Total space in Resolution is at least width squared

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- natural complexity measure analogue to *space* in Turing Machines, introduced in [ET '01] and [ABRW '02]

Why space?

- natural complexity measure analogue to *space* in Turing Machines, introduced in [ET '01] and [ABRW '02]
- (lower bounds for space usage of SAT-solvers)

Given an unsatisfiable CNF formula φ, find a proof of its unsatisfiability, *i.e.* a derivation of $\bot$, using the inference rule:

$$\frac{A \vee x, \;\; B \vee \neg x}{A \vee B} \qquad A,B \text{ clauses, } x \text{ variable}$$

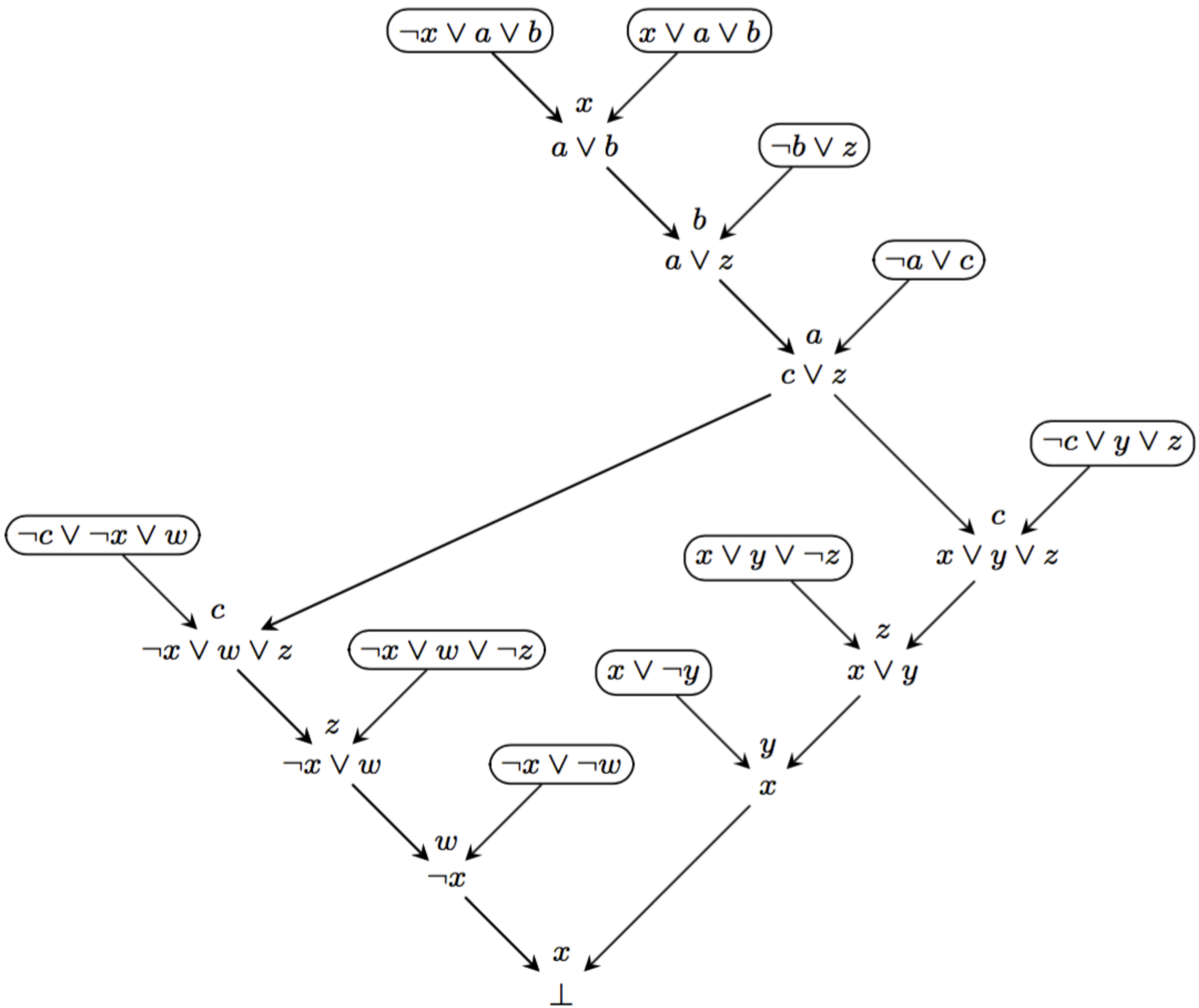
Given an unsatisfiable CNF formula φ, find a proof of its unsatisfiability, *i.e.* a derivation of $\bot$, using the inference rule:

$$\frac{A \vee x, \;\; B \vee \neg x}{A \vee B} \qquad A, B \text{ clauses, } x \text{ variable}$$

e.g. what is a Resolution proof of:

$$(\neg x \vee a \vee b) \wedge (x \vee a \vee b) \wedge (\neg b \vee z) \wedge (\neg a \vee c) \wedge (\neg c \vee y \vee z) \wedge$$
$$\wedge (x \vee y \vee \neg z) \wedge (x \vee \neg y) \wedge (\neg x \vee \neg w) \wedge (\neg x \vee w \vee \neg z) \wedge (\neg c \vee x \vee w)?$$

[Huang, Yu '87]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$x \vee a \vee b$
x
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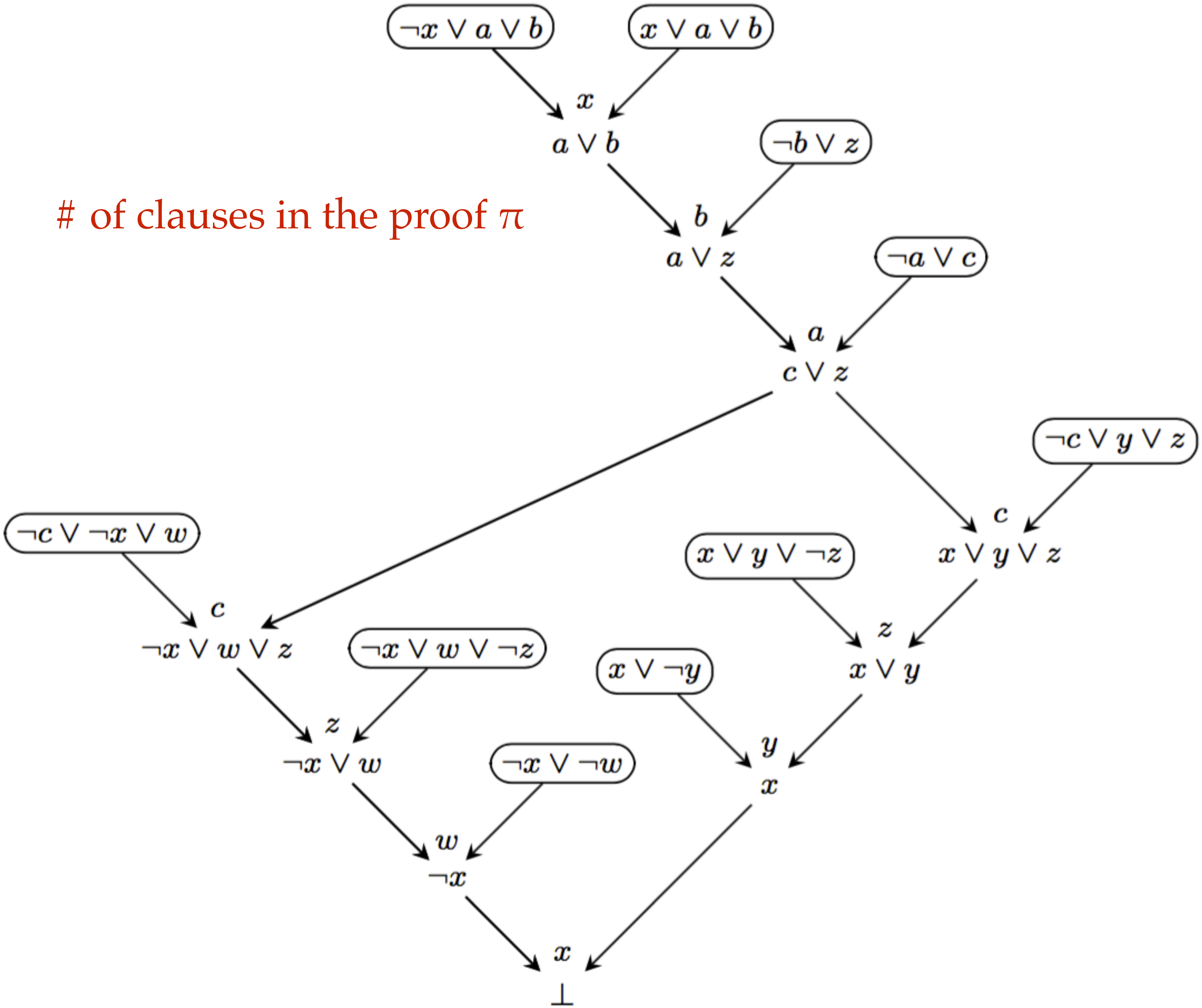
$\neg b \vee z$
b
$a \vee z$
$\neg a \vee c$
a
$c \vee z$
$\neg c \vee y \vee z$
c
$x \vee y \vee z$
$\neg c \vee \neg x \vee w$
c
$\neg x \vee w \vee z$
$x \vee y \vee \neg z$
z
$x \vee y$
$\neg x \vee w \vee \neg z$
z
$\neg x \vee w$
$x \vee \neg y$
y
x
$\neg x \vee \neg w$
w
$\neg x$
x
$\bot$

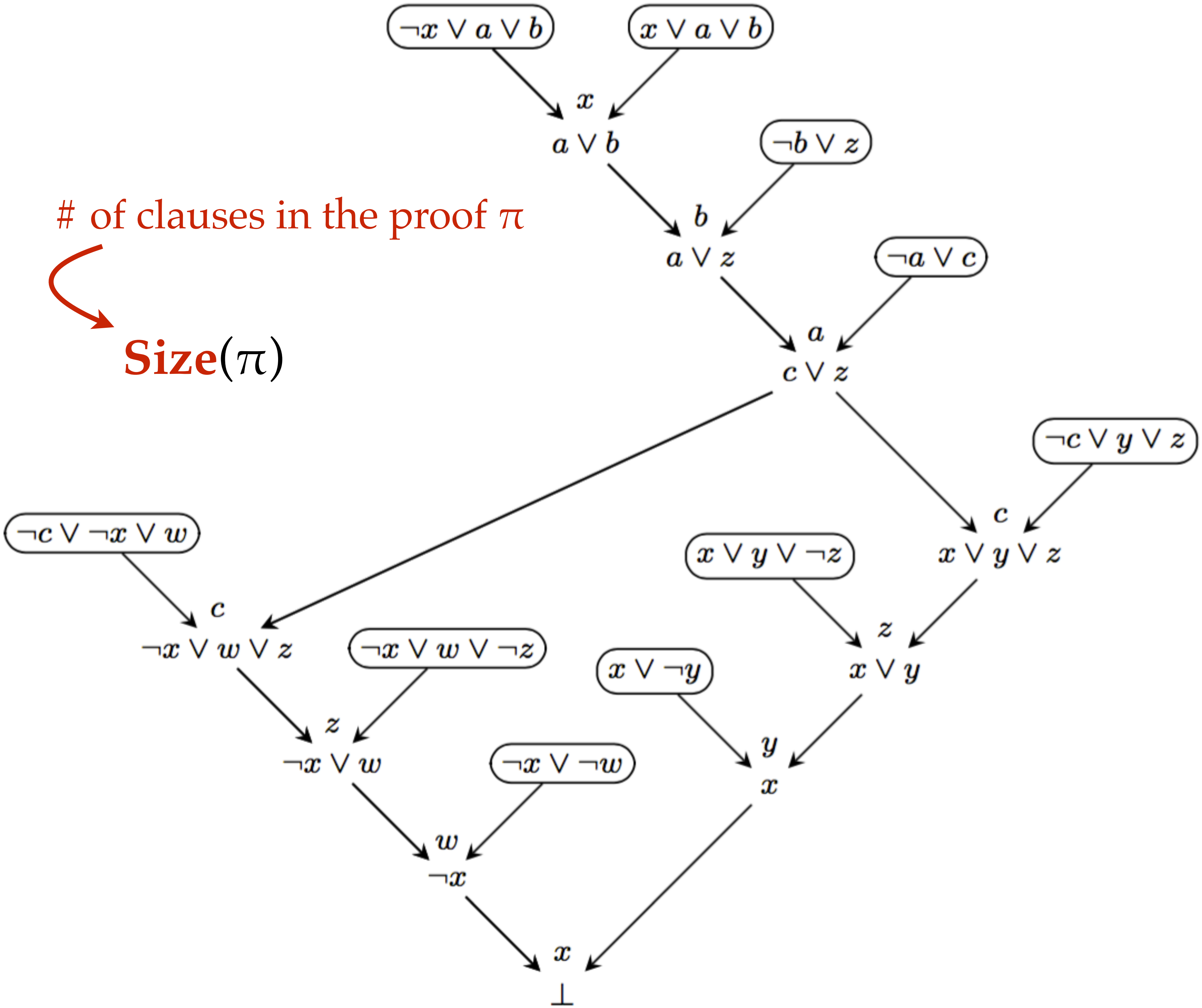
# of clauses in the proof π



Size

of clauses in the proof π

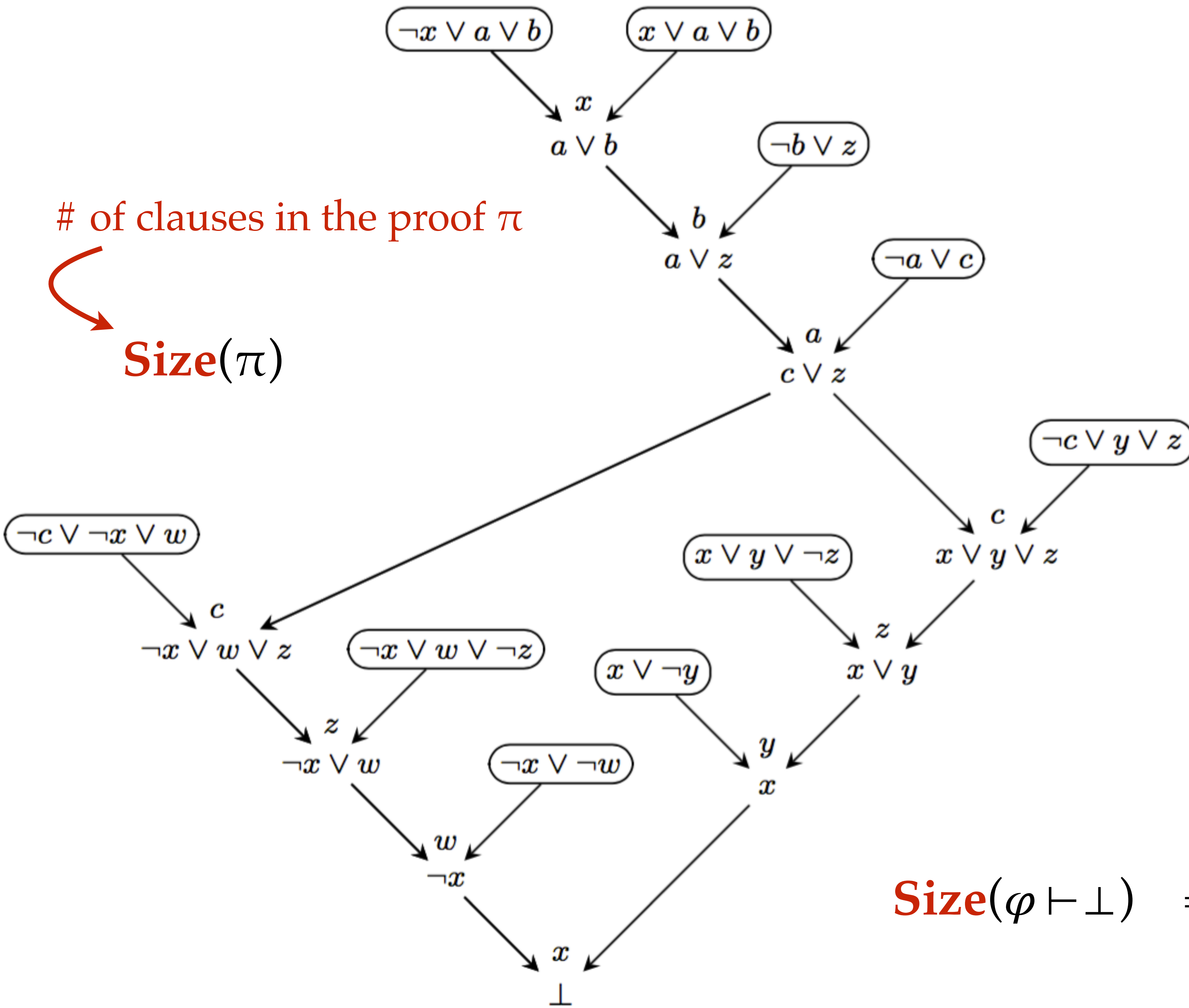
Size(π)



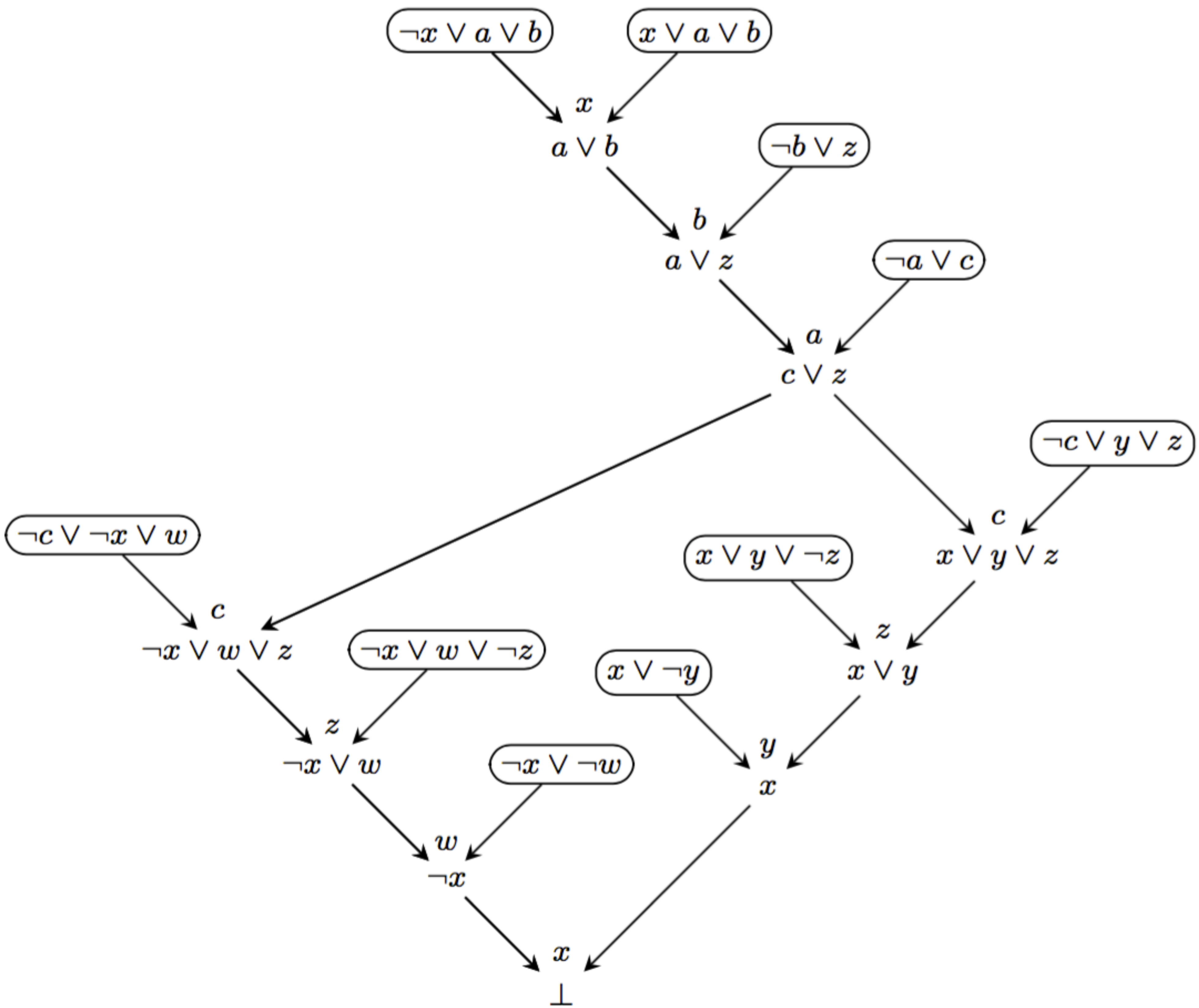
Size

of clauses in the proof π

Size(π)



Size$(\varphi \vdash \bot) \quad = \min_\pi$ **Size**(π)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x
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$\neg b \vee z$
b
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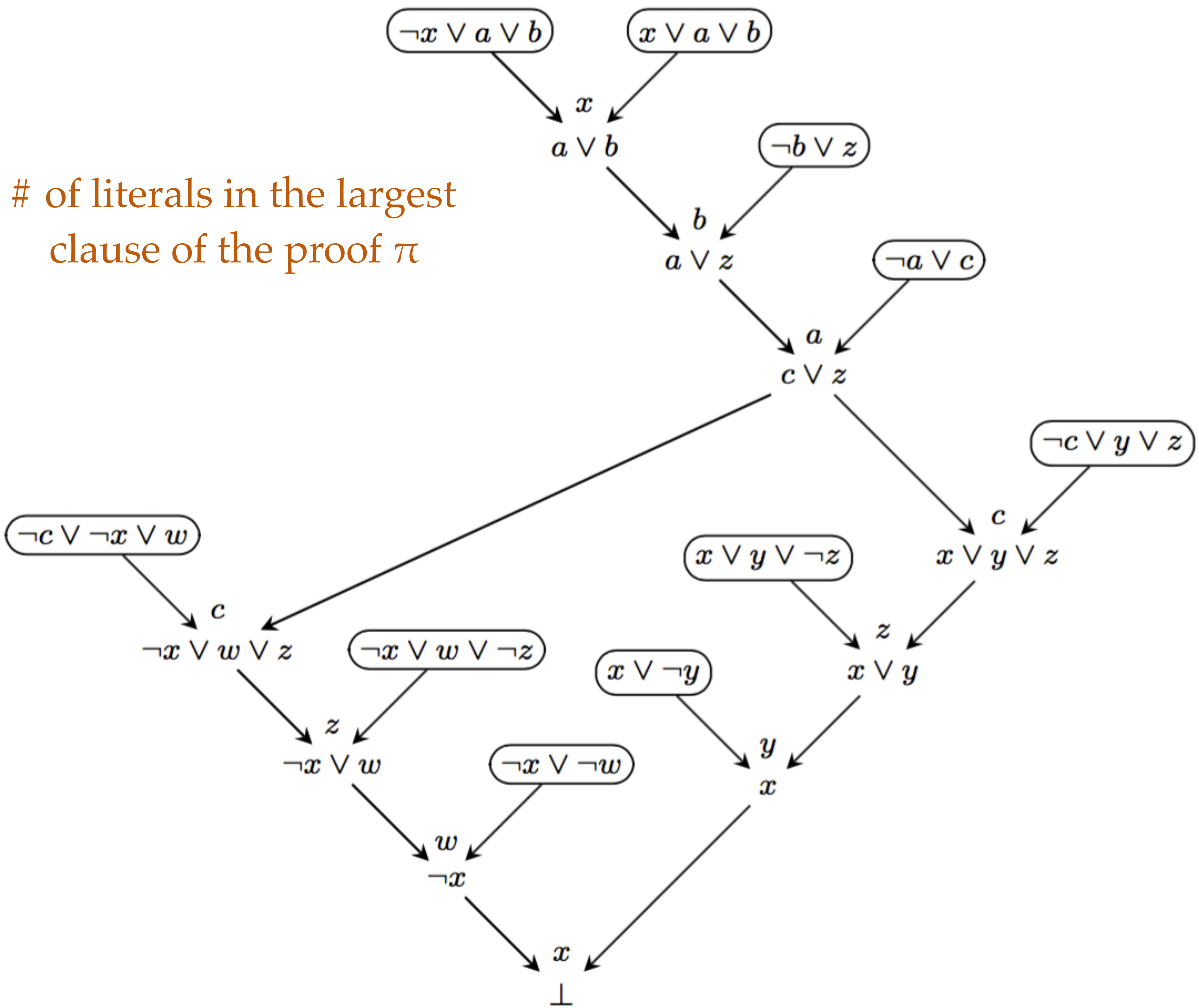
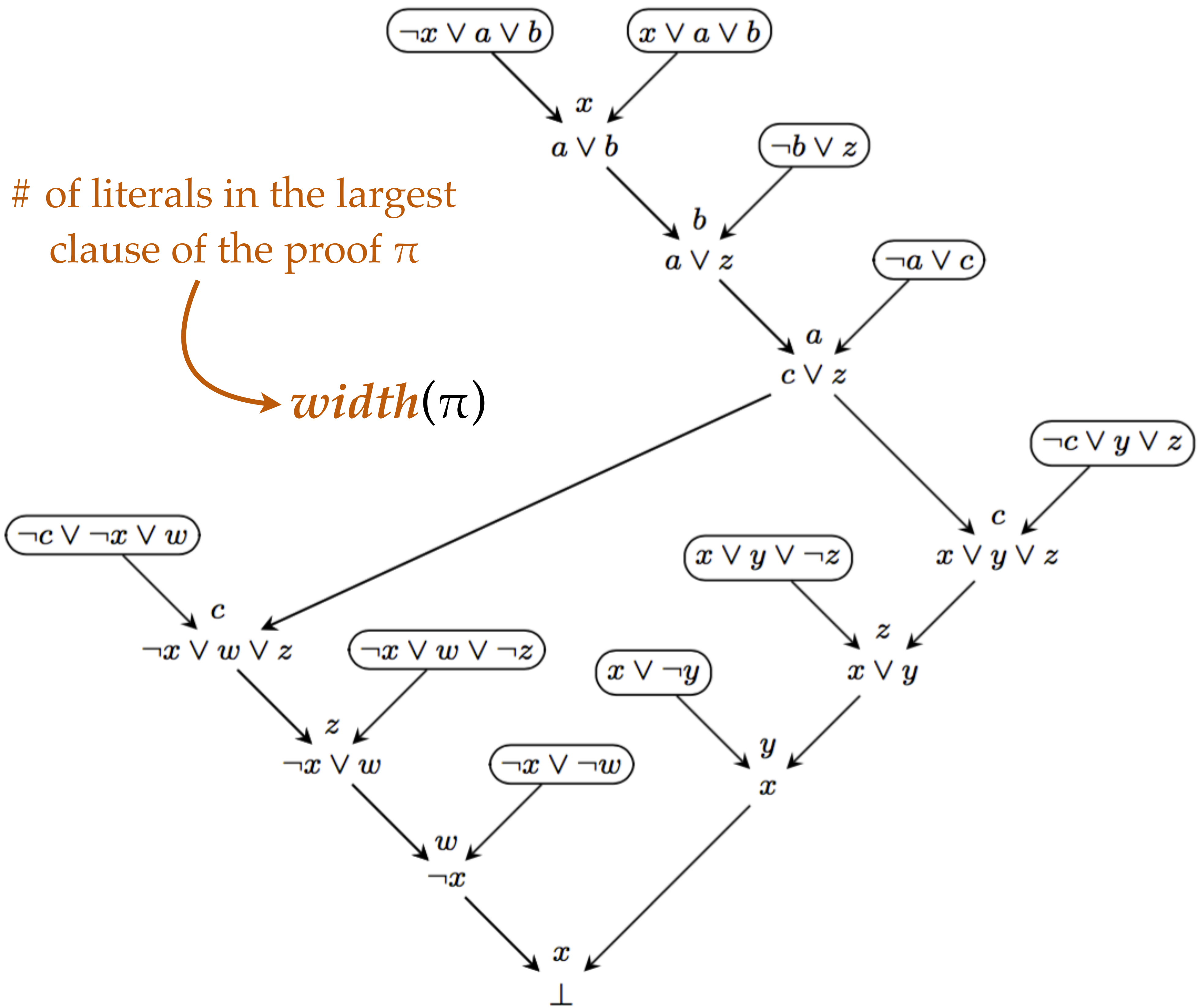
$\neg a \vee c$
a
$c \vee z$
$\neg c \vee y \vee z$
$\neg c \vee \neg x \vee w$
c
$x \vee y \vee \neg z$
$x \vee y \vee z$
c
$\neg x \vee w \vee z$
$\neg x \vee w \vee \neg z$
z
$x \vee \neg y$
$x \vee y$
z
$\neg x \vee w$
$\neg x \vee \neg w$
y
x
w
$\neg x$
x
$\bot$

# of literals in the largest clause of the proof π



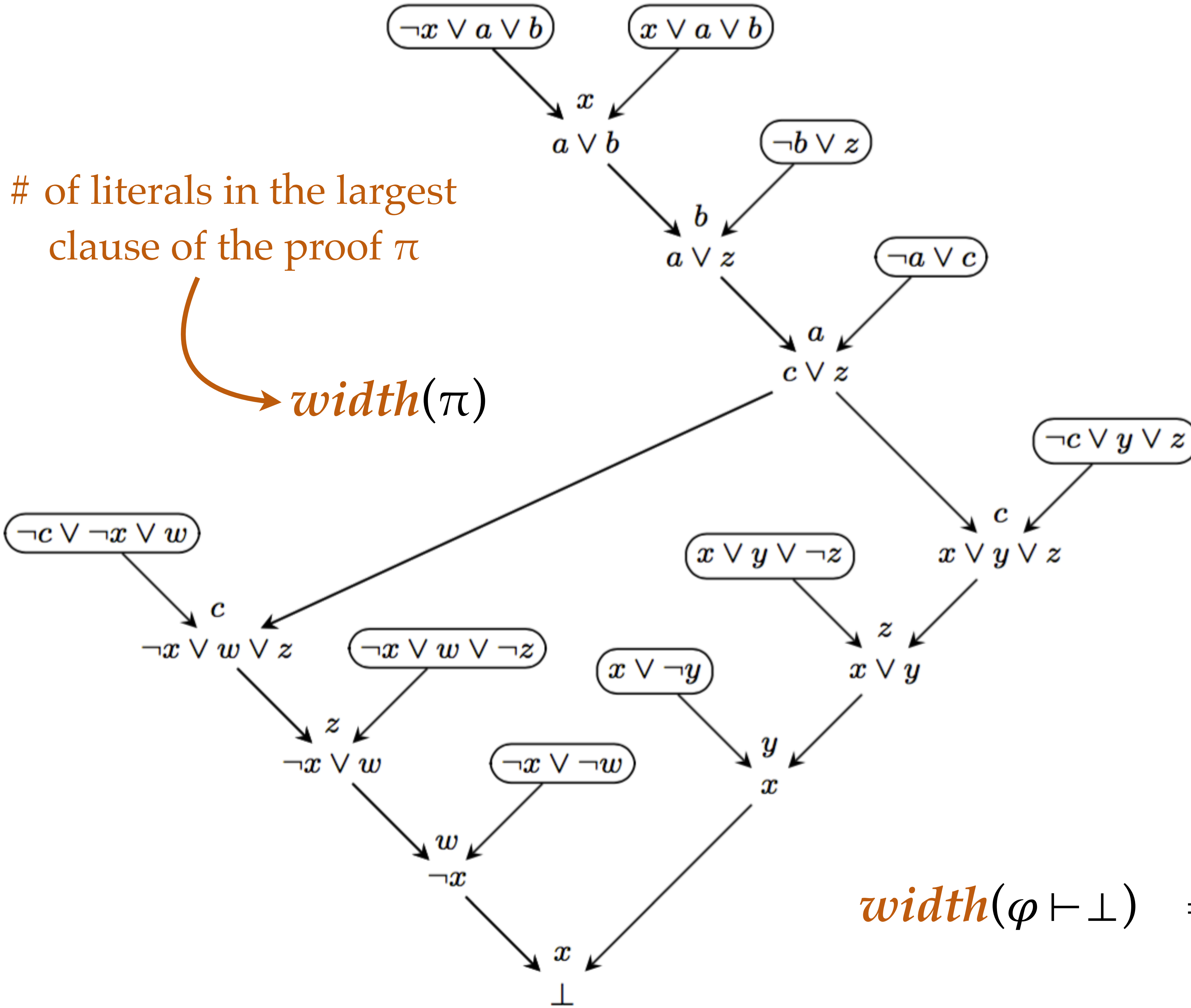
# of literals in the largest clause of the proof π

width(π)

