Complexity of tropical Schur polynomials

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Abstract

We study the complexity of computation of a tropical Schur polynomial Ts_λ where λ is a partition, and of a tropical polynomial Tm_λ obtained by the tropicalization of the monomial symmetric function m_λ. Then Ts_λ and Tm_λ coincide as tropical functions (so, as convex piece-wise linear functions), while differ as tropical polynomials. We prove the following bounds on the complexity of computing over the tropical semi-ring $(\mathbb{R}, \max, +)$:

- a polynomial upper bound for Ts_λ and
- an exponential lower bound for Tm_λ.

Also the complexity of tropical skew Schur polynomials is discussed.

Introduction

We study computations (i. e. circuits, see e. g. [2]) over a *tropical semi-ring* $(\mathbb{R}, \max, +)$ where max plays a role of addition, and + plays a role of multiplication (see e. g. [9]). Actually, computations over $(\mathbb{R}, \max, +)$ were considered in Computer Science earlier than tropical algebra and geometry (and even the term "tropical" itself) have emerged (see e. g. [10] and further references there).

The *tropicalization* of a polynomial $f = \sum_I a_I x_1^{i_1} \cdots x_n^{i_n} \in \mathbb{R}[x_1, \dots, x_n]$ is a *tropical polynomial* $Trop(f) := \max_I\{i_1x_1 + \cdots + i_nx_n\}$ defined over the tropical semi-ring $(\mathbb{R}, \max, +)$ (see e. g. [9]). One can treat a tropical polynomial as a convex piece-wise linear function.

We study a *tropical Schur polynomial* $Ts_\lambda = Trop(s_\lambda)$ (see Section 1) being the tropicalizations of the Schur function s_λ, where $\lambda = \{\lambda_1, \dots, \lambda_n\}$ is a partition.

Since Ts_λ is a convex piece-wise linear function $\max_W\{w_1x_1 + \cdots + w_nx_n\}$ where the multiindices W range over all integer points of the Newton polyhedron of s_λ, it coincides with a function $Tm_\lambda := \max_J\{j_1x_1 + \cdots + j_nx_n\}$ where the multiindices J range over all the vertices of the Newton polyhedron of s_λ. Note that Tm_λ are the tropicalizations of the monomial symmetric functions m_λ which form (as well as s_λ) a

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basis in the ring of symmetric functions (see [11]). On the other hand, Ts_λ and Tm_λ differ as the elements of the *semi-ring of tropical polynomials* [9].

We exhibit (see Theorem 1) a polynomial complexity algorithm which computes Ts_λ over $(\mathbb{R}, \max, +)$. On the contrary, we prove (see Theorem 2) an exponential lower bound on the complexity of computing Tm_λ over $(\mathbb{R}, \max, +)$. This demonstrates an interesting phenomenon: while Ts_λ and Tm_λ coincide as tropical functions, their complexities as tropical polynomials differ considerably.

Observe that in [7] there was designed a polynomial complexity *subtraction-free* algorithm (relying on the cluster transformations), in other words a computation over $(\mathbb{R}, +, \times, /)$ for Schur polynomials. The tropicalization of this algorithm provides a polynomial complexity computation of Ts_λ over a *tropical semi-field* $(\mathbb{R}, \max, +, -)$. Thus, the algorithm from Theorem 1 is better because it avoids subtraction (viewed as a tropical analog of division). It is unclear, whether the complexity of computation of Tm_λ over $(\mathbb{R}, \max, +, -)$ is polynomial?

On the other hand, from the tropicalization of the results of [7] we conclude that the tropical polynomial expressing the maximal weight directed spanning tree in the complete graph has a polynomial complexity over $(\mathbb{R}, \max, +, -)$, while its complexity over $(\mathbb{R}, \max, +)$ is exponential. In the proofs of complexity lower bounds we make use of technical tools developed in [12], [10], where some exponential complexity lower bounds were established for computations over $(\mathbb{R}, +, \times)$ as well as over the tropical semi-ring $(\mathbb{R}, \max, +)$.

