$ it holds $y_{j+d} - y_j = -sd$, $j \in \mathbb{Z}$.

In a certain (informal) sense periodic tropical recurrent sequences are similar to classical recurrent sequences. On the other hand, for some vectors a there exist non-periodic tropical recurrent sequences satisfying a. We study for which a they exist.

In section 1 vectors a are considered such that all the finite points (i, a_i), $a_i < \infty$ lie on a single (bounded) edge of $P(a)$. We show that all the tropical recurrent sequences satisfying a are periodic iff the points (i, a_i), $a_i < \infty$ form an arithmetic progression with some difference d. In the latter case any tropical recurrent sequence satisfying a has period d.

In section 2 a more general situation is studied when $P(a)$ has a single (bounded) edge. First, we note that if points (i, a_i) lying on this edge do not form an arithmetic progression then there exists a non-periodic tropical recurrent sequence satisfying a. When, on the contrary, these points form an arithmetic progression with the difference 2 we prove that all the tropical recurrent sequences satisfying a are periodic iff points (i, a_i) not lying on the edge, also form an arithmetic progression.

In section 3 vectors a are considered with an arithmetic progression having the difference 3 of points (i, a_i) lying on the (single) edge. We provide two examples of a: one having only periodic tropical recurrent sequences satisfying a, and another one with non-periodic sequences. These two examples demonstrate that the existence of non-periodic tropical recurrent sequences satisfying a can not be expressed just in terms of arithmetic progressions. It would be interesting to give an explicit answer to the question of existence of non-periodic tropical recurrent sequences satisfying a.

In section 4 we design an algorithm which tests for vector $a = (a_0, \dots, a_n)$ with integer coordinates $a_i \in \mathbb{Z}$, $0 \le i \le n$ whether there exists a non-periodic tropical recurrent sequence satisfying a.

In section 5 we introduce a tropical entropy $H(a)$ and a tropical minimal entropy $h(a)$, they fulfil inequalities $0 \le h(a) \le H(a)$. We provide an upper bound on $H(a)$ and calculate $h(a)$ and $H(a)$ for some examples of vectors a.

In section 6 we extend the concepts of the tropical (respectively, minimal) entropy to tropical multivariable recurrent sequences. Again we provide an upper bound on $H(a)$ and calculate $H(a)$ for an example of vector a.

1 Tropical recurrent sequences satisfying a vector lying on a single bounded edge of Newton polygon

Remark 1.1 *We say that a tropical recurrent sequence $y = \{y_j\}_{j \in \mathbb{Z}}$ satisfying vector a is minimal if for any $j \in \mathbb{Z}$ there exists $k \in \mathbb{Z}$, $j-n \le k \le j$ such that $a_{j-k} + y_j = \min_{0 \le i \le n}\{a_i + y_{i+k}\}$. In other words, one can not diminish y_j keeping all the rest y_l, $l \ne j$. Throughout the paper (except for sections 5, 6) we consider only minimal tropical recurrent sequences.*

One can plot a_i as point (i, a_i) on the plane, respectively, y_j as (j, y_j).

In this section we study the case when all the points (i, a_i) are located on a single bounded edge of Newton polygon $P(a)$. Making a suitable affine transformation of the plane one can assume w.l.o.g. that this edge is situated on the abscissas axis, so either $a_i = 0$ or $a_i = \infty$ for all $i \in \mathbb{Z}$.

Note that if $y^{(1)}$, $y^{(2)}$ are two tropical recurrent sequences satisfying a then $\min\{b_1 + y^{(1)}, b_2 + y^{(2)}\}$ is also a tropical recurrent sequence satisfying a, where $b_1, b_2 \in \mathbb{R}$.

There is always the trivial infinite tropical recurrent sequence $\{y_j = \infty, j \in \mathbb{Z}\}$, so we suppose that tropical recurrent sequences under consideration differ from the infinite one.

Proposition 1.2 *Let for vector $a = (a_0, \dots, a_n)$ hold $a_i = 0$ for all finite a_i. Then all the tropical recurrent sequences satisfying a are periodic iff all i for which $a_i = 0$ form an arithmetic progression. If d is the difference of this progression then every tropical recurrent sequence satisfying a is periodic with the period d.*

Proof. First assume that set $S := \{i \in \mathbb{Z} : a_i = 0\}$ does not form an arithmetic progression. We claim that for any $0 < b \in \mathbb{R} \cup \{\infty\}$ and $k \in \mathbb{Z}$ a tropical recurrent sequence $y_i = b$ for $i - k \in S$ and $y_i = 0$ otherwise, satisfies a.

Suppose the contrary. To simplify notations assume w.l.o.g. that $k = 0$. Then there exists $0 \neq s \in \mathbb{Z}$ such that minimum $\min_{0 \le i \le n}\{a_i + y_{i+s}\}$ is attained once. When $s > 0$ we have $0 = a_n + y_{n+s} = \min_{0 \le i \le n}\{a_i + y_{i+s}\}$. According to the supposition for each $0 \le i < n$, $i \in S$ it holds $a_i + y_{i+s} = b$, therefore S forms an arithmetic progression with the difference s, we get a contradiction. A similar argument works when $s < 0$, in this case $0 = a_0 + y_s = \min_{0 \le i \le n}\{a_i + y_{i+s}\}$. The claim is proved.

Sequence y is minimal (see Remark 1.1). Indeed, for each $i \in S$ we have $b = a_i + y_i = \min_{0 \le l \le n}\{a_l + y_l\}$. On the other hand, for each $i \notin S$ we have $0 = a_0 + y_i = \min_{0 \le l \le n}\{a_l + y_{i+l}\}$ due to the proved above claim.

