Complexity of tropical Schur polynomials

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Abstract

We study the complexity of computation of a tropical Schur polynomial Ts_λ where λ is a partition, and of a tropical polynomial Tm_λ obtained by the tropicalization of the monomial symmetric function m_λ. Then Ts_λ and Tm_λ coincide as tropical functions (so, as convex piece-wise linear functions), while differ as tropical polynomials. We prove the following bounds on the complexity of computing over the tropical semi-ring $(\mathbb{R}, \max, +)$:

- a polynomial upper bound for Ts_λ and
- an exponential lower bound for Tm_λ.

Also the complexity of tropical skew Schur polynomials is discussed.

Keywords: tropical Schur polynomials, complexity over the tropical semi-ring

Introduction

We study computations (i. e. circuits, see e. g. [2]) over a *tropical semi-ring* $(\mathbb{R}, \max, +)$ where max plays a role of addition, and + plays a role of multiplication (see e. g. [12]). Actually, computations over $(\mathbb{R}, \max, +)$ were considered in Computer Science earlier than tropical algebra and geometry (and even the term "tropical" itself) have emerged (see e. g. [13] and further references there).

The *tropicalization* of a polynomial $f = \sum_I a_I x_1^{i_1} \cdots x_n^{i_n} \in \mathbb{R}[x_1, \dots, x_n]$ is a *tropical polynomial* $Trop(f) := \max_I\{i_1x_1 + \cdots + i_nx_n\}$ defined over the tropical semi-ring $(\mathbb{R}, \max, +)$ (see e. g. [12]). One can treat a tropical polynomial as a convex piece-wise linear function.

We study a *tropical Schur polynomial* $Ts_\lambda = Trop(s_\lambda)$ (see Section 1) being the tropicalizations of the Schur function s_λ, where $\lambda = \{\lambda_1, \dots, \lambda_n\}$, $\lambda_1 \geq \lambda_2 \geq \cdots$ is a partition.

Schur functions is an important class in combinatorics, representation theory and geometry (see, for example, [10]). These functions form a distinguished basis in the ring of symmetric polynomials. Schur functions considered as functions of λ's satisfy

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Plücker relations [10], and such hipostasis of Schur functions is of importance for construction of polynomial solutions to discrete Hirotra equation [18]. Tropical Plücker relations take the form of the tropical octahedron recursion [5] and tropical Schur polynomials satisfy this recursion. Tropicalization of polynomials relates to optimization problems on the corresponding Newton polytopes. For Schur functions the Newton polytopes are permutohedrons, a subclass of important class of polytopes in combinatorial optimization, base-polytopes, see Section 3 below.

Since Ts_λ is a convex piece-wise linear function $\max_W\{w_1x_1 + \cdots + w_nx_n\}$ where the multiindices W range over all integer points of the Newton polyhedron of s_λ, it coincides with a function $Tm_\lambda := \max_J\{j_1x_1 + \cdots + j_nx_n\}$ where the multiindices J range over all the vertices of the Newton polyhedron of s_λ. Note that Tm_λ are the tropicalizations of the monomial symmetric functions m_λ which form (as well as s_λ) a basis in the ring of symmetric functions (see [16]). On the other hand, Ts_λ and Tm_λ differ as the elements of the *semi-ring of tropical polynomials* [12].

We exhibit (see Theorem 2.1) a polynomial complexity $O(n^2 \cdot \lambda_1)$ algorithm which computes Ts_λ over $(\mathbb{R}, \max, +)$. On the contrary, we prove (see Theorem 4.1) an exponential lower bound on the complexity of computing Tm_λ over $(\mathbb{R}, \max, +)$. This demonstrates an interesting phenomenon: while Ts_λ and Tm_λ coincide as tropical functions, their complexities as tropical polynomials differ considerably.

Observe that in [8] there was designed a polynomial complexity *subtraction-free* algorithm (relying on the cluster transformations), in other words a computation over $(\mathbb{R}, +, \times, /)$ for Schur polynomials. The tropicalization (see [1]) of this algorithm provides a polynomial complexity computation of Ts_λ over a *tropical semi-field* $(\mathbb{R}, \max, +, -)$. Thus, the algorithm from Theorem 2.1 is better because it avoids subtraction (viewed as a tropical analog of division). It is unclear, whether the complexity of computation of Tm_λ over $(\mathbb{R}, \max, +, -)$ is polynomial?