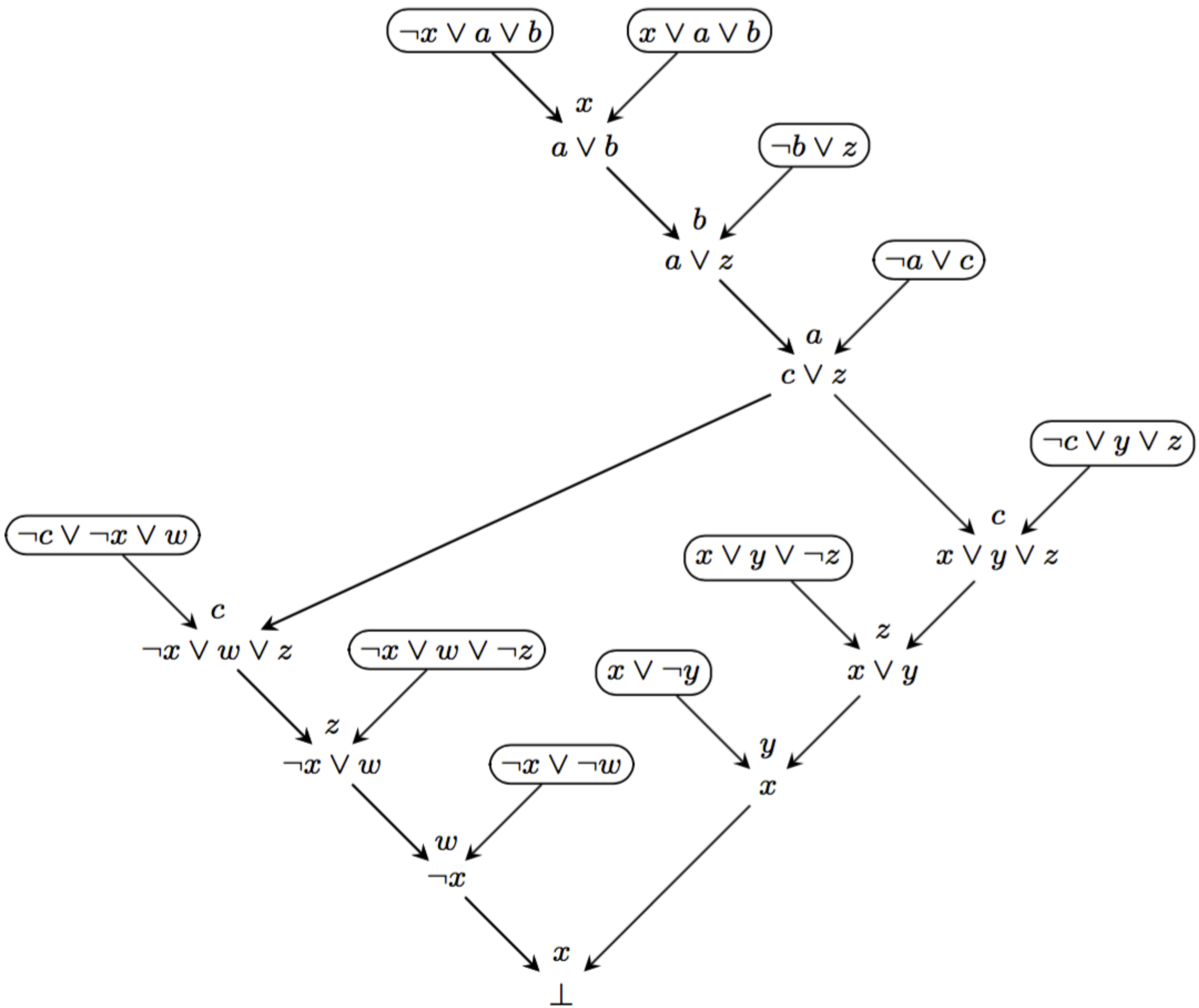


# of literals in the largest clause of the proof π

$\textit{width}(\pi)$



$$\textit{width}(\varphi \vdash \bot) \quad = \min_\pi \textit{width}(\pi)$$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x ∨ a ∨ b
x
a ∨ b
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b
a ∨ z
¬a ∨ c
a
c ∨ z
¬c ∨ y ∨ z
¬c ∨ ¬x ∨ w
c
x ∨ y ∨ ¬z
c
x ∨ y ∨ z
c
¬x ∨ w ∨ z
¬x ∨ w ∨ ¬z
z
x ∨ ¬y
x ∨ y
z
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¬x ∨ ¬w
y
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¬x
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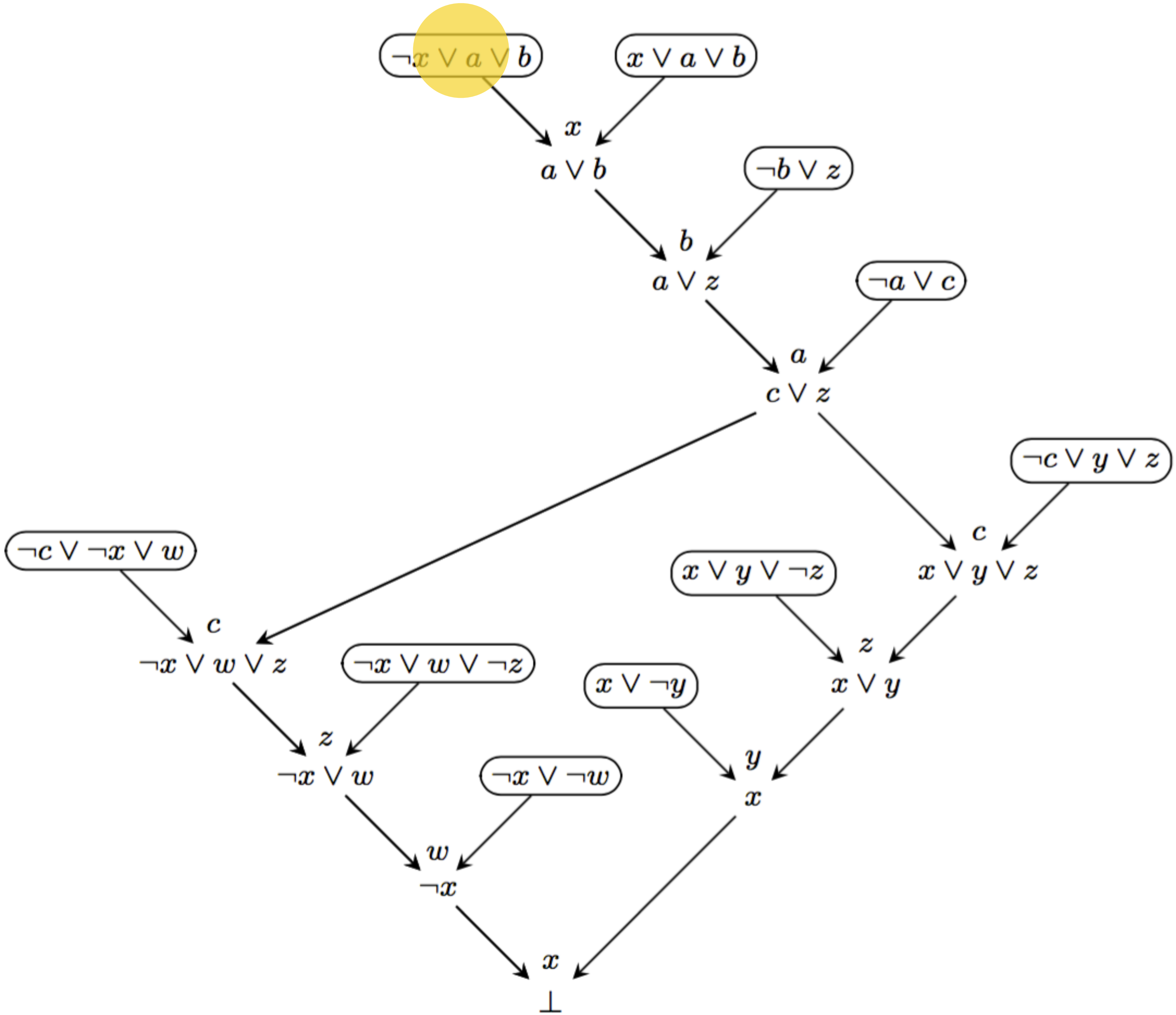
⊥



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$\neg b \lor z$
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a
$c \lor z$
$\neg c \lor y \lor z$
$\neg c \lor \neg x \lor w$
c
$x \lor y \lor \neg z$
$x \lor y \lor z$
c
$\neg x \lor w \lor z$
$\neg x \lor w \lor \neg z$
z
$x \lor \neg y$
$x \lor y$
z
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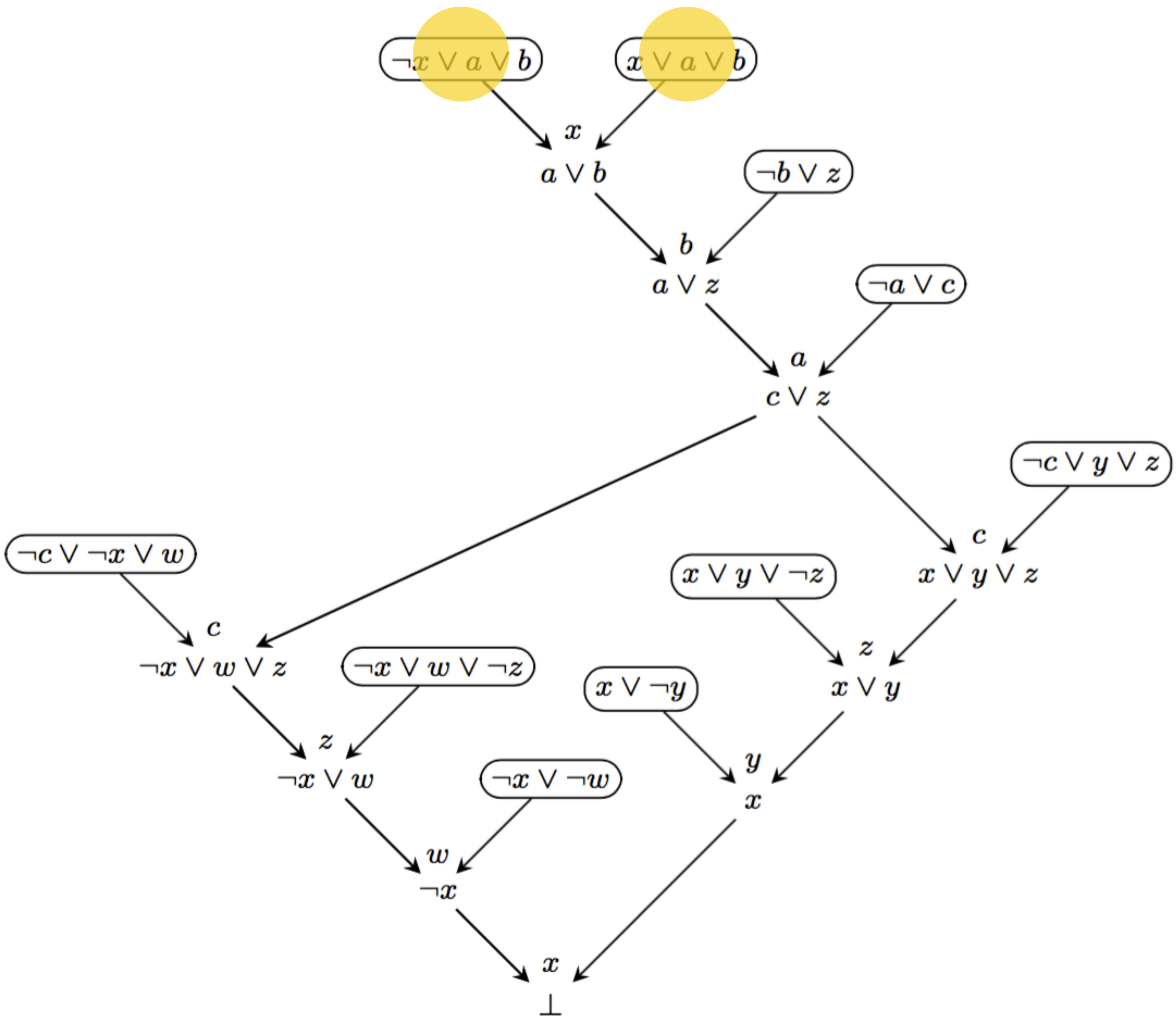
$\neg x \lor \neg w$
y
x
w
$\neg x$
x
$\bot$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c
x ∨ y ∨ ¬z
c
x ∨ y ∨ z
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¬x ∨ w ∨ ¬z
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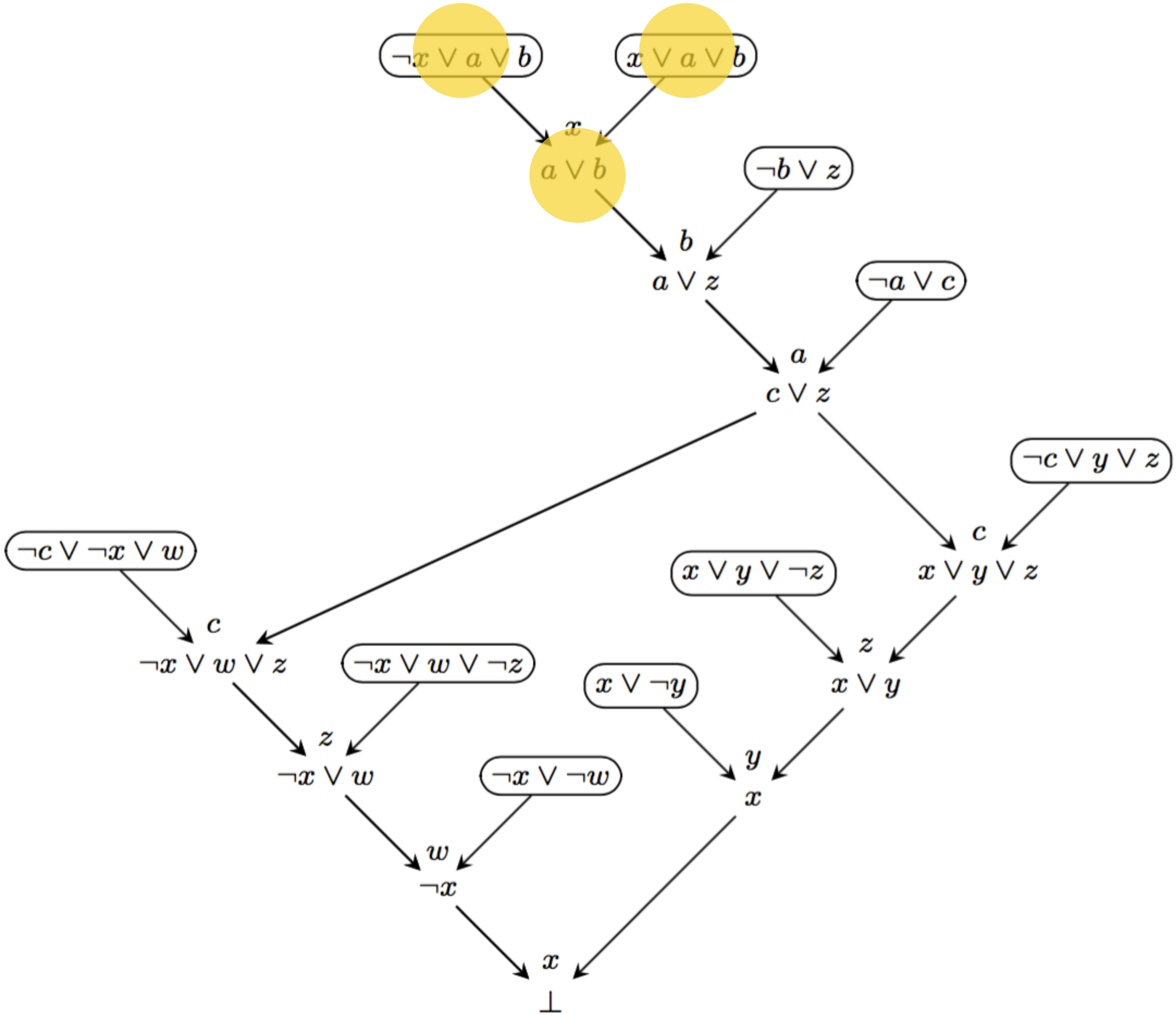
⊥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x ∨ y ∨ z
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z
x ∨ y
x ∨ ¬y
y
x
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¬x ∨ w
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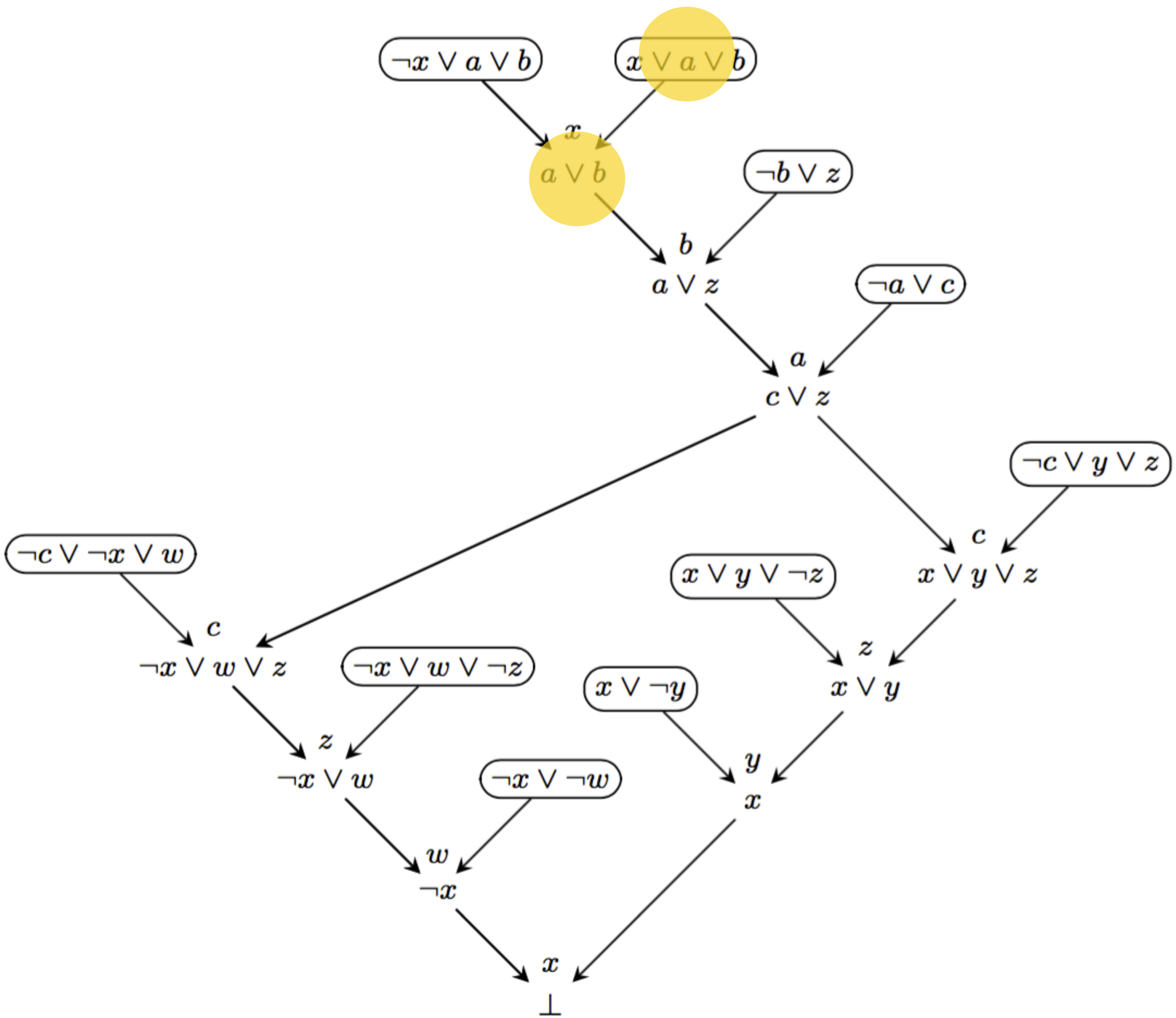
w
¬x
x
⊥



$\neg x \vee a \vee b$
$x \vee a \vee b$
x
$a \vee b$
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a
$c \vee z$
$\neg c \vee y \vee z$
c
$x \vee y \vee z$
$x \vee y \vee \neg z$
$\neg c \vee \neg x \vee w$
c
$\neg x \vee w \vee z$
$\neg x \vee w \vee \neg z$
z
$x \vee y$
$x \vee \neg y$
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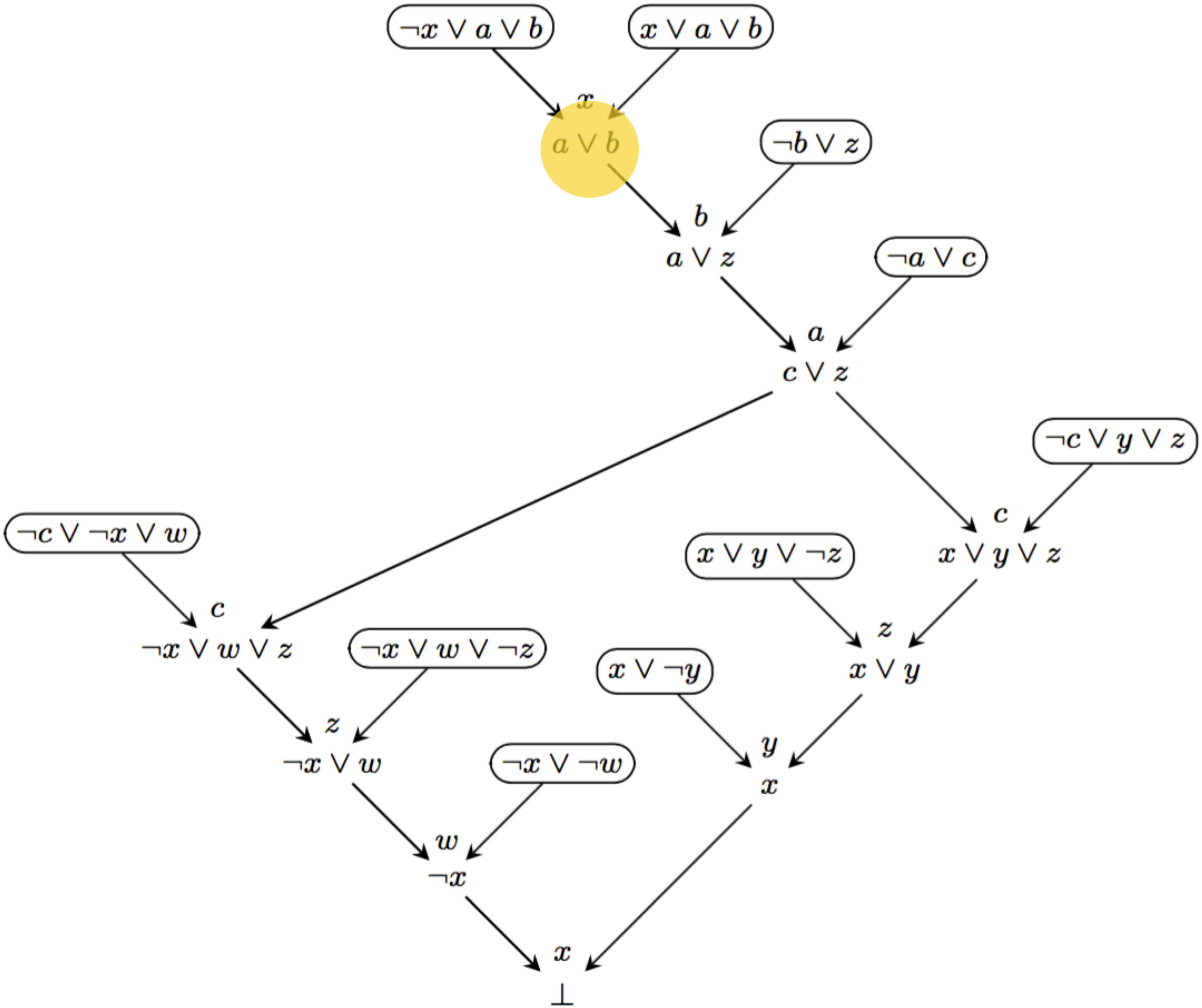
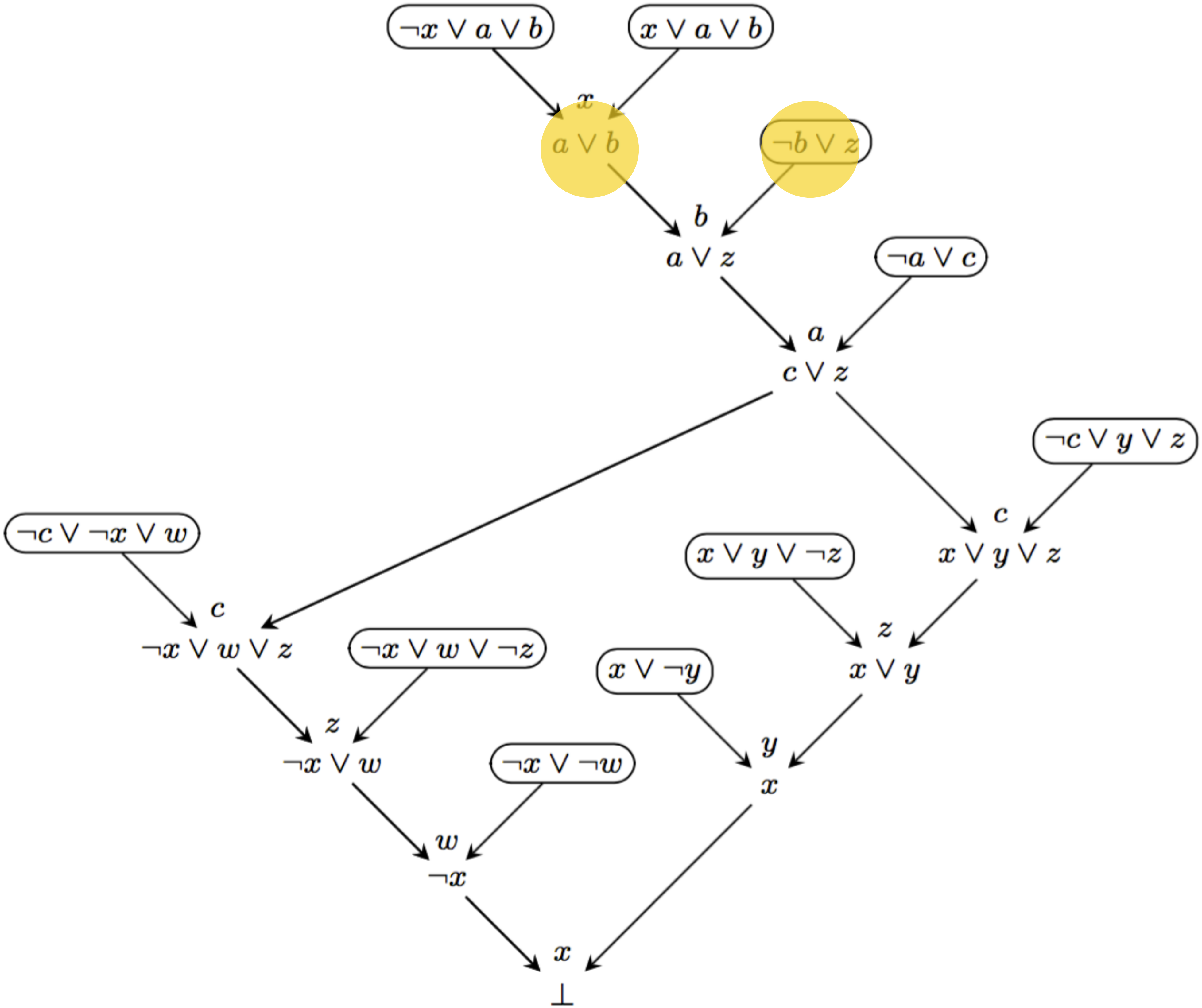
$\neg x \vee w$
$\neg x \vee \neg w$
y
x
w
$\neg x$
x
$\bot$