Remark 1.3 *Thus, in case when S does not form an arithmetic progression one can modify the zero tropical recurrent sequence $\{y_i = 0, i \in \mathbb{Z}\}$ increasing y_{i+k} for $i \in S$ by $b_k = b > 0$. Moreover, one can take arbitrary integers $\dots, k_{-1}, k_0, k_1, \dots$ such that $k_{l+1} - k_l > 2n$, $l \in \mathbb{Z}$, and for each k_l modify the zero solution by $b_{k_l} > 0$ as described above. Thus, one obtains an uncountable number of modifications (just by choosing $\{k_l\}_{l \in \mathbb{Z}}$ regardless of b_{k_l}) of the zero tropical recurrent sequence, satisfying a.*

Coming back to the proof of Proposition 1.2, let now S form an arithmetic progression with a difference d. Let a tropical recurrent sequence y satisfy a. For each $0 \le i_0 < d$ the subsequence $\{y_{id+i_0} : i \in \mathbb{Z}\}$ of y constitutes a tropical recurrent sequence satisfying a. Therefore, one can consider each of these d subsequences instead of y, thus assuming that $d = 1$. To prove the required last statement in the Proposition on periodicity it suffices to show that y_i is constant for $i \in \mathbb{Z}$ when $d = 1$. Denote $c := \min_{0 \le i \le n}\{y_i\} = \min_{0 \le i \le n}\{a_i + y_i\}$. Then this minimum is attained for two different $0 \le i_1 < i_2 \le n$. Observe that for every $i_1 < i < i_2$ it holds $y_i = c$ as well due to the minimality of y (see Remark 1.1). In addition, observe that $c = \min_{-1 \le i \le n+1}\{y_i\}$. Indeed, if on the contrary $y_{n+1} < c$ then minimum $\min_{0 \le i \le n}\{a_i + y_{i+1}\} = y_{n+1}$ is attained only once, which contradicts that y is a tropical recurrent sequence satisfying a. Similarly, one shows that $y_{-1} \ge c$. Repeating this argument recursively one deduces that $c = \min_{-\infty < i < \infty}\{y_i\}$. Considering $c = a_0 + y_{i_2} = \min_{0 \le i \le n}\{a_i + y_{i+i_2}\}$ one gets that there exists $i_2 < i_3 \le i_2 + n$ such that $y_{i_3} = c$. Then as above one obtains that $y_i = c$ for any $i_2 < i < i_3$. Similarly, there exists $i_1 - n \le i_4 < i_1$ such that $y_{i_4} = c$. Hence $y_i = c$ for any $i_4 < i < i_1$. Repeating this argument one concludes that $y_i = c$ for any $i \in \mathbb{Z}$.

This completes the proof that y is periodic with the period d. $\Box$

2 Tropical recurrent sequences for a Newton polygon with a single edge

In the previous section we studied the case when points (i, a_i), $0 \le i \le n$ are located on a line. Note that in general, Newton polygon $P(a)$ has two unbounded edges and several bounded ones. In the present section we suppose that $P(a)$ has a single bounded edge. Similar to the previous section, making a suitable affine transformation of the plane one can assume w.l.o.g. that this edge lies on the abscissas axis. Again as in the previous section we consider set $S := \{0 \le i \le n : a_i = 0\}$. In particular, $0, n \in S$.

Remark 2.1 *First, consider the case when S does not form an arithmetic progression. Then similar to the proof of Proposition 1.2 one can modify the zero tropical recurrent sequence $y_j = 0$, $j \in \mathbb{Z}$ by replacing $y_j = b$ for $j \in S$, while $b > 0$ should be taken less than $\min\{a_i : a_i > 0\}$. Thus, again one obtains an uncountable number of periodic tropical recurrent sequences satisfying a.*

So, from now on we assume that S forms an arithmetic progression with a difference d. In the present section we study the case $d = 2$ and investigate when there is a non-periodic tropical recurrent sequence satisfying a. In particular, in this case n is even.

Theorem 2.2 *Let Newton polygon $P(a)$ have a single bounded edge on the abscissas axis, and the points S of a on this edge form an arithmetic progression with the difference 2. Then any tropical recurrent sequence satisfying a is periodic iff all a_i with odd i are equal. In the latter case any tropical recurrent sequence satisfying a, has period 2.*

Remark 2.3 *In other words, under the conditions of the Theorem any sequence is periodic iff points (i, a_i) are located on two parallel lines: one for even $0 \le i \le n$ and the second for odd $1 \le i \le n-1$.*

Proof of the theorem. First consider the case when not all a_i with odd i are equal. Denote $c := \min\{a_i : \text{odd}\, i\} > 0$, $e := \min\{a_i : \text{odd}\, i,\ a_i > c\} > c$ and $C := \{i : a_i = c\}$. Take a periodic (with the period 2) tropical recurrent sequence $y^{(0)} := \{y^{(0)}_{2i} = c,\ y^{(0)}_{2i+1} = 0,\ i \in \mathbb{Z}\}$ satisfying a. Let us modify it (denote the modified sequence by $y := \{y_i\ i \in \mathbb{Z}\}$) putting

- $y_{2i} := e$, $0 \le 2i \le n$;
- $y_{2i+1} := e - c$, $2i + 1 \in C$, $1 \le 2i + 1 \le n - 1$;
- $y_{2i+1} = 0$, $2i + 1 \notin C$, $1 \le 2i + 1 \le n - 1$,

while keeping the rest of the coordinates unchanged.

Let us verify that y satisfies a. For any odd $k \ge 3$ minimum $0 = a_n + y_{n+k} = a_{n-2} + y_{n-2+k} = \min_{0 \le i \le n}\{a_i + y_{i+k}\}$ is attained at least twice. Similarly, for any odd $k \le -3$ minimum $0 = a_0 + y_k = a_2 + y_{k+2} = \min_{0\le i\le n}\{a_i + a_{i+k}\}$ is attained at least twice as well. For $k = \pm 1$ minimum $\min_{0\le i\le n}\{a_i + y_{i+k}\} = 0$ is attained at least twice since there is an odd $1 \le i \le n-1$ such that $i \notin C$. For $k = 0$ minimum $\min_{0\le i\le n}\{a_i + y_i\} = e$ is attained (for any even $0 \le i \le n$, for any $i \in C$ and for any odd $i \notin C$ such that $a_i = e$) at least twice as well. For an even $k \ne 0$ minimum $\min_{0\le i\le n}\{a_i + y_{i+k}\} = c$. For an even $k \ge 4$ this minimum is attained at least twice for $c = a_n + y_{n+k} = a_{n-2} + y_{n-2+k} = \min_{0\le i\le n}\{a_i + y_{i+k}\}$. Similarly, for an even $k \le -4$ this minimum is also attained at least twice for $c = a_0 + y_k = a_2 + y_{k+2} = \min_{0\le i\le n}\{a_i + y_{i+k}\}$. Consider the case $k = 2$ (the case $k = -2$ can be considered in a similar way). We have $a_n + y_{n+2} = 0 + c$. We claim that there is an odd $1 \le 2i-1 \le n-1$ such that $a_{2i-1} + y_{2i+1} = c$. Assume that there exists $2i-1 \in C$, $1 \le 2i-1 \le n-3$ for which $2i+1 \notin C$. Then $a_{2i-1} = c$, $y_{2i+1} = 0$, which proves the claim under the assumption. If such $2i-1 \in C$ does not exist then $n-1 \in C$, hence $a_{n-1} + y_{n+1} = c + 0$, which proves the claim.