We conjecture that in the algebraic setup, it is exponential hard to calculate s_λ without division, i.e. over $(\mathbb{R}, +, \times)$.

On the other hand, from the tropicalization of the results of [8] we conclude that the tropical polynomial expressing the maximal weight directed spanning tree in the complete graph has a polynomial complexity over $(\mathbb{R}, \max, +, -)$, while its complexity over $(\mathbb{R}, \max, +)$ is exponential. In the proofs of complexity lower bounds we make use of technical tools developed in [19], [13], where some exponential complexity lower bounds were established for computations over $(\mathbb{R}, +, \times)$ as well as over the tropical semi-ring $(\mathbb{R}, \max, +)$.

In Section 3 we provide some necessary concepts and results on base-polytopes and submodular functions.

In Section 5, we speculate that the complexity of a skew Schur polynomial $Ts_{\lambda/\mu}$ in n variables (being the tropicalization of the skew Schur polynomial $s_{\lambda/\mu}$) might depend on the shapes of the partitions λ, μ, and we conjecture that for some shapes its complexity over the semi-ring $(\mathbb{R}, \max, +)$ is exponential, while over the semi-field $(\mathbb{R}, \max, +, -)$ the complexity is (polynomial) $O(n^5)$ due to the tropicalization of the subtraction-free algorithm from [8] which computes skew Schur polynomials.

1 Tropical Schur polynomials and monomial symmetric polynomials

Recall that for a fixed alphabet $[n] := \{1, \ldots, n\}$ and a partition $\lambda = (\lambda_1 \geq \ldots \geq \lambda_n)$ the Schur polynomial s_λ [16] is defined as follows. Consider a matrix Δ having n rows and infinite number of columns with the entry x_i^j at i-th row and j-th column, $1 \leq i \leq n$, $j \geq 0$. By Δ_λ denote the $n \times n$ subdeterminant of Δ formed by the columns $\lambda_n, \lambda_{n-1} + 1, \ldots, \lambda_2 + n - 2, \lambda_1 + n - 1$. In particular, $\Delta_{(0,\ldots,0)} = \prod_{1 \leq i < j \leq n}(x_j - x_i)$ is the Vandermond determinant. Thus, the symmetric polynomial s_λ is defined as the quotient $\Delta_\lambda / \Delta_{(0,\ldots,0)}$.

For example, for $n = 3$ and $\lambda = (2, 1, 0)$, we get

$$s_{(2,1,0)}(x_1, x_2, x_3) = \frac{\det \begin{pmatrix} 1 & x_3^2 & x_3^4 \\ 1 & x_2^2 & x_2^4 \\ 1 & x_1^2 & x_1^4 \end{pmatrix}}{\det \begin{pmatrix} 1 & x_3 & x_3^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_1 & x_1^2 \end{pmatrix}} = \frac{x_1^4 x_3^2 + x_2^4 x_1^2 + x_3^4 x_2^2 - x_1^4 x_2^2 - x_2^4 x_3^2 - x_3^4 x_1^2}{(x_2 - x_1)(x_3 - x_1)(x_3 - x_2)} =$$

$$x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_2^2 x_3 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3.$$

The coefficients $K_{\mu,\lambda}$ of the expansion

$$s_\lambda(x) = \sum_{\mu \in ch(w(\lambda),\, w \in S_n)} K_{\mu,\lambda} x^\mu, \tag{1}$$

are called Kostka numbers, where $x = (x_1, \ldots, x_n)$, $x^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n}$, $(\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n$, S_n is the group of permutations of the finite set $[n]$, $w(\lambda) = (\lambda_{w(1)}, \ldots, \lambda_{w(n)})$, and $ch(w(\lambda),\, w \in S_n)$ denotes the convex hull of the points $w(\lambda)$, $w \in S_n$, we introduce a relation $\mu \preceq \lambda$ if $\mu \in ch(w(\lambda),\, w \in S_n)$.