Clause Space

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$\neg x \vee w \vee z$
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z
$x \vee \neg y$
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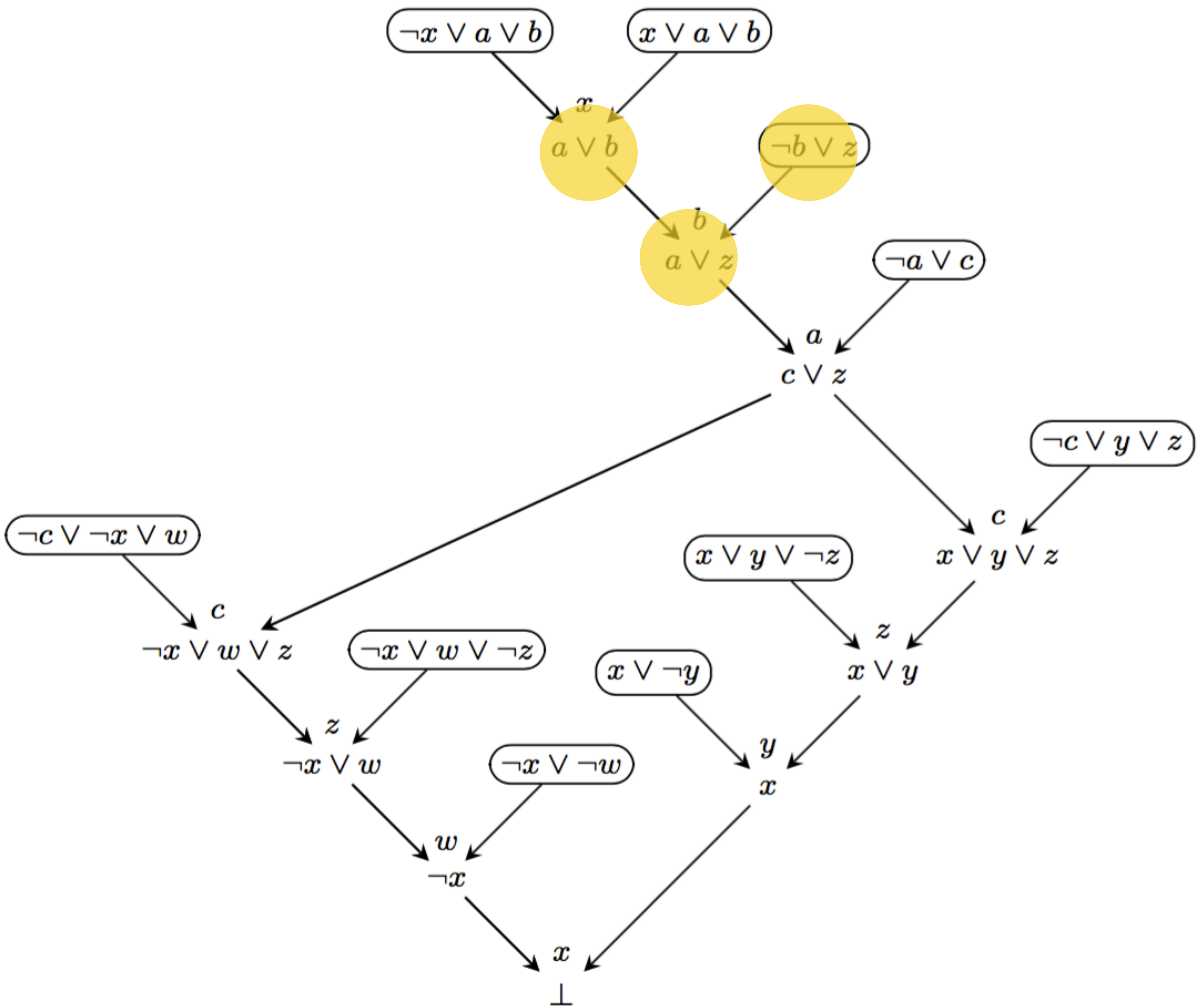
$\neg x$
x
$\bot$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$\neg x \vee w \vee z$
$\neg x \vee w \vee \neg z$
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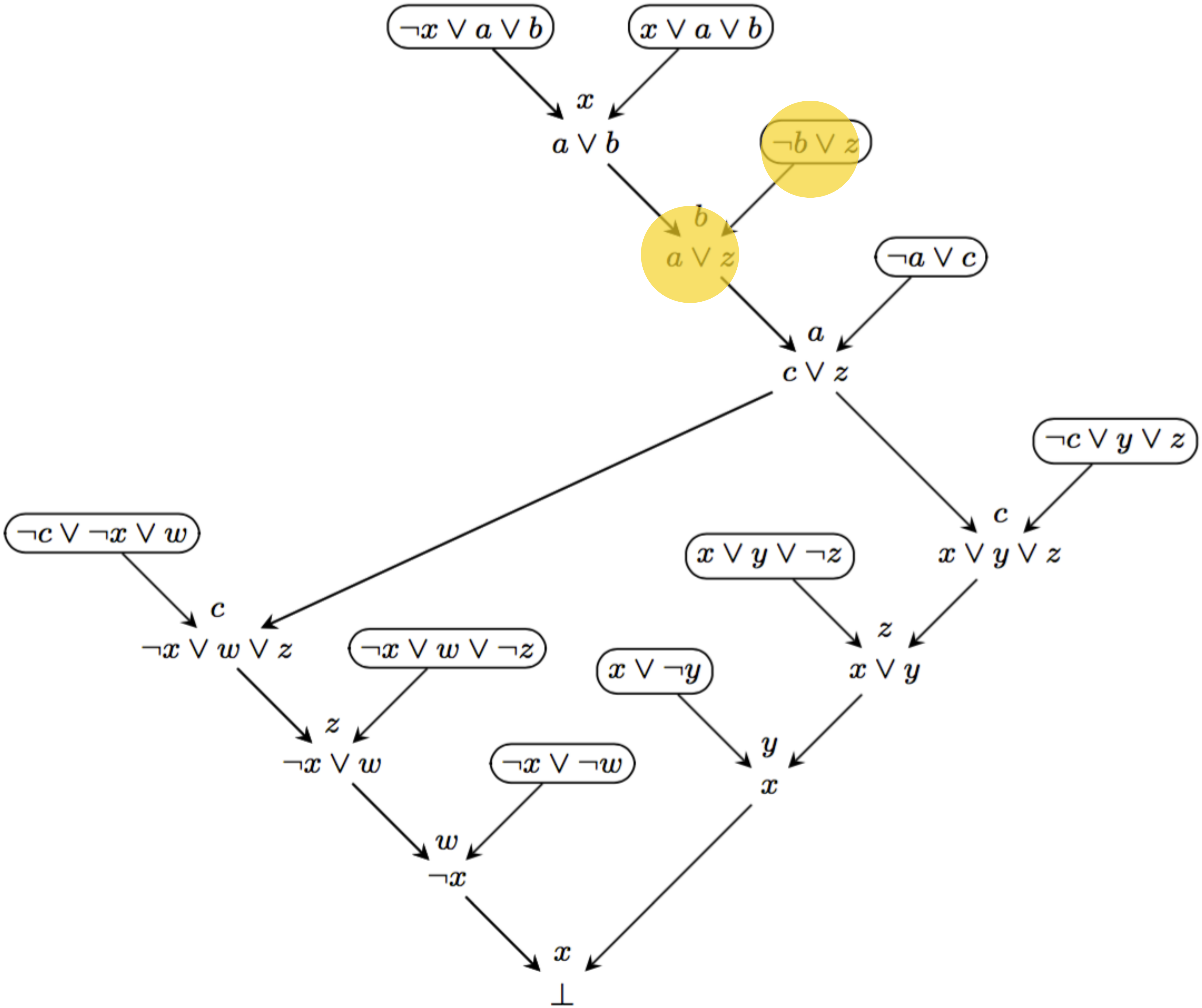
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$\neg x \vee w \vee z$
$\neg x \vee w \vee \neg z$
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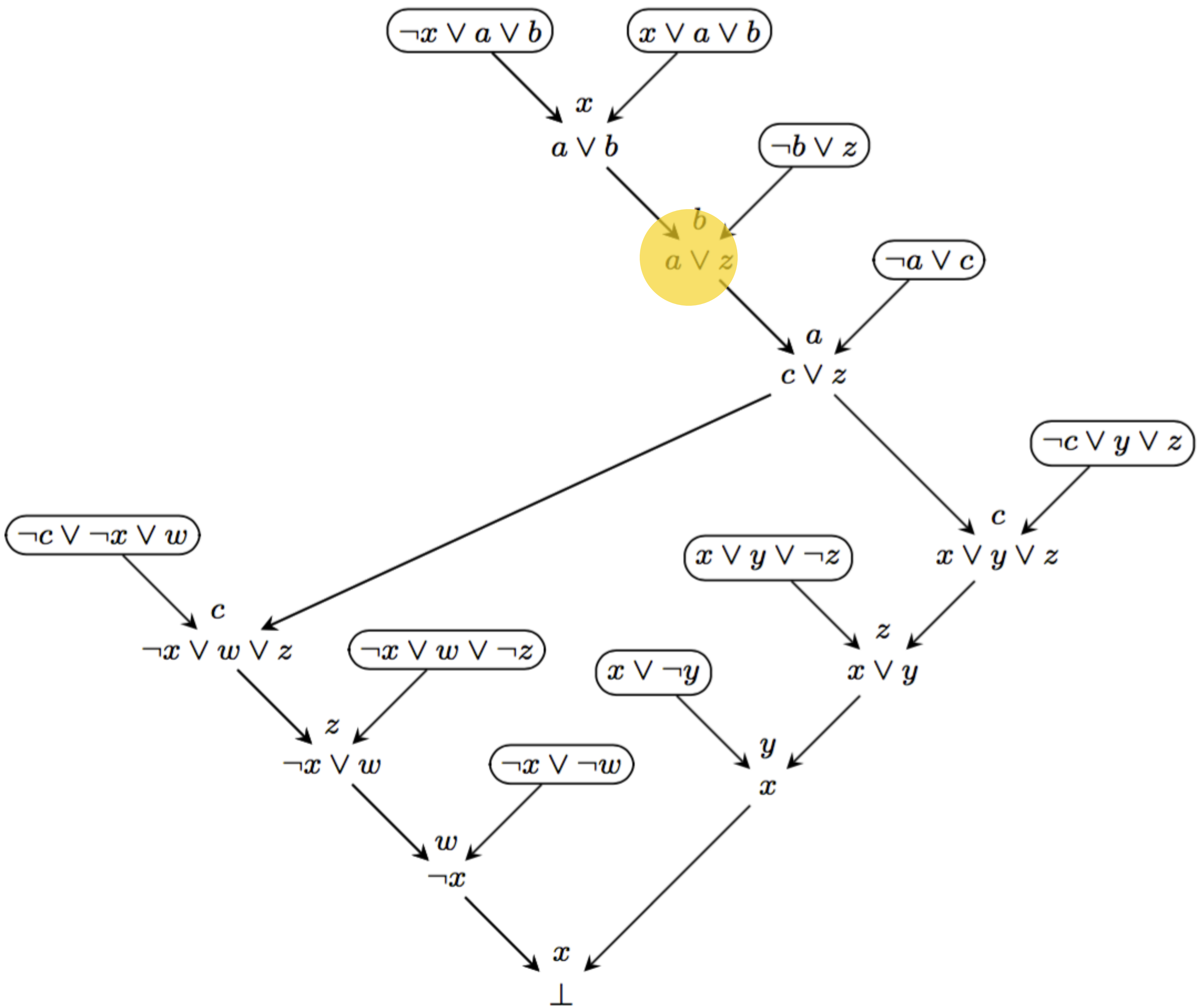
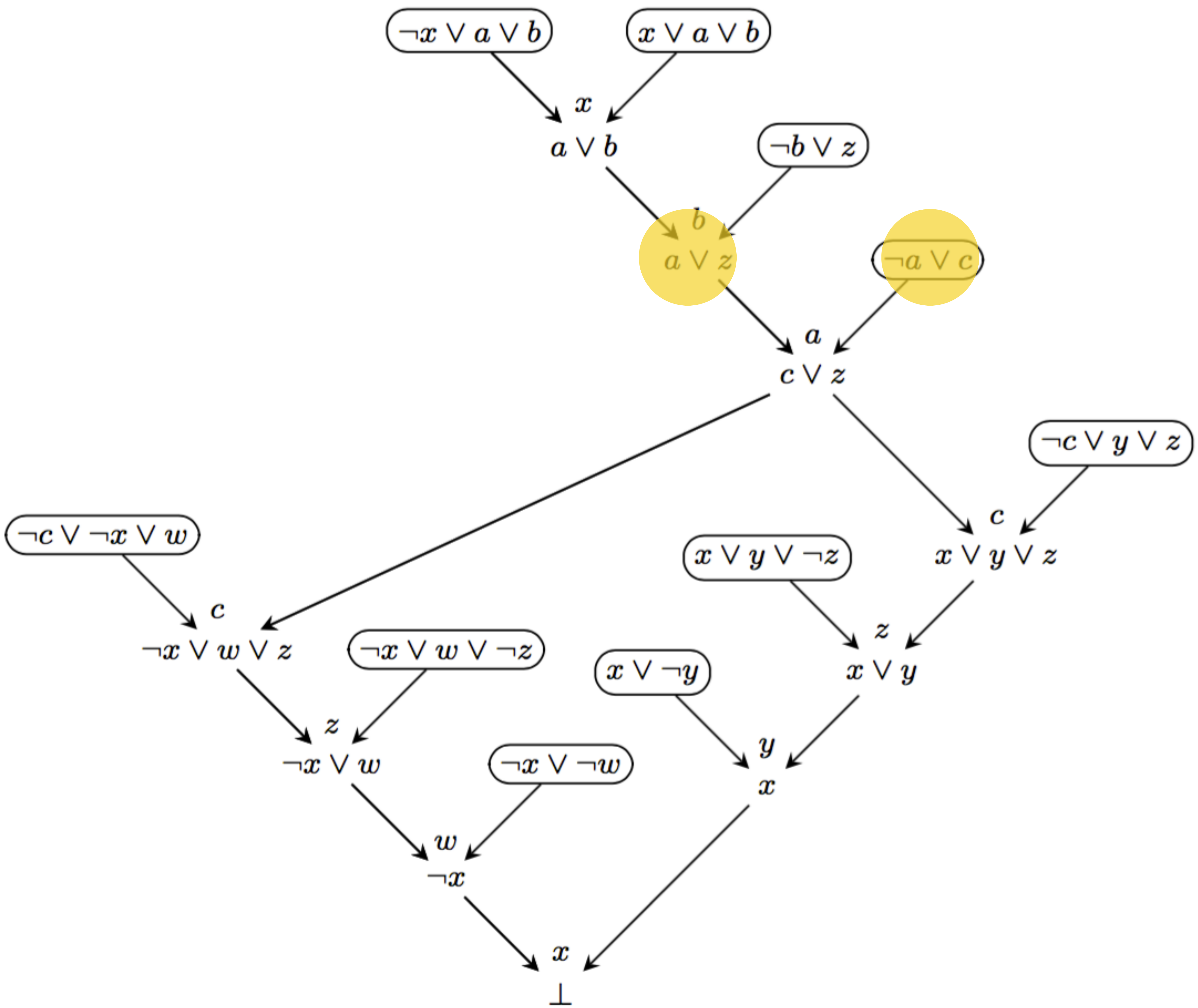
$\bot$

Clause Space

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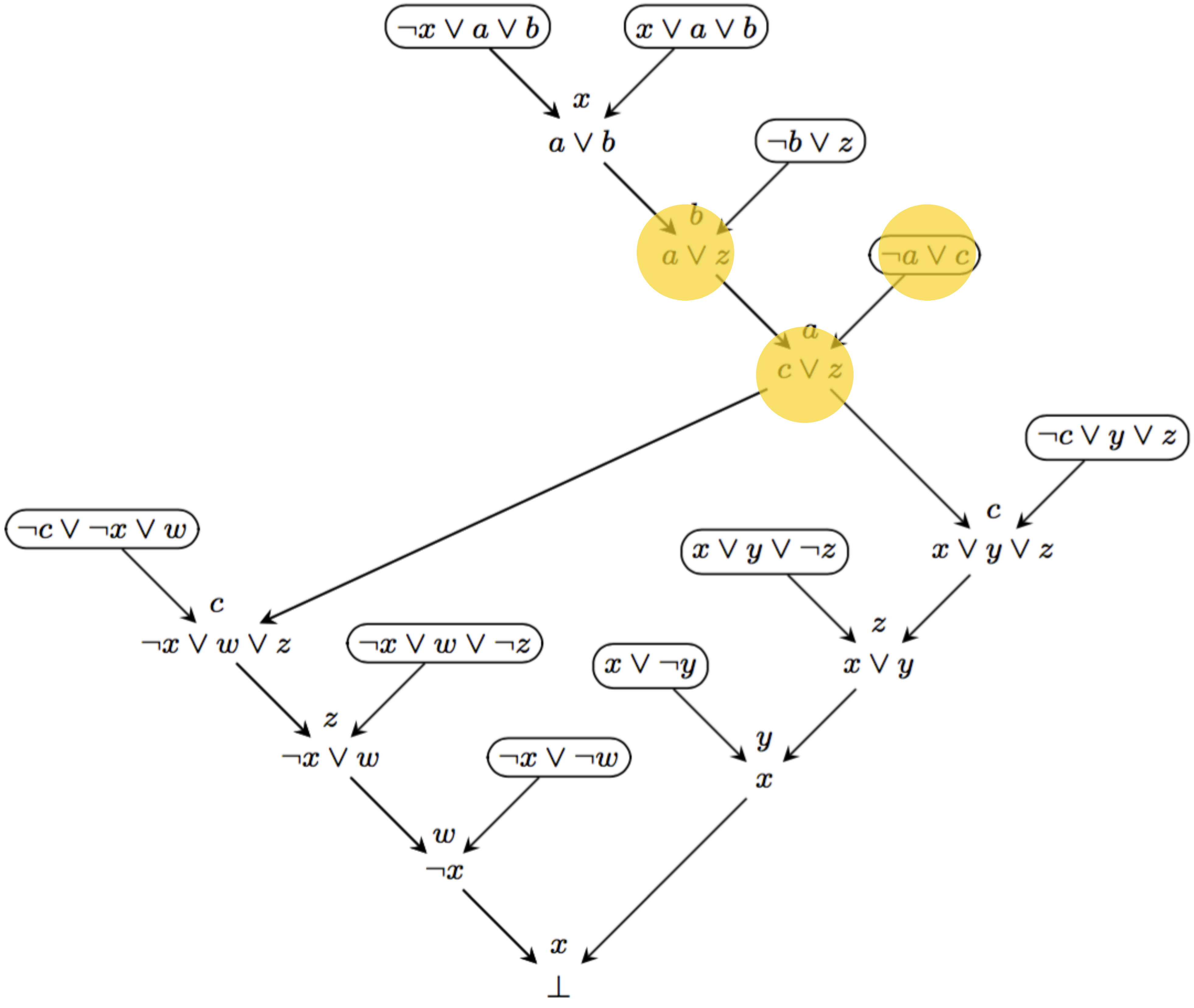

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$\neg c \vee y \vee z$
$\neg c \vee \neg x \vee w$
c
$x \vee y \vee z$
$x \vee y \vee \neg z$
c
$\neg x \vee w \vee z$
$\neg x \vee w \vee \neg z$
z
$x \vee y$
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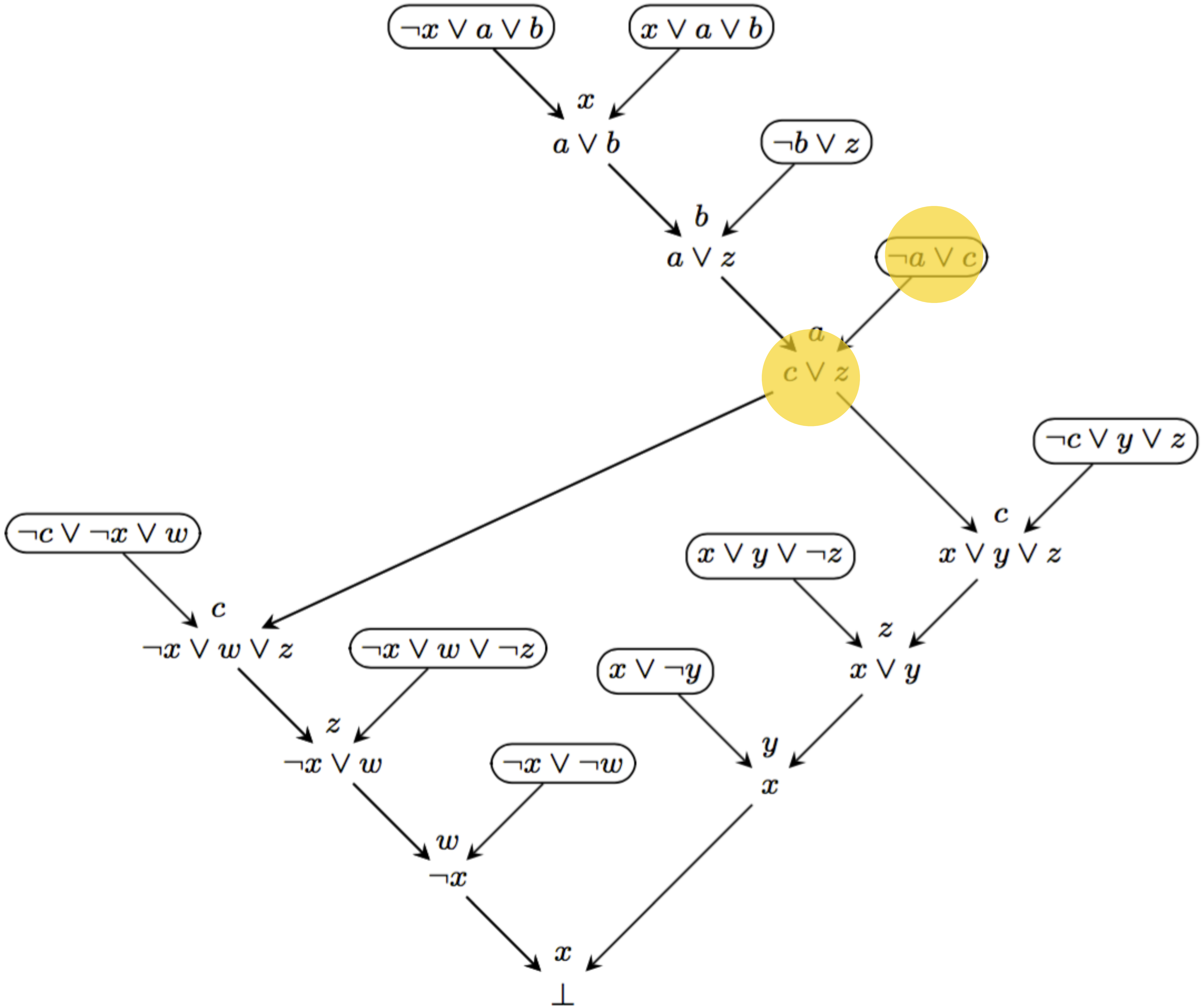
$\bot$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a ∨ b
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¬a ∨ c
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c ∨ z
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c
x ∨ y ∨ z
¬c ∨ ¬x ∨ w
x ∨ y ∨ ¬z
c
¬x ∨ w ∨ z
¬x ∨ w ∨ ¬z
z
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z
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y
x
w
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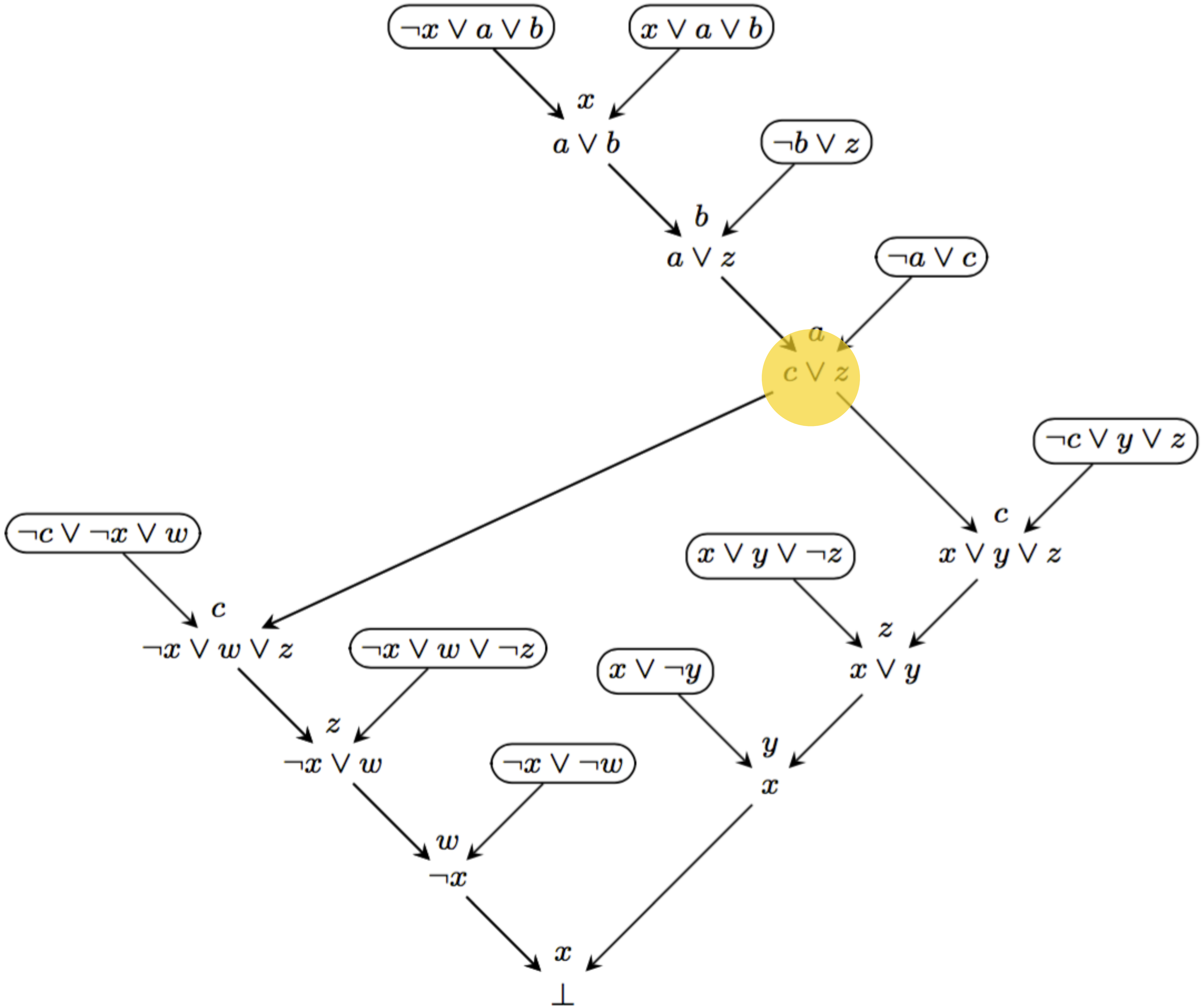
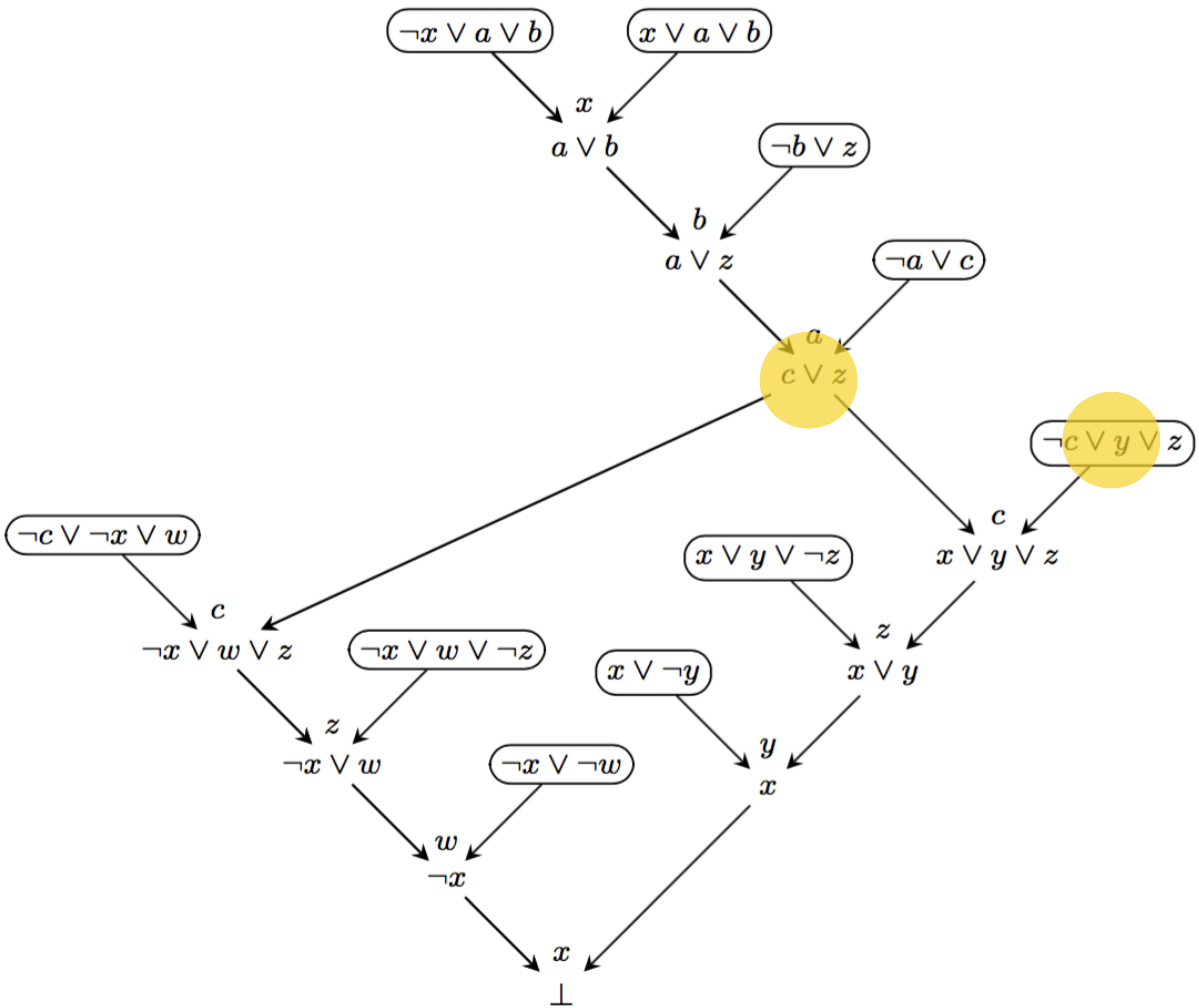
x
⊥