In Sections 2, we speculate that the complexity of a skew Schur polynomial $Ts_{\lambda/\mu}$ in n variables (being the tropicalization of the skew Schur polynomial $s_{\lambda/\mu}$) might depend on the shapes of the partitions λ, μ, and we conjecture that for some shapes its complexity over the semi-ring $(\mathbb{R}, \max, +)$ is exponential, while over the semi-field $(\mathbb{R}, \max, +, -)$ the complexity is (polynomial) $O(n^5)$ due to the tropicalization of the subtraction-free algorithm from [7] which computes skew Schur polynomials.

In the Appendix we provide some necessary concepts and results on base-polytopes and submodular functions.

1 Tropical Schur polynomials

For a fixed alphabet $[n] := \{1, \dots, n\}$ and a partition $\lambda = (\lambda_1 \geq \dots \geq \lambda_n)$, we consider a tropical Schur polynomial Ts_λ in the form of maximization of a linear function over the set of integer points of the Newton polytope of the usual Schur polynomial [11]

$$s_\lambda(x) = \sum_{\mu \in ch(w(\lambda),\, w \in S_n)} K_{\mu,\lambda} x^\mu,$$

where $x = (x_1, \dots, x_n)$, $x^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n}$, S_n denotes the group of permutations of the finite set $[n]$, $w(\lambda) = (\lambda_{w(1)}, \dots, \lambda_{w(n)})$, and $ch(w(\lambda),\, w \in S_n)$ denotes the convex hull of the points $w(\lambda)$, $w \in S_n$, we denote $\mu \preceq \lambda$ if $\mu \in ch(w(\lambda),\, w \in S_n)$, and $K_{\mu,\lambda}$ are the Kostka numbers. For details see [11].

Thus, the tropicalization of Schur polynomial $s_\lambda(x)$ is

$$Ts_\lambda(x) = \max_{\mu \in ch(w(\lambda),\, w \in S_n)} x(\mu),$$

here we consider x as a linear functional on $\mathbb{R}^n$, and $x(\mu)$ denotes the value of the functional at $\mu \in \mathbb{Z}^n$.

1.1 Complexity: upper bound

The tropicalization (see [1]) of the cluster algorithm in [7] provides an algorithm for computing tropical polynomial $Ts_\lambda(x)$ within bit-complexity $O(k^3)$, $k := \lambda_1 + n$, over the tropical semi-field $(\mathbb{R}, \max, +, -)$ (in the algebraic setup in [7] we consider $\mathbb{R}$ with addition, multiplication and division).

We conjecture that in the algebraic setup, it is exponential hard to calculate s_λ without division, i.e. over $(\mathbb{R}, +, \times)$.

However, the situation drastically changes in the tropical setup. Namely, we can calculate Ts_λ over the tropical semi-ring $(\mathbb{R}, \max, +)$ within bit-complexity $O(n^2 \cdot \lambda_1)$.

Let us recall that the Newton polytope $NP(e_k)$ of an elementary symmetric function

$$e_k(x_1, \dots, x_n) = \sum_{i_1 < i_2 < \dots < i_k} x_{i_1} x_{i_2} \dots x_{i_k},$$

is a *hypersimplex*, that is the convex hull of the set

$$\binom{[n]}{k} = \{I \subset [n],\, |I| = k\},$$

where a subset I is naturally identified with a vertex of the hypercube $2^{[n]}$.

A hypersimplex is a matroid, a subclass of base-polytopes. The useful facts on base polytopes are collected in the Appendix.