For $s_{(2,1,0)}(x_1, x_2, x_3)$, we get 7 points in $ch(w(2, 1, 0),\, w \in S_3)$, $(1, 1, 1)$ and 6 permutations of $(2, 1, 0)$. The Kostka numbers are $K_{w(2,1,0),(2,1,0)} = 1$, $w \in S_3$, and $K_{(1,1,1),(2,1,0)} = 2$.

The Newton polytope of s_λ is defined as the convex hull of all the vectors $\mu \in \mathbb{R}^n$ with non-zero coefficients $K_{\mu,\lambda} \neq 0$, that is $ch(w(\lambda),\, w \in S_n)$. The tropicalization Ts_λ of the Schur polynomial s_λ can be treated as the maximum of a linear functional over the Newton polytope. Thus, the tropical Schur polynomial $s_\lambda(x)$ is

$$Ts_\lambda(x) = \max_{\mu \in ch(w(\lambda),\, w \in S_n)} x(\mu),$$

here we consider x as a linear functional on $\mathbb{R}^n$, and $x(\mu)$ denotes the value of the functional at $\mu \in \mathbb{Z}^n$.

The maximal value Ts_λ at a linear functional over the Newton polytope equals the maximal value at a linear functional over the vertices of the Newton polytope, i, e. the value of the function

$$Tm_\lambda(x) = \max_{w \in S_n} x(w(\lambda)),$$

being the tropicalization of the monomial symmetric function $m_\lambda = \sum_{w \in S_n} x^{w(\lambda)}$, therefore Ts_λ coincides with Tm_λ as functions.

2 Complexity: upper bound for tropical Schur polynomials

The purpose of this Section is to prove the following theorem.

Theorem 2.1 . *A tropical Schur polynomial Ts_λ can be calculated within (polynomial) $O(n^2 \cdot \lambda_1)$ bit complexity over* $(\mathbb{R}, \max, +)$.

Proof. First let us recall that the Newton polytope $NP(e_k)$ of an elementary symmetric function

$$e_k(x_1, \dots, x_n) = \sum_{i_1 < i_2 < \dots < i_k} x_{i_1} x_{i_2} \dots x_{i_k},$$

is a *hypersimplex*, that is the convex hull of the set

$$\binom{[n]}{k} = \{I \subset [n], \, |I| = k\},$$

where a subset I is naturally identified with a vertex of the hypercube $2^{[n]}$.

A hypersimplex is a matroid, a subclass of base-polytopes. The useful facts on base polytopes are collected in Section 3.

Denote by λ' the dual partition to λ, that is $\lambda'_i = |\{j : \lambda_j \geq i\}, \, i = 1, \dots, \lambda_1\}|$. From the Littlewood formula (see [16]) it follows

$$\prod_k e_{\lambda'_k} = s_\lambda + \sum_{\mu \prec \lambda} K_{\lambda', \mu'} s_\mu.$$

Hence the Newton polytope $NP(Ts_\lambda)$ of the Schur polynomial s_λ coincides with the Minkowski sum of the Newton polytopes $\sum_k NP(e_{\lambda'_k})$. Moreover, since the hypersymplexes are matroids, the directions of edges of any hypersimplex take the form $\{e_i - e_j\}$. The latter set is unimodular, and from [4] we get

$$NP(Ts_\lambda)(\mathbb{Z}) = \sum_{1 \leq k \leq \lambda_1} NP(e_{\lambda'_k})(\mathbb{Z}), \qquad (2)$$

where, for a polytope P, $P(\mathbb{Z})$ denotes the set of integer points in P.

Due to (2), in order to calculate Ts_λ, one needs first to calculate tropical elementary Schur functions $Te_{\lambda'_k}$, $1 \leq k \leq \lambda_1$. Since

$$e_k(x_1, \dots, x_n) = e_k(x_1, \dots, x_{n-1}) + x_n e_{k-1}(x_1, \dots, x_{n-1}),$$

and the tropicalization of this identity holds, the complexity of computation of a tropical elementary Schur function is quadratic in n (to this end, one can use this identity as a recursion on n, k following the Pascal triangle). ☐

3 Base-polytopes

Here we recall some basic facts on base-polytopes. For details see [6, 9].