One can check the minimality of y (see Remark 1.1) for any (odd) $2i+1$ such that $y_{2i+1} = 0$ with the help of an appropriate even k. For an odd $2i+1 \in C$ (in this case $y_{2i+1} = e - c$) one uses $k = 0$. Also $k = 0$ is used for even $0 \le 2i \le n$ (in this case $y_{2i} = e$). For an even $|2i| \ge n+2$ an appropriate odd k is involved.

Actually, one can take any $0 < q < d - c$ and modify $y^{(0)}$ putting

- $y_{2i} = c + q$, $0 \le 2i \le n$;
- $y_{2i+1} = q$, $2i+1 \in C$, $1 \le 2i+1 \le n-1$;
- $y_{2i+1} = 0$, $2i+1 \notin C$, $1 \le 2i+1 \le n-1$,

while keeping the rest of the coordinates unchanged.

Again as in the proof of Proposition 1.2 one can modify $y^{(0)}$ changing $y_{k_l}, y_{k_l+1}, \dots, y_{k_l+n}$ as described above for integers $\dots < k_{-1} < k_0 < k_1 < \dots$ such that $k_{l+1} - k_l > 2n$ for all integers l. Thus, there is an uncountable number of non-periodic tropical recurrent sequences satisfying a.

Now we consider a such that $a_{2i} = 0$, $0 \le 2i \le n$; $a_{2i+1} = c > 0$, $1 \le 2i+1 \le n-1$ and prove that any tropical recurrent sequence y satisfying a, is periodic with the period 2.

Denote $b := \min_{-n\le i\le n}\{y_i\}$. Similar to the proof of Proposition 1.2 one can deduce that $b = \min_{-n-1\le i\le n+1}\{y_i\}$ and further by recursion that $b = \min_{-\infty<i<\infty}\{y_i\}$. Let $y_i = b$ for some even $-n \le i \le n$ (an odd i can be considered in a similar way). Denote $B := \{2i : y_{2i} = b\}$. For any pair of adjacent elements $2i_1 < 2i_2$ of B one has $2(i_2 - i_1) \le n$, because otherwise, minimum $\min_{0\le i\le n}\{a_i + y_{i+2i_1}\} = b$ is attained only once for $i = 0$. Therefore, for every even k we have $\min_{0\le i\le n}\{a_i + y_{i+k}\} = b$, and for every odd l we have $\min_{0\le i\le n}\{a_i + y_{i+l}\} \le b + c$, in addition $y_l \le b + c$ due to the minimality of y (see Remark 1.1). Again due to the minimality $y_k = b$ for every even k.

Denote $p := \min\{y_l : -n \leq l \leq n, \text{odd}\, l\} \leq b + c$. If $p = b + c$ then one deduces that $y_l = b + c$ for any odd l arguing as above by recursion on $|l|$, thus y is periodic with the period 2. So, assume that $p < b + c$. Arguing as above we conclude that $p = \min\{y_l : \text{odd}\, l \in \mathbb{Z}\}$ and that for all odd integers $\ldots < l_{-1} < l_0 < l_1 < \ldots$ for which $y_{l_j} = p$, we have $l_{j+1} - l_j \leq n$. Hence due to the minimality of y we get that $y_l = p$ for any odd l. Thus, y is periodic with the period 2. Theorem is proved. $\square$

3 Newton polygon with a single edge and period greater than 2

So far, we considered vector a such that its Newton polygon $P(a)$ had a single bounded edge (recall that w.l.o.g. one can assume that this edge is situated on the abscissas axis). Moreover, one can suppose that points $(i, a_i = 0)$ on this edge form an arithmetic progression with a difference d (otherwise, as we have shown above, there would be a non-periodic tropical recurrent sequence satisfying a). We have given a complete answer to the question of existence of non-periodic sequences for $d = 1, 2$. In the present section we provide examples for $d = 3$ which demonstrate that the answer is more complicated in this case and depends not only on the properties to be arithmetic progressions as for $d = 1, 2$. It would be interesting to give a complete answer for $d \geq 3$.

Proposition 3.1 *Let vector* $a : a_0 = a_3 = 0, a_1 = b, a_2 = c; b, c > 0, b \neq c$. *Any tropical recurrent sequence satisfying* a *is periodic iff either* $b < c \leq 2b$ *or* $c < b \leq 2c$. *In this case any sequence is periodic with the period 3.*

Proof. First consider the case $c > 2b$ (the case $b > 2c$ can be studied in a similar way). Take the following periodic tropical recurrent sequence satisfying a: $y^{(0)}_{3i} := 0$; $y^{(0)}_{3i+1} = 2b$; $y^{(0)}_{3i+2} := b$; $i \in \mathbb{Z}$. Consider a real $0 < e \leq c - 2b$. We modify $y^{(0)}$ resulting in a non-periodic tropical recurrent sequence y satisfying a: $y_1 = 2b + e$, $y_2 = b + e$, $y_4 = 2b + e$ and keeping the rest of the coordinates of $y^{(0)}$ unchanged.

Similar to the proofs of Proposition 1.2 and Theorem 2.2 one can choose integers $\ldots < k_{-1} < k_0 < k_1 < \ldots$ and reals $0 < e_l \leq c - 2b$ such that $k_{l+1} - k_l \geq 3$, and modify $y^{(0)}$ putting $y_{3k_l+1} = 2b + e_l$; $y_{3k_l+2} = b + e_l$; $y_{3k_l+4} = 2b + e_l$ for all integers l. Thus, one achieves a non-countable number of non-periodic tropical recurrent sequences satisfying a.

Now we study a fulfilling inequalities $b < c \leq 2b$ (the case of inequalities $c < b \leq 2c$ is considered in a similar way). Let a tropical recurrent sequence y satisfy a. Denote $q := \min_{-3 \leq i \leq 3}\{y_i\}$. Arguing as in the proofs of Proposition 1.2 and Theorem 2.2, we conclude that $q = \min_{-4 \leq i \leq 4}\{y_i\}$, and continuing this argument we get by induction that $q = \min_{-\infty<i<\infty}\{y_i\}$. One can assume w.l.o.g. that $y_{3i_0} = q$ for some integer i_0. Since minimum $\min_{0\leq i\leq 3}\{a_i + y_{3i_0+i}\}$ is attained at least twice, we deduce that $y_{3i_0+3} = q$. Continuing in this way, we deduce that $y_{3j} = q$ for every integer j.