Clause Space

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c
$x \lor y \lor z$
$\neg c \lor \neg x \lor w$
$x \lor y \lor \neg z$
c
$\neg x \lor w \lor z$
$\neg x \lor w \lor \neg z$
z
$x \lor \neg y$
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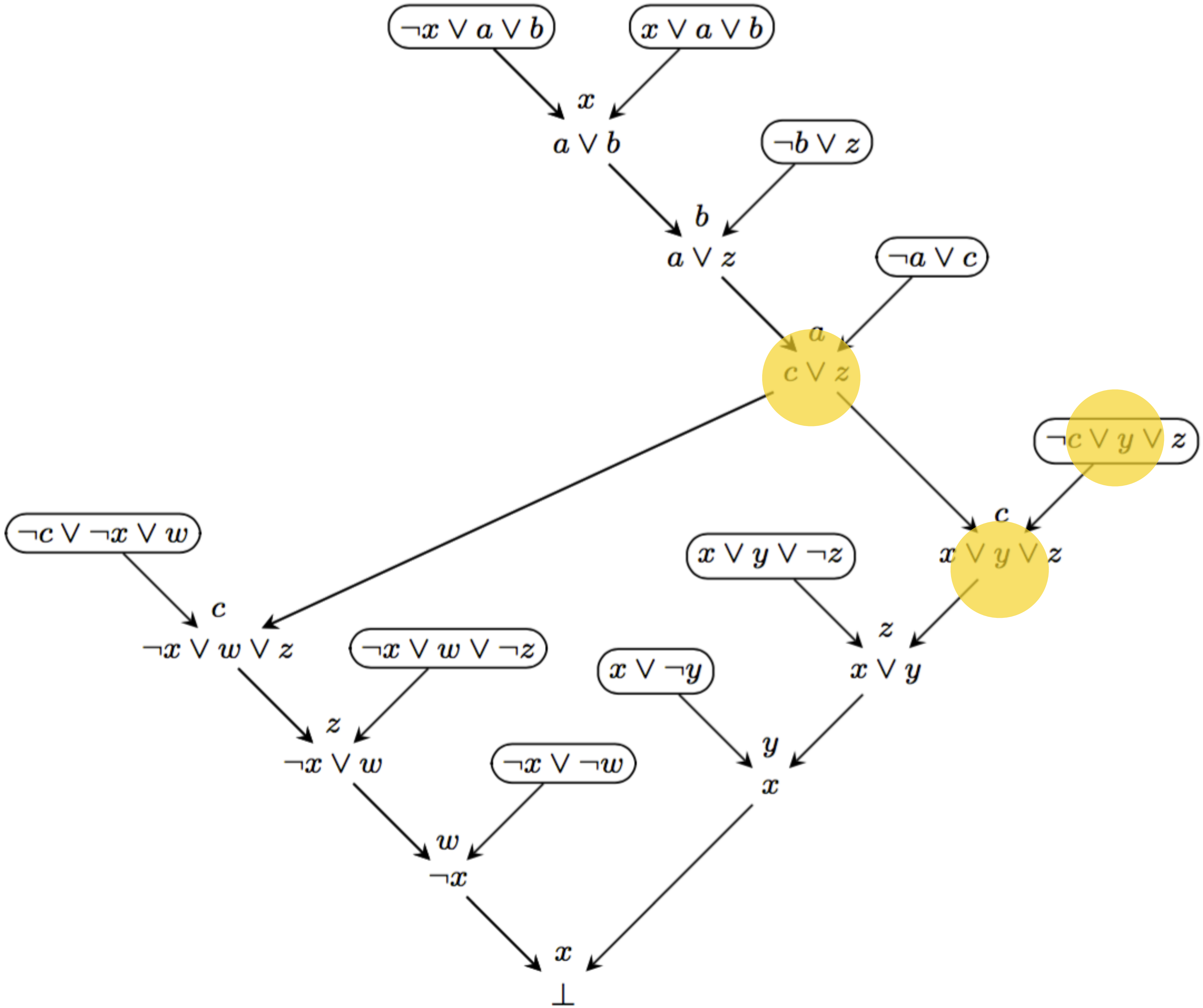
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a ∨ b
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¬a ∨ c
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c ∨ z
¬c ∨ y ∨ z
c
x ∨ y ∨ z
¬c ∨ ¬x ∨ w
c
¬x ∨ w ∨ z
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z
¬x ∨ w ∨ ¬z
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¬x ∨ ¬w
w
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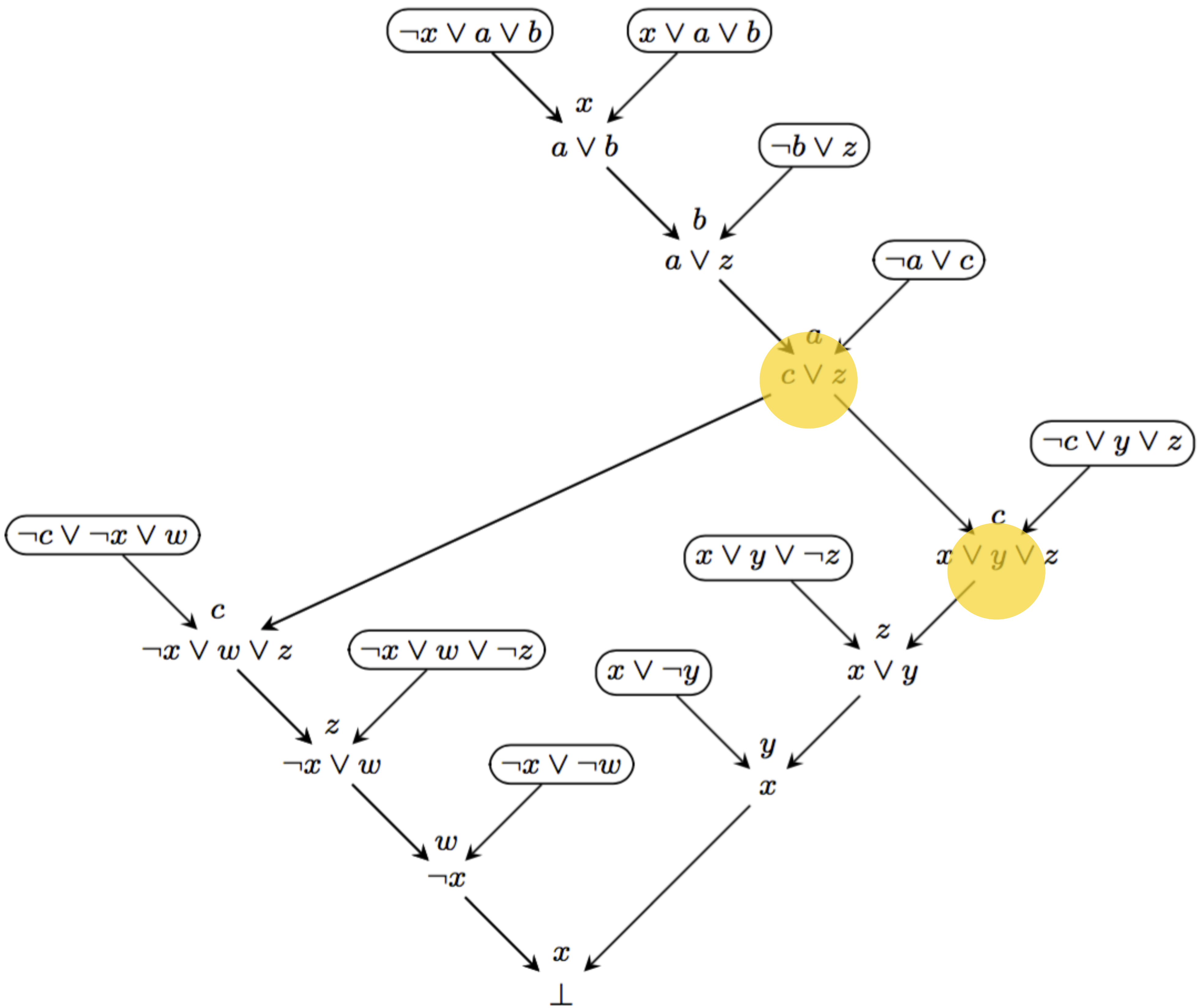
x
⊥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x ∨ y ∨ z
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z
x ∨ y
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y
x
¬c ∨ ¬x ∨ w
c
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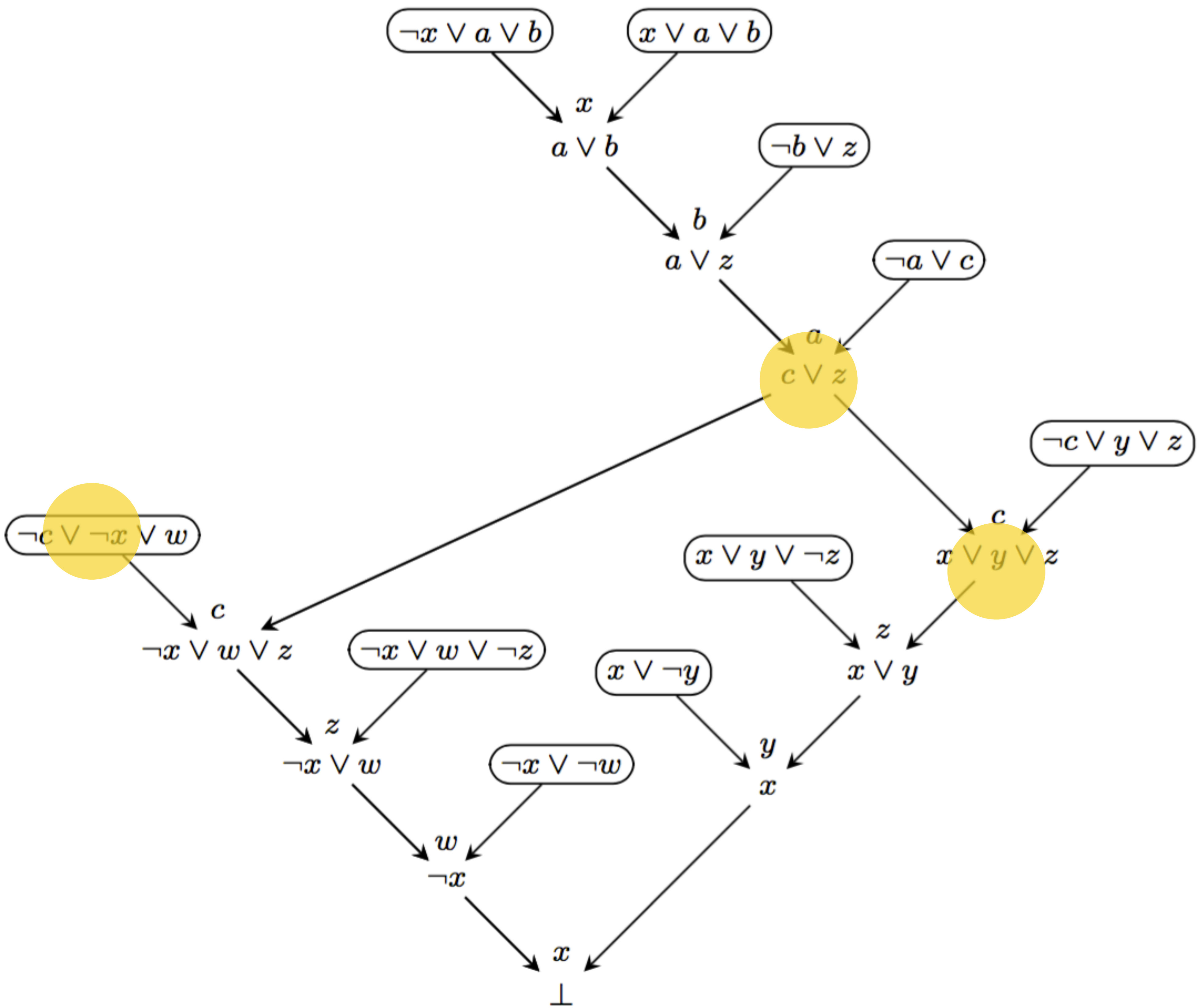
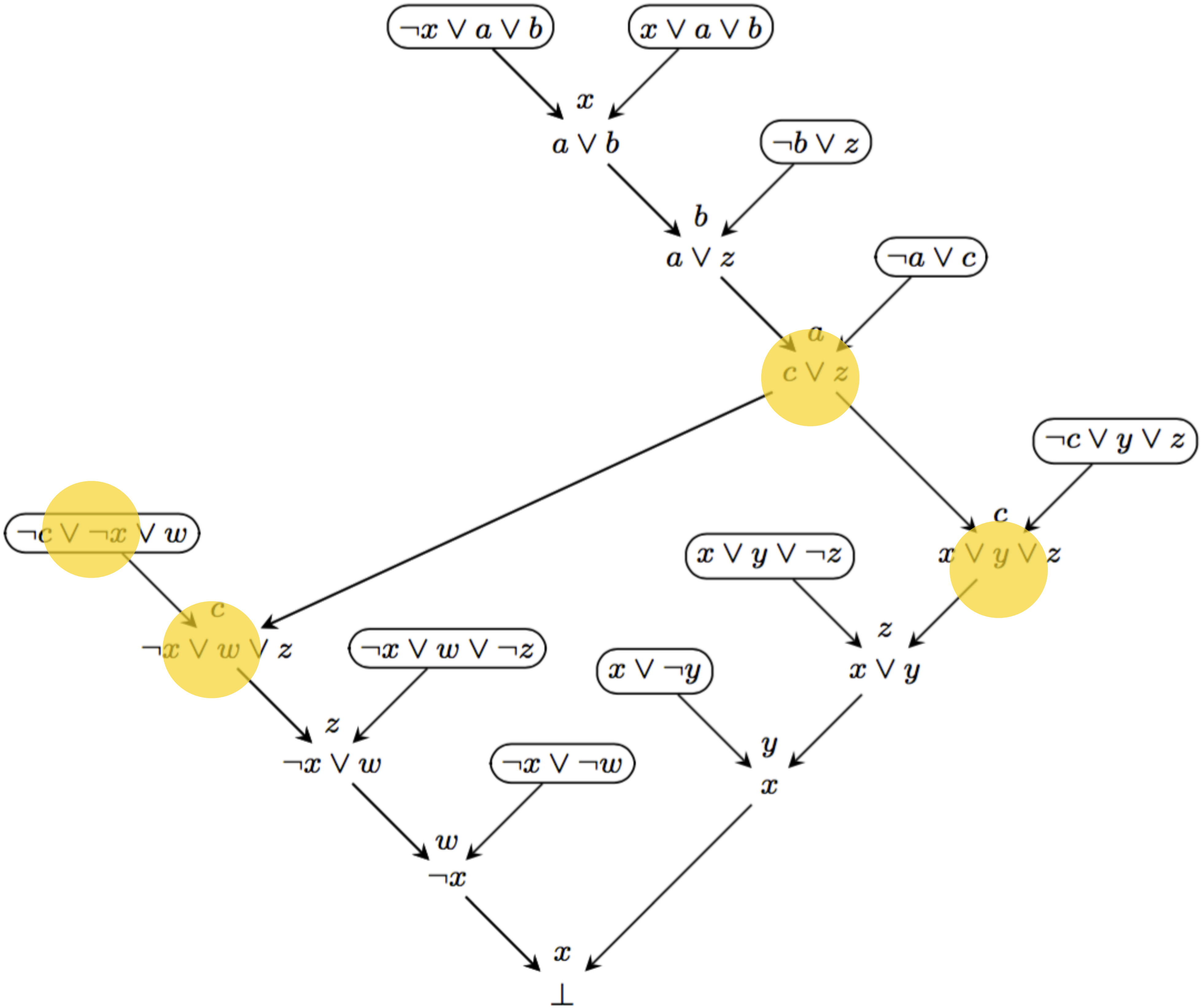
⊥

Clause Space

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c
$x \vee y \vee \neg z$
$x \vee y \vee z$
c
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y
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x
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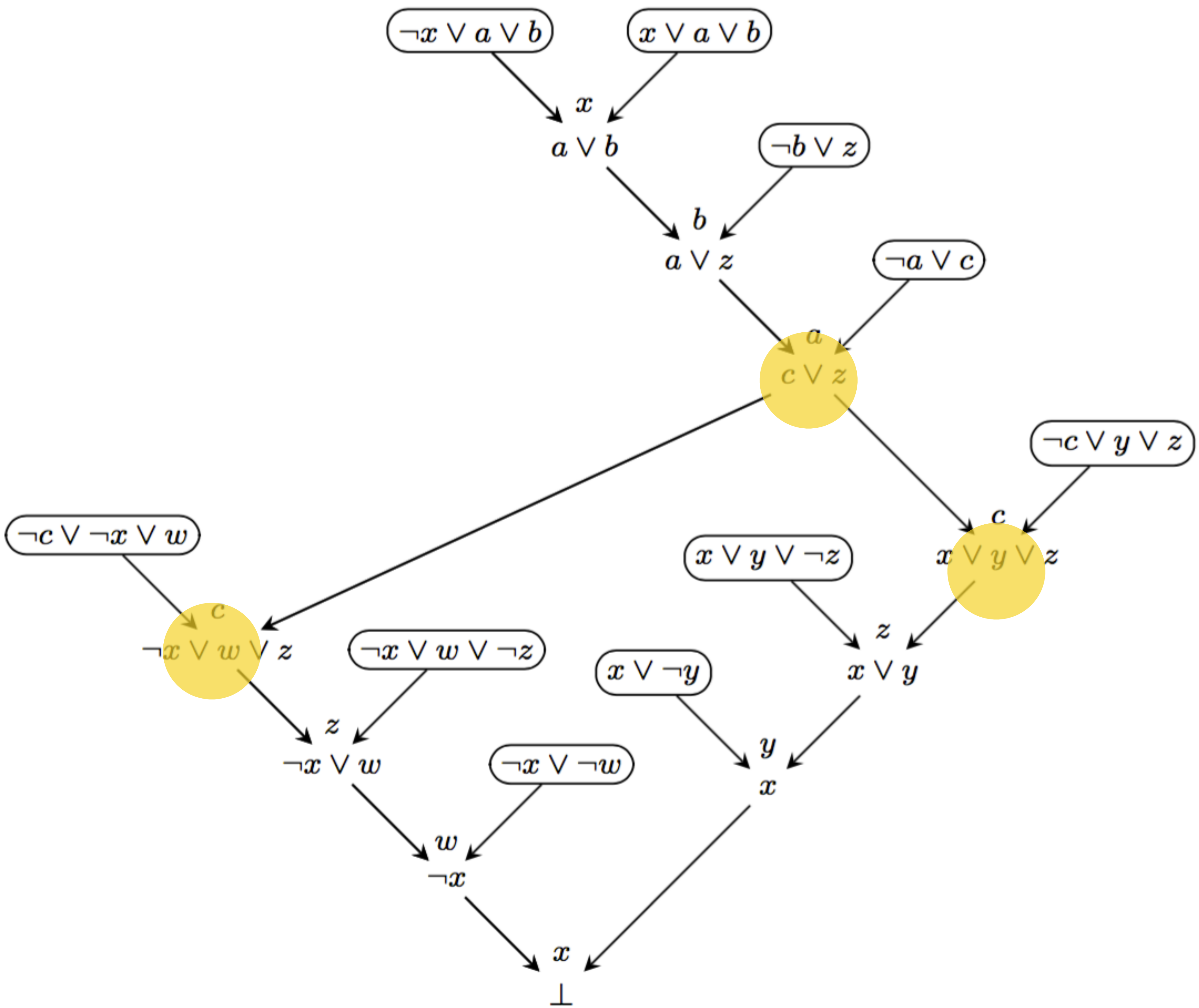
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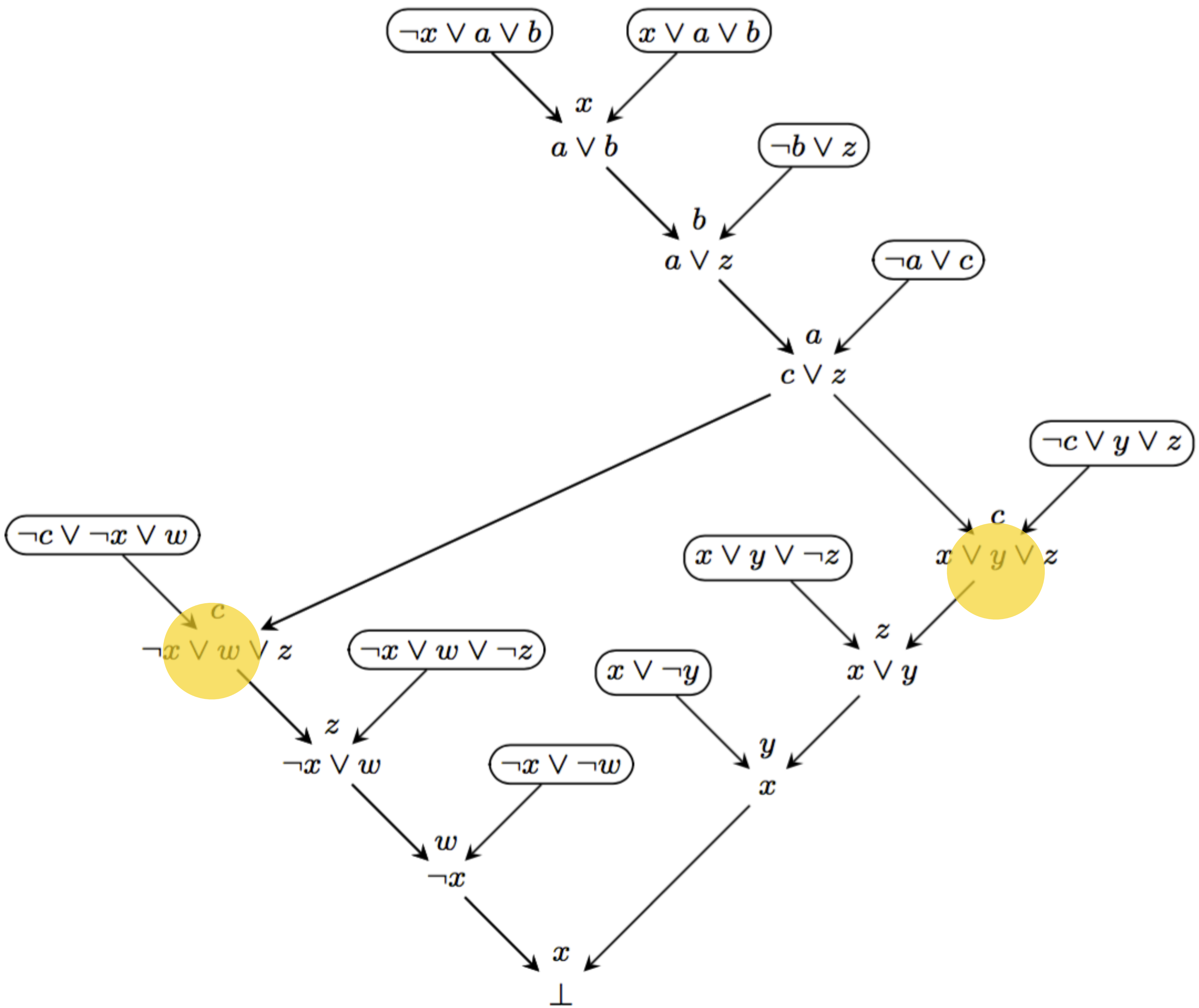
⊥



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$x \vee a \vee b$
x
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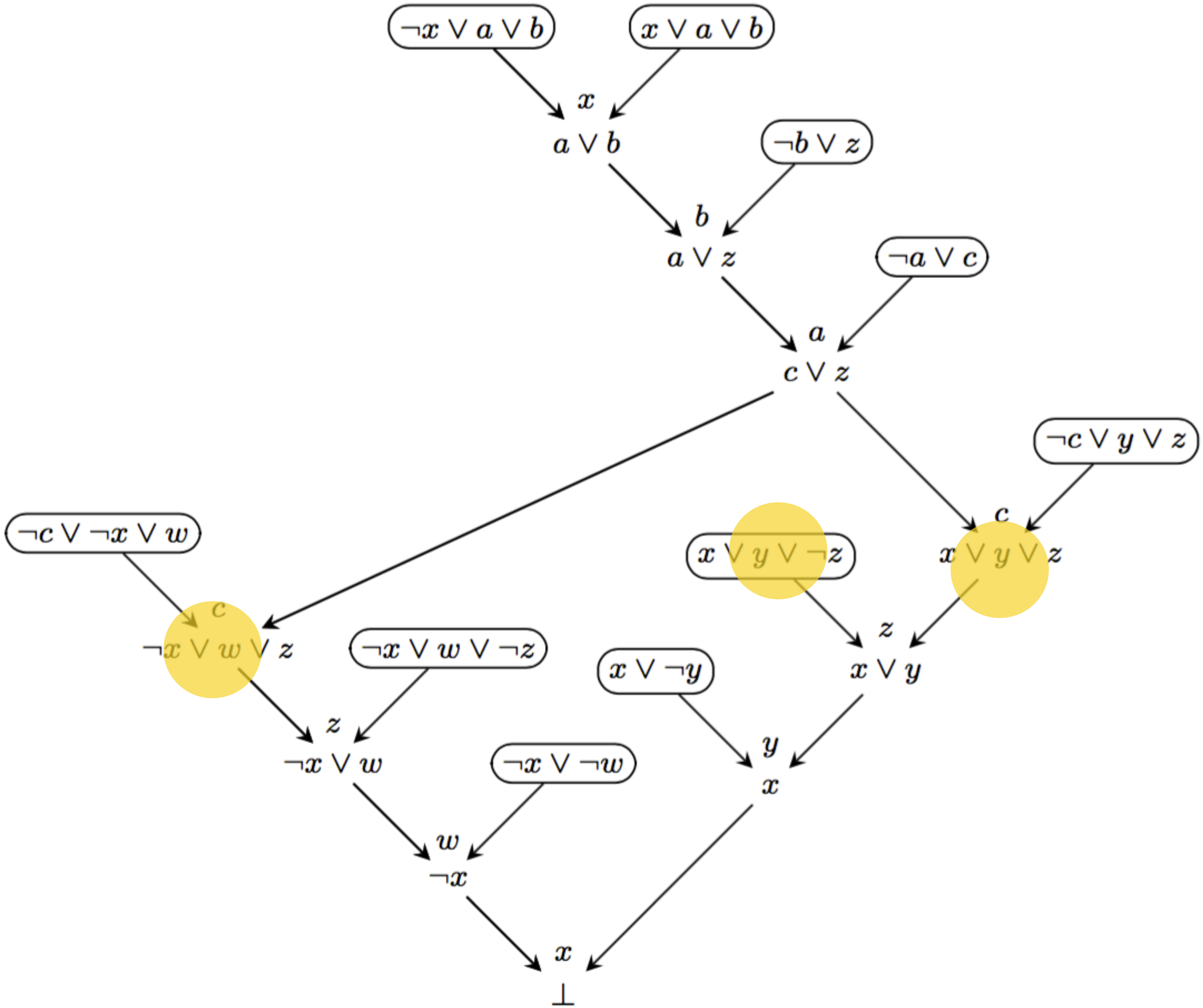
b
$a \vee z$
$\neg a \vee c$
a
$c \vee z$
$\neg c \vee y \vee z$
$\neg c \vee \neg x \vee w$
c
$x \vee y \vee \neg z$
$x \vee y \vee z$
c
$\neg x \vee w \vee z$
$\neg x \vee w \vee \neg z$
z
$x \vee \neg y$
$x \vee y$
z
$\neg x \vee w$
$\neg x \vee \neg w$
y
x
w
$\neg x$
x
$\bot$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$\neg c \vee y \vee z$
$\neg c \vee \neg x \vee w$
c
$x \vee y \vee \neg z$
c
$x \vee y \vee z$
$\neg x \vee w \vee z$
$\neg x \vee w \vee \neg z$
z
$x \vee \neg y$
$x \vee y$
z
$\neg x \vee w$
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y
x
w
$\neg x$
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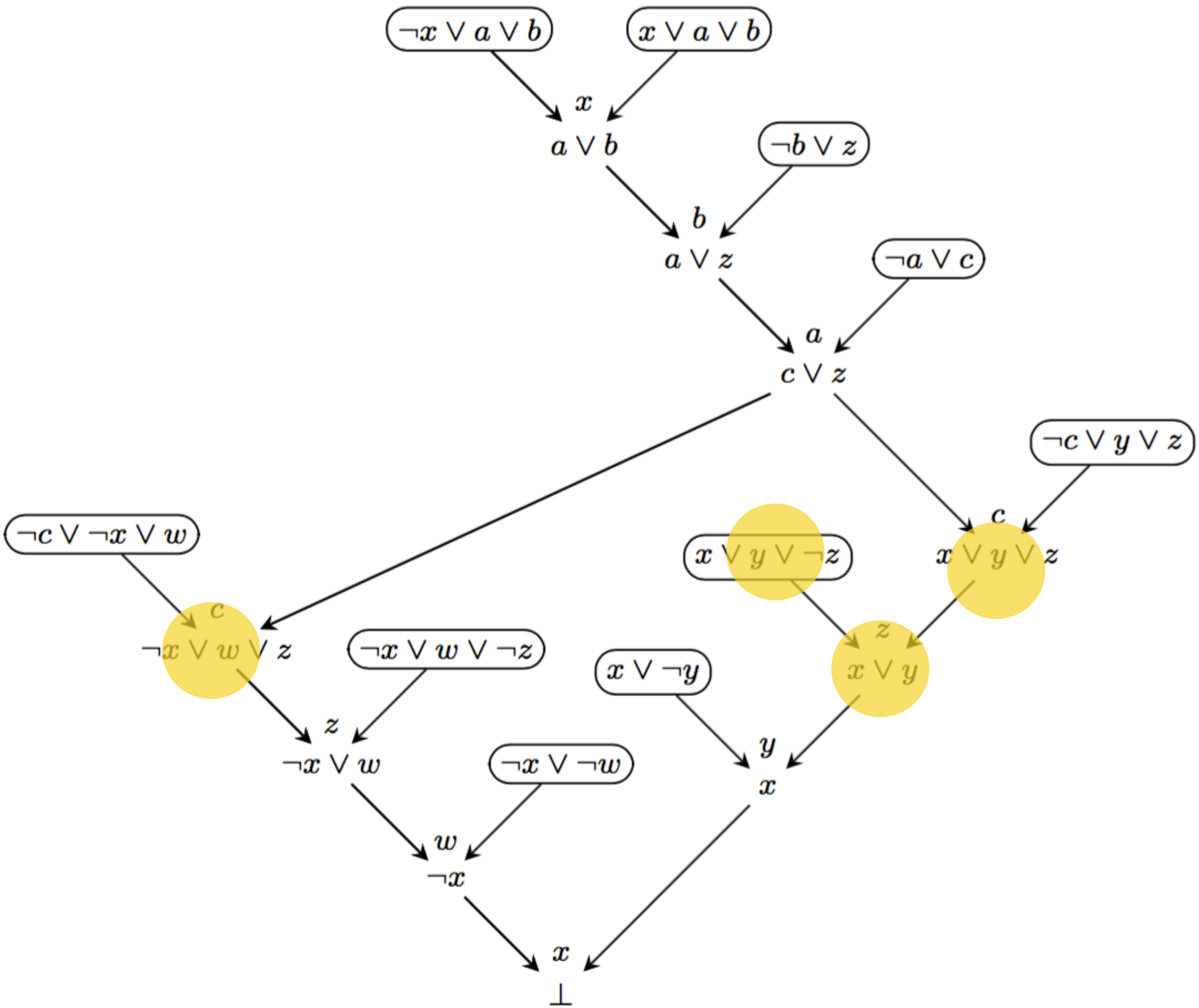
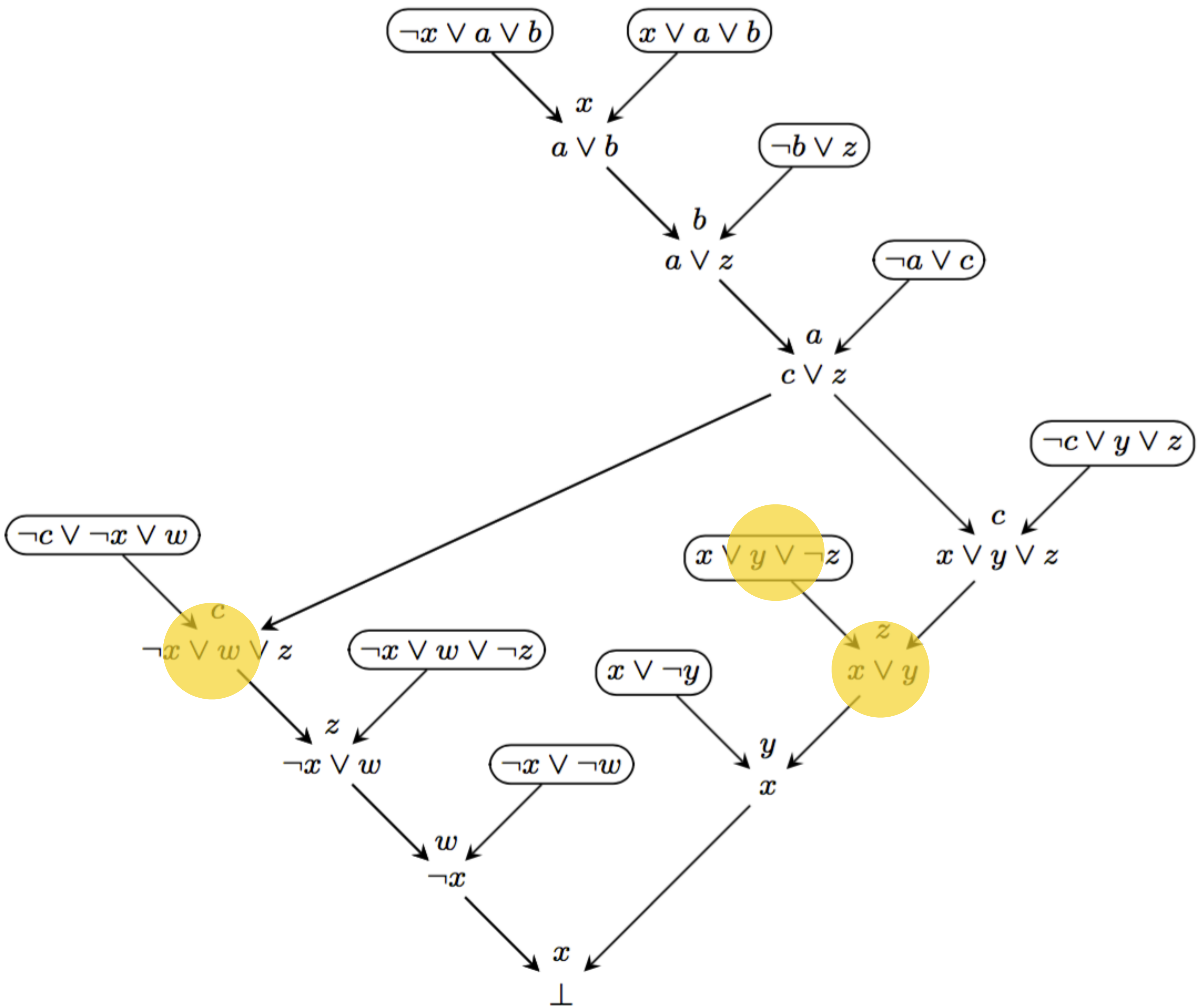
$\bot$