A function $f : 2^{[n]} :\to \mathbb{R}$ is *submodular* if, for any S, $T \subseteq [n]$, it holds

$$f(S) + f(T) \geq f(S \cup T) + f(S \cap T).$$

To a submodular function f is associated a base-polytope B_f in $\mathbb{R}^n$

$$B_f := \{x \in \mathbb{R}^n \,:\, x(S) \leq f(S), x([n]) = f([n])\},$$

where $x(S)$ denotes the sum $\sum_{i \in S} x_i$.

This polytope is located in the hyperplane $x([n]) = f([n])$. Edges of such a polytope are parallel to 'roots' $\alpha_i - \alpha_j$, where α_i denotes the i-th basis vector in $\mathbb{R}^n$ ([11]).

The Edmonds greedy algorithm [6] implies that the vertices of the base-polytope are labeled by permutations from S_n. Namely, for a permutation $w \in S_n$, the corresponding vertex $x_w(f)$ has coordinates defined by the rule $x_{w(1)} = f(\{w(1)\})$, $x_{w(2)} = f(\{w(1), w(2)\}) - f(\{w(1)\}), \ldots,$

$$x_{w(i)} = f(\{w(1), \ldots, w(i)\}) - f(\{w(1), \ldots, w(i-1)\}).$$

There holds

Theorem 3.1 *([6]) B_f is the convex hull of $x_w(f)$, $w \in S_n$.* □

Any facet (a face of codimension 1) of a base-polytope is a direct product of two base-polytopes (one of polytopes can degenerate to a point). Moreover, each facet is labeled by a subset $W \subset [n]$ and there holds

Proposition 3.2 *([9]) The facet of B_f labeled by $W \subset [n]$ is the product of the base-polytope $B_{f|_W} := \{x \in \mathbb{R}^W \,:\, x(S) \leq f(S), S \subset W, x(W) = f(W)\}$ and the base-polytope $B_{f^W} := \{x \in \mathbb{R}^{[n] \setminus W} \,:\, x(T) \leq f(T \cup W) - f(W), T \subset [n] \setminus W, x([n] \setminus W) = f([n]) - f(W)\}$.* □

The polytope $B_{f|_W}$ is a subset of $\mathbb{R}^W$, and the polytope B_{f^W} is a subset of $\mathbb{R}^{[n] \setminus W}$.

Let us note that the facet labeled by the complementary set $[n] \setminus W$, is the product of the polytope $B_{f|_{[n] \setminus W}}$ in $\mathbb{R}^{[n] \setminus W}$ and the polytope $B_{f^{[n] \setminus W}}$ in $\mathbb{R}^W$. In other words, these facets are parallel and decomposed as the product of polytopes in $\mathbb{R}^W$ and $\mathbb{R}^{[n] \setminus W}$.

Thus, a facet labeled by a subset W of cardinality k has at most $k! \times (n-k)!$ vertices. Moreover, this bound on the number of vertices is valid for any 'cut'

$$B_f \cap \{x \in \mathbb{R}^{[n]} \,:\, x(W) = a, x_i = 0, i \notin W\},$$

where a is in the segment $f([n]) - f([n] \setminus W) \leq a \leq f(W)$. (From the submodularity it holds that $f(W) + f([n] \setminus W) \geq f([n])$.)

Lemma 3.3 *For a subset W of cardinality k, the polytope*

$$B_f \cap \{x \in \mathbb{R}^{[n]} \,:\, x(W) = a, x_i = 0, i \notin W\}$$

has at most $k! \times (n-k)!$ vertices.

Proof. The intersection of a base polytope and halfspaces of the form $x_i \le 0$, $x_i \ge 0$, $i \in [n]$, or $x([n]) \le a$, $a \in \mathbb{R}$ remains a base polytope ([9]).

Because of this, such a cut is a facet of the base polytope

$$B_f \cap \{x \in \mathbb{R}^{[n]} : x(W) \le a, x_i = 0, i \notin W\}.$$

□

Let us warn that in general the intersection of base-polytopes may be not a base-polytope.