Denote by λ' the dual partition to λ, that is $\lambda'_i = |\{j : \lambda_j \geq i\}, i = 1, \dots, \lambda_1\}|$. From the Littlewood formula (see [11]) it follows

$$\prod_k e_{\lambda'_k} = s_\lambda + \sum_{\mu \prec \lambda} K_{\lambda', \mu'} s_\mu.$$

Hence the Newton polytope $NP(Ts_\lambda)$ of the Schur polynomial s_λ coincides with the Minkowski sum of the Newton polytopes $\sum_k NP(e_{\lambda'_k})$. Moreover, since the hypersymplexes are matroids, the directions of edges of any hypersimplex take the form $\{e_i - e_j\}$. The latter set is unimodular, and from [4] we get

$$NP(Ts_\lambda)(\mathbb{Z}) = \sum_{1 \leq k \leq \lambda_1} NP(e_{\lambda'_k})(\mathbb{Z}), \qquad (1)$$

where, for a polytope P, $P(\mathbb{Z})$ denotes the set of integer points in P.

Because of this, we have

Theorem 1. A tropical Schur polynomial Ts_λ can be calculated within (polynomial) $O(n^2 \cdot \lambda_1)$ bit complexity over $(\mathbb{R}, \max, +)$.

Proof. Due to (1), in order to calculate Ts_λ, one needs first to calculate tropical elementary Schur functions $Te_{\lambda'_k}$, $1 \leq k \leq \lambda_1$. Since

$$e_k(x_1, \dots, x_n) = e_k(x_1, \dots, x_{n-1}) + x_n e_{k-1}(x_1, \dots, x_{n-1}),$$

and a similar identity holds in the tropical setup, the complexity of computation of a tropical elementary Schur function is quadratic in n (to this end, one can use the Pascal triangle). $\square$

1.2 Complexity: lower bound

Since tropical Schur function takes the form of maximization of a linear functional over a polytope, it suffices to consider only the vertices of such a polytope. However, over the semi-ring $(\mathbb{R}, \max +)$ the complexity of such a modification can increase exponentially. We demonstrate this phenomenon for a tropical Schur function.

Namely, let us consider the tropicalization of the monomial symmetric functions $m_\lambda = \sum_{w \in S_n} x^{w(\lambda)}$,

$$Tm_\lambda(x) = \max_{w \in S_n} x(w(\lambda)).$$

Observe that Ts_λ and Tm_λ coincide as *tropical functions*, while they differ as the elements of the *semi-ring of tropical polynomials*, and the complexity of computation in the latter semi-ring is polynomial for Ts_λ (Theorem 1), while the complexity of Tm_λ is exponential as we prove in the following theorem.

Theorem 2. For λ with the ith part of the form $\lambda_{n-i+1} := ni + i^2$, $i = 1, \dots, n$, the complexity of computation of Tm_λ over the tropical semiring $(\mathbb{R}, \max, +)$ is exponential.

Proof. Throughout the proof we omit the adjective "tropical" for tropical polynomials and utilize for the latter the customary notations $+, \times$ for tropical operations $\max, +$, respectively. For a (homogeneous) polynomial P by $mon(P)$ denote the set of monomials of P. We will use the following result from [12], [10]. If for any homogeneous polynomials R, Q such that $mon(P) \supset mon(RQ)$, and of the powers $1/3 \deg P \leq \deg R, \deg Q \leq 2/3 \deg P$, we have $\frac{|mon P|}{|mon(RQ)|} > c_1^n$, for some $c_1 > 1$, then the complexity of computation of P over $(\mathbb{R}, \max, +)$ is exponential. We mention that a similar complexity lower bound holds as well for computations over $(\mathbb{R}, +, \times)$ [12], [10].

In our case we have to show that R and Q have exponentially small deal of monomials wrt $n!$ (which equals the number of monomials in $P := Tm_\lambda$).

Let us explain our choice of such a specific λ. The parts of λ form a Golomb ruler ([6]), that is $\lambda_i + \lambda_j = \lambda_k + \lambda_l$ iff $\{i, j\} = \{k, l\}$.