Now we consider coordinates for an arithmetic progression $\{y_{3i+2} : i \in \mathbb{Z}\}$. Denote $r := \min_{0\leq i\leq 1}\{y_{3i+2}\}$. Hence $r \leq q + b$ since minimum $\min_{0\leq i\leq 3}\{a_i + y_{i+2}\} \leq a_1 + y_3 = b + q < a_2 + q \leq c + y_4$ should be attained at least twice. Arguing as above, we deduce that $r = \min_{-\infty<i<\infty}\{y_{3i+2}\}$. If $r < q + b$ then since $\min_{0\leq i\leq 3}\{a_i+y_{3j+i+2}\} \leq a_0+r = r < b+q = a_1+y_{3j+3} < c+q \leq a_2+y_{3j+4}$ for any integer j, we conclude that $y_{3i+2} = r$ for every integer i. Now let $r = q + b$. Since $\min_{0\leq i\leq 3}\{a_i + y_{3j+i+1}\} \leq a_2 + q = c + q \leq 2b + q = b + r \leq a_1 + y_{3j+2}$, we get that if $y_{3j+2} > r$ then $a_1 + y_{3j+2}$ does not attain minimum in $\min_{0\leq i\leq 3}\{a_i + y_{3j+i+1}\}$, hence $a_0 + y_{3j+2} = y_{3j+2}$ attains the minimum in $\min_{0\leq i\leq 3}\{a_i + y_{3j+i+2}\} \leq a_1 + y_{3j+3} = b + q = r$ according to the minimality of y at y_{3j+2} (cf. Remark 1.1). Thus, $y_{3i+2} = r$ for every integer i.

Finally, we consider coordinates for an arithmetic progression $\{y_{3i+1} : i \in \mathbb{Z}\}$. Due to the minimality of y one deduces that $y_{3i+1} \leq t_0 := \min\{b+r, c+q\}$ for every integer i. Denote $t := \min_{0\leq i\leq 1}\{y_{3i+1}\} \leq t_0$. Arguing as above, we conclude that $t = \min_{-\infty<i<\infty}\{y_{3i+1}\}$. When $t = t_0$, we have obviously, $y_{3i+1} = t$ for every integer i. When $t < t_0$, arguing as above we also show that $y_{3i+2} = t$ for every integer i. This completes the proof of the Proposition. $\square$

Remark 3.2 *When $P(a)$ has several edges $g_1, \ldots, g_k$ with slopes $s_1 < \cdots < s_k$, respectively, then as a tropical recurrent sequence satisfying a one can take points $\{(i, y_i) : i \in \mathbb{Z}\}$ lying on the edges of an (infinite in both directions) convex polygon having edges $g'_k, g'_{k-1}, \ldots, g'_1$ with the slopes $-s_k, -s_{k-1}, \ldots, -s_1$, respectively, such that g'_j is not shorter than g_j, $1 \leq j \leq k$.*

Conversely, in section 4 [2] it is proved, in fact, that if for each $0 \leq i \leq n$ point (i, a_i) lies on the boundary of $P(a)$ then any tropical recurrent sequence satisfying a has the described form.

4 Algorithm testing existence of a non-periodic tropical recurrent sequence

Let $a = (a_0, \ldots, a_n) \in \mathbb{Z}$ be a vector with integer coordinates whose Newton polygon has a single bounded edge which is located on the abscissas axis. In this section we prove the following theorem.

Theorem 4.1 *There is an algorithm which for a vector $a = (a_0, \ldots, a_n)$, $0 \leq a_i \leq M$, $0 \leq i \leq n$, $a_0 = a_n = 0$ tests whether there ex-*

ists a non-periodic tropical recurrent sequence satisfying a. The complexity of the algorithm does not exceed $M^{O(M^n)}$.

Proof. It suffices to consider sequences $y = \{y_i, i \in \mathbb{Z}\}$ with integer coordinates y_i. Denote $q := \min_{-n\le i\le n}\{y_i\}$. Replacing y_i by $y_i - q$, $i \in \mathbb{Z}$ one can assume w.l.o.g. that $q = 0$. Arguing as in the proofs of Proposition 1.2 and of Theorem 2.2 above, we conclude that $\min_{-\infty<i<\infty}\{y_i\} = 0$.

Denote $S := \{i \in \mathbb{Z} : y_i = 0\}$. Again arguing as in the proofs of Proposition 1.2 and of Theorem 2.2 above, we deduce that for any pair of adjacent element $i_1 < i_2$ of S inequality $i_2 - i_1 \le n$ holds.

For each $j \in \mathbb{Z}$ due to the minimality of y (see Remark 1.1) there exists an integer k, $j - n \le k \le j$ such that $a_{j-k} + y_j = \min_{0\le i\le n}\{a_i + y_{k+i}\}$. On the other hand, there is $0 \le l \le n$ such that $y_{k+l} = 0$, hence $y_j \le a_l$. Thus, $y_j \le M$, $j \in \mathbb{Z}$.

We say that a *vector* $z = (z_0, \dots, z_N) \in \{0, \dots, M\}^{N+1}$ *satisfies* a if for each $0 \le k \le N - n$ minimum $\min_{0\le i\le n}\{a_i + z_{i+k}\}$ is attained at least twice and for each $n \le j \le N - n$ there exists $j - n \le k \le j$ such that $a_{j-k} + z_j = \min_{0\le i\le n}\{a_i + z_{k+i}\}$ (cf. the definition of a tropical recurrent sequence and Remark 1.1). Treating z as a word in the alphabet $\{0, \dots, M\}$ we call $(z_0, \dots, z_{3n-1})$ (of length $3n$) the prefix of z and $(z_{N-3n+1}, \dots, z_N)$ (also of length $3n$) its suffix. Note that the prefix and the suffix can overlap.

For a vector $u \in \{0, \dots, M\}^{3n}$ we say that z is *u-word* if $z = uw = vu$ for some words w, v. We call a u-word z *closed* [1] if there are no occurrences of u as a subword in z other than its prefix and its suffix.