Clause Space

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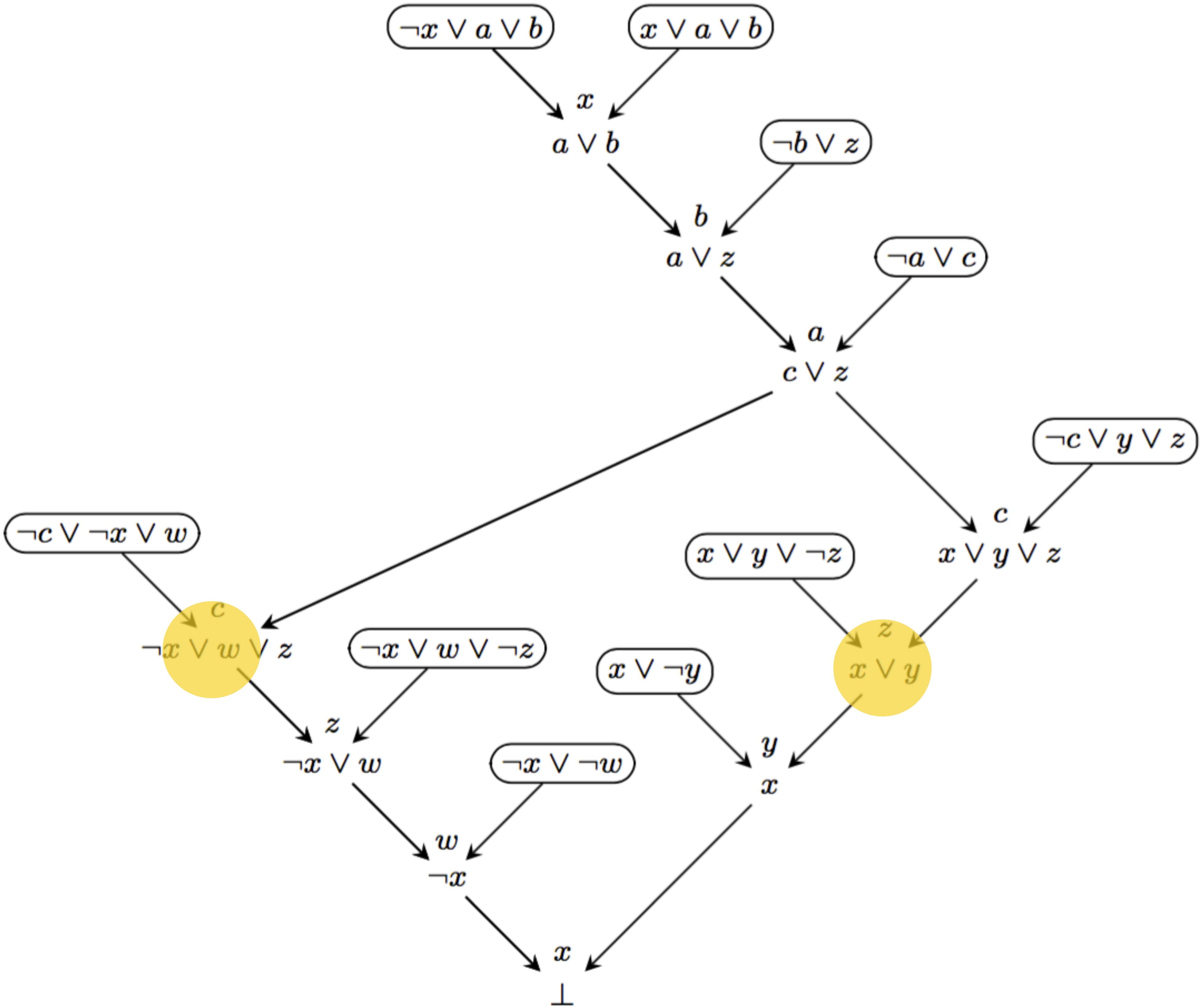
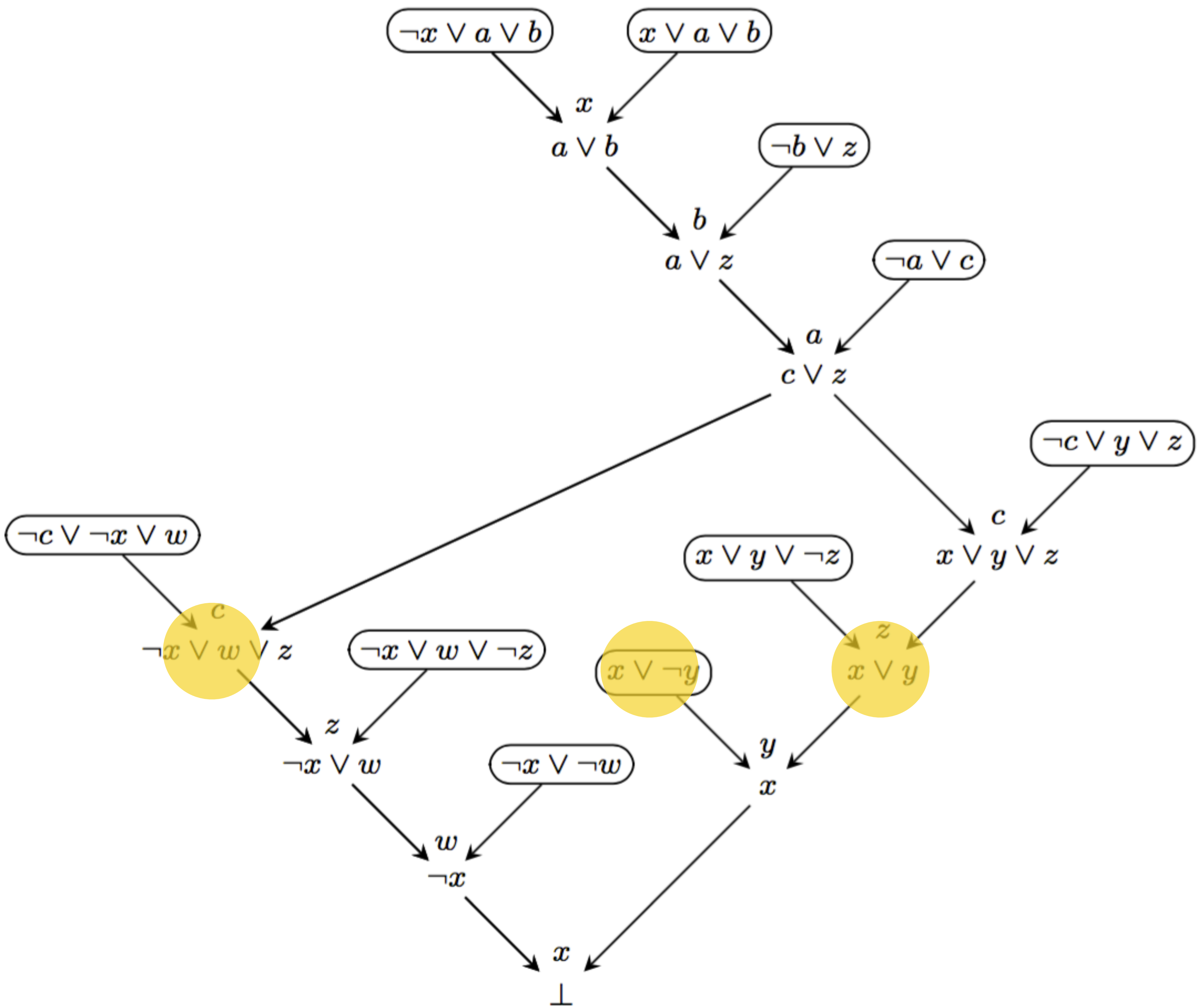
$\neg a \lor c$
a
$c \lor z$
$\neg c \lor y \lor z$
c
$x \lor y \lor z$
$\neg c \lor \neg x \lor w$
$x \lor y \lor \neg z$
c
$\neg x \lor w \lor z$
$\neg x \lor w \lor \neg z$
z
$x \lor y$
$x \lor \neg y$
z
$\neg x \lor w$
$\neg x \lor \neg w$
y
x
w
$\neg x$
x
$\bot$



$\neg x \vee a \vee b$
$x \vee a \vee b$
x
$a \vee b$
$\neg b \vee z$
b
$a \vee z$
$\neg a \vee c$
a
$c \vee z$
$\neg c \vee y \vee z$
c
$x \vee y \vee z$
$\neg c \vee \neg x \vee w$
$x \vee y \vee \neg z$
c
$\neg x \vee w \vee z$
$\neg x \vee w \vee \neg z$
$x \vee \neg y$
z
$x \vee y$
z
$\neg x \vee w$
$\neg x \vee \neg w$
y
x
w
$\neg x$
x
$\bot$



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$x \vee a \vee b$
x
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$\neg b \vee z$
b
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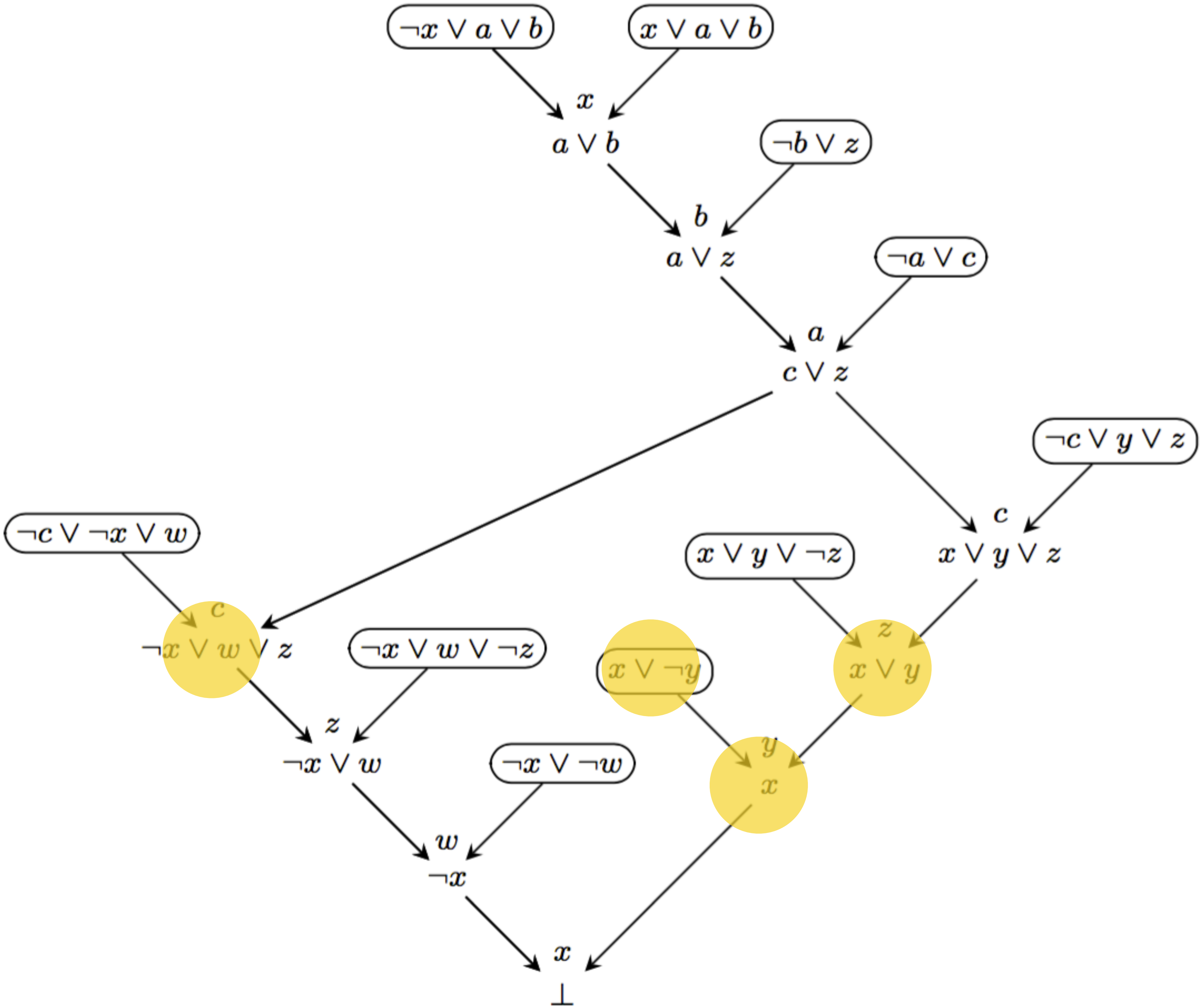
$\neg a \vee c$
a
$c \vee z$
$\neg c \vee y \vee z$
c
$x \vee y \vee z$
$\neg c \vee \neg x \vee w$
$x \vee y \vee \neg z$
c
$\neg x \vee w \vee z$
$\neg x \vee w \vee \neg z$
$x \vee \neg y$
z
$x \vee y$
z
$\neg x \vee w$
$\neg x \vee \neg w$
y
x
w
$\neg x$
x
$\bot$



$\neg x \vee a \vee b$
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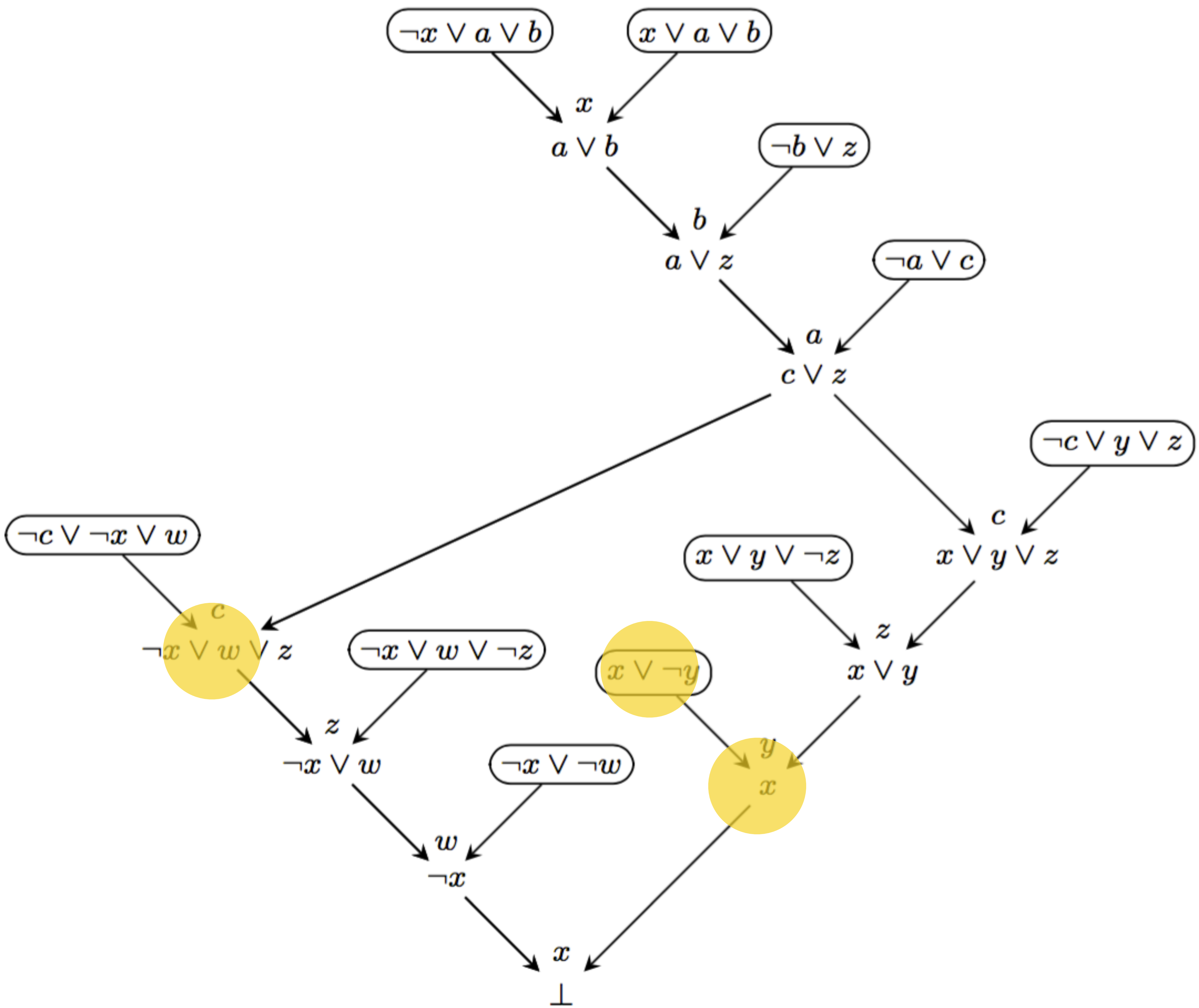
x
$a \vee b$
$\neg b \vee z$
b
$a \vee z$
$\neg a \vee c$
a
$c \vee z$
$\neg c \vee y \vee z$
$\neg c \vee \neg x \vee w$
c
$x \vee y \vee \neg z$
$x \vee y \vee z$
c
$\neg x \vee w \vee z$
$\neg x \vee w \vee \neg z$
$x \vee \neg y$
z
$x \vee y$
z
$\neg x \vee w$
$\neg x \vee \neg w$
y
x
w
$\neg x$
x
$\bot$



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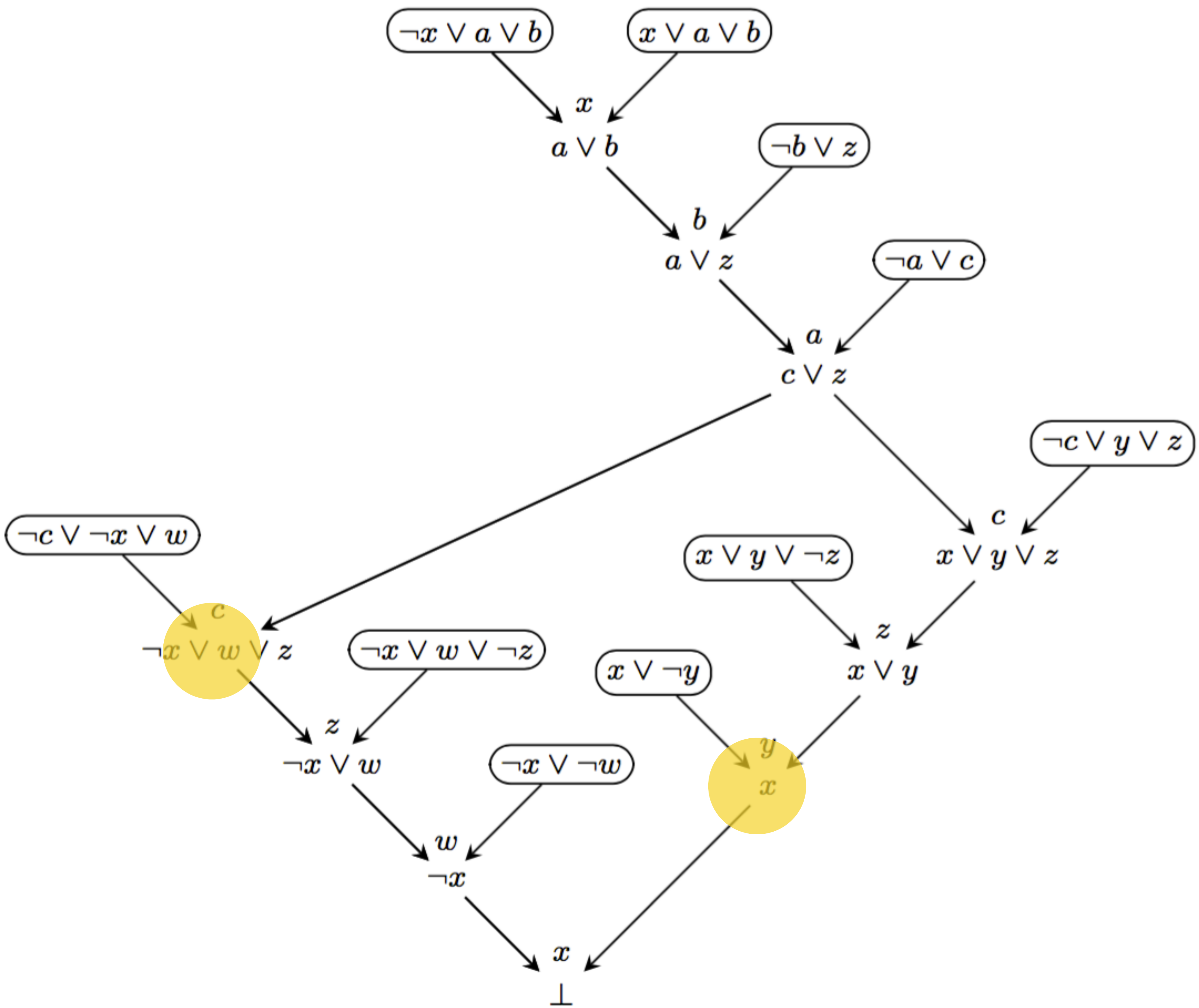
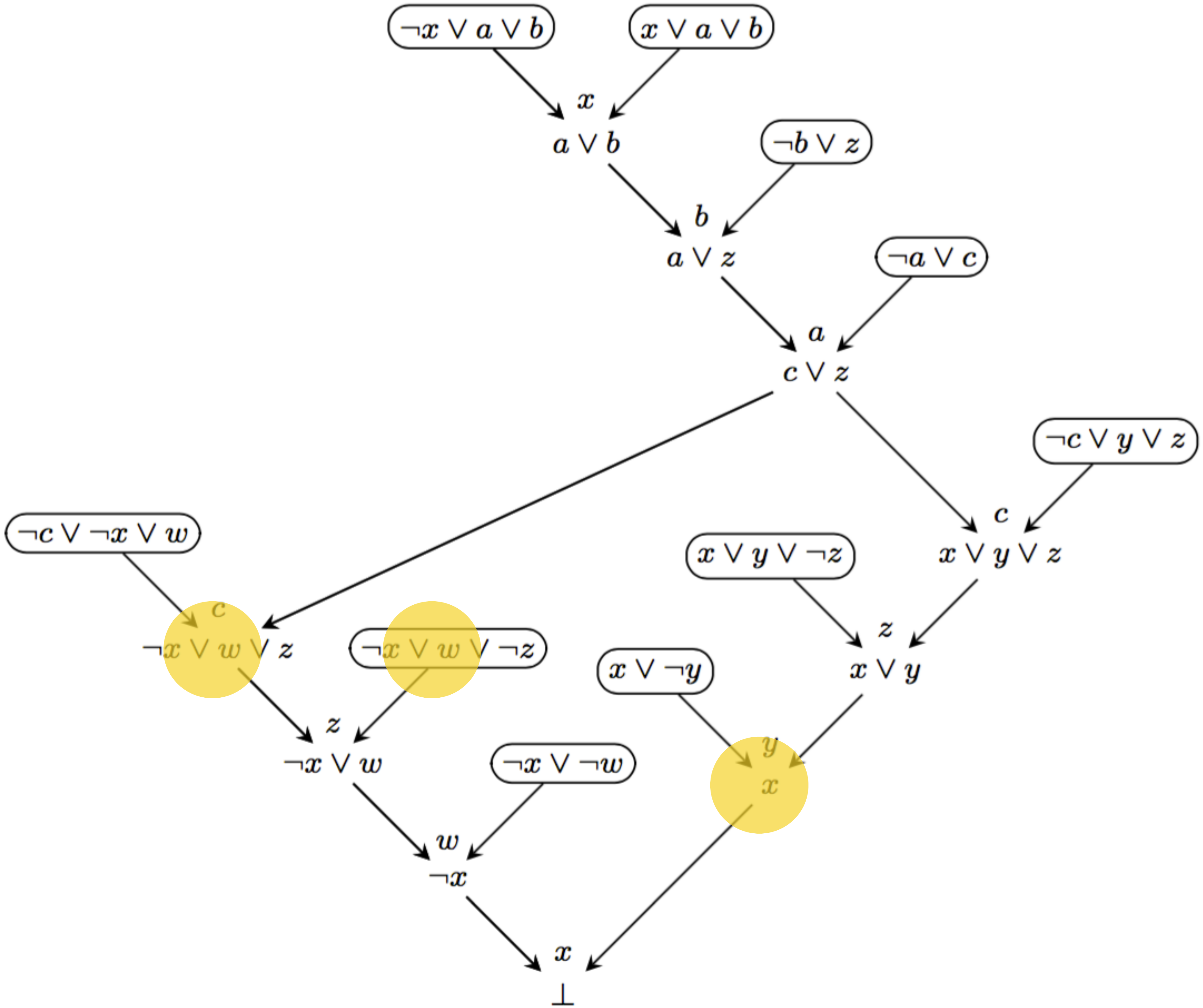
$x \vee a \vee b$
x
$a \vee b$
$\neg b \vee z$
b
$a \vee z$
$\neg a \vee c$
a
$c \vee z$
$\neg c \vee y \vee z$
$\neg c \vee \neg x \vee w$
c
$x \vee y \vee \neg z$
c
$x \vee y \vee z$
$\neg x \vee w \vee z$
$\neg x \vee w \vee \neg z$
$x \vee \neg y$
z
$x \vee y$
z
$\neg x \vee w$
$\neg x \vee \neg w$
y
x
w
$\neg x$
x
$\bot$

Clause Space

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$\neg b \vee z$
b
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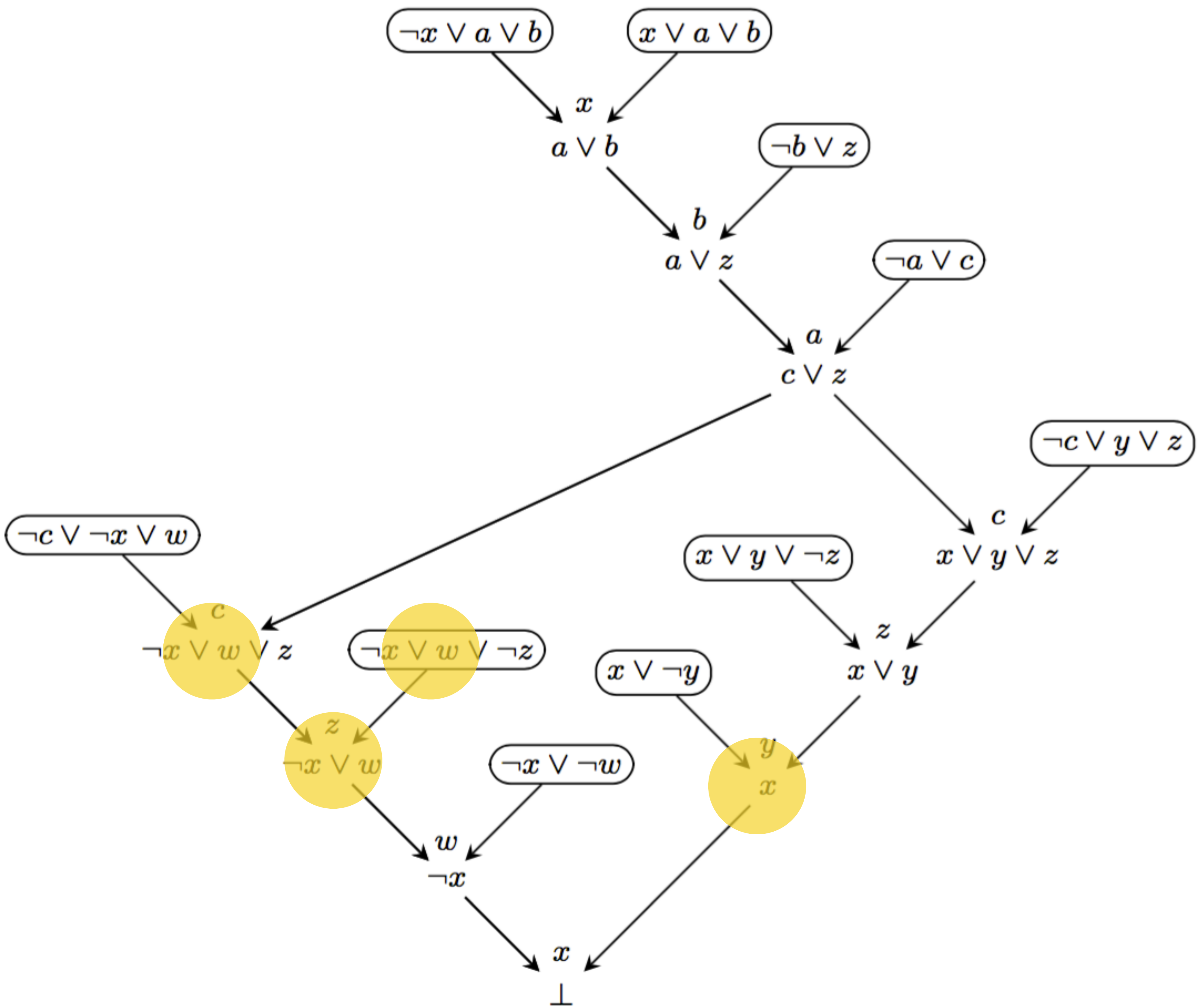
$\neg a \vee c$
a
$c \vee z$
$\neg c \vee y \vee z$
$\neg c \vee \neg x \vee w$
c
$x \vee y \vee \neg z$
$x \vee y \vee z$
c
$\neg x \vee w \vee z$
$\neg x \vee w \vee \neg z$
z
$x \vee \neg y$
$x \vee y$
z
$\neg x \vee w$
$\neg x \vee \neg w$
y
x
w
$\neg x$
x
$\bot$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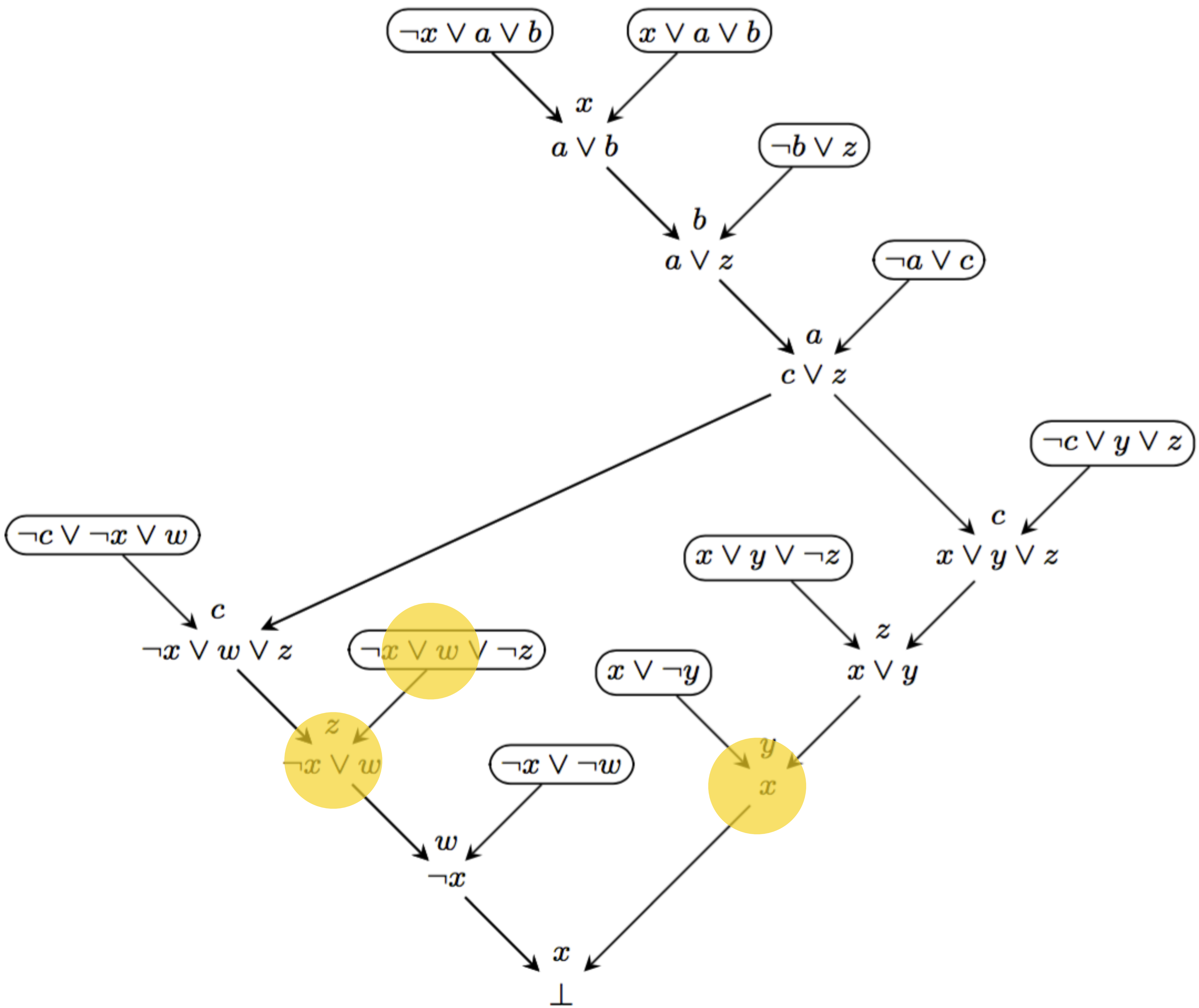
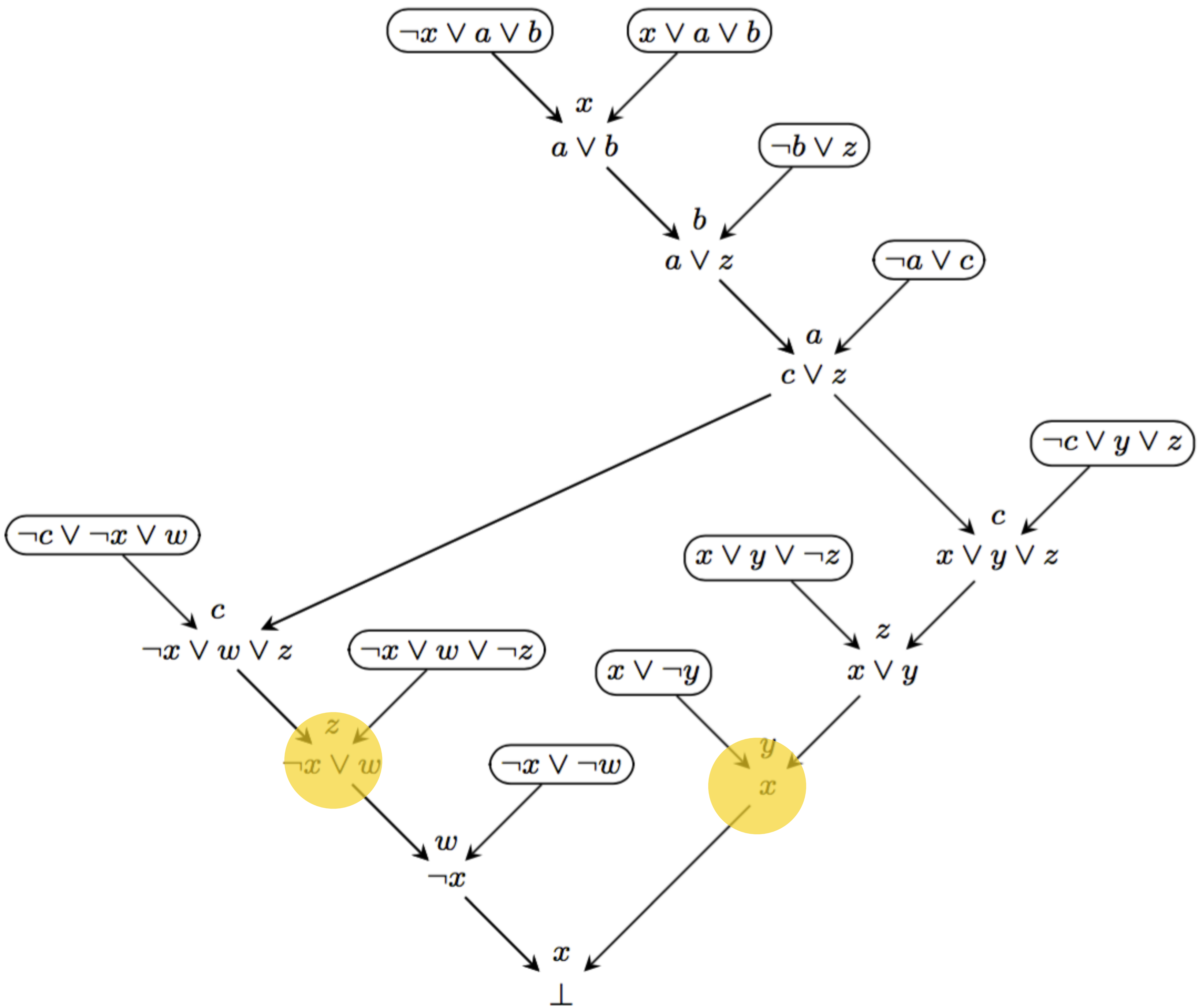
x ∨ a ∨ b
x
a ∨ b
¬b ∨ z
b
a ∨ z
¬a ∨ c
a
c ∨ z
¬c ∨ y ∨ z
c
x ∨ y ∨ z
¬c ∨ ¬x ∨ w
x ∨ y ∨ ¬z
c
¬x ∨ w ∨ z
¬x ∨ w ∨ ¬z
z
x ∨ y
x ∨ ¬y
z
¬x ∨ w
¬x ∨ ¬w
y
x
w
¬x
x
⊥