This property allows us to separate variables, namely we have $Q = Q'(x_i, i \in S) M(x_j, j \in [n] \setminus S)$ and $R = N(x_i, i \in S) R'(x_j, j \in [n] \setminus S)$, where M and N are monomials in variables $x_j, j \in [n] \setminus S$ and $x_i, i \in S$, respectively. Indeed, assume the contrary. Then there exists $m \in [n]$ and four monomials

$$q_1 = \cdots x_m^\alpha \cdots, \; q_2 = \cdots x_m^\beta \cdots \in mon(Q); \; r_1 = \cdots x_m^\gamma \cdots, \; r_2 = \cdots x_m^\delta \cdots \in mon(R)$$

such that $\alpha \neq \beta$, $\gamma \neq \delta$. Since

$$r_1 q_1, r_2 q_2, r_1 q_2, r_2 q_1 \in mon(RQ) \subset mon(P)$$

there are $i, j, k, l \in [n]$ for which $\alpha+\gamma = \lambda_i$, $\beta+\delta = \lambda_j$, $\alpha+\delta = \lambda_k$, $\beta+\gamma = \lambda_l$. Hence $\lambda_i + \lambda_j = \lambda_k + \lambda_l$, and we get a contradiction with the Golomb property.

Thus, we have a separation of variables. We get two polynomials $A := NQ'$ and $B := MR'$ in variable x_i, $i \in S$, and x_j, $j \in [n] \setminus S$, respectively.

At the beginning we consider a case of no separation of variables. This means that either Q or R is a monomial. Let for definiteness Q be a monomial.

Then we claim that if $c := \frac{\deg Q}{\deg P} \in [\frac{1}{4}, \frac{3}{4}]$, then R has exponentially small number of monomials wrt $n!$. Throughout this Section we assume in all the bounds n to be sufficiently big.

Let us prove this claim.

Let $Q = x_1^{\nu_1} \cdots x_n^{\nu_n}$. Firstly, we observe that w.l.o.g. one can suppose that for any i there exists j such that $\nu_i = \lambda_j$. Indeed, if at least two ν_{i_1}, ν_{i_2} among $\{\nu_i\}_i$ violate this condition, we can increase ν_{i_1} by 1 and decrease ν_{i_2} also by 1, thereby not decreasing $|mon(R)|$ for which $mon(QR) \subset mon(P)$. Observe that herein $|mon(R)|$ could increase only if $\nu_{i_2} = \lambda_j + 1$ for some j. If just a single $\lambda_j > \nu_i > \lambda_{j+1}$ violates the condition under discussion, we can preserve inequalities $\frac{\deg Q}{\deg P} \in [\frac{1}{4}, \frac{3}{4}]$ as follows: either replace ν_i by λ_j which keeps $|mon(R)|$ or replace by λ_{j+1} which does not decrease $|mon(R)|$.

Let $b_j := \{i : \nu_i = \lambda_j\}$, $j = n, \dots, 1$. Then the number of monomials in $R(x)$ is equal to

$$M := b_n(b_n + b_{n-1} - 1) \cdots (b_n + \ldots + b_1 - (n-1)).$$

We have

$$\sum b_i \lambda_i = c \sum \lambda_i.$$

Then, we have

$$\sum_i \lambda_i - \sum b_i \lambda_i + \lambda_1 - \lambda_n = \sum_{j=0}^{n-2} (b_n + \ldots + b_{n-j} - j)(\lambda_{n-j-1} - \lambda_{n-j}).$$

Thus

$$M \prod (\lambda_{j-1} - \lambda_j) \le \Big(\frac{(1-c)\sum \lambda_i + \lambda_1 - \lambda_n}{n} \Big)^n.$$

We have $\sum \lambda_i \sim \frac{5n^3}{6}$, $\prod(\lambda_{j-1} - \lambda_j) \sim 2^n \frac{(3/2n)!}{(1/2n)!} \sim (\frac{3^{3/2}}{e} n)^n$.