Lemma 4.2 *If $uw_1 = v_1u =: z^{(1)} \ne z^{(2)} := uw_2 = v_2u$ are different closed u-words both satisfying a, then a tropical recurrent sequence*

$$y = \cdots w_{l_{-1}} w_{l_0} w_{l_1} \cdots = \cdots v_{l_{-1}} v_{l_0} v_{l_1} \cdots$$

satisfies a where $0 \le l_j \le 1$, $j \in \mathbb{Z}$. Moreover, there exists a non-periodic y.

Proof. Observe that

$$\cdots v_{l_{-2}} v_{l_{-1}} u w_{l_0} w_{l_1} w_{l_2} \cdots = \cdots v_{l_{-2}} v_{l_{-1}} v_{l_0} u w_{l_1} w_{l_2} \cdots = \quad (1)$$

$$\cdots v_{l_{-2}} v_{l_{-1}} v_{l_0} v_{l_1} u w_{l_2} \cdots = \cdots v_{l_{-2}} v_{l_{-1}} v_{l_0} v_{l_1} v_{l_2} v_{l_3} \cdots = y \quad (2)$$

by "pulling" word u to the right. "Pulling" u to the left we obtain

$$\cdots v_{l_{-3}} v_{l_{-2}} v_{l_{-1}} u w_{l_0} w_{l_1} w_{l_2} \cdots = \cdots v_{l_{-3}} v_{l_{-2}} u w_{l_{-1}} w_{l_0} w_{l_1} w_{l_2} \cdots =$$

$$\cdots v_{l_{-3}} u w_{l_{-2}} w_{l_{-1}} w_{l_0} w_{l_1} w_{l_2} \cdots = \cdots w_{l_{-3}} w_{l_{-2}} w_{l_{-1}} w_{l_0} w_{l_1} w_{l_2} \cdots = y.$$

Therefore, for each $j \in \mathbb{Z}$ there is an occurrence in y of u with the rightmost letter at the rightmost letter of w_{l_j}:

$$\cdots w_{l_{j-1}} w_{l_j} w_{l_{j+1}} = \cdots v_{l_{j-1}} v_{l_j} u w_{l_{j+1}}$$

(cf. above). Symmetrically, there is (the same) occurrence in y of u with the leftmost letter at the leftmost letter of $v_{l_{j+1}}$.

We claim that there are no other occurrences in y of u. Indeed, suppose the contrary and assume for definiteness that the rightmost letter of an occurrence is inside subword w_{l_0} and does not coincide with the rightmost letter of w_{l_0}. Then since $y = \cdots v_{l_{-1}} u w_{l_0} \cdots$ we get a contradiction with the closedness of $z^{(l_0)} = u w_{l_0}$, which proves the claim.

Now we show that one can choose $\{0 \le l_j \le 1 : j \in \mathbb{Z}\}$, so that y is non-periodic. Assume that by recursion on d bits $l_0, \dots, l_s$ for some $s \ge 0$ are already produced such that word $v_{l_0} \cdots v_{l_s}$ is not t-periodic with any period $1 \le t \le d$ (the base of recursion for $d = 1$ is obvious). Pick bits $l_{s+1}, \dots, l_r$ in an arbitrary way until the sum of the lengths $L := |v_{l_{s+1}}| + \cdots + |v_{l_r}|$ becomes for the first time greater or equal to $d+1$. If $L > d+1$ then $v_{l_0} \cdots v_{l_s} v_{l_{s+1}} \cdots v_{l_r}$ is not $(d+1)$-periodic due to the claim above. Now let $L = d+1$. Then we choose the next $l_{r+1} \ne l_{s+1}$. Observe that whatever we continue to produce l_{r+2}, $l_{r+3}, \dots$, we have

$$v_{l_0} \cdots v_{l_s} v_{l_{s+1}} \cdots v_{l_r} v_{l_{r+1}} \cdots = v_{l_0} \cdots v_{l_s} u w_{l_{s+1}} w_{l_{s+2}} \cdots w_{l_r} w_{l_{r+1}} w_{l_{r+2}} \cdots =$$

$$v_{l_0} \cdots v_{l_s} v_{l_{s+1}} \cdots v_{l_r} u w_{l_{r+1}} w_{l_{r+2}} \cdots,$$

and the occurrences $u w_{l_{s+1}} = z^{(l_{s+1})}$ and $u w_{l_{r+1}} = z^{(l_{r+1})}$ are located in y on the distance $d+1$. Therefore, the produced word is not $(d+1)$-periodic taking into the account that both $z^{(1)}$ and $z^{(2)}$ are not the prefixes of each other, which follows from the closedness of $z^{(1)}$, $z^{(2)}$.

Note that one can continue $\cdots v_{l_{-1}} v_{l_0} v_{l_1} \cdots$ to the left from v_{l_0} in an arbitrary way.

We prove that $y = \{y_i : i \in \mathbb{Z}\}$ satisfies a. We have to verify that for any integer k minimum $\min_{0 \le i \le n}\{a_i + y_{i+k}\}$ is attained at least twice. Assume for definiteness that y_{n+k} is located in subword w_{l_0} of $y = \cdots w_{l_{-1}} w_{l_0} w_{l_1} \cdots$. Then $y = \cdots v_{l_{-2}} v_{l_{-1}} u w_{l_0} \cdots$ and thus, $u w_{l_0} = z^{(l_0)}$ satisfies a.

Finally, we show that y satisfies the condition of minimality (see Remark 1.1). To this end, we pick an arbitrary letter e of y (by a letter we mean also its position in y). Suppose for definiteness that e belongs to a subword w_{l_0} of y (cf. the left-hand side of (1)). Let $u w_{l_0} =: E_l e E_r$ for appropriate words E_l, E_r. First, assume that the length $|E_r| \ge n$. Since the subword $u w_{l_0} = (z_0 \dots z_N)$ of y (being one of two words $z^{(1)}$, $z^{(2)}$) where $z_k = e$ for suitable $0 \le k \le N$, satisfies a, there exists $0 \le j \le n$ such that $e + a_j = z_k + a_j = \min_{0 \le i \le n}\{z_{i+k-j} + a_i\}$ (we call the latter property the *attainability of* e).