4 Complexity: lower bound for tropical monomial symmetric functions

Since tropical Schur function takes the form of maximization of a linear functional over a polytope, it suffices to consider only the vertices of such a polytope. However, over the semi-ring $(\mathbb{R}, \max +)$ the complexity of such a modification can increase exponentially. We demonstrate this phenomenon for a tropical Schur function.

Observe that Ts_λ and Tm_λ (see Section 1) coincide as *tropical functions*, while they differ as the elements of the *semi-ring of tropical polynomials*, and the complexity of computation in the latter semi-ring is polynomial for Ts_λ (Theorem 2.1), while the complexity of Tm_λ is exponential as we prove in the following theorem.

Theorem 4.1 *For λ with the ith part of the form $\lambda_{n-i+1} := 2ni + \{i^2 mod\, n\}$, $i = 1, \dots, n$ for prime n, the complexity of computation of Tm_λ over the tropical semiring $(\mathbb{R}, \max, +)$ is exponential.*

Proof. Throughout the proof we omit the adjective "tropical" for tropical polynomials and utilize for the latter the customary notations $+, \times$ for tropical operations $\max, +$, respectively. For a (homogeneous) polynomial P by $mon(P)$ denote the set of monomials of P. We will use the following result.

Lemma 4.2 *[19], [13]. Let P be a homogeneous polynomial in n variables. If for any homogeneous polynomials R, Q such that $mon(P) \supset mon(RQ)$, and of the degrees $1/3 \deg P \le \deg R, \deg Q \le 2/3 \deg P$, we have $\frac{|monP|}{|mon(RQ)|} > c_1^n$, for some $c_1 > 1$, then the complexity of computation of P over $(\mathbb{R}, \max, +)$ is exponential.*

We mention that a similar complexity lower bound holds as well for computations over $(\mathbb{R}, +, \times)$.

In our case we have to show that a polynomial RQ has exponentially small deal of monomials wrt $n!$ (which equals the number of monomials in $P = Tm_\lambda$).

Let us explain our choice of such a specific λ. The parts of λ form a *Golomb ruler* ([7]), that is $\lambda_i + \lambda_j = \lambda_k + \lambda_l$ iff $\{i, j\} = \{k, l\}$.

Lemma 4.3 *[7]. Partition λ specified in Theorem 4.1 forms a Golomb ruler.*

This property allows us to separate variables. Below P denotes Tm_λ for a partition λ forming a Golomb ruler.

Lemma 4.4 *Let $mon(QR) \subset mon(P)$ for a partition λ which forms a Golomb ruler and homogeneous polynomials Q, R. Then there is a subset $S \subset [n]$ such that $Q = Q'(x_i, i \in S)M(x_j, j \in [n] \setminus S)$ and $R = N(x_i, i \in S)R'(x_j, j \in [n] \setminus S)$, where M and N are monomials in variables $x_j, j \in [n] \setminus S$ and $x_i, i \in S$, respectively.*

Proof. Indeed, assume the contrary. Then there exists $m \in [n]$ and four monomials

$$q_1 = \cdots x_m^\alpha \cdots, \ q_2 = \cdots x_m^\beta \cdots \in mon(Q); \ r_1 = \cdots x_m^\gamma \cdots, \ r_2 = \cdots x_m^\delta \cdots \in mon(R)$$

such that $\alpha \neq \beta$, $\gamma \neq \delta$. Since

$$r_1 q_1, r_2 q_2, r_1 q_2, r_2 q_1 \in mon(RQ) \subset mon(P)$$

there are $i, j, k, l \in [n]$ for which $\alpha + \gamma = \lambda_i$, $\beta + \delta = \lambda_j$, $\alpha + \delta = \lambda_k$, $\beta + \gamma = \lambda_l$. Hence $\lambda_i + \lambda_j = \lambda_k + \lambda_l$, and we get a contradiction with the Golomb property. $\square$

Thus, we have a separation of variables. We get two polynomials $A := NQ'$ and $B := MR'$ in variable x_i, $i \in S$, and x_j, $j \in [n] \setminus S$, respectively.

