Complexity of tropical Schur polynomials

Dima Grigoriev* and Gleb Koshevoy†

Abstract

We study the complexity of computation of a tropical Schur polynomial $Ts_\lambda$ where $\lambda$ is a partition, and of a tropical polynomial $Tm_\lambda$ obtained by the tropicalization of the monomial symmetric function $m_\lambda$. Then $Ts_\lambda$ and $Tm_\lambda$ coincide as tropical functions (so, as convex piece-wise linear functions), while differ as tropical polynomials. We prove the following bounds on the complexity of computing over the tropical semi-ring $(\mathbb{R}, \max, +)$:

- a polynomial upper bound for $Ts_\lambda$
- an exponential lower bound for $Tm_\lambda$.

Also the complexity of tropical skew Schur polynomials is discussed.

Introduction

We study computations (i.e. circuits, see e.g. [2]) over a tropical semi-ring $(\mathbb{R}, \max, +)$ where $\max$ plays a role of addition, and $+$ plays a role of multiplication (see e.g. [9]). Actually, computations over $(\mathbb{R}, \max, +)$ were considered in Computer Science earlier than tropical algebra and geometry (and even the term "tropical" itself) have emerged (see e.g. [10] and further references there).

The tropicalization of a polynomial $f = \sum_f a_I x_1^{i_1} \cdots x_n^{i_n} \in \mathbb{R}[x_1, \ldots, x_n]$ is a tropical polynomial $\text{Trop}(f) := \max_I \{i_1x_1 + \cdots + i_nx_n\}$ defined over the tropical semi-ring $(\mathbb{R}, \max, +)$ (see e.g. [9]). One can treat a tropical polynomial as a convex piece-wise linear function.

We study a tropical Schur polynomial $Ts_\lambda = \text{Trop}(s_\lambda)$ (see Section 1) being the tropicalizations of the Schur function $s_\lambda$, where $\lambda = \{\lambda_1, \ldots, \lambda_n\}$ is a partition.

Since $Ts_\lambda$ is a convex piece-wise linear function $\max_W \{w_1x_1 + \cdots + w_nx_n\}$ where the multiindices $W$ range over all integer points of the Newton polyhedron of $s_\lambda$, it coincides with a function $Tm_\lambda := \max_J \{j_1x_1 + \cdots + j_nx_n\}$ where the multiindices $J$ range over all the vertices of the Newton polyhedron of $s_\lambda$. Note that $Tm_\lambda$ are the tropicalizations of the monomial symmetric functions $m_\lambda$ which form (as well as $s_\lambda$) a

*CNRS, Mathématiques, Université de Lille, Villeneuve d’Ascq, 59655, France E-mail address: Dmitry.Grigoryev@math.univ-lille1.fr
†Central Institute of Economics and Mathematics RAS, 117418, Moscow, Russia; email: koshevoy@cemi.rssi.ru
basis in the ring of symmetric functions (see [11]). On the other hand, \( T_{S\lambda} \) and \( T_{M\lambda} \) differ as the elements of the semi-ring of tropical polynomials [9].

We exhibit (see Theorem 1) a polynomial complexity algorithm which computes \( T_{S\lambda} \) over \((\mathbb{R}, \max, +)\). On the contrary, we prove (see Theorem 2) an exponential lower bound on the complexity of computing \( T_{M\lambda} \) over \((\mathbb{R}, \max, +)\). This demonstrates an interesting phenomenon: while \( T_{S\lambda} \) and \( T_{M\lambda} \) coincide as tropical functions, their complexities as tropical polynomials differ considerably.

Observe that in [7] there was designed a polynomial complexity subtraction-free algorithm (relying on the cluster transformations), in other words a computation over \((\mathbb{R}, +, \times, /)\) for Schur polynomials. The tropicalization of this algorithm provides a polynomial complexity computation of \( T_{S\lambda} \) over a tropical semi-field \((\mathbb{R}, \max, +, -)\). Thus, the algorithm from Theorem 1 is better because it avoids subtraction (viewed as a tropical analog of division). It is unclear, whether the complexity of computation of \( T_{M\lambda} \) over \((\mathbb{R}, \max, +, -)\) is polynomial?