Now assume that $|E_r| < n$. Then $uw_{l_0} = v_{l_0}u$ and e belongs to the occurrence of u in the right-hand side of (1), in particular, $uw_{l_1} =: E_l^{(1)} e E_r^{(1)}$ for appropriate words $E_l^{(1)}$, $E_r^{(1)}$ with $|E_l^{(1)}| \ge 2n$. If $|E_r^{(1)}| \ge n$ we use that the subword uw_{l_1} satisfies a and deduce the attainability of e in this case. If $|E_r^{(1)}| < n$ then the subword $uw_{l_2} =: E_l^{(2)} e E_r^{(2)}$ of y (cf. the left-hand side of (2)) where $|E_r^{(2)}| = |E_r^{(1)}| + |w_{l_2}|$ and $|E_l^{(2)}| = |E_l^{(1)}| - |w_{l_1}| \ge |E_l^{(1)}| - |E_r^{(1)}|$, hence $|E_r^{(2)}| \ge n + 1$. Again, if $|E_r^{(2)}| \ge n$ then we use that the word uw_{l_2} satisfies a and therefore, e is attainable. Assume that $|E_r^{(2)}| < n$, in this case $|E_l^{(2)}| \ge 2n$.

Continuing in this way, we arrive for the first time to a subword $uw_{l_s} =: E_l^{(s)} e E_r^{(s)}$ of y when $|E_r^{s)}| \ge n$, in this case $|E_l^{(s)}| \ge n + 1$. Again use that the word uw_{l_s} satisfies a and conclude that e is attainable. Lemma is proved. $\square$

In the following lemma which one can directly verify, we describe a procedure of shortening u-word $z := (z_0 \dots, z_N)$, $N > 3n$.

Lemma 4.3 *Assume that* $z \in \{0, \dots, M\}^{N+1}$ *satisfying* $a \in \{0, \dots, M\}^{n+1}$ *has 3 occurrences of one subword, i. e. for some* $0 \le k_1 < k_2 < k_3 \le N - 3n + 1$ *it holds*

$$(z_{k_1}, \dots, z_{k_1+3n-1}) = (z_{k_2}, \dots, z_{k_2+3n-1}) = (z_{k_3}, \dots, z_{k_3+3n-1}).$$

Then both words

$$(z_0, \dots, z_{k_1+3n-1}, z_{k_2+3n}, \dots, z_N) \quad \textit{and} \quad (z_0, \dots, z_{k_2+3n-1}, z_{k_3+3n}, \dots, z_N)$$

satisfy a. *If* z *is* u*-word then both latter words are* u*-words as well. If moreover,* z *is a closed* u*-word then both words are also closed* u*-words.*

Corollary 4.4 *Under the conditions of Lemma 4.3 one can shorten word* z *when its length* $N + 1 > 2(M + 1)^{3n} + 3n - 1$.

Observe that one can shorten u-word similar to Lemma 4.3 when there are just two occurrences of the same subword, but under the conditions of Lemma 4.3 one can guarantee below an upper bound on the length of resulting words.

Assume that there exists a non-periodic tropical recurrent sequence $y = \{y_i : i \in \mathbb{Z}\}$ satisfying a. Then there exist two its different closed u-subwords for an appropriate $u \in \{0, \dots, M\}^{3n}$ of the forms

$$y^{(1)} = (y_{k_1}, \dots, y_{k_2}),\ y^{(2)} = (y_{k_3}, \dots, y_{k_4})$$

for suitable integers k_1, k_2, k_3, k_4 (moreover, one can assume that $y^{(1)}$ is adjacent to $y^{(2)}$, i. e. $k_2 = k_3 + 3n - 1$, although we don't use it).

First let it be impossible to shorten neither $y^{(1)}$ nor $y^{(2)}$ (see Lemma 4.3). In this case the lengths of these words do not exceed $2(M+1)^{3n}+3n-1$ due to Corollary 4.4. One can apply Lemma 4.2 to $y^{(1)}$, $y^{(2)}$ and conclude that there exists a non-periodic tropical recurrent sequence satisfying a.

Now assume that one can shorten $y^{(2)}$ relying on Lemma 4.3. Also one applies the shortening procedure from Lemma 4.3 to $y^{(1)}$ and obtains a word $\overline{y^{(1)}}$ which one can not shorten further (the case $\overline{y^{(1)}} = y^{(1)}$ is not excluded). Then the length $|\overline{y^{(1)}}| \leq 2(M+1)^{3n} + 3n - 1$ due to Corollary 4.4. Employing the shortening procedure from Lemma 4.3 to $y^{(2)}$ and at each its step choosing a word among two considered in Lemma 4.3 which is not shorter, we terminate one step before we reach a word $\overline{y^{(2)}}$ which can't be shorten further. The resulting word after termination we denote by $\widetilde{y^{(2)}}$. Then $|\widetilde{y^{(2)}}| \leq 2|\overline{y^{(2)}}| \leq 4(M+1)^{3n} + 6n - 2$ because of Lemma 4.3 and Corollary 4.4, taking into the account that we choose a not shorter word among two possible words. In addition, $\widetilde{y^{(2)}} \neq \overline{y^{(1)}}$ since one can't shorten $\overline{y^{(1)}}$. Again we can apply Lemma 4.2 to the words $\widetilde{y^{(2)}}$, $\overline{y^{(1)}}$. Thus, we have established the following corollary.

Corollary 4.5 *If there exists a non-periodic tropical recurrent sequence satisfying* $a \in \{0, \ldots, M\}^{n+1}$ *then there are two different closed* u*-words satisfying* a *with lengths at most* $4(M+1)^{3n} + 6n - 2$ *for some* $u \in \{0, \ldots, M\}^{3n}$.

The algorithm checks all possible words from $\{0, \ldots, M\}^{N+1}$ where $N := 4(M+1)^{3n} + 6n - 2$ and tests whether among them there are two different closed u-words satisfying a for some $u \in \{0, \ldots, M\}^{3n}$. The correctness of the designed algorithm follows from Corollary 4.5 and Lemma 4.2. The complexity of the algorithm is bounded by $M^{O(N)} \leq M^{O(M^n)}$, which completes the proof of the Theorem. $\square$

Remark 4.6 *(i) More generally, one can design an algorithm similar to Theorem 4.1 for vectors* $a = (a_0, \ldots, a_n)$ *with rational coordinates* $a_i \in \mathbb{Q}$, $0 \leq i \leq n$ *looking for tropical recurrent sequences of the form* $y = \{y_i/q : i \in \mathbb{Z}, y_i \in \mathbb{Z}\}$ *where* q *being the common denominator of* $a_0, \ldots, a_n$*;*

(ii) it would be interesting to design an algorithm similar to Theorem 4.1 for vectors a *allowing real algebraic and infinite coordinates.*

5 Tropical entropy

Let $a = (a_0, \ldots, a_n) \in \mathbb{R}^{n+1}$. For $0 \leq s \in \mathbb{Z}$ denote by $D_s \subset \mathbb{R}^s$ (respectively, $M_s \subset \mathbb{R}^s$) the set of vectors satisfying a (respectively, satisfying in addition the minimality condition, see Remark 1.1 and section 4). Both D_s and M_s are polyhedral complexes [4]. Denote $d_s := \dim D_s$, $m_s := \dim M_s$.