Clause Space

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x ∨ a ∨ b
x
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¬b ∨ z
b
a ∨ z
¬a ∨ c
a
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¬c ∨ y ∨ z
c
¬c ∨ ¬x ∨ w
x ∨ y ∨ z
x ∨ y ∨ ¬z
c
¬x ∨ w ∨ z
¬x ∨ w ∨ ¬z
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z
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y
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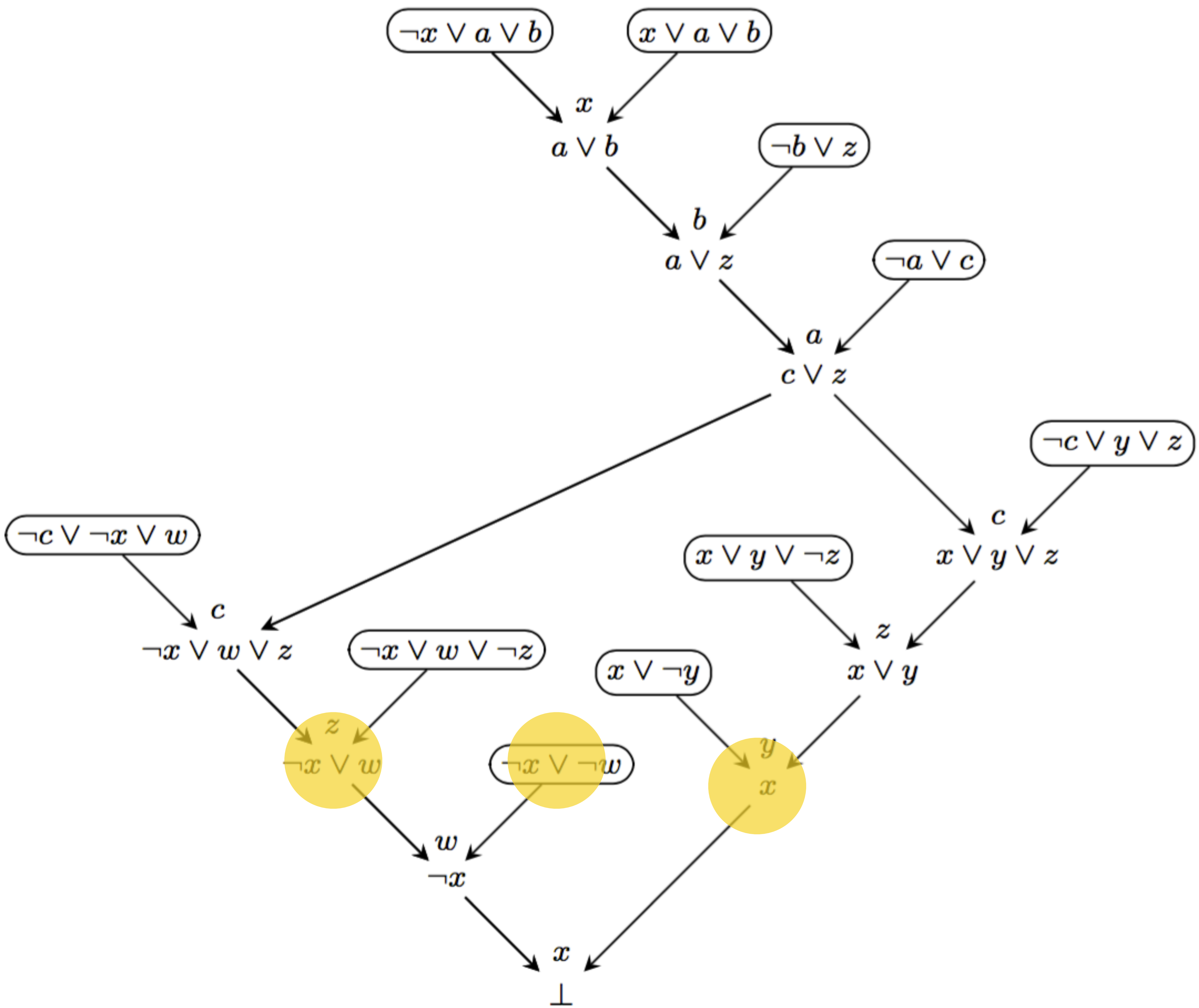
w
¬x
x
⊥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x
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$\neg b \lor z$
b
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a
$c \lor z$
$\neg c \lor y \lor z$
c
$x \lor y \lor z$
$\neg c \lor \neg x \lor w$
c
$\neg x \lor w \lor z$
$x \lor y \lor \neg z$
z
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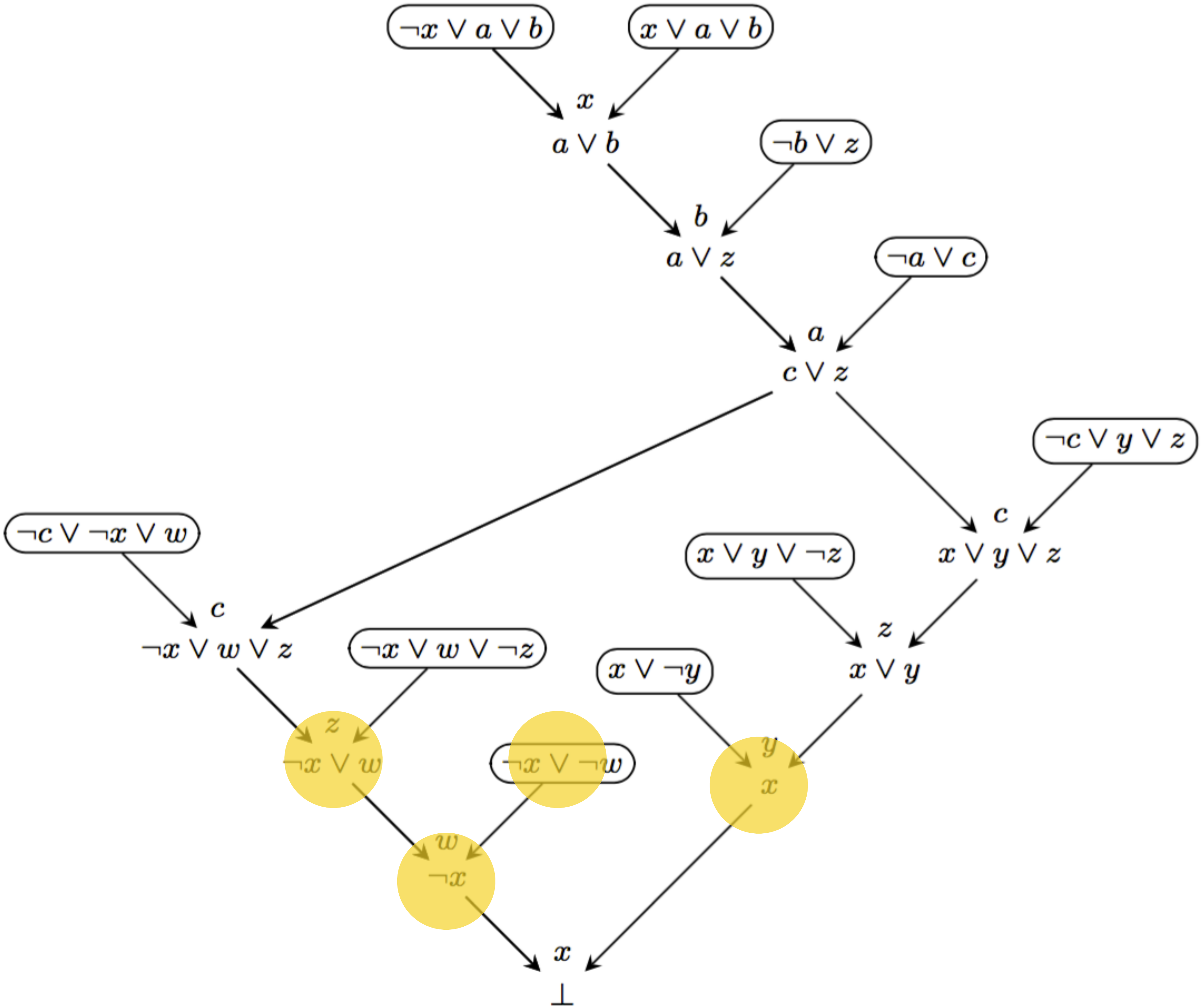
$\neg x \lor w \lor \neg z$
$x \lor \neg y$
z
$\neg x \lor w$
$\neg x \lor \neg w$
y
x
w
$\neg x$
x
$\bot$



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a
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c
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$x \vee y \vee z$
c
z
$\neg x \vee w \vee z$
$\neg x \vee w \vee \neg z$
$x \vee \neg y$
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z
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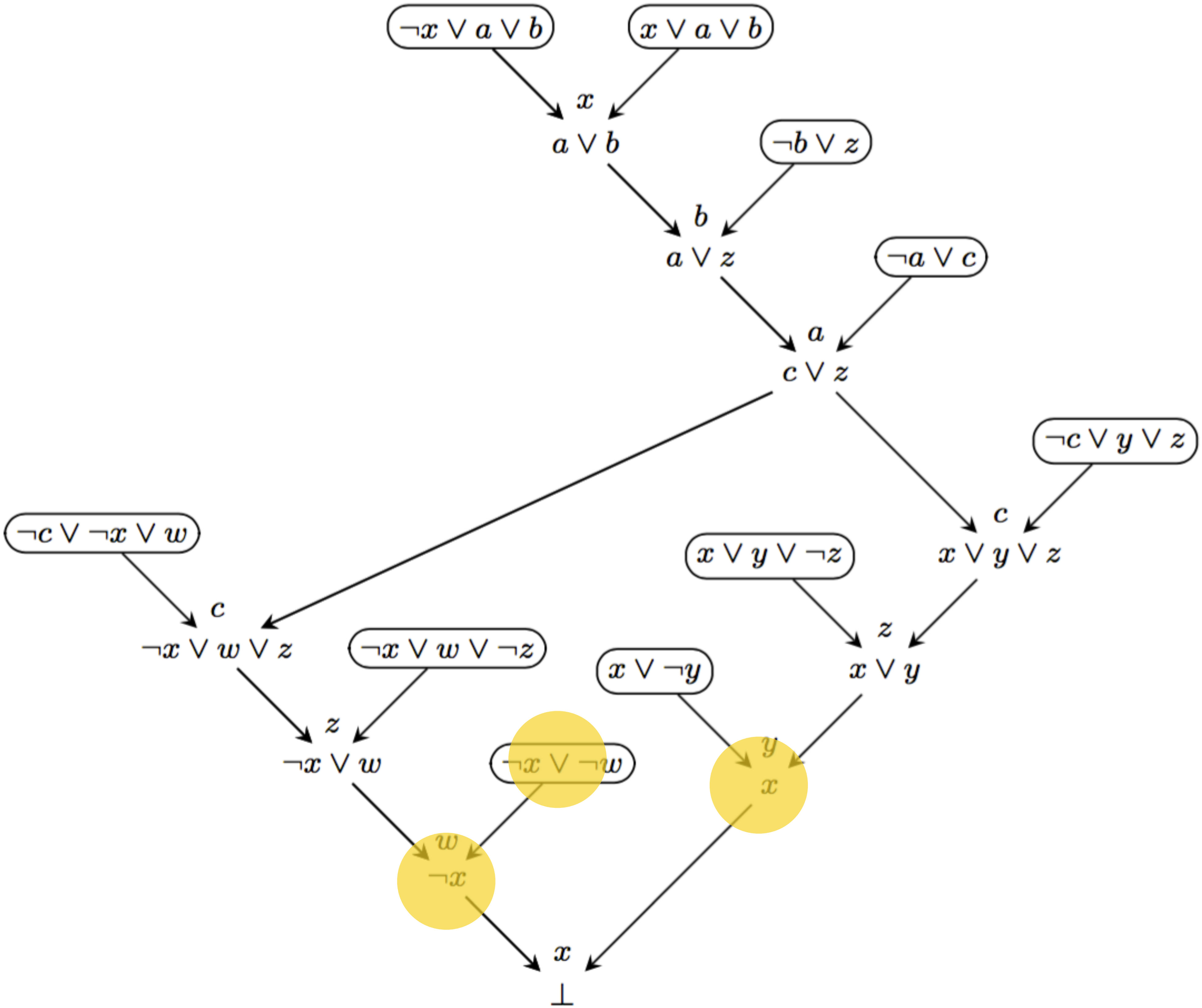
y
x
w
$\neg x$
x
$\bot$



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$\neg c \vee y \vee z$
c
$\neg c \vee \neg x \vee w$
$x \vee y \vee \neg z$
$x \vee y \vee z$
c
$\neg x \vee w \vee z$
$\neg x \vee w \vee \neg z$
z
$x \vee \neg y$
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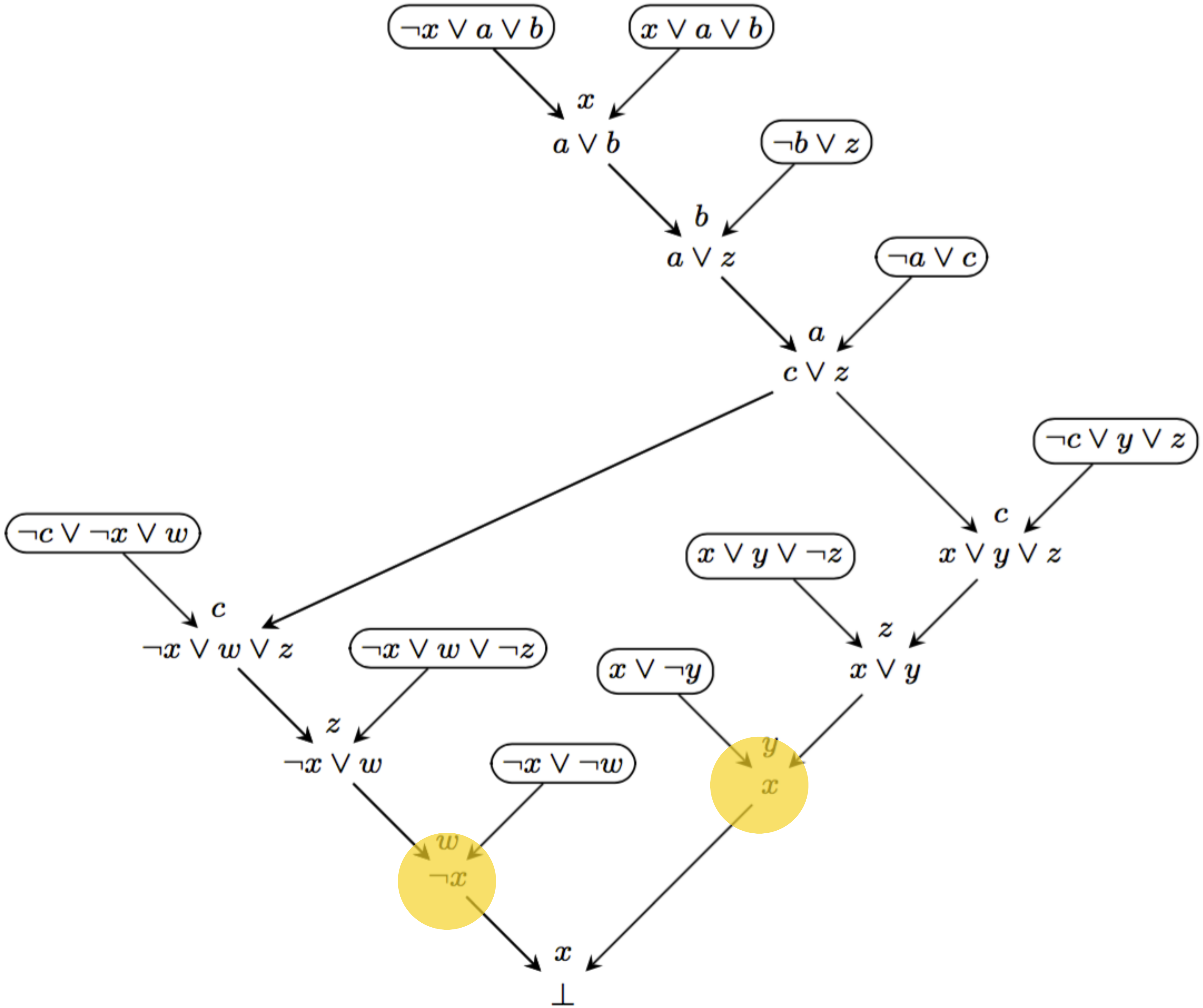
y
x
w
$\neg x$
x
$\bot$



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a
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$\neg c \vee y \vee z$
c
$x \vee y \vee z$
$\neg c \vee \neg x \vee w$
c
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$x \vee y \vee \neg z$
z
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$\neg x \vee w \vee \neg z$
$x \vee \neg y$
z
$\neg x \vee w$
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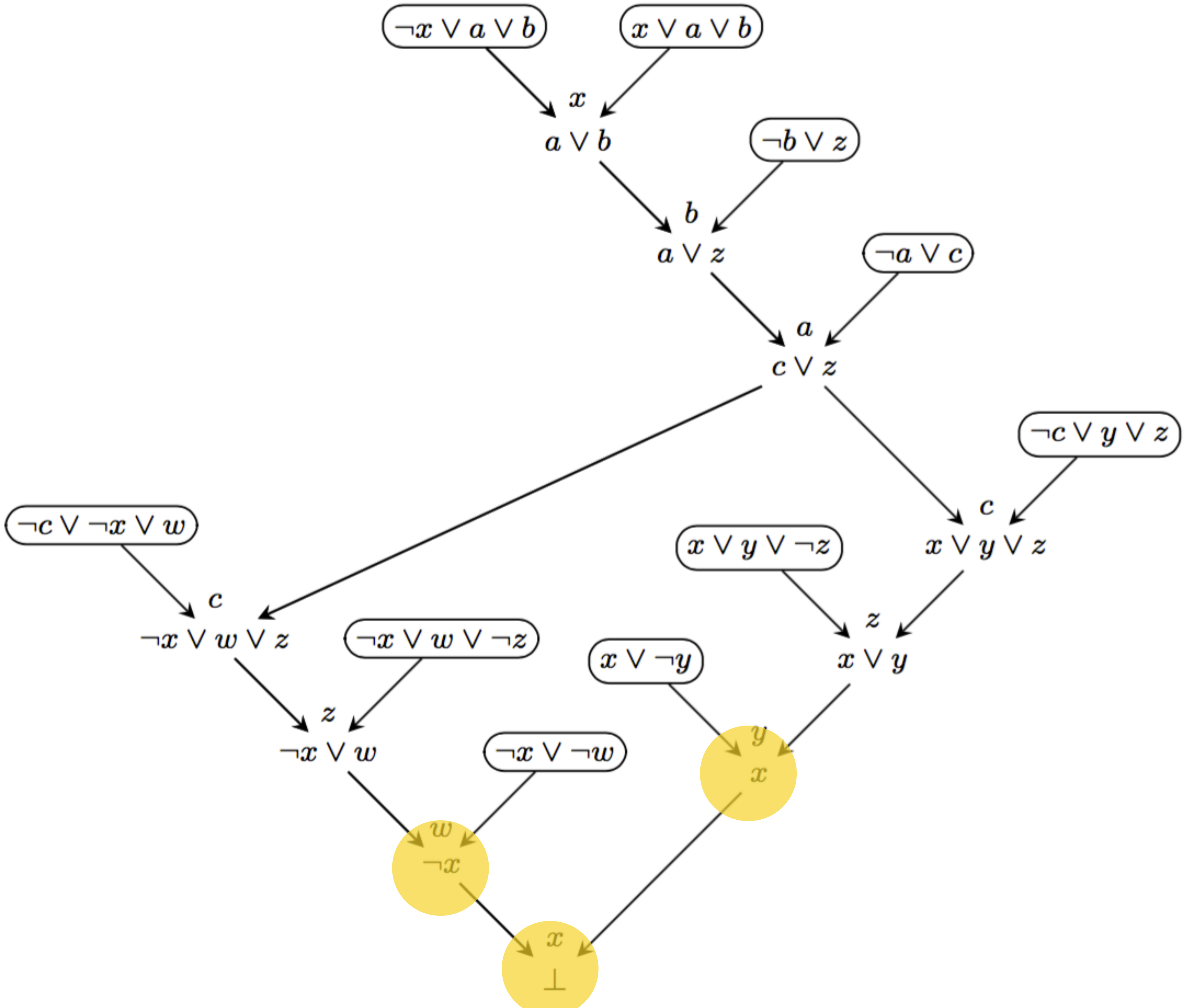
y
x
w
$\neg x$
x
$\bot$