At the beginning we consider a case of $S = \emptyset$. This means that Q is a monomial.

Lemma 4.5 *Assume that $mon(QR) \subset mon(P = Tm_\lambda)$ where λ forms a Golomb ruler and Q is a monomial such that $c := \frac{\deg Q}{\deg P} \in [\frac{1}{4}, \frac{3}{4}]$. Then R has exponentially small number of monomials wrt $n!$.*

Proof. Throughout this Section we assume in all the bounds n to be sufficiently big.

Let $Q = x_1^{\nu_1} \cdots x_n^{\nu_n}$. Firstly, we observe that w.l.o.g. one can suppose that for any i there exists j such that $\nu_i = \lambda_j$. Indeed, if at least two ν_{i_1}, ν_{i_2} among $\{\nu_i\}_i$ violate this condition, we can increase ν_{i_1} by 1 and decrease ν_{i_2} also by 1, thereby not decreasing $|mon(R)|$ for which $mon(QR) \subset mon(P)$. Observe that herein $|mon(R)|$ could increase only if $\nu_{i_2} = \lambda_j + 1$ for some j. If just a single $\lambda_j > \nu_i > \lambda_{j+1}$ violates the condition under discussion, we can preserve inequalities $\frac{\deg Q}{\deg P} \in [\frac{1}{4}, \frac{3}{4}]$ as follows: either replace ν_i by λ_j which keeps $|mon(R)|$ or replace by λ_{j+1} which does not decrease $|mon(R)|$.

Let $b_j := \{i : \nu_i = \lambda_j\}$, $j = n, \ldots, 1$. Then the number of monomials in R is equal to

$$t := b_n(b_n + b_{n-1} - 1) \cdots (b_n + \ldots + b_1 - (n-1)).$$

We have

$$\sum b_i \lambda_i = c \sum \lambda_i.$$

Then, we have

$$\sum_i \lambda_i - \sum b_i \lambda_i + \lambda_1 - \lambda_n = \sum_{j=0}^{n-2} (b_n + \ldots + b_{n-j} - j)(\lambda_{n-j-1} - \lambda_{n-j}).$$

Thus

$$t\prod(\lambda_{j-1}-\lambda_j) \le \Big(\frac{(1-c)\sum\lambda_i+\lambda_1-\lambda_n}{n}\Big)^n.$$

We have $\sum\lambda_i \sim \frac{5n^3}{6}$, $\prod(\lambda_{j-1}-\lambda_j) \sim 2^n\frac{(3/2n)!}{(1/2n)!} \sim (\frac{3^{3/2}}{e}n)^n$.

Therefore it holds (taking into account that due to the choice of λ_i, the degree of P is $5/6n^3+O(n^2)$) that

$$t \le (\frac{5e(1-c)n}{3^{3/2}6})^n. \tag{3}$$

Thus, for $1-c < \frac{6\cdot3^{3/2}}{5e^2} < \frac{31.14}{38.64}$, the number of monomials in R is exponentially small wrt $n!$. For $c\in[1/4,3/4]$, this is the case. □

Now we complete the proof of Theorem 4.1. Let $mon(QR)\subset mon(P)$.

Recall that the polytope $Per_n := ch(w(\lambda), w\in S_n)$ is a base-polytope (see Section 3) which is set by a submodular function $b_\lambda(T)=\sum_{i=1,\dots|T|}\lambda_{n-i+1}$, $T\subset[n]$. Thus, a pair of parallel facets (we agree that a facet is a face of codimension 1) labeled by a subset $W\subset[n]$, $|W|=k$, are defined by $x(W)=b_\lambda(W)=\sum_{i=1}^k\lambda_{n-i+1}$ and $x([n]\setminus W)=b_\lambda([n]\setminus W)=\sum_{i=1}^{n-k}\lambda_{n-i+1}$, respectively, and any cut with the same separation of coordinates is defined by $x(W)=a$, $a\in[\sum_{i=1}^k\lambda_{n-i+1},\sum_{j=1}^k\lambda_j]$ (see Proposition 3.2 and Lemma 3.3). Because of symmetry of b_λ wrt permutations of coordinates, facets of Per_n are labeled by numbers in $[n]$ (cardinality of set labeling facet). The number of the vertices of a facet labeled by $k\in[n]$ (recall that k corresponds to separation of variables in groups of $k=|S|$ and $n-k$ variables taking into the account Lemma 4.4) is