On the other hand, from the tropicalization of the results of [7] we conclude that the tropical polynomial expressing the maximal weight directed spanning tree in the complete graph has a polynomial complexity over \((\mathbb{R}, \max, +, -)\), while its complexity over \((\mathbb{R}, \max, +)\) is exponential. In the proofs of complexity lower bounds we make use of technical tools developed in [12], [10], where some exponential complexity lower bounds were established for computations over \((\mathbb{R}, +, \times)\) as well as over the tropical semi-ring \((\mathbb{R}, \max, +)\).

In Sections 2, we speculate that the complexity of a skew Schur polynomial \( T_{S\lambda/\mu} \) in \( n \) variables (being the tropicalization of the skew Schur polynomial \( s_{\lambda/\mu} \)) might depend on the shapes of the partitions \( \lambda, \mu \), and we conjecture that for some shapes its complexity over the semi-ring \((\mathbb{R}, \max, +)\) is exponential, while over the semi-field \((\mathbb{R}, \max, +, -)\) the complexity is (polynomial) \( O(n^5) \) due to the tropicalization of the subtraction-free algorithm from [7] which computes skew Schur polynomials.

In the Appendix we provide some necessary concepts and results on base-polytopes and submodular functions.

## 1 Tropical Schur polynomials

For a fixed alphabet \([n] := \{1, \ldots, n\}\) and a partition \( \lambda = (\lambda_1 \geq \ldots \geq \lambda_n) \), we consider a tropical Schur polynomial \( T_{S\lambda} \) in the form of maximization of a linear function over the set of integer points of the Newton polytope of the usual Schur polynomial [11]

\[
s_\lambda(x) = \sum_{\mu \in \text{ch}(w(\lambda), w \in S_n)} K_{\mu, \lambda} x^\mu,
\]

where \( x = (x_1, \ldots, x_n) \), \( x^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n} \), \( S_n \) denotes the group of permutations of the finite set \([n]\), \( w(\lambda) = (\lambda_{w(1)}, \ldots, \lambda_{w(n)}) \), and \( \text{ch}(w(\lambda), w \in S_n) \) denotes the convex hull of the points \( w(\lambda) \), \( w \in S_n \); we denote \( \mu \preceq \lambda \) if \( \mu \in \text{ch}(w(\lambda), w \in S_n) \), and \( K_{\mu, \lambda} \) are the Kostka numbers. For details see [11].

Thus, the tropicalization of Schur polynomial \( s_\lambda(x) \) is

\[
T_{S\lambda}(x) = \max_{\mu \in \text{ch}(w(\lambda), w \in S_n)} x(\mu),
\]

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here we consider \( x \) as a linear functional on \( \mathbb{R}^n \), and \( x(\mu) \) denotes the value of the functional at \( \mu \in \mathbb{Z}^n \).

1.1 Complexity: upper bound

The tropicalization (see [1]) of the cluster algorithm in [7] provides an algorithm for computing tropical polynomial \( Ts_\lambda(x) \) within bit-complexity \( O(k^3) \), \( k := \lambda_1 + n \), over the tropical semi-field \((\mathbb{R}, \max, +, -)\) (in the algebraic setup in [7] we consider \( \mathbb{R} \) with addition, multiplication and division).

We conjecture that in the algebraic setup, it is exponential hard to calculate \( s_\lambda \) without division, i.e. over \((\mathbb{R}, +, \times)\).

However, the situation drastically changes in the tropical setup. Namely, we can calculate \( Ts_\lambda \) over the tropical semi-ring \((\mathbb{R}, \max, +)\) within bit-complexity \( O(n^2 \cdot \lambda_1) \).

Let us recall that the Newton polytope \( NP(e_k) \) of an elementary symmetric function

\[
e_k(x_1, \ldots, x_n) = \sum_{i_1 < i_2 < \ldots < i_k} x_{i_1}x_{i_2} \ldots x_{i_k},
\]

is a hypersimplex, that is the convex hull of the set

\[
\left( \begin{array}{c} [n] \\ k \end{array} \right) = \{ I \subset [n], |I| = k \},
\]

where a subset \( I \) is naturally identified with a vertex of the hypercube \( 2^{[n]} \).

A hypersimplex is a matroid, a subclass of base-polytopes. The useful facts on base polytopes are collected in the Appendix.