When $i+j=s$ denote by $p:\mathbb{R}^s \to \mathbb{R}^i$, $q:\mathbb{R}^s \to \mathbb{R}^j$ the projection onto the first i coordinates and respectively, onto the last j coordinates. Since $p(D_s) \subset D_i$, $q(D_s) \subset D_j$, $p(M_s) \subset M_i$, $q(M_s) \subset M_j$ we have $d_{i+j} \le d_i + d_j$, $m_{i+j} \le m_i + m_j$. Therefore, there exist limits

$$H = H(a) := \lim_{s\to\infty} d_s/s, \quad h = h(a) := \lim_{s\to\infty} m_s/s$$

which we call the *tropical entropy of* a (respectively, the *tropical minimal entropy of* a). Clearly, $0 \le h \le H$.

Proposition 5.1 $H \le 1 - 1/n$

Proof. The polyhedral complex D_s is a union of polyhedra such that each of these polyhedra Q satisfies the following conditions. For every $0 \le j \le s-n$ there exists a pair $0 \le i_1 < i_2 \le n$ such that $z_{j+i_1} + a_{i_1} = z_{j+i_2} + a_{i_2} = \min_{0\le i\le n}\{z_{j+i} + a_i\}$ for any $(z_1, \dots, z_s) \in D_s$. For $j = i_1 + 1$ there exists a pair $0 \le i_3 < i_4 \le n$ fulfilling the similar conditions, hence $(i_1+1+i_3)-(j+i_1) \le n$. Therefore, there are at least $\lfloor s/n \rfloor$ such pairs. Each such pair $(j+i_1,\, j+i_2)$ imposes a linear restriction $z_{j+i_1} - z_{j+i_2} = a_{i_2} - a_{i_1}$ on Q. Thus, $d_s \le s - \lfloor s/n \rfloor$. $\square$

Example 5.2 *Let now vector* a *be with* $a_0 = \cdots = a_n = 0$. *Consider a polyhedron consisting of vectors* $z = (z_1, \dots, z_s) \in \mathbb{R}^s$ *such that* $z_{(n+1)i+1} = z_{(n+1)i+2} = 0$ *for every* i *and the rest of the coordinates being arbitrary real non-negative. Then* $z \in D_s$, *hence* $d_s \ge s(1-2/(n+1))$, *thus,* $H(a) \ge 1-2/(n+1)$.

Now we prove an upper bound on H. *Let* $y = (y_1, \dots, y_s) \in D_s$. *Suppose w.l.o.g. that* $\min_{1\le i\le s}\{y_i\} = 0$. *Denote* $I := \{1 \le i \le s : y_i = 0\}$ *and* $i_1 < i_2 < \cdots$ *being the consecutive elements of* I. *Observe that* $i_{j+2} - i_j \le n$ *for every* j *since otherwise,* $\min_{0\le l\le n}\{a_l + y_{l+i_j+1}\}$ *is attained only once for* $l = i_{j+1} - i_j - 1$. *Therefore,* y *belongs to a linear space* $\{y_i = 0,\, i \in I\}$ *with the dimension at most* $\lceil s(1-2/(n+1)) \rceil$. *Thus,* $H(a) = 1 - 2/(n+1)$.

Due to Proposition 1.2 any vector satisfying a *and the condition of the minimality (see Remark 1.1) has all equal coordinates, so* $m_s = 1$, *hence* $h(a) = 0$.

Remark 5.3 *There is a gap between an upper bound on* H *from Proposition 5.1 and the latter example. The conjecture is that* $1 - 2/(n+1)$ *is an upper (sharp) bound on* H.

We call a vector a *regular* if the set $J := \{i \,|\, a_i < \infty\}$ is an arthmetic progression and each point $(i,\, a_i)$, $i \in J$ is a vertex of the Newton polygon $P(a)$.

Theorem 5.4 *If a vector a is not regular then $H(a) \geq 1/6$.*

Proof. First consider the case when at least three points of a lie on a (bounded) edge of $P(a)$. Similar to the beginnings of sections 1, 2 making suitable affine transformations one can suppose w.l.o.g. that an edge containing at least three points of a lies on the abscissas axis. Consider the points of a located on this edge: $\{(i,\, 0) \,:\, i \in I\}$ where $|I| \geq 3$. One can assume w.l.o.g. that the greatest common divisor $GCD(I)$ of the differences $i_1 - i_2$ of all the pairs of the elements $i_1,\, i_2 \in I$ of I equals 1. Otherwise, one can consider separately all $GCD(I)$ arithmetic progressions with the difference $GCD(I)$.

Pick any three elements of I not all with the same parity, say $0,\, 2i,\, j$ w.l.o.g. where $i \geq 1$ and j being odd. Consider the following tropical recurrent sequence z satisfying a. For odd indices $2l+1$ we put $z_{2l+1} = 0$. For even indices $2(2si+t)$, $s \in \mathbb{Z}$, $0 \leq t < i$ we put $z_{2(2si+t)} = 0$ and $z_{2((2s+1)i+t)}$ we put arbitrarily non-negative. Taking finite fragments $(z_0, \dots, z_N)$ with growing N we conclude that $H(a) \geq 1/4$ in this case.

Now assume that no edge of $P(a)$ contains a point of a other than two vertices of this edge. Take an edge of $P(a)$ with the vertices $(i,\, a_i)$, $(j,\, a_j)$ with the maximal difference $j - i > 0$. Again one can suppose w.l.o.g. that these vertices are $(0,\, 0)$, and $(n,\, 0)$. There exists $i \in J$ such that n does not divide i since a is not regular. Among such i pick i_0 for which $c := a_{i_0}$ is minimal, then $c > 0$. Denote $k := GCD(n, i_0)$.