$$k!(n-k)!.$$

Because of this, the cardinality of monomials of the product $A\cdot B=Q\cdot R$ is bounded by $k(A)!(n-k(A))!$, where $k(A)=k$. Note that $\deg(A)=\lambda_{i_1}+\dots+\lambda_{i_k}$ for suitable $1\le i_1<\dots<i_k\le n$ satisfies

$$\deg(A)\in[\sum_{i=1}^k\lambda_{n-i+1},\sum_{j=1}^k\lambda_j].$$

There are two cases.

Case 1. $\deg A, \deg B \ge c'\cdot\deg P$, for some sufficiently small constant c' which we choose later. In such a case, $k=k(A)$, $n-k=k(B)\ge c''\cdot n$ for some sufficiently small constant c'' depending on c' (since $\deg P$ is cubic in n). This implies that $A\cdot B$ has at most $k!(n-k)!$ number of monomials, so exponentially small wrt $n!$ and we are done.

Case 2. Either $\deg A < c'\deg P$ or $\deg B < c'\deg P$. Let for definiteness $\deg A < c'\deg P$. Then, the degree of the monomial M satisfies $\frac{\deg M}{\deg B}\in[\frac14,\frac34]$ since c' is sufficiently small.

Then, the same reasoning as above in the proof of Lemma 4.5, provides a bound $|mon(R')|\le(c_0(n-k))^{n-k}$ for any fixed $c_0>\frac{5e(1-c)}{3^{3/2}6}$ (see (3)) due to an appropriate choice of sufficiently small c' in Case 1. We take $c_0<1/e$. Because of this and that A has at most $k!$ monomials we get that

$$|mon(RQ)|=|mon(AB)|\le k!(c_0(n-k))^{n-k}<n!/c_1^n$$

for some $c_1>1$. This finishes the proof of Theorem 4.1 making use of Lemma 4.2. □

5 Tropical skew Schur polynomials

Skew Schur functions form an important class of symmetric functions and is of importance in many areas, especially in physics, see, for example [14]).

Recall that, for a skew Young diagram $\lambda \setminus \mu$ (where $\mu \leq \lambda$, which denotes the coordinate-wise inequality of the partitions), a semi-standard Young tableaux (SSYT) of a shape $\lambda \setminus \mu$ (in the alphabet $[n]$) is a filling of the Young diagram $\lambda \setminus \mu$ with entries from $[n]$ strictly increasing along the columns and non-decreasing along the rows ([16]). We accept the French style to draw Young diagram. Here is an example of a skew SSYT of shape $(5,3,3,1) \setminus (2,1)$

$$\begin{array}{ccccc} 3 & & & & \\ 2 & 2 & 4 & & \\ & 1 & 2 & & \\ & & 1 & 1 & 2 \end{array}$$

The weight of such a tableau T is the tuple $wt(T) := (\#1(T), \#2(T), \dots, \#n(T))$, where $\#i(T)$ denotes the number of times integer i occurs in T. The skew Schur polynomial $s_{\lambda\setminus\mu}$ is defined by (see [16])

$$s_{\lambda\setminus\mu}(x) = \sum_T x^{wt(T)},$$

where the sum runs over the set of all skew semistandard Young tableaux of shape $\lambda \setminus \mu$.

A skew Schur function $s_{\lambda\setminus\mu}(x)$ is symmetric function and its decomposition in the basis of Schur functions involve famous Littlewood-Richardson coefficients:

$$s_{\lambda\setminus\mu}(x) = \sum_\nu c^\lambda_{\mu,\nu} s_\nu(x). \tag{4}$$

The Littlewood-Richardson coefficients $c^\lambda_{\mu,\nu}$ are the structure constants in the algebra of symmetric functions with respect to the basis of Schur functions

$$s_\mu(x)s_\nu(x) = \sum_\lambda c^\lambda_{\mu,\nu} s_\lambda(x).$$

Another instance of LR-coefficients is as the multiplicities of irreducible representationsin the induced tensor product of representations of the symmetric group. The fourth occurrence is as intersection numbers in the Schubert calculus on a Grassmanian. (For details on LR-coefficients see [10]).