Denote by \( \lambda' \) the dual partition to \( \lambda \), that is \( \lambda'_i = |\{ j : \lambda_j \geq i \}, i = 1, \ldots, \lambda_1 \}|. \)

From the Littlewood formula (see [11]) it follows

\[
\prod_k e_{\lambda'_k} = s_\lambda + \sum_{\mu < \lambda} K_{\lambda', \mu} s_\mu.
\]

Hence the Newton polytope \( NP(Ts_\lambda) \) of the Schur polynomial \( s_\lambda \) coincides with the Minkowski sum of the Newton polytopes \( \sum_k NP(e_{\lambda'_k}) \). Moreover, since the hypersymplexes are matroids, the directions of edges of any hypersimplex take the form \( \{e_i - e_j\} \). The latter set is unimodular, and from [4] we get

\[
NP(Ts_\lambda)(\mathbb{Z}) = \sum_{1 \leq k \leq \lambda_1} NP(e_{\lambda'_k})(\mathbb{Z}), \tag{1}
\]

where, for a polytope \( P \), \( P(\mathbb{Z}) \) denotes the set of integer points in \( P \).

Because of this, we have

**Theorem 1.** A tropical Schur polynomial \( Ts_\lambda \) can be calculated within (polynomial) \( O(n^2 \cdot \lambda_1) \) bit complexity over \((\mathbb{R}, \max, +)\).

**Proof.** Due to (1), in order to calculate \( Ts_\lambda \), one needs first to calculate tropical elementary Schur functions \( Te_{\lambda'_k}, 1 \leq k \leq \lambda_1 \). Since

\[
e_k(x_1, \ldots, x_n) = e_k(x_1, \ldots, x_{n-1}) + x_n e_{k-1}(x_1, \ldots, x_{n-1}),
\]

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and a similar identity holds in the tropical setup, the complexity of computation of a tropical elementary Schur function is quadratic in \( n \) (to this end, one can use the Pascal triangle).

\[ \Box \]

### 1.2 Complexity: lower bound

Since tropical Schur function takes the form of maximization of a linear functional over a polytope, it suffices to consider only the vertices of such a polytope. However, over the semi-ring \( (\mathbb{R}, \max +) \) the complexity of such a modification can increase exponentially. We demonstrate this phenomenon for a tropical Schur function.

Namely, let us consider the tropicalization of the monomial symmetric functions
\[
m_\lambda = \sum_{w \in S_n} x^{w(\lambda)} , \quad Tm_\lambda(x) = \max_{w \in S_n} x(w(\lambda)).
\]

Observe that \( Ts_\lambda \) and \( Tm_\lambda \) coincide as tropical functions, while they differ as the elements of the semi-ring of tropical polynomials, and the complexity of computation in the latter semi-ring is polynomial for \( Ts_\lambda \) (Theorem 1), while the complexity of \( Tm_\lambda \) is exponential as we prove in the following theorem.

**Theorem 2.** For \( \lambda \) with the \( i \)th part of the form \( \lambda_{n-i+1} = ni + i^2, i = 1, \ldots, n \), the complexity of computation of \( Tm_\lambda \) over the tropical semiring \( (\mathbb{R}, \max, +) \) is exponential.

**Proof.** Throughout the proof we omit the adjective ”tropical” for tropical polynomials and utilize for the latter the customary notations +, × for tropical operations max, +, respectively. For a (homogeneous) polynomial \( P \) by \( \text{mon}(P) \) denote the set of monomials of \( P \). We will use the following result from [12], [10]. If for any homogeneous polynomials \( R, Q \) such that \( \text{mon}(P) \supset \text{mon}(RQ) \), and of the powers \( 1/3 \deg P \leq \deg R, \deg Q \leq 2/3 \deg P \), we have \( \frac{\text{mon}(P)}{\text{mon}(RQ)} > c_1^n \), for some \( c_1 > 1 \), then the complexity of computation of \( P \) over \( (\mathbb{R}, \max, +) \) is exponential. We mention that a similar complexity lower bound holds as well for computations over \( (\mathbb{R}, +, \times) \) [12], [10].

In our case we have to show that \( R \) and \( Q \) have exponentially small deal of monomials wrt \( n! \) (which equals the number of monomials in \( P := Tm_\lambda \)).