Therefore it holds (taking into account that due to the choice of λ_i, the degree of P is $5/6n^3 + O(n^2)$) that

$$M \le (\frac{5e(1-c)n}{3^{3/2}6})^n. \qquad (2)$$

Thus, for $1 - c < \frac{6 \cdot 3^{3/2}}{5e^2} < \frac{31.14}{38.64}$, the number of monomials in R is exponentially small wrt $n!$. For $c \in [1/4, 3/4]$, this is the case.

Now consider the case of a non-monomial Q. In such a case we have a separation of variables.

Let us recall that the polytope $Per_n := ch(\sigma(\lambda), \sigma \in S_n)$ is a base-polytope (see the Appendix) which is set by a submodular function $b_\lambda(T) = \sum_{i=1,\ldots|T|} \lambda_{n-i}$, $T \subset [n]$. Thus, a pair of parallel facets (we agree that a facet is a face of codimension 1) labeled by a subset $W \subset [n]$, $|W| = k$, are defined by $x(W) = b_\lambda(W) = \sum_{i=1,\ldots,k} \lambda_i$ and

$x([n]-W)=b_\lambda([n]-W)=\sum_{i=1,\dots,n-k}\lambda_i$, respectively, and any cut with the same separation of coordinates is defined by $x(W)=a$, $a\in[\sum_{j=1}^k\lambda_j,\sum_{i=1}^k\lambda_{n-i}]$ (for details see the Appendix). Because of symmetry of b_λ wrt permutations of coordinates, facets of Per_n are labeled by numbers in $[n]$. The number of the vertices of a facet labeled by $k\in[n]$ (recall that k corresponds to separation of variables in groups of k and $n-k$ variables) is

$$k!(n-k)!.$$

Because of this, the cardinality of monomials of the product $A\cdot B$ is bounded by $k(A)!(n-k(A))!$, where $k:=k(A)=|S|$. Note that $\deg(A)=\lambda_{i_1}+\cdots+\lambda_{i_k}$ for suitable $1\le i_1<\cdots<i_k\le n$ satisfies

$$\deg(A)\in[\sum_{j=1}^k\lambda_j,\sum_{i=1}^k\lambda_{n-i}].$$

There are two cases.

Case 1. $\deg A,\deg B\ge c'\cdot\deg P$, for some sufficiently small constant c' which we choose later. In such a case, $k=k(A)$, $n-k=k(B)\ge c''\cdot n$ for some sufficiently small constant c'' depending on c' (since $\deg P$ is cubic in n). This implies that $A\cdot B$ has at most $k!(n-k)!$ number of monomials, so exponentially small wrt $n!$ and we are done.

Case 2. Either $\deg A<c'\deg P$ or $\deg B<c'\deg P$. Let for definiteness $\deg A<c'\deg P$. Then, the degree of the monomial M satisfies $\frac{\deg M}{\deg B}\in[\frac{1}{4},\frac{3}{4}]$ since c' is sufficiently small.

Then, the same reasoning as above in the case of no separation with a single monomial, provides a bound $|mon(R')|\le(c_0(n-k))^{n-k}$ for any fixed $c_0>\frac{5e(1-c)}{3^{3/2}6}$ (see (2)) due to an appropriate choice of sufficiently small c' in Case 1. We take $c_0<1/e$. Because of this and that A has at most $k!$ monomials we get that

$$|mon(RQ)|=|mon(AB)|\le k!(c_0(n-k))^{n-k}<c_2^n n!$$

for some $c_2<1$. This finishes the proof of Theorem 2. □

2 Tropical skew Schur polynomials

In this Section we discuss a conjecture that for a tropical skew Schur polynomial its complexity over the tropical semi-ring might depend on the shape of the corresponding diagram and could be exponential. While over the tropical semi-field the complexity is always polynomial.