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a
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$\neg c \lor y \lor z$
c
$\neg c \lor \neg x \lor w$
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c
z
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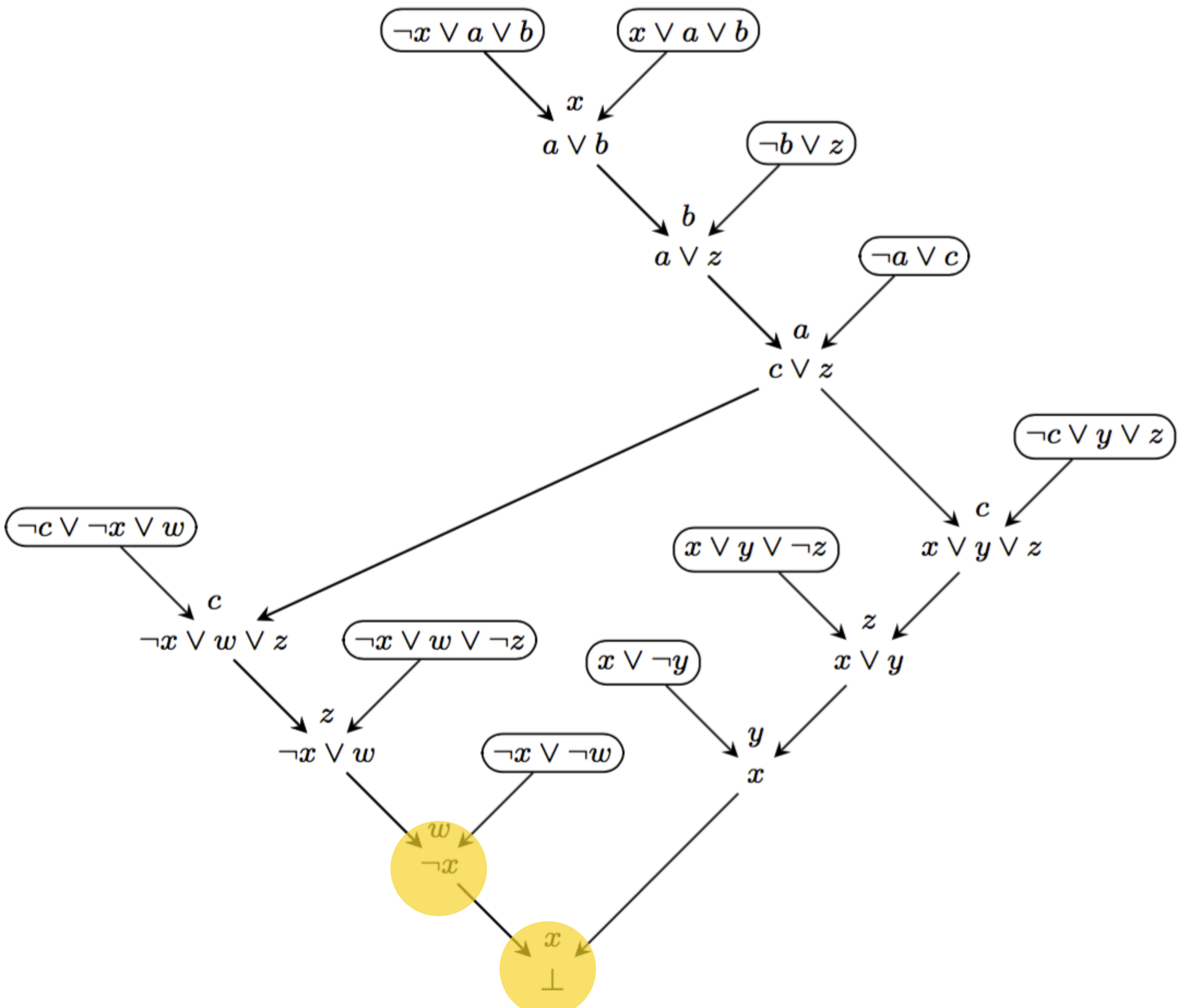
w
$\neg x$
x
$\bot$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x ∨ a ∨ b
x
a ∨ b
¬b ∨ z
b
a ∨ z
¬a ∨ c
a
c ∨ z
¬c ∨ y ∨ z
c
x ∨ y ∨ z
¬c ∨ ¬x ∨ w
c
¬x ∨ w ∨ z
x ∨ y ∨ ¬z
z
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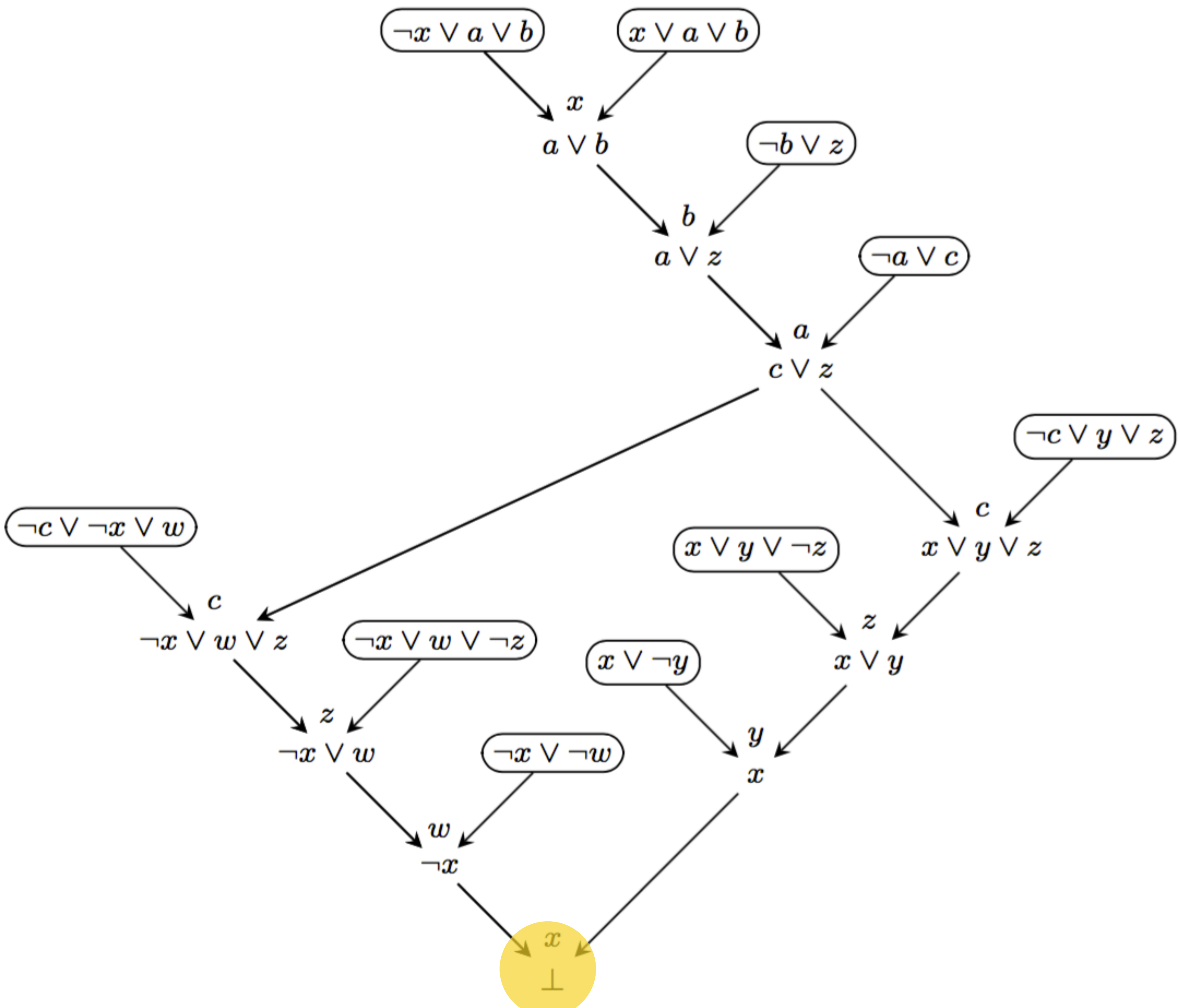
¬x ∨ ¬w
w
¬x
x
⊥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$\neg a \vee c$
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$c \vee z$
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$\neg c \vee \neg x \vee w$
c
$x \vee y \vee \neg z$
c
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$\neg x \vee w \vee z$
$\neg x \vee w \vee \neg z$
z
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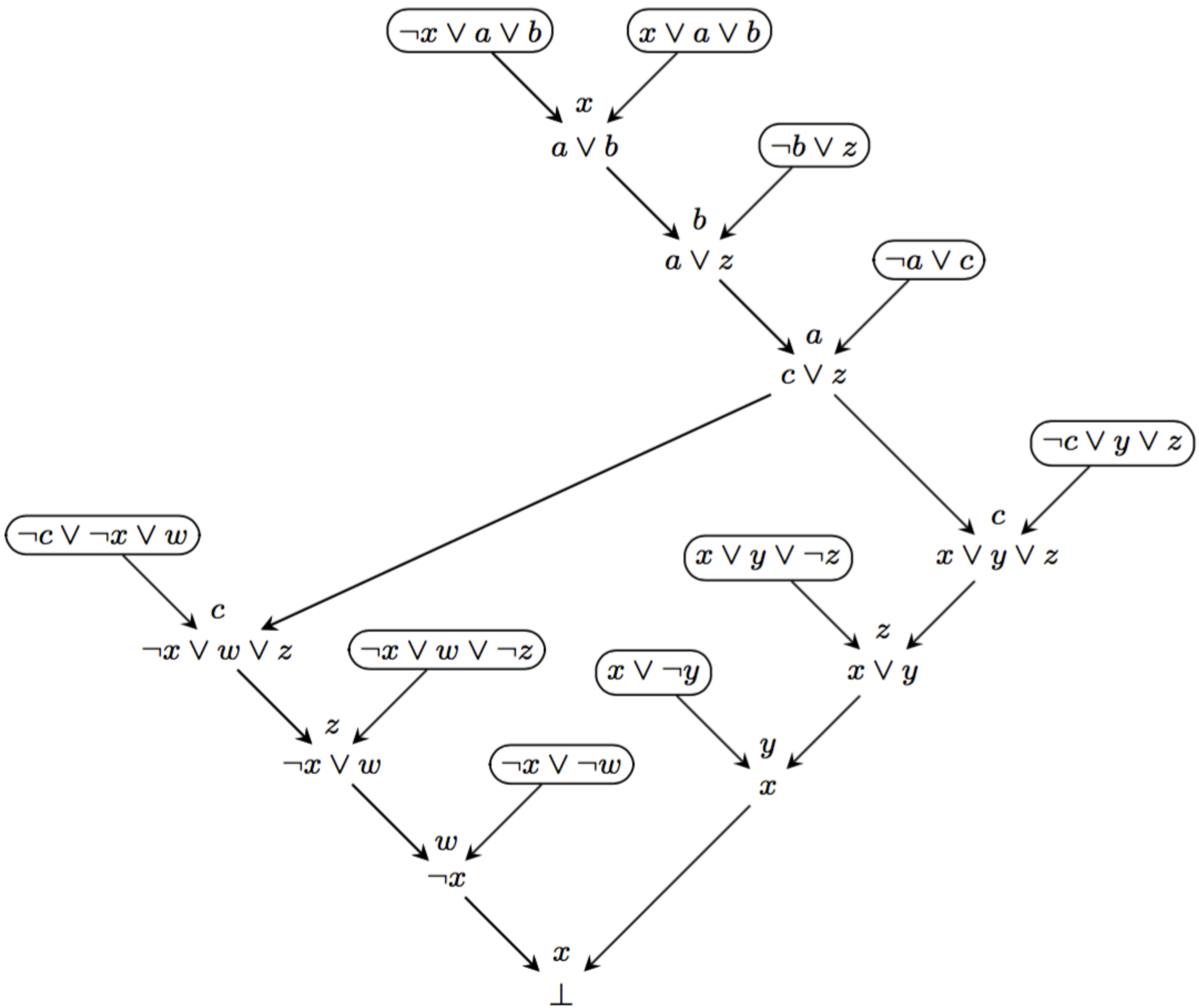
y
x
w
$\neg x$
x
$\bot$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x ∨ a ∨ b
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a ∨ b
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b
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a
c ∨ z
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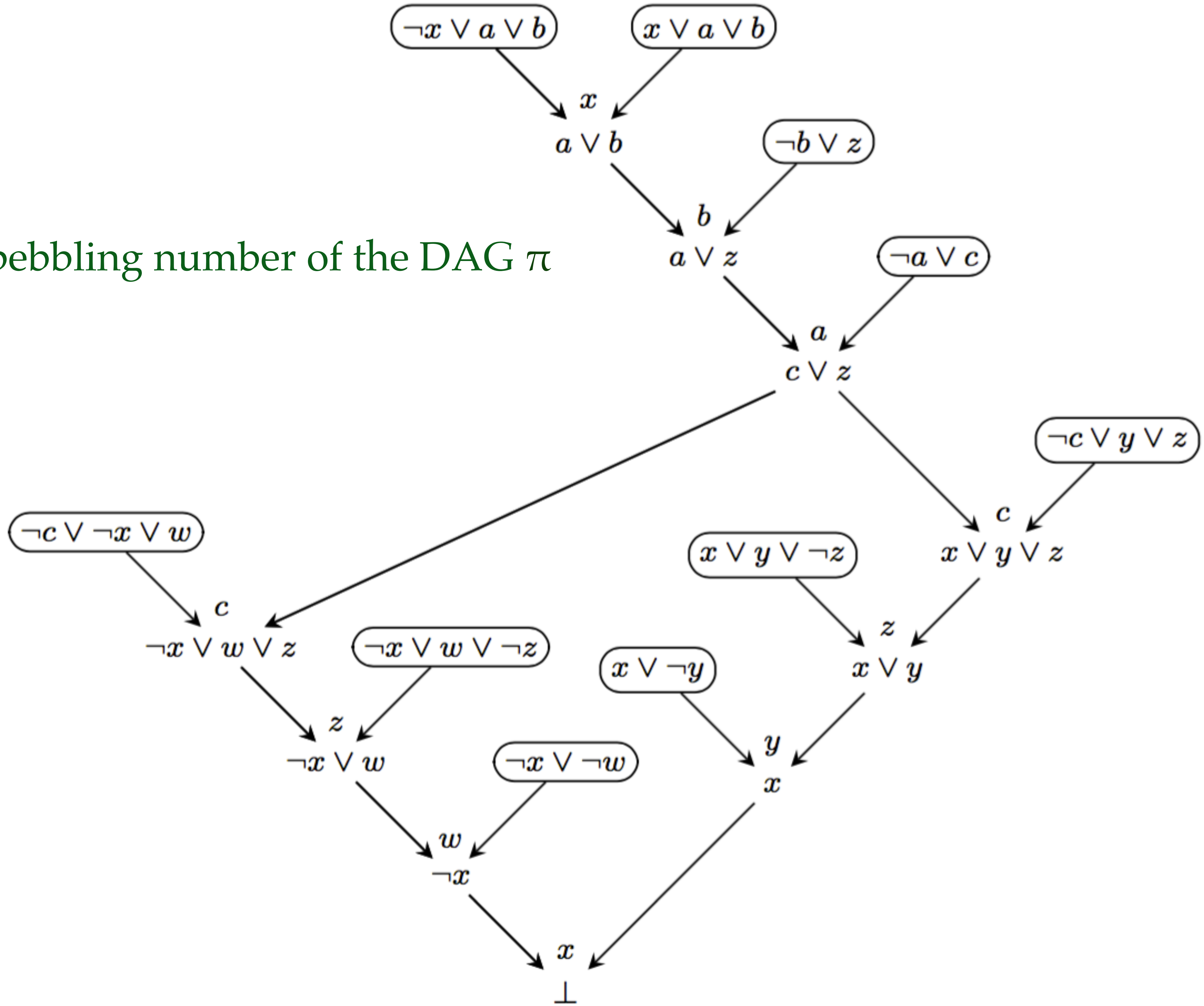
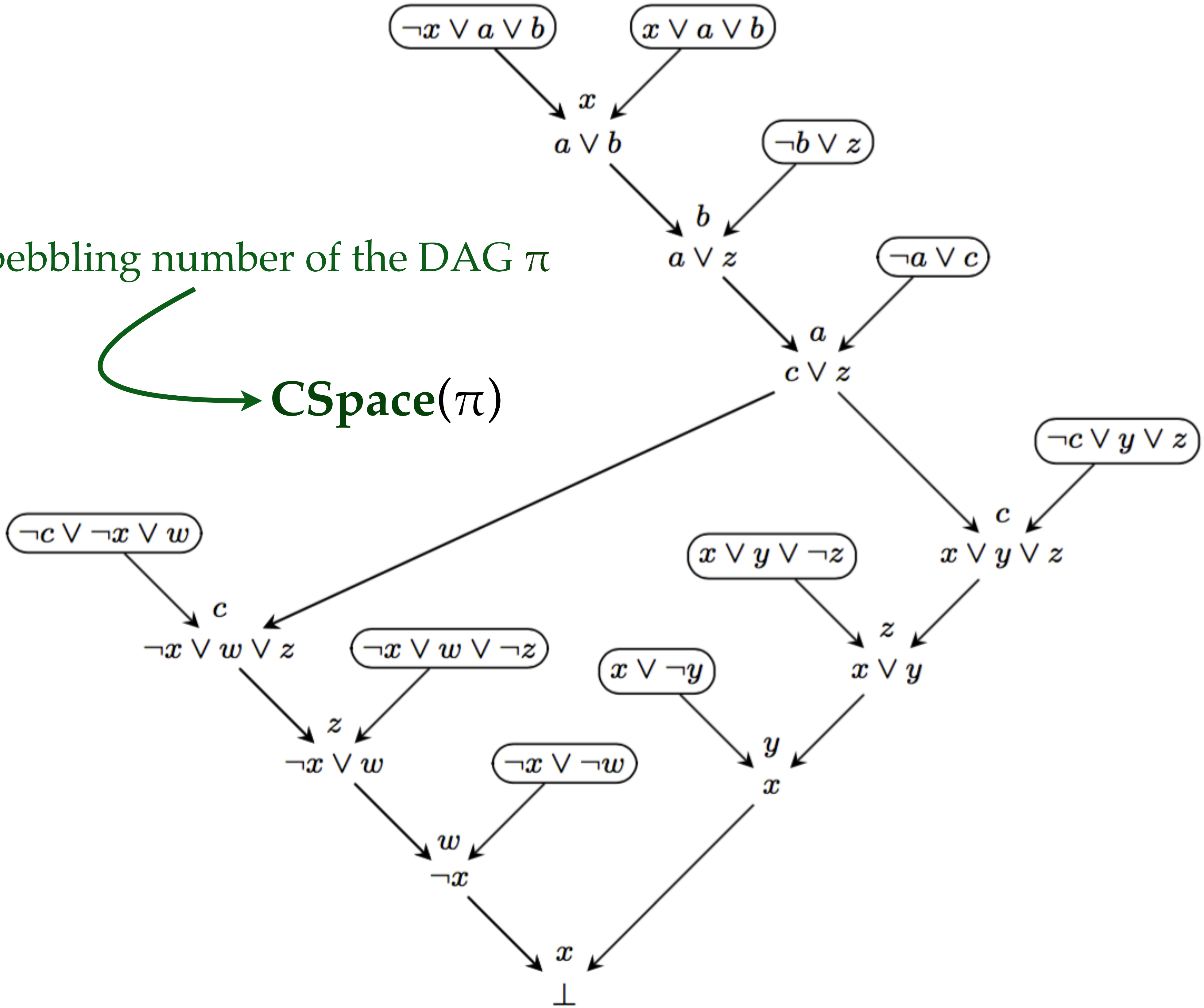
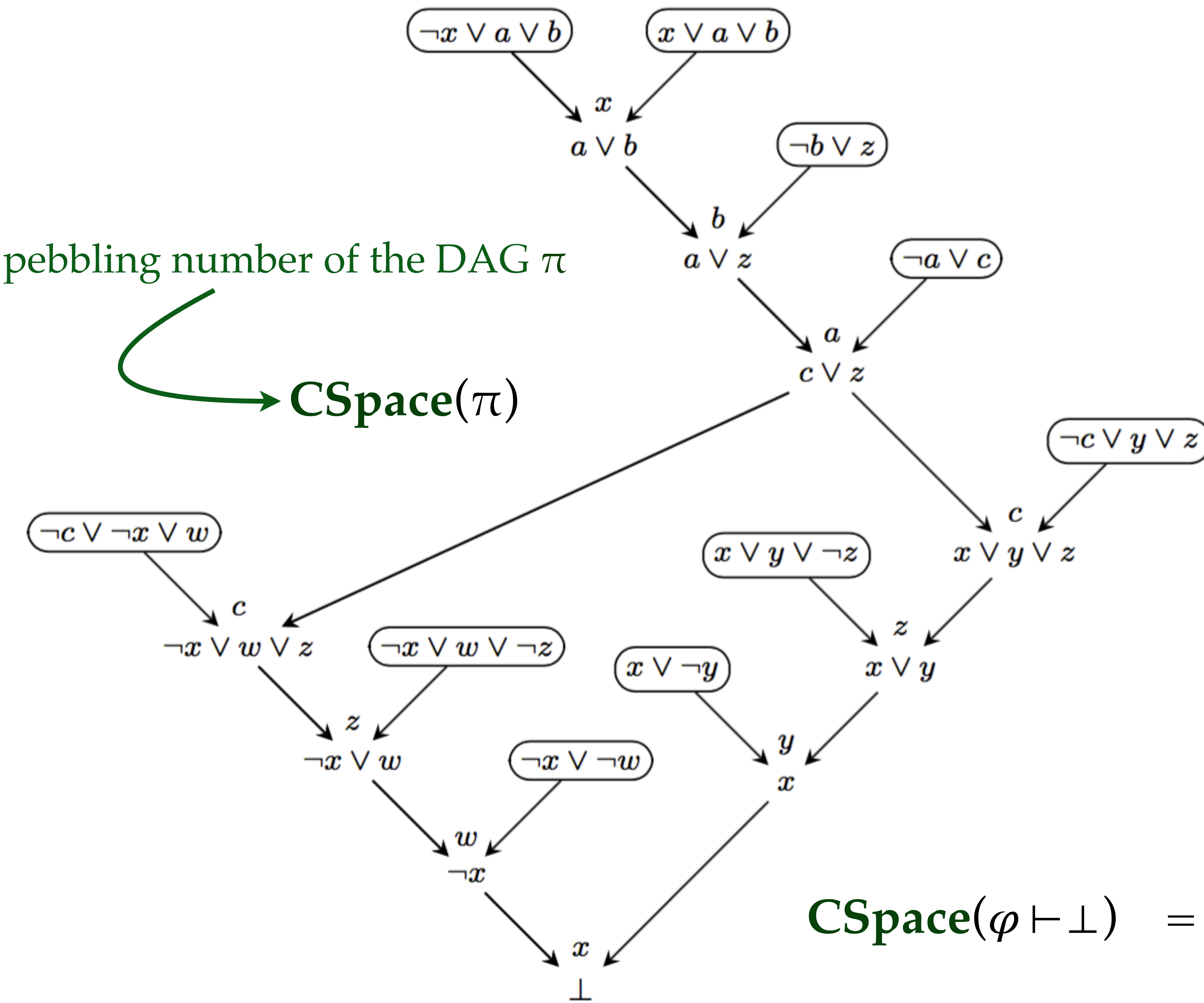
¬c ∨ ¬x ∨ w
c
x ∨ y ∨ ¬z
x ∨ y ∨ z
c
¬x ∨ w ∨ z
¬x ∨ w ∨ ¬z
z
x ∨ ¬y
x ∨ y
z
¬x ∨ w
¬x ∨ ¬w
y
x
w
¬x
x
⊥

pebbling number of the DAG π

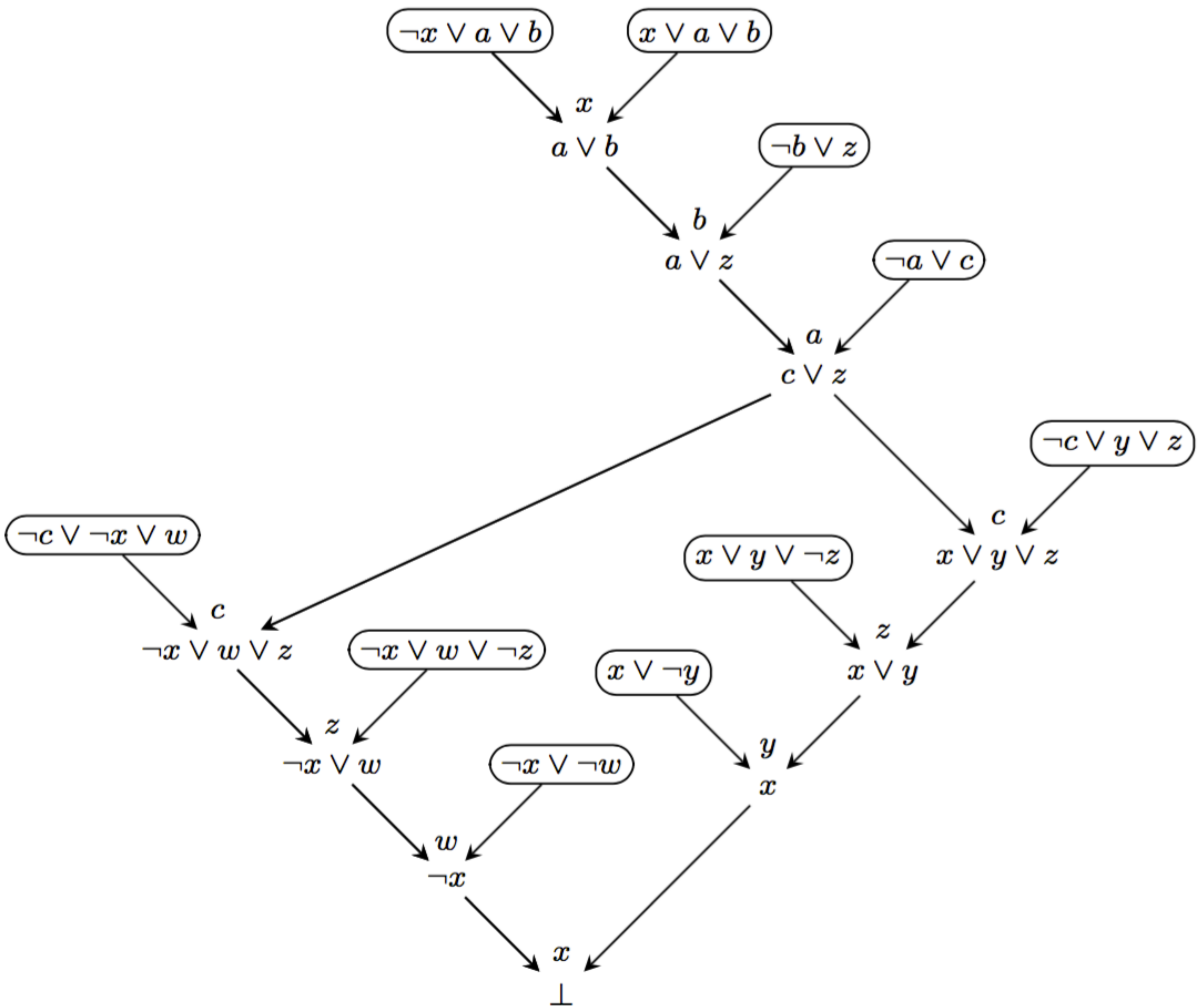




pebbling number of the DAG π → **CSpace**(π)



CSpace$(\varphi \vdash \bot)$ $= \min_\pi$ **CSpace**(π)



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x ∨ a ∨ b
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a ∨ b
¬b ∨ z
b
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a
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c
x ∨ y ∨ z
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c
¬x ∨ w ∨ z
x ∨ y ∨ ¬z
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y
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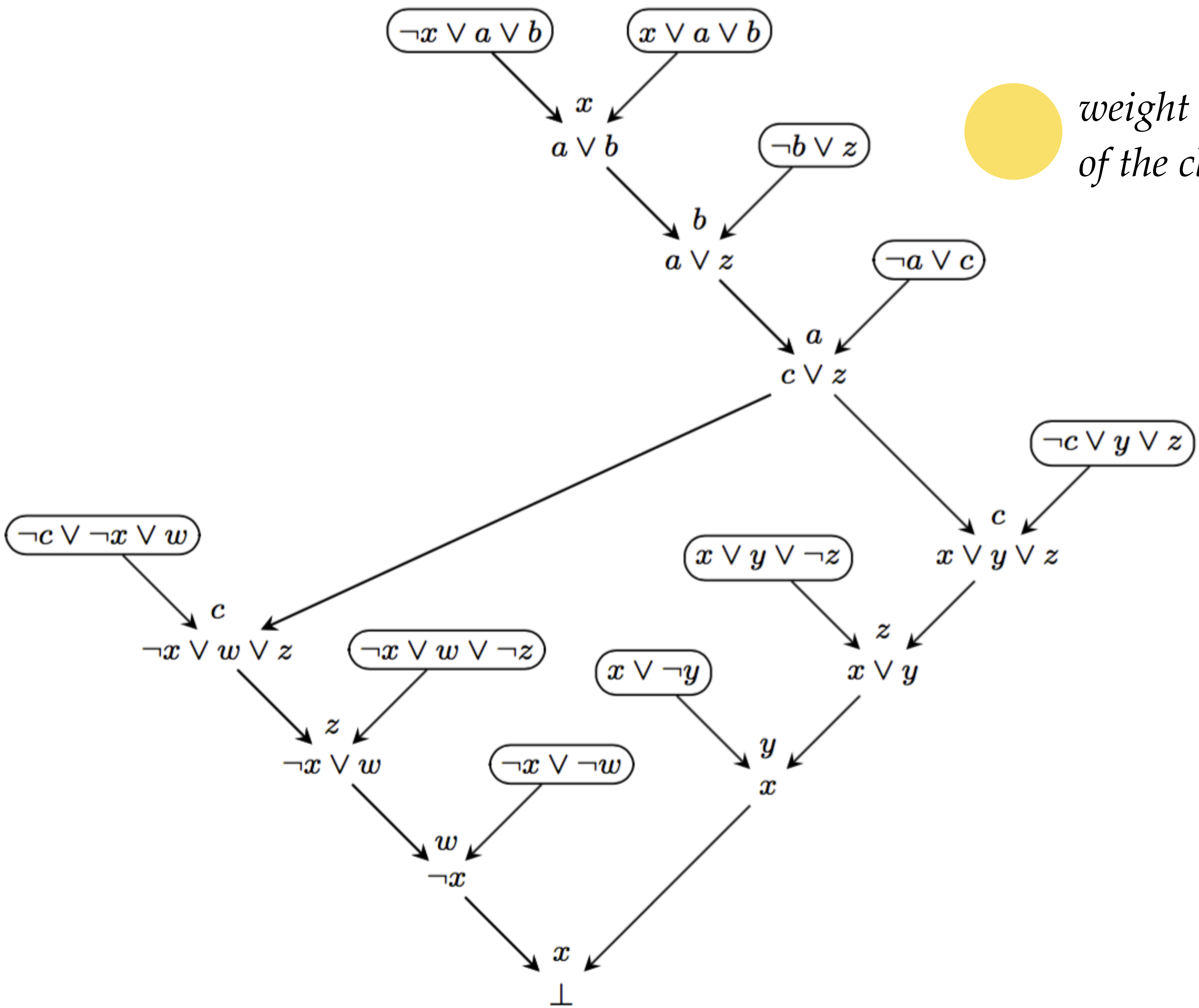
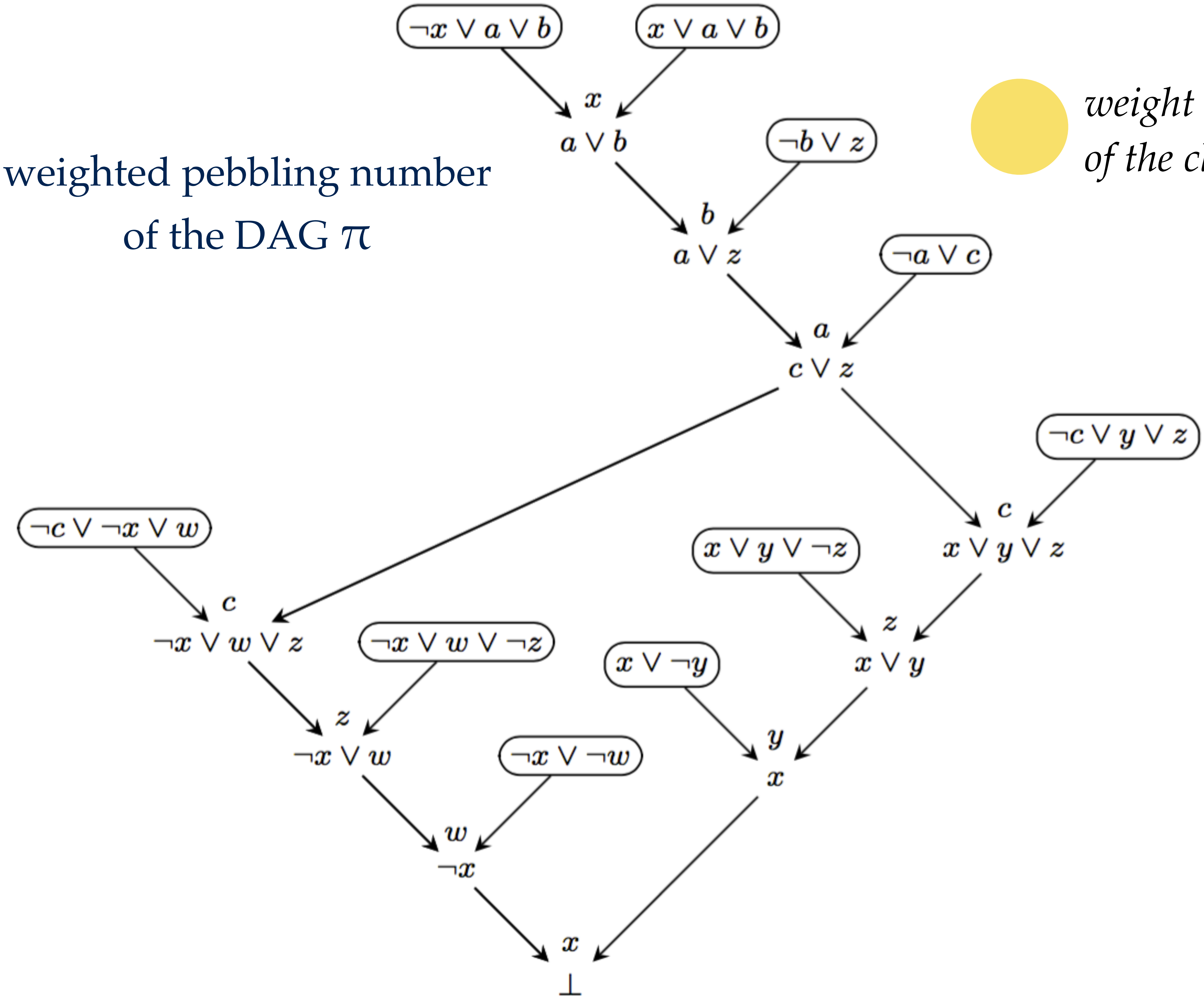
¬x ∨ ¬w
w
¬x
x
⊥



weight as the # of literals of the clause covered

Total Space

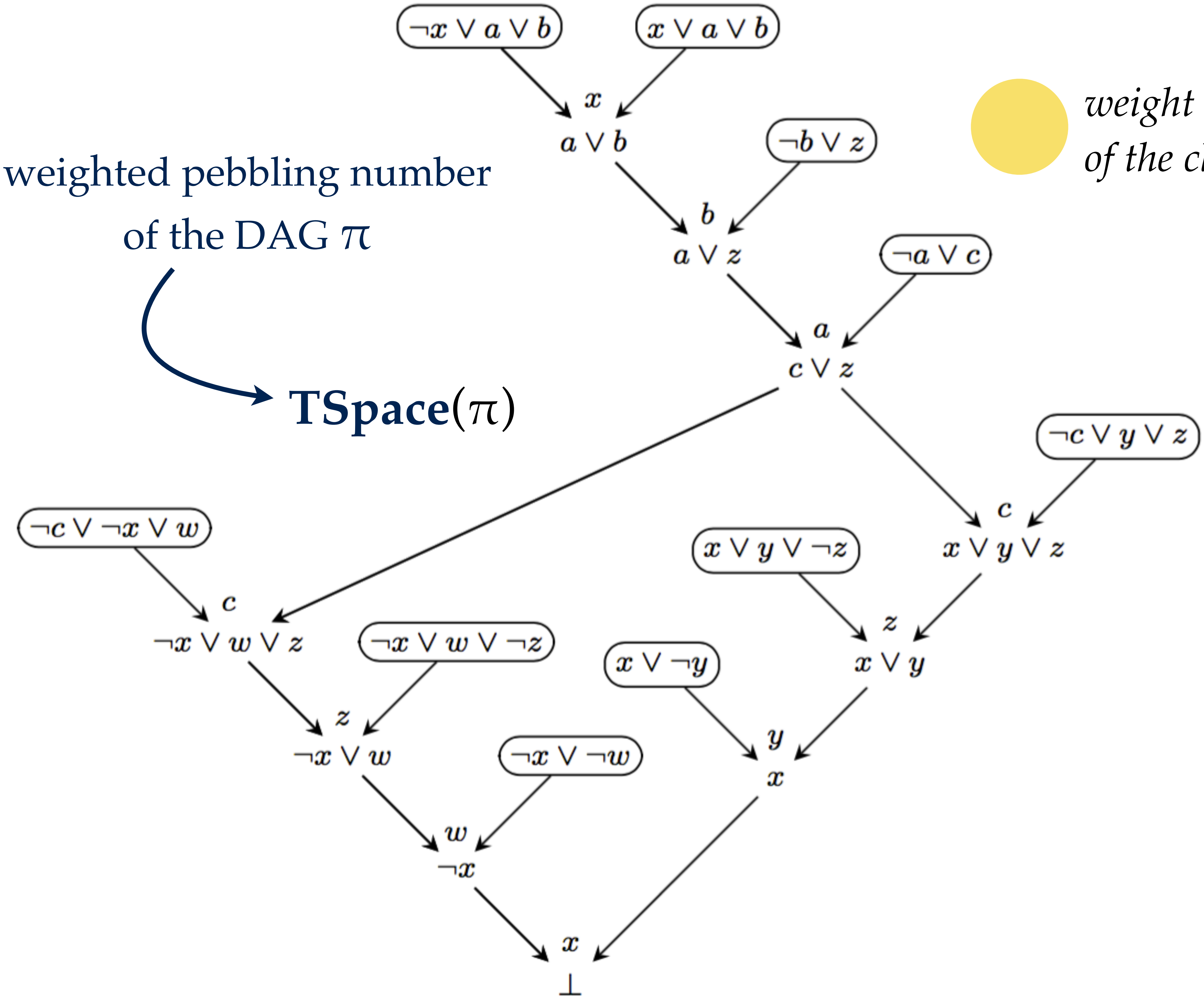
weighted pebbling number
of the DAG π

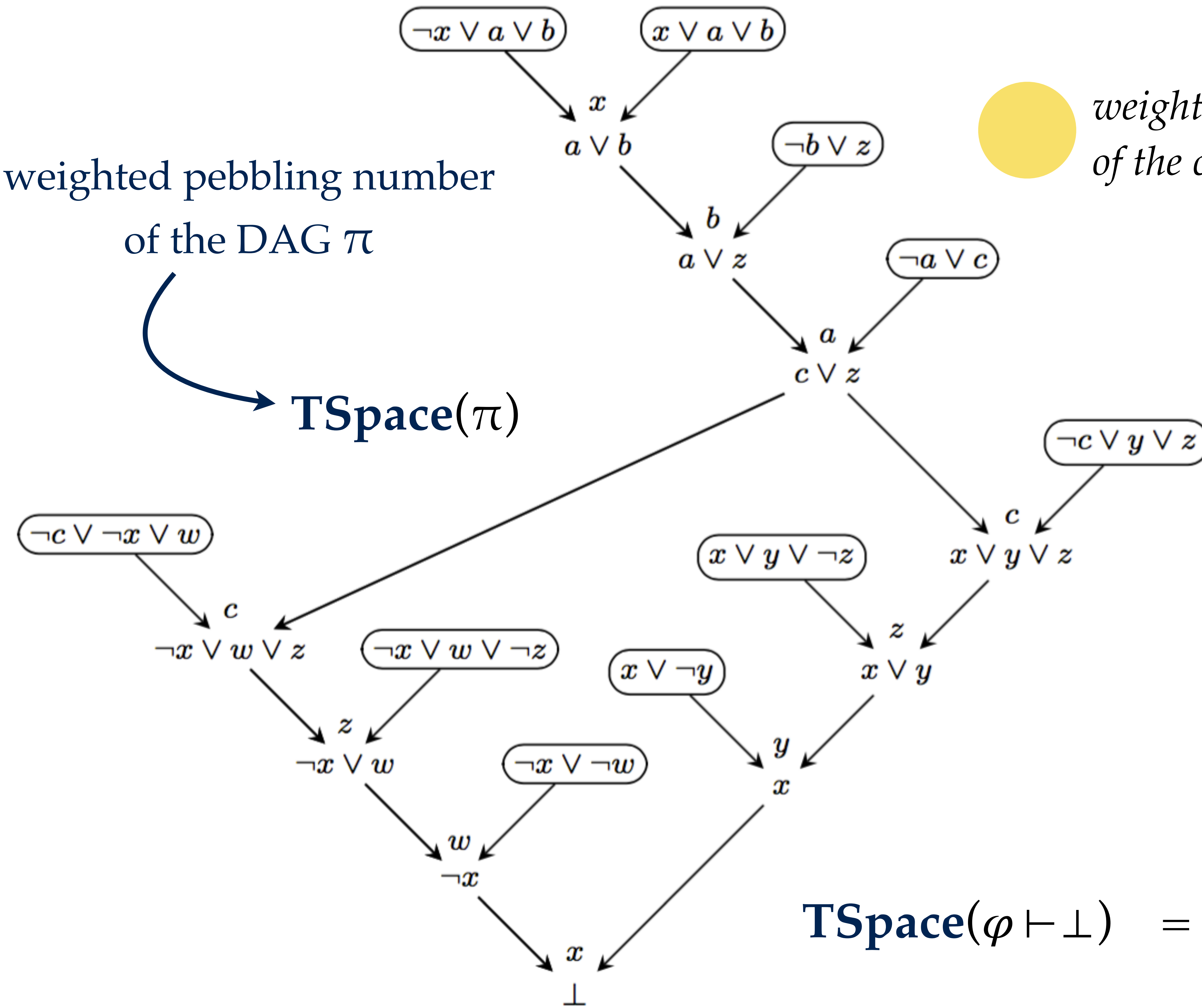


weight as the # of literals of the clause covered weighted pebbling number of the DAG π

TSpace(π)

weight as the # of literals of the clause covered





weighted pebbling number of the DAG π → **TSpace**(π)

weight as the # of literals of the clause covered

TSpace$(\varphi \vdash \bot) \quad = \min_\pi$ **TSpace**(π)