Let us explain our choice of such a specific \( \lambda \). The parts of \( \lambda \) form a Golomb ruler ([6]), that is \( \lambda_i + \lambda_j = \lambda_k + \lambda_l \) iff \( \{i,j\} = \{k,l\} \).

This property allows us to separate variables, namely we have \( Q = Q'(x_i, i \in S)M(x_j, j \in [n] \setminus S) \) and \( R = N(x_i, i \in S)R'(x_j, j \in [n] \setminus S) \), where \( M \) and \( N \) are monomials in variables \( x_j, j \in [n] \setminus S \) and \( x_i, i \in S \), respectively. Indeed, assume the contrary. Then there exists \( m \in [n] \) and four monomials
\[
q_1 = \cdots x_m^\alpha \cdots , \quad q_2 = \cdots x_m^\beta \cdots \in \text{mon}(Q) ; \quad r_1 = \cdots x_m^\gamma \cdots , \quad r_2 = \cdots x_m^\delta \cdots \in \text{mon}(R)
\]
such that \( \alpha \neq \beta, \gamma \neq \delta \). Since
\[
r_1q_1, r_2q_2, r_1q_2, r_2q_1 \in \text{mon}(RQ) \subset \text{mon}(P)
\]

\[ \Box \]
there are \( i, j, k, l \in [n] \) for which \( \alpha + \gamma = \lambda_i, \beta + \delta = \lambda_j, \alpha + \delta = \lambda_k, \beta + \gamma = \lambda_l \). Hence \( \lambda_i + \lambda_j = \lambda_k + \lambda_l \), and we get a contradiction with the Golomb property.

Thus, we have a separation of variables. We get two polynomials \( A := NQ' \) and \( B := MR' \) in variable \( x_i, i \in S \), and \( x_j, j \in [n] \setminus S \), respectively.

At the beginning we consider a case of no separation of variables. This means that either \( Q \) or \( R \) is a monomial. Let for definiteness \( Q \) be a monomial.

Then we claim that if \( c := \frac{\deg Q}{\deg P} \in \left[ \frac{1}{3}, \frac{3}{4} \right] \), then \( R \) has exponentially small number of monomials wrt \( n ! \). Throughout this Section we assume in all the bounds \( n \) to be sufficiently big.

Let us prove this claim.

Let \( Q = x_1^{\nu_1} \cdots x_n^{\nu_n} \). Firstly, we observe that w.l.o.g. one can suppose that for any \( i \) there exists \( j \) such that \( \nu_i = \lambda_j \). Indeed, if at least two \( \nu_{i_1}, \nu_{i_2} \) among \( \{ \nu_i \} \) violate this condition, we can increase \( \nu_{i_1} \) by 1 and decrease \( \nu_{i_2} \) also by 1, thereby not decreasing \( |\text{mon}(R)| \) for which \( \text{mon}(QR) \subset \text{mon}(P) \). Observe that herein \( |\text{mon}(R)| \) could increase only if \( \nu_{i_2} = \lambda_j + 1 \) for some \( j \). If just a single \( \lambda_j > \nu_i > \lambda_{j+1} \) violates the condition under discussion, we can preserve inequalities \( \frac{\deg Q}{\deg P} \in \left[ \frac{1}{3}, \frac{3}{4} \right] \) as follows: either replace \( \nu_i \) by \( \lambda_j \) which keeps \( |\text{mon}(R)| \) or replace by \( \lambda_{j+1} \) which does not decrease \( |\text{mon}(R)| \).

Let \( b_j := \{ i : \nu_i = \lambda_j \}, j = n, \ldots, 1 \). Then the number of monomials in \( R(x) \) is equal to
\[
M := b_n (b_n + b_{n-1} - 1) \cdots (b_n + \ldots + b_1 - (n - 1)).
\]
We have
\[
\sum b_i \lambda_i = c \sum \lambda_i.
\]
Then, we have
\[
\sum \lambda_i - \sum b_i \lambda_i + \lambda_1 - \lambda_n = \sum_{j=0}^{n-2} (b_n + \ldots + b_{n-j} - j) (\lambda_{n-j} - \lambda_{n-j}).
\]
Thus
\[
M \prod (\lambda_{j-1} - \lambda_j) \leq \left( \frac{(1 - c) \sum \lambda_i + \lambda_1 - \lambda_n}{n} \right)^n.
\]
We have \( \sum \lambda_i \sim \frac{5n^3}{6}, \prod (\lambda_{j-1} - \lambda_j) \sim 2^n \left( \frac{3/2n!}{[1/2n!]} \right) \sim (2^{3/2}n^n) \).
Therefore it holds (taking into account that due to the choice of \( \lambda_i \), the degree of \( P \) is \( 5/6 n^3 + O(n^2) \)) that
\[
M \leq \left( \frac{5e(1-c)n}{3^{3/2}} \right)^n.
\]
(2)
Thus, for \( 1 - c < \frac{6.3^{3/2}}{5e} < \frac{31.14}{39.64} \), the number of monomials in \( R \) is exponentially small wrt \( n ! \). For \( c \in [1/4, 3/4] \), this is the case.