Recall that, for a skew Young diagram $\lambda\setminus\mu$ (where $\mu\le\lambda$, which denotes the coordinate-wise inequality of the partitions), a semi-standard Young tableaux (SSYT) of a shape $\lambda\setminus\mu$ (in the alphabet $[n]$) is a filling of the Young diagram $\lambda\setminus\mu$ with entries from $[n]$ strictly increasing along the columns and non-decreasing along the rows ([11]). We accept the French style to draw Young diagram. Here is an example of a skew SSYT of shape $(5,3,3,1)\setminus(2,1)$

$$\begin{array}{ccccc} 3 & & & & \\ 2 & 2 & 4 & & \\ & 1 & 2 & & \\ & & 1 & 1 & 2 \end{array}$$

The weight of such a tableau T is the tuple $wt(T) := (\#1(T), \#2(T), \ldots, \#n(T))$, where $\#i(T)$ denotes the number of times integer i occurs in T. The skew Schur polynomial $s_{\lambda\setminus\mu}$ is defined by (see [11])

$$s_{\lambda\setminus\mu} = \sum_T x^{wt(T)},$$

where the sum runs over the set of all skew semistandard Young tableaux of shape $\lambda\setminus\mu$.

The tropical Schur polynomial $Ts_{\lambda\setminus\mu}(x)$ is a piece-wise linear function defined by the tropicalization of the above formula in the tropical semi-ring, that is

$$Ts_{\lambda\setminus\mu}(x) = \max_T(x, wt(T)).$$

where max is taken over all SSYT T of shape $\lambda\setminus\mu$. For $\mu = 0$, we obtain a usual tropical Schur polynomial (cf. Section 1).

Thus, $Ts_{\lambda\setminus\mu}(x)$ is a piece-wise linear function of the form of the maximum of a linear function $(x,\cdot)$ over the set of points $\nu := wt(T)$, while T runs over the set of all skew semistandard Young tableaux of shape $\lambda\setminus\mu$.

This set of weights constitute the set of integer points of the polytope $\mathcal{GC}(\lambda,\mu)$ defined by the inequalities

$$\lambda([1,|I|]) - \Delta_{|I|} \geq \nu(I), \quad \lambda([n]) - \Delta_n = \nu([n]),$$

where $\lambda([1,|I|]) = \lambda_1 + \cdots \lambda_{|I|}$, $\nu(I) = \sum_{i\in I}\nu_i$, $\Delta_{|I|} = \Delta_1 + \ldots \Delta_{|I|}$, $\Delta_k := \max\{0, \mu_1 - \lambda_{k+1}\} + \max\{0, \mu_2 - \lambda_{k+2}\} + \cdots + \max\{0, \mu_{n-k} - \lambda_n\}$ (for details see [3]).

For given λ and μ we get a function $\Lambda : 2^{[n]} \to \mathbb{R}$, $\Lambda(I) = \lambda([1,|I|]) - \Delta_{|I|}$, $I \subseteq [n]$.

The properties of this function depend on shape $\lambda\setminus\mu$. For example, for $\mu = 0$, this function is submodular (see the Appendix below). Let λ and μ be such that the function Λ is submodular. That is, for any $|I|$, it holds

$$\lambda([1,|I|]) - \Delta_{|I|} - \lambda([1,|I|+1]) - \Delta_{|I|+1} \geq \lambda([1,|I|+1]) - \Delta_{|I|+1} - \lambda([1,|I|+2]) - \Delta_{|I|+2}.$$

In such a case, the polytope $\mathcal{GC}(\lambda,\mu)$ is a base-polytope, and the complexity of computation of $Ts_{\lambda\setminus\mu}(x)$ as a tropical function using the greedy algorithm (see [5] and the Appendix) is polynomial in n.

While, for λ and μ, for which Λ fails to be submodular, the problem of finding maximum can be hard, since some of the vertices of $\mathcal{GC}(\lambda,\mu)$ do not even corresponds to the weights of SSYT. Because of this we conjecture that the complexity of computation of the tropical polynomial $Ts_{\lambda\setminus\mu}(x)$ is exponential as well over the semi-ring $(\mathbb{R}, +, \max)$.