Consider a sequence $z := \{z_i\}_{i \in \mathbb{Z}}$ such that

- $z_{sn-2ji_0+i} = 0$ when $0 \leq 2j < n/k$;
- $z_{2sn-(2j+1)i_0+i} = c$ when $0 < 2j+1 < n/k$;
- $z_{(2s+1)n-(2j+1)i_0+i} \geq c$ when $0 < 2j+1 < n/k$

for $s \in \mathbb{Z}$, $0 \leq i < k$. Then z satisfies a. Taking finite fragments $(z_0, \dots, z_N)$ with growing N we conclude that $H(a) \geq 1/4$ for even n/k and $H(a) \geq \frac{k\lfloor n/(2k) \rfloor}{2n} \geq 1/6$ (the latter inequality becomes an equality when $n/k = 3$). □

Example 5.5 *Let vector $a := (0,\, 1,\, 0)$. Then for any sequence $(z_0, \dots, z_N)$ satisfying a denote $c := \min\{z_0, \dots, z_N\}$. Let $z_i = c$ for some $0 \leq i \leq N$. For definiteness assume that i is even (the case of an odd i is considered in a similar way). Then $z_l = c$ for any even $0 \leq l \leq N$. If $m := \min_{0<2j+1\leq N}\{z_{2j+1}\} < c+1$ then $z_{2j+1} = m$ for any $0 < 2j+1 \leq N$. If $m \geq c+1$ then $m = c+1$. For any odd $0 < 2j-1 \leq N-2$ we have that either $z_{2j-1} = c+1$ or $z_{2j+1} = c+1$. Therefore, the number of odd $0 < 2j+1 \leq N$ for which $z_{2j+1} > c+1$ does not exceed $\lceil N/4 \rceil$, thereby $H(a) \leq 1/4$.*

On the other hand, from the proof of Theorem 5.4 we conclude that $H(a) \geq 1/4$ (in the notations of the proof of Theorem 5.4 $n = 2$, $i_0 = 1$).

Remark 5.6 *If all the points (i, a_i), $a_i < \infty$, $0 \leq i \leq n$ are the vertices of the Newton polygon $P(a)$ (thus, a being regular) then section 4 [2] implies* *that $H(a)=0$ (cf. also Remark 3.2). Our conjecture is that a stronger than in Theorem 5.4 bound $H(a)\geq 1/4$ holds for not regular vector a.*

Corollary 5.7 *A vector a is regular iff $H(a)=0$. For non-regular a the inequality $H(a)\geq 1/6$ holds.*

Example 5.8 *Now we give an example of a vector $a=(a_0, a_1, a_2, a_3)$ with a positive $h(a)$. Put $a_0=a_3=0$, $a_2>2a_1>0$ (see Proposition 3.1). Then at the beginning of the proof of Proposition 3.1 a family of vectors satisfying a is constructed for an arbitrary $k_1<k_2<\cdots$ with $k_{i+1}-k_i=3$ for all i. which demonstrates that $h(a)\geq 1/9$.*

On the contrary, if $2a_1\geq a_2>a_1>0$ then $h(a)=0$ (cf. the proof of Proposition 3.1).

Remark 5.9 *(i) For a with a single bounded edge of its Newton polygon (cf. sections 1, 2, 3) is it true that $h(a)=0$ iff all the tropical minimal recurrent sequences satisfying a are periodic?*

(ii) For a vector a from Example 5.2 it was, actually, established that when $(n+1)|(s-r)$, $0\leq r\leq n$, function $d_s=(s-r)(n-1)/(n+1)+r$ in case if $0\leq r<n$ and $d_s=(s-n)(n-1)/(n+1)+n-1$ if $r=n$. Is it true that for an arbitrary vector a function d_s is linear with the leading coefficient $H(a)$ for s from each fixed arithmetic progression with the difference $n+1$?

6 Tropical multivariable recurrent sequences

For a vector $a=\{a_I\in\mathbb{R}\cup\{\infty\}\,:\,I\in\mathbb{Z}^m\}$ with a finite number of I such that $a_I\in\mathbb{R}$ (the set of such I we call the *support* of a) we say that a *tropical multivariable recurrent sequence* $\{z_I\in\mathbb{R}\,:\,I\in\mathbb{Z}^m\}$ *satisfies* a if for any $J\in\mathbb{Z}^m$ the minimum $\min_I\{a_I+z_{I+J}\}$ is attained at least for two different I. Similar to Remark 1.1 one can also define tropical multivariable *minimal* recurrent sequences.

For a parallelepiped in the lattice $Q\subset\mathbb{Z}^m$ with the sides $q_1,\dots,q_m$, respectively, we say that $\{z_I\,:\,I\in Q\}$ satisfies a if for any $J\in\mathbb{Z}^m$ the minimum $\min_I\{a_I+z_{I+J}\}$ is attained at least twice, provided that $I+J\in Q$ for each I from the support of a. Clearly, one could consider an arbitrary subset of $\mathbb{Z}^m$ rather than just a parallelepiped.

Similar to section 5 define $d_{q_1,\dots,q_m}:=d_{q_1,\dots,q_m}(a)$ to be the dimension of the tropical linear prevariety $\{z_I\,:\,I\in Q\}$ satisfying a. Analogously one defines $m_{q_1,\dots,q_m}$ with respect to tropical multivariable minimal recurrent sequences. There exist limits

$$H(a):=\lim_{q_1,\dots,q_m\to\infty} d_{q_1,\dots,q_m}/(q_1\cdots q_m),\quad h(a):=\lim_{q_1,\dots,q_m\to\infty} m_{q_1,\dots,q_m}/(q_1\cdots q_m)$$

which we also call the *tropical entropy* (respectively, the *tropical minimal entropy*) of a. Again $0 \leq h(a) \leq H(a)$. One can prove the following proposition similarly to the proof of Proposition 5.1.

Proposition 6.1 *Let the support of a be located in a cube with the side r. Then $H(a) \leq 1 - 1/r^m$.*

Example 6.2 *Let a be a vector with $a_I = 0$ for all $I = (i_1, \dots, i_m)$, $1 \leq i_1, \dots, i_m \leq r$, so its support is a cube with the side $r-1$. Then the following $z = \{z_I\}$ is a tropical multivariable recurrent sequence satisfying a. Put $z_{l_1 r, l_2 r, \dots, l_m r} = z_{l_1 r+1, l_2 r, \dots, l_m r} = 0$ for all integers $l_1, \dots, l_m$, and the rest of the coordinates of z being arbitrary non-negative. Therefore, $H(a) \geq 1 - 2/r^m$.*

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