Now consider the case of a non-monomial \( Q \). In such a case we have a separation of variables.

Let us recall that the polytope \( Per_n := ch(\sigma(\lambda), \sigma \in S_n) \) is a base-polytope (see the Appendix) which is set by a submodular function \( b_{\lambda}(T) = \sum_{i=1, \ldots, |T|} \lambda_{n-i}, T \subset [n] \). Thus, a pair of parallel facets (we agree that a facet is a face of codimension 1) labeled by a subset \( W \subset [n], |W| = k \), are defined by \( x(W) = b_{\lambda}(W) = \sum_{i=1, \ldots, k} \lambda_i \) and
\[x([n] - W) = b_\lambda([n] - W) = \sum_{i=1,\ldots,n-k} \lambda_i\text{, respectively, and any cut with the same separation of coordinates is defined by } x(W) = a, a \in [\sum_{j=1}^k \lambda_j, \sum_{i=1}^k \lambda_{n-i}]\text{ (for details see the Appendix). Because of symmetry of } b_\lambda \text{ wrt permutations of coordinates, facets of } \text{Per}_n \text{ are labeled by numbers in } [n].\] The number of the vertices of a facet labeled by } k \in [n] \text{ (recall that } k \text{ corresponds to separation of variables in groups of } k \text{ and } n - k \text{ variables) is } k!(n-k)!.

Because of this, the cardinality of monomials of the product } A \cdot B \text{ is bounded by } k\lambda((A)!!(n-k-\lambda(A)))!, \text{ where } k := k(A) = |S|. \text{ Note that } \deg(A) = \lambda_{i_1} + \cdots + \lambda_{i_k} \text{ for suitable } 1 \leq i_1 < \cdots < i_k \leq n \text{ satisfies } \deg(A) \in [\sum_{j=1}^k \lambda_j, \sum_{i=1}^k \lambda_{n-i}].

There are two cases.

Case 1. \(\deg A, \deg B \geq c' \cdot \deg P\), for some sufficiently small constant \(c'\) which we choose later. In such a case, \(k = k(A), n - k = k(B) \geq c'' \cdot n\) for some sufficiently small constant \(c''\) depending on \(c'\) (since \(\deg P\) is cubic in \(n\)). This implies that \(A \cdot B\) has at most \(k!(n-k)!\) number of monomials, so exponentially small wrt \(n\)! and we are done.

Case 2. Either \(\deg A < c' \cdot \deg P\) or \(\deg B < c' \cdot \deg P\). Let for definiteness \(\deg A < c' \cdot \deg P\). Then, the degree of the monomial \(M\) satisfies \(\deg M \deg B \in [\frac{1}{4}, \frac{3}{4}]\) since \(c'\) is sufficiently small.

Then, the same reasoning as above in the case of no separation with a single monomial, provides a bound \(|\text{mon}(R')| \leq (c_0(n-k))^{n-k}\) for any fixed \(c_0 > \frac{5e(1-c)}{3^{7/4}e^6}\) (see (2)) due to an appropriate choice of sufficiently small \(c'\) in Case 1. We take \(c_0 < 1/e\). Because of this and that \(A\) has at most \(k!\) monomials we get that

\[|\text{mon}(RQ)| = |\text{mon}(AB)| \leq k!(c_0(n-k))^{n-k} < c_2 n!\]

for some \(c_2 < 1\). This finishes the proof of Theorem 2.

\[\square\]

2 Tropical skew Schur polynomials

In this Section we discuss a conjecture that for a tropical skew Schur polynomial its complexity over the tropical semi-ring might depend on the shape of the corresponding diagram and could be exponential. While over the tropical semi-field the complexity is always polynomial.