However, over the semi-field $(\mathbb{R}, \max, +, -)$, the complexity of the tropical skew Schur polynomial $Ts_{\lambda\backslash\mu}(x)$ is polynomial independently of λ and μ. This follows from the tropicalization of the subtraction-free algorithm in [7] which computes skew Schur polynomials.

Appendix

Here we recall some basic facts on base-polytopes. For details see [5, 8].

A function $f : 2^{[n]} :\rightarrow \mathbb{R}$ is *submodular* if, for any S, $T \subseteq [n]$, it holds

$$f(S) + f(T) \geq f(S \cup T) + f(S \cap T).$$

To a submodular function f is associated a base-polytope B_f in $\mathbb{R}^n$

$$B_f := \{x \in \mathbb{R}^n \,:\, x(S) \leq f(S), x([n]) = f([n])\},$$

where $x(S)$ denotes the sum $\sum_{i \in S} x_i$.

This polytope is located in the hyperplane $x([n]) = f([n])$. Edges of such a polytope are parallel to 'roots' $\alpha_i - \alpha_j$, where α_i denotes the i-th basis vector in $\mathbb{R}^n$.

The Edmonds greedy algorithm [5] implies that the vertices of the base-polytope are labeled by permutations from S_n. Namely, for a permutation $\sigma \in S_n$, the corresponding vertex has coordinates defined by the rule $x_{\sigma(1)} = f(\{\sigma(1)\})$, $x_{\sigma(2)} = f(\{\sigma(1), \sigma(2)\}) - f(\{\sigma(1)\}), \ldots,$

$$x_{\sigma(i)} = f(\{\sigma(1), \ldots, \sigma(i)\}) - f(\{\sigma(1), \ldots, \sigma(i-1)\}).$$

Any facet of a base-polytope is a direct product of two base-polytopes. Moreover, each facet is labeled by a subset $W \subset [n]$ and is the product of the base-polytope $B_{f|_W} := \{x \in \mathbb{R}^W \,:\, x(S) \leq f(S), S \subset W, x(W) = f(W)\}$ and the base-polytope $B_{f^W} := \{x \in \mathbb{R}^{[n]\setminus W} \,:\, x(T) \leq f(T \cup W) - f(W), T \subset [n] \setminus W, x([n] \setminus W) = f([n]) - f(W)\}$. The polytope $B_{f|_W}$ is a subset of $\mathbb{R}^W$, and the polytope B_{f^W} is a subset of $\mathbb{R}^{[n]-W}$. Remark that the facet labeled by the complementary set $[n] - W$, is the product of the polytope $B_{f|_{[n]-W}}$ in $\mathbb{R}^{[n]-W}$ and the polytope $B_{f^{[n]-W}}$ in $\mathbb{R}^W$. In other words, these facets are parallel and decomposed as the product of polytopes in $\mathbb{R}^W$ and $\mathbb{R}^{[n]-W}$.

Thus, a facet labeled by a subset W of cardinality k has at most $k! \times (n-k)!$ vertices. Moreover, this bound on the number of vertices is valid for any 'cut'

$$B_f \cap \{x \in \mathbb{R}^{[n]} \,:\, x(W) = a, x_i = 0, i \notin W\},$$

where a is in the segment $f([n]) - f([n] - W) \leq a \leq f(W)$. (From the submodularity it holds that $f(W) + f([n] - W) \geq f([n])$.) In fact, such a cut is a facet of the base polytope

$$B_f \cap \{x \in \mathbb{R}^{[n]} \,:\, x(W) \leq a, x_i = 0, i \notin W\}.$$

Let us warn that in general the intersection of base-polytopes may be not a base-polytope, but the intersection of a base-polytope with a half-space $\{x \in \mathbb{R}^{[n]} \,:\, x(W) \leq a, x_i = 0, i \notin W\}$ is always a base-polytope.

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