The computation of LR-coefficients is $\#P$-complete [17], while verification $c^\lambda_{\mu,\nu} \neq 0$ is polynomial (the problem $c^\lambda_{\mu,\nu} \neq 0$ is equivalent to existence a solution in a linear system with $3n^2$ equations, see for example [15]).

The tropical Schur polynomial $Ts_{\lambda\setminus\mu}(x)$ is a piece-wise linear function defined by the tropicalization of the above formula in the tropical semi-ring, that is

$$Ts_{\lambda\setminus\mu}(x) = \max_T(x, wt(T)).$$

where max is taken over all SSYT T of shape $\lambda \setminus \mu$. For $\mu = 0$, we obtain a usual tropical Schur polynomial (cf. Section 1).

Thus, $Ts_{\lambda\setminus\mu}(x)$ is a piece-wise linear function of the form of the maximum of a linear function $(x, \cdot)$ over the set of points $\nu := wt(T)$, while T runs over the set of all skew semistandard Young tableaux of shape $\lambda \setminus \mu$.

This set of weights constitute the set of integer points of the polytope $\mathcal{GC}(\lambda, \mu)$ defined by the inequalities

$$\lambda([1, |I|]) - \Delta_{|I|} \geq \nu(I), \quad \lambda([n]) - \Delta_n = \nu([n]),$$

where $\lambda([1, |I|]) = \lambda_1 + \cdots \lambda_{|I|}$, $\nu(I) = \sum_{i \in I} \nu_i$, $\Delta_{|I|} = \Delta_1 + \ldots \Delta_{|I|}$, $\Delta_k := \max\{0, \mu_1 - \lambda_{k+1}\} + \max\{0, \mu_2 - \lambda_{k+2}\} + \cdots + \max\{0, \mu_{n-k} - \lambda_n\}$ (for details see [3]).

For given λ and μ we get a function $\Lambda : 2^{[n]} \to \mathbb{R}$, $\Lambda(I) = \lambda([1, |I|]) - \Delta_{|I|}$, $I \subseteq [n]$.

The properties of this function depend on shape $\lambda \setminus \mu$. For example, for $\mu = 0$, this function is submodular (see Section 3).

Proposition 5.1 *Let λ and μ be such that the function Λ is submodular. That is, for any $|I|$, it holds*

$$\lambda([1, |I|]) - \Delta_{|I|} - \lambda([1, |I|+1]) - \Delta_{|I|+1} \geq \lambda([1, |I|+1]) - \Delta_{|I|+1} - \lambda([1, |I|+2]) - \Delta_{|I|+2}.$$

Then the complexity of computation of $Ts_{\lambda\setminus\mu}(x)$ is polynomial in n.

Proof. In such a case, the polytope $\mathcal{GC}(\lambda, \mu)$ is a base-polytope. Hence the complexity of computation of $Ts_{\lambda\setminus\mu}(x)$ as a tropical function is polynomial in n due to the greedy algorithm (see Theorem 3.1). □

For λ and μ, for which Λ fails to be submodular, the problem of finding maximum can be hard, since some of the vertices of $\mathcal{GC}(\lambda, \mu)$ do not even corresponds to the weights of skew semi-standard Young tableaux.

Our conjecture is that the complexity of computation of the tropical polynomial $Ts_{\lambda\setminus\mu}(x)$ is exponential as well over the semi-ring $(\mathbb{R}, +, \max)$.

Modulo this conjecture, we speculate that difference of computational complexities between the Schur functions and skew Schur functions reflects in 'complexities' of domains of summations in (1) and (4).

Let us note, over the semi-field $(\mathbb{R}, \max, +, -)$, the complexity of the tropical skew Schur polynomial $Ts_{\lambda\setminus\mu}(x)$ is polynomial independently of λ and μ. This follows from the tropicalization of the subtraction-free algorithm in [8] which computes skew Schur polynomials.

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