Recall that, for a skew Young diagram \(\lambda \setminus \mu\) (where \(\mu \leq \lambda\), which denotes the coordinate-wise inequality of the partitions), a semi-standard Young tableaux (SSYT) of a shape \(\lambda \setminus \mu\) (in the alphabet \([n]\)) is a filling of the Young diagram \(\lambda \setminus \mu\) with entries from \([n]\) strictly increasing along the columns and non-decreasing along the rows ([11]). We accept the French style to draw Young diagram. Here is an example
of a skew SSYT of shape \((5, 3, 3, 1) \setminus (2, 1)\)

\[
\begin{array}{cccc}
3 \\
2 & 2 & 4 \\
1 & 2 \\
1 & 1 & 2
\end{array}
\]

The weight of such a tableau \(T\) is the tuple \(wt(T) := (\#_1(T), \#_2(T), \ldots, \#_n(T))\), where \(\#_i(T)\) denotes the number of times integer \(i\) occurs in \(T\). The skew Schur polynomial \(s_{\lambda \setminus \mu}\) is defined by (see [11])

\[s_{\lambda \setminus \mu} = \sum_T x^{wt(T)},\]

where the sum runs over the set of all skew semistandard Young tableaux of shape \(\lambda \setminus \mu\).

The tropical Schur polynomial \(T s_{\lambda \setminus \mu}(x)\) is a piece-wise linear function defined by the tropicalization of the above formula in the tropical semi-ring, that is

\[T s_{\lambda \setminus \mu}(x) = \max_T (x, wt(T)),\]

where \(\max\) is taken over all SSYT \(T\) of shape \(\lambda \setminus \mu\). For \(\mu = 0\), we obtain a usual tropical Schur polynomial (cf. Section 1).

Thus, \(T s_{\lambda \setminus \mu}(x)\) is a piece-wise linear function of the form of the maximum of a linear function \((x, \cdot)\) over the set of points \(\nu := wt(T)\), while \(T\) runs over the set of all skew semistandard Young tableaux of shape \(\lambda \setminus \mu\).

This set of weights constitute the set of integer points of the polytope \(GC(\lambda, \mu)\) defined by the inequalities

\[
\lambda([1, |I|]) - \Delta_{|I|} \geq \nu(I), \quad \lambda([n]) - \Delta_n = \nu([n]),
\]

where \(\lambda([1, |I|]) = \lambda_1 + \cdots \lambda_{|I|}, \nu(I) = \sum_{i \in I} \nu_i, \Delta_{|I|} = \Delta_1 + \cdots \Delta_{|I|}, \Delta_k := \max\{0, \mu_1 - \lambda_{k+1}\} + \max\{0, \mu_2 - \lambda_{k+2}\} + \cdots + \max\{0, \mu_{n-k} - \lambda_n\}\) (for details see [3]).

For given \(\lambda\) and \(\mu\) we get a function \(\Lambda : 2^{|n|} \to \mathbb{R}, \Lambda(I) = \lambda([1, |I|]) - \Delta_{|I|}, I \subseteq [n]\).

The properties of this function depend on shape \(\lambda \setminus \mu\). For example, for \(\mu = 0\), this function is submodular (see the Appendix below). Let \(\lambda\) and \(\mu\) be such that the function \(\Lambda\) is submodular. That is, for any \(|I|\), it holds

\[
\lambda([1, |I|]) - \Delta_{|I|} - \lambda([1, |I| + 1]) - \Delta_{|I|+1} \geq \lambda([1, |I| + 1]) - \Delta_{|I|+1} - \lambda([1, |I| + 2]) - \Delta_{|I|+2}.
\]

In such a case, the polytope \(GC(\lambda, \mu)\) is a base-polytope, and the complexity of computation of \(T s_{\lambda \setminus \mu}(x)\) as a tropical function using the greedy algorithm (see [5] and the Appendix) is polynomial in \(n\).