A bigger picture

[Atserias, Dalmau '02]
[Ben-Sasson, Wigderson '01]
[Esteban, Toran '01]

For *every* unsatisfiable k-CNF φ in n variables:

[Atserias, Dalmau '02]
[Ben-Sasson, Wigderson '01]
[Esteban, Toran '01]

For *every* unsatisfiable k-CNF φ in n variables:

$$\mathbf{O}(\mathit{width}(\varphi \vdash \bot)\log n) = \log \mathbf{Size}(\varphi \vdash \bot) = \mathbf{\Omega}((\mathit{width}(\varphi \vdash \bot) - k)^2 / n)$$

[Atserias, Dalmau '02]
[Ben-Sasson, Wigderson '01]
[Esteban, Toran '01]

For *every* unsatisfiable k-CNF φ in n variables:

$$\mathbf{O}(\mathit{width}(\varphi \vdash \bot)\log n) = \log \mathbf{Size}(\varphi \vdash \bot) = \mathbf{\Omega}((\mathit{width}(\varphi \vdash \bot) - k)^2 / n)$$

$$n + 1 \geq \mathbf{CSpace}(\varphi \vdash \bot) = \mathbf{\Omega}(\mathit{width}(\varphi \vdash \bot) - k)$$

[Atserias, Dalmau '02]
[Ben-Sasson, Wigderson '01]
[Esteban, Toran '01]

For *every* unsatisfiable k-CNF φ in n variables:

$$\mathbf{O}(\textit{width}(\varphi \vdash \bot)\log n) = \log \mathbf{Size}(\varphi \vdash \bot) = \mathbf{\Omega}((\textit{width}(\varphi \vdash \bot) - k)^2 / n)$$

$$n + 1 \geq \mathbf{CSpace}(\varphi \vdash \bot) = \mathbf{\Omega}(\textit{width}(\varphi \vdash \bot) - k)$$

$$n(n+1) \geq \mathbf{TSpace}(\varphi \vdash \bot) = \,??$$

[Atserias, Dalmau '02]
[Ben-Sasson, Wigderson '01]
[Esteban, Toran '01]

For *every* unsatisfiable k-CNF φ in n variables:

$$\mathbf{O}(\textit{width}(\varphi \vdash \bot)\log n) = \log \mathbf{Size}(\varphi \vdash \bot) = \mathbf{\Omega}((\textit{width}(\varphi \vdash \bot) - k)^2 / n)$$

$$n + 1 \geq \mathbf{CSpace}(\varphi \vdash \bot) = \mathbf{\Omega}(\textit{width}(\varphi \vdash \bot) - k)$$

$$n(n+1) \geq \mathbf{TSpace}(\varphi \vdash \bot) = \mathbf{\Omega}((\textit{width}(\varphi \vdash \bot) - k)^2)$$

[**B** ’16 (to appear)]

[Atserias, Dalmau ’02]
[Ben-Sasson, Wigderson ’01]
[Esteban, Toran ’01]

For every unsatisfiable ***k***-CNF formula φ

$$\mathbf{TSpace}(\varphi \vdash \bot) \geq (\mathit{width}(\varphi \vdash \bot) - \boldsymbol{k} - 4)^2/16$$

or more precisely

$$\mathbf{TSpace}(\varphi \vdash \bot) \geq (\underline{\mathit{awidth}}(\varphi \vdash \bot) - 2)^2/4$$

or more precisely

every pebbling of a Resolution refutation of φ *must contain a configuration of at least* $(\underline{\mathit{awidth}}(\varphi \vdash \bot) - 2)/2$ *pebbles each covering a clause of at least* $(\underline{\mathit{awidth}}(\varphi \vdash \bot) - 2)/2$ *many literals.*

[**B** ICALP'16 (to appear)]

For every unsatisfiable k-CNF formula φ

$$\textbf{TSpace}(\varphi \vdash \bot) \geq \boldsymbol{\Omega}((\textit{width}(\varphi \vdash \bot) - k)^2)$$

There are (*many*!) CNF formulas φ in n variables s.t.

$$\textbf{CSpace}(\varphi \vdash \bot) = \Theta(n)$$

$$\textbf{TSpace}(\varphi \vdash \bot) = \Theta(n^2)$$

For every unsatisfiable k-CNF formula φ

$$\textbf{TSpace}(\varphi \vdash \bot) \geq \boldsymbol{\Omega}((\textit{width}(\varphi \vdash \bot) - k)^2)$$

There are (*many*!) CNF formulas φ in n variables s.t.

$$\textbf{CSpace}(\varphi \vdash \bot) = \Theta(n)$$

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[Alekhnovic, Ben-Sasson, Razborov, Wigderson '02]
[**B**, Galesi, Thapen '14]
[Bennett, **B,** Galesi, Huynh, Molloy, Wollan]
[**B** ICALP'16 (to appear)]

For every unsatisfiable k-CNF formula φ

$$\textbf{TSpace}(\varphi \vdash \bot) \geq \Omega((\mathit{width}(\varphi \vdash \bot) - k)^2)$$

There are (*many*!) CNF formulas φ in n variables s.t.

$$\textbf{CSpace}(\varphi \vdash \bot) = \Theta(n)$$

$$\textbf{TSpace}(\varphi \vdash \bot) = \Theta(n^2)$$

for example:

[Alekhnovic, Ben-Sasson, Razborov, Wigderson '02]
[**B**, Galesi, Thapen '14]
[Bennett, **B**, Galesi, Huynh, Molloy, Wollan]
[**B** ICALP'16 (to appear)]

For every unsatisfiable k-CNF formula φ

$$\textbf{TSpace}(\varphi \vdash \bot) \geq \Omega((\mathit{width}(\varphi \vdash \bot) - k)^2)$$

There are (*many*!) CNF formulas φ in n variables s.t.

$$\textbf{CSpace}(\varphi \vdash \bot) = \Theta(n)$$

$$\textbf{TSpace}(\varphi \vdash \bot) = \Theta(n^2)$$

for example:

- for φ any random k-CNF formula with n variables and $O(n)$ clauses

[Alekhnovic, Ben-Sasson, Razborov, Wigderson '02]
[**B**, Galesi, Thapen '14]
[Bennett, **B,** Galesi, Huynh, Molloy, Wollan]
[**B** ICALP'16 (to appear)]

For every unsatisfiable k-CNF formula φ

$$\mathbf{TSpace}(\varphi \vdash \bot) \geq \mathbf{\Omega}((\mathit{width}(\varphi \vdash \bot) - k)^2)$$

There are (*many*!) CNF formulas φ in n variables s.t.

$$\mathbf{CSpace}(\varphi \vdash \bot) = \Theta(n)$$

$$\mathbf{TSpace}(\varphi \vdash \bot) = \Theta(n^2)$$

for example:

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width

For every unsatisfiable k-CNF formula φ

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width →

[Kullmann '99]

For every unsatisfiable k-CNF formula φ

TSpace$(\varphi \vdash \bot) \geq$ (***width***$(\varphi \vdash \bot) - k - 4)^2 / 16$

width → ***<u>awidth</u>***

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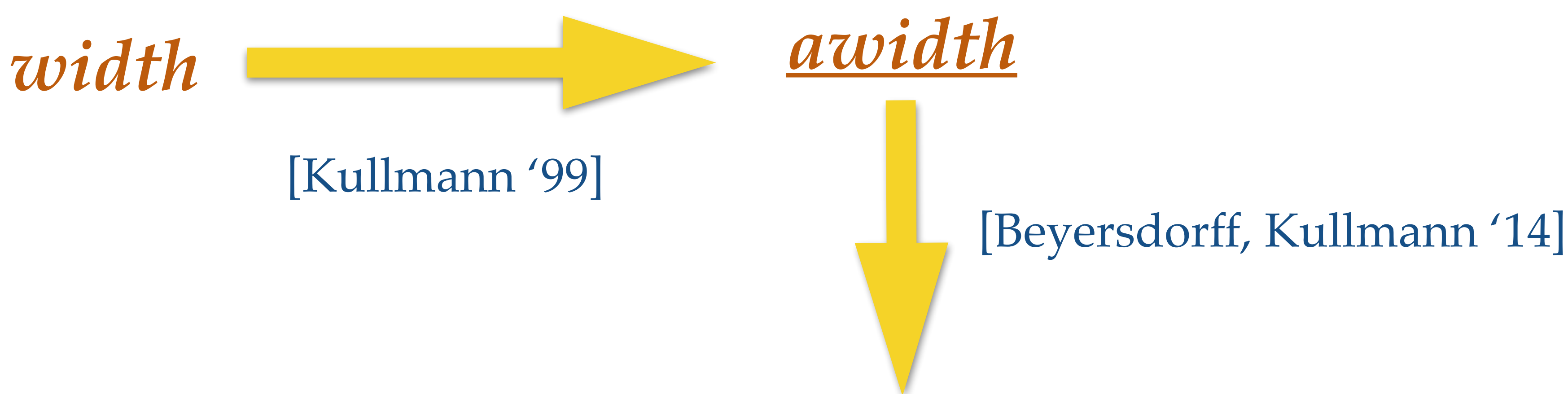

Proof scheme

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width → *awidth*

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combinatorial characterization of *awidth*

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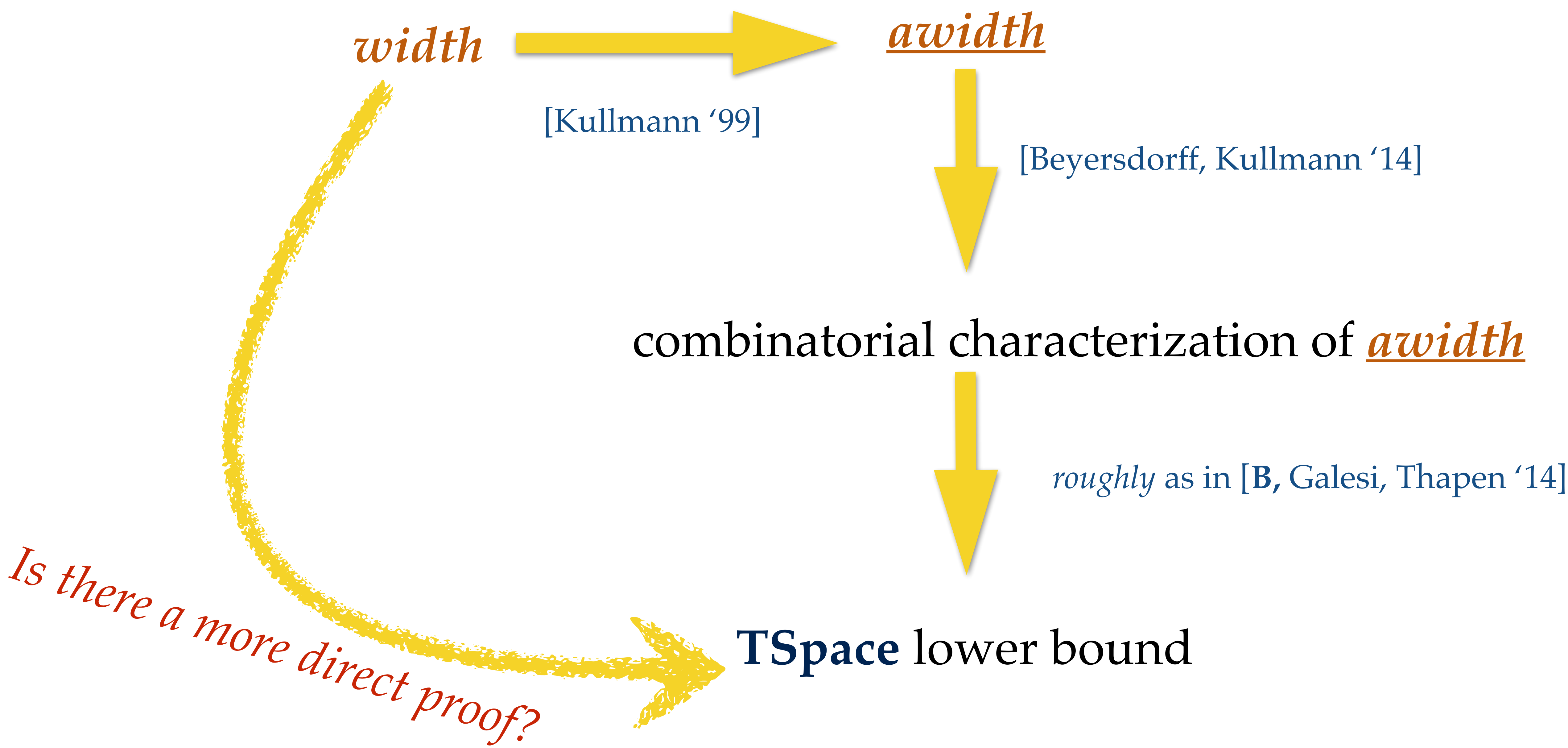
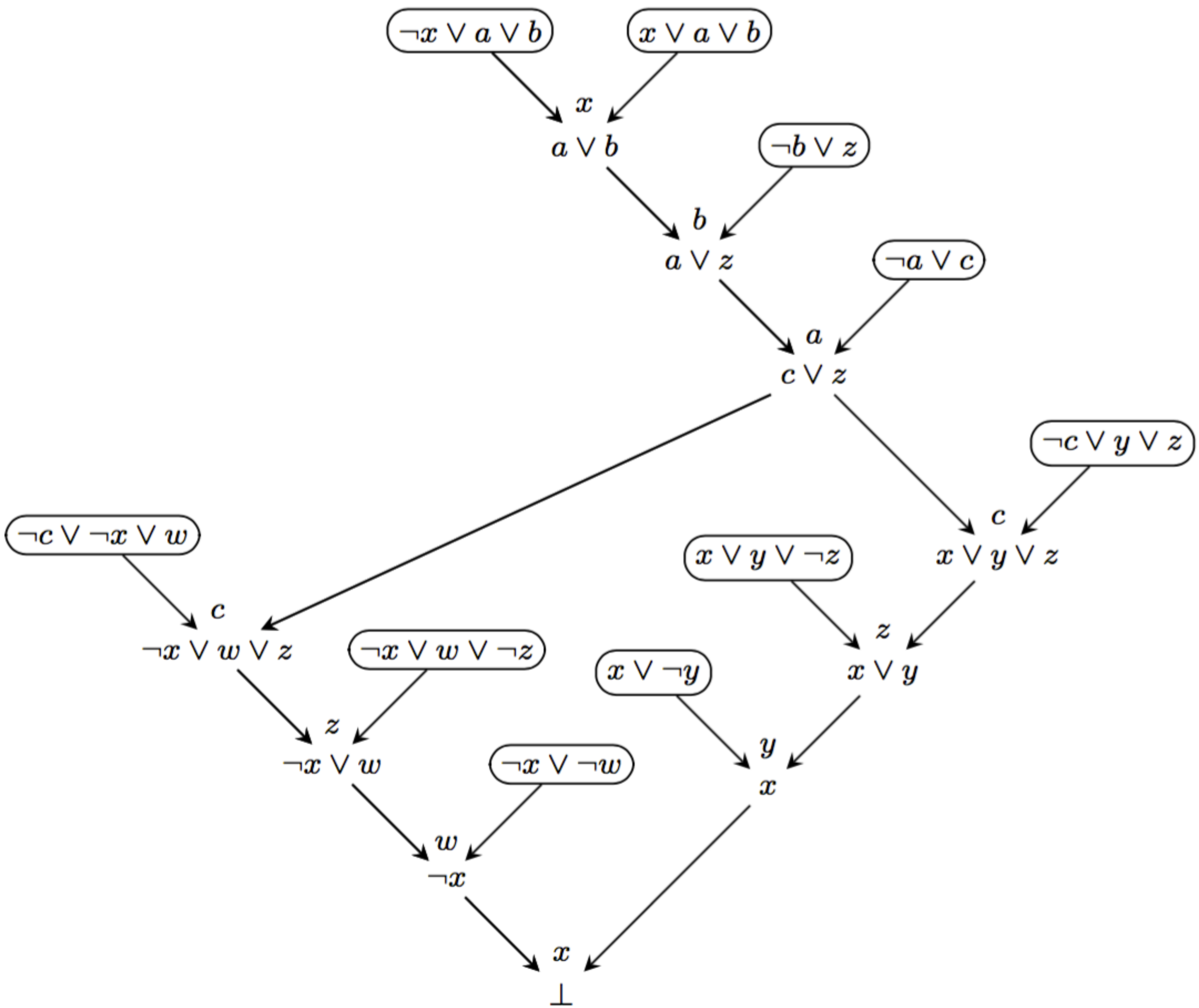


Asymmetric width



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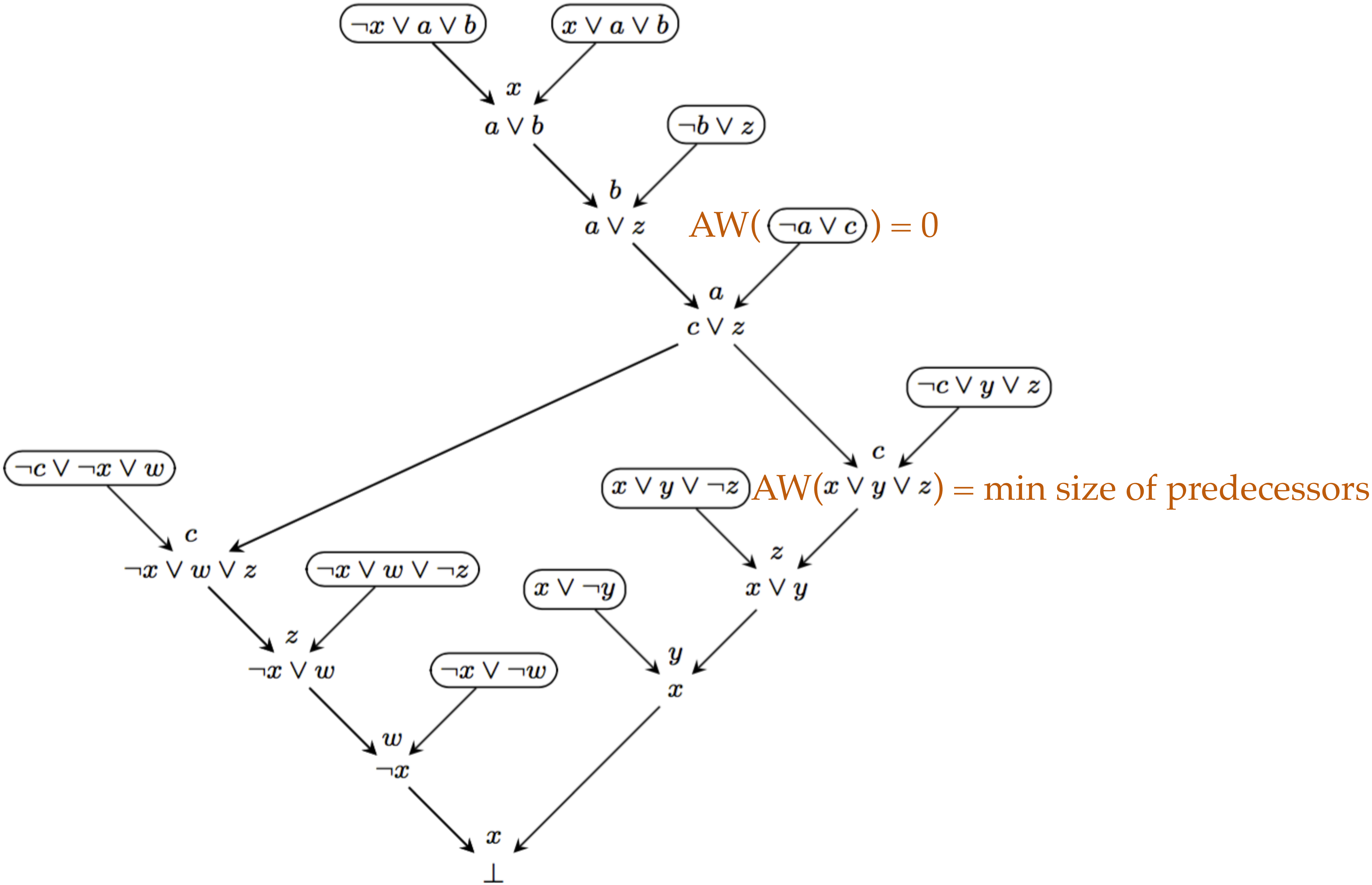

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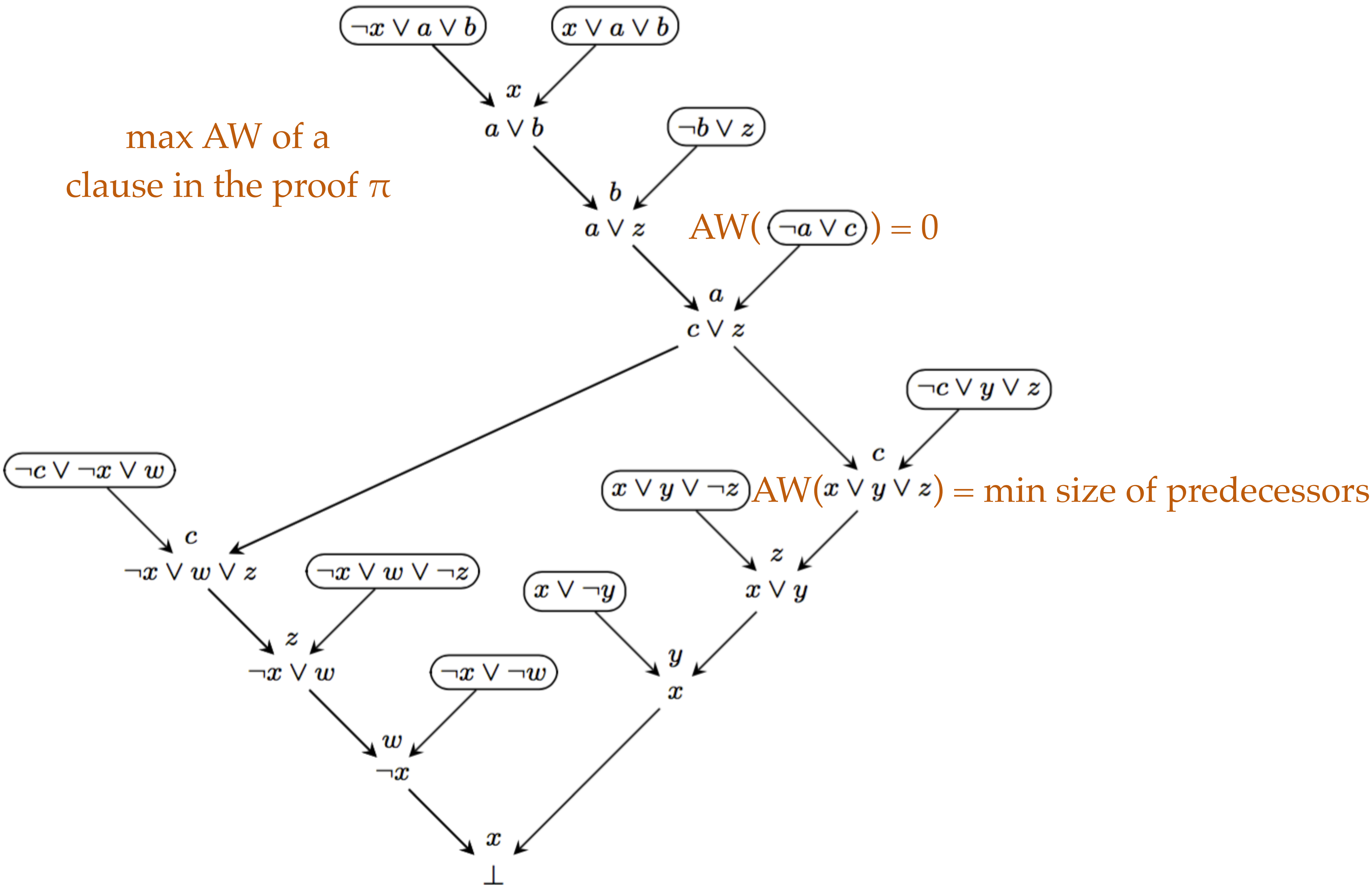

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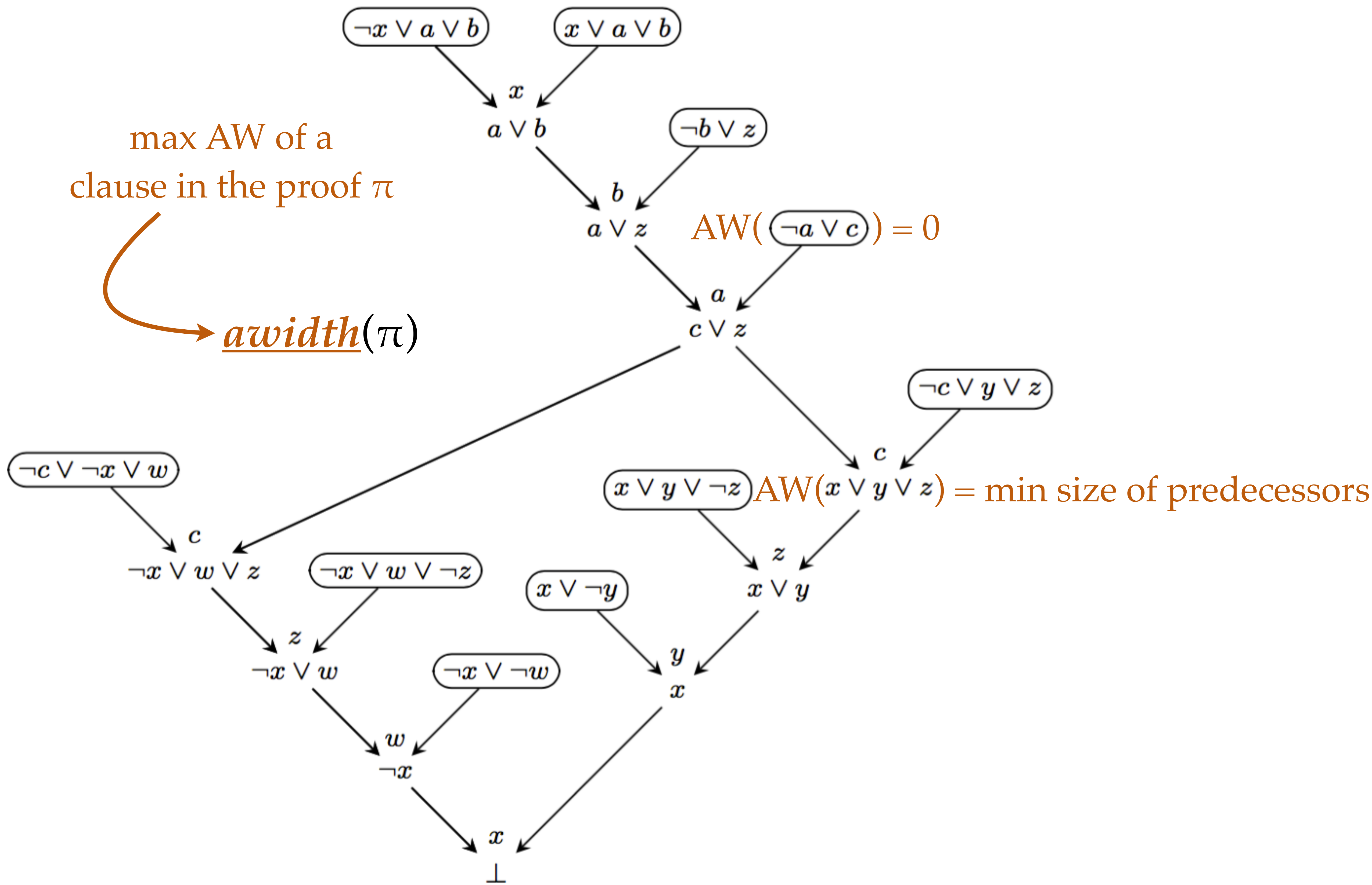

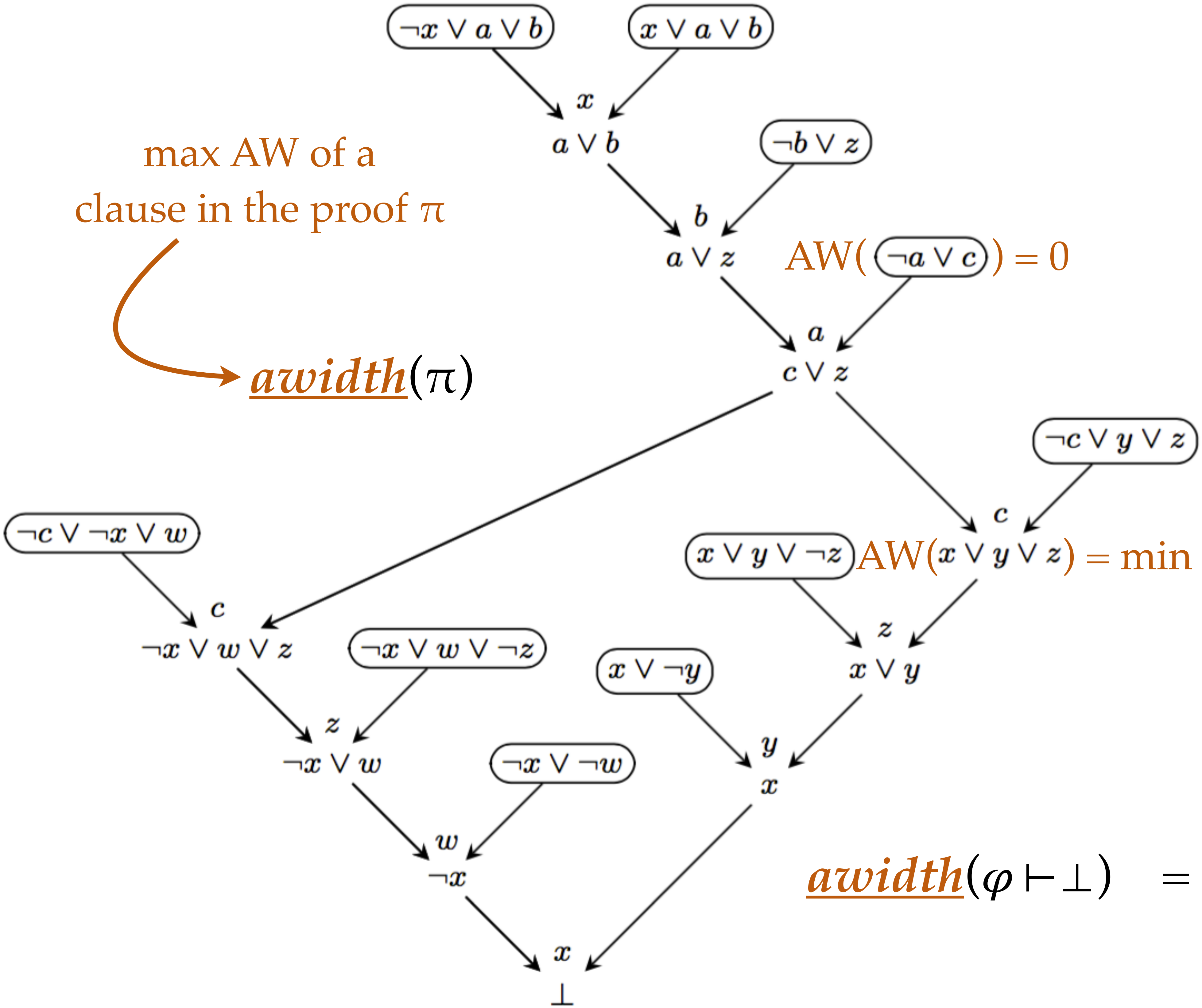

max AW of a clause in the proof π

$\to$ ***awidth***(π)

awidth$(\varphi \vdash \bot) \quad = \min_\pi$ ***awidth***(π)

[Kullmann '99]

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TSpace lower bound

Combinatorial characterizations

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[Atserias, Dalmau '01]

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TSpace lower bound

Beyond Resolution

[Clegg, Edmonds, Impagliazzo '99]
[Alekhnovic, Ben-Sasson, Razborov, Wigderson '02]
[**B**, Galesi '13]
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$$\mathbf{TSpace}_{\mathsf{PC}}(\mathrm{CT}_n \vdash \bot) = \mathbf{\Theta}(n^2)$$

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Open problems

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- Is there any unsatisfiable CNF φ in n variables and $n^{O(1)}$ clauses with $n^{O(1)}$ Resolution refutations s.t. **TSpace**$(\varphi \vdash \bot) = \Theta(n^2)$?

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Thanks! 🙂

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