While, for \(\lambda\) and \(\mu\), for which \(\Lambda\) fails to be submodular, the problem of finding maximum can be hard, since some of the vertices of \(GC(\lambda, \mu)\) do not even corresponds to the weights of SSYT. Because of this we conjecture that the complexity of computation of the tropical polynomial \(T s_{\lambda \setminus \mu}(x)\) is exponential as well over the semi-ring \((\mathbb{R}, +, \max)\).
However, over the semi-field \((\mathbb{R}, \max, +, -)\), the complexity of the tropical skew Schur polynomial \(T_{\lambda \mu}(x)\) is polynomial independently of \(\lambda\) and \(\mu\). This follows from the tropicalization of the subtraction-free algorithm in [7] which computes skew Schur polynomials.

**Appendix**

Here we recall some basic facts on base-polytopes. For details see [5, 8].

A function \(f : 2^{[n]} \rightarrow \mathbb{R}\) is submodular if, for any \(S, T \subseteq [n]\), it holds

\[
f(S) + f(T) \geq f(S \cup T) + f(S \cap T).
\]

To a submodular function \(f\) is associated a base-polytope \(B_f\) in \(\mathbb{R}^n\)

\[
B_f := \{ x \in \mathbb{R}^n : x(S) \leq f(S), x([n]) = f([n]) \},
\]

where \(x(S)\) denotes the sum \(\sum_{i \in S} x_i\).

This polytope is located in the hyperplane \(x([n]) = f([n])\). Edges of such a polytope are parallel to ‘roots’ \(\alpha_i - \alpha_j\), where \(\alpha_i\) denotes the \(i\)-th basis vector in \(\mathbb{R}^n\).

The Edmonds greedy algorithm [5] implies that the vertices of the base-polytope are labeled by permutations from \(S_n\). Namely, for a permutation \(\sigma \in S_n\), the corresponding vertex has coordinates defined by the rule

\[
x_{\sigma(1)} = f(\{\sigma(1)\}), \ x_{\sigma(2)} = f(\{\sigma(1), \sigma(2)\}) - f(\{\sigma(1)\}), \ldots,
\]

\[
x_{\sigma(i)} = f(\{\sigma(1), \ldots, \sigma(i)\}) - f(\{\sigma(1), \ldots, \sigma(i-1)\}).
\]

Any facet of a base-polytope is a direct product of two base-polytopes. Moreover, each facet is labeled by a subset \(W \subset [n]\) and is the product of the base-polytope \(B_{f|W} := \{ x \in \mathbb{R}^W : x(S) \leq f(S), S \subset W, x(W) = f(W) \}\) and the base-polytope \(B_{f\setminus W} := \{ x \in \mathbb{R}^{[n]\setminus W} : x(T) \leq f(T \cup W) - f(W), T \subset [n] \setminus W, x([n] \setminus W) = f([n]) - f(W) \}\). The polytope \(B_{f|W}\) is a subset of \(\mathbb{R}^W\), and the polytope \(B_{f\setminus W}\) is a subset of \(\mathbb{R}^{[n]-W}\). Remark that the facet labeled by the complementary set \([n] - W\), is the product of the polytope \(B_{f|[n]-W}\) in \(\mathbb{R}^{[n]-W}\) and the polytope \(B_{f\setminus [n]-W}\) in \(\mathbb{R}^W\). In other words, these facets are parallel and decomposed as the product of polytopes in \(\mathbb{R}^W\) and \(\mathbb{R}^{[n]-W}\).

Thus, a facet labeled by a subset \(W\) of cardinality \(k\) has at most \(k! \times (n-k)!\) vertices. Moreover, this bound on the number of vertices is valid for any ‘cut’

\[
B_f \cap \{ x \in \mathbb{R}^{[n]} : x(W) = a, x_i = 0, i \notin W \},
\]

where \(a\) is in the segment \(f([n]) - f([n] - W) \leq a \leq f(W)\). (From the submodularity it holds that \(f(W) + f([n] - W) \geq f([n]).\) In fact, such a cut is a facet of the base polytope

\[
B_f \cap \{ x \in \mathbb{R}^{[n]} : x(W) \leq a, x_i = 0, i \notin W \}.
\]

Let us warn that in general the intersection of base-polytopes may be not a base-polytope, but the intersection of a base-polytope with a half-space \(\{ x \in \mathbb{R}^{[n]} : x(W) \leq a, x_i = 0, i \notin W \}\) is always a base-polytope.

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References


