Proof Complexity Modulo the Polynomial Hierarchy: Understanding Alternation as a Source of Hardness

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St. Petersburg, May 2016

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Note: SAT treated as a black-box oracle by QBF solvers
(e.g. QBF solver *sKizzo* - Benedetti '05)

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- Connection to separation of complexity classes

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This also clashes with the QBF view of SAT as an oracle.

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- An ensemble has polynomially bounded proofs if it *contains* a proof system where all false QBFs have polysize proofs
- Result: straightforward to define ensembles that have poly bd proofs on any set of QBFs with bounded alternation

  So, proof size lower bounds address the ability to handle alternation

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This talk: focus on 1 and 2.

Act: Framework

Proof system ensemble

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Def (simplified): A proof system ensemble for a language L is a sequence $(L_k)_{k\geqslant 1}$ of langs in PH such that:

$$(\forall k \geqslant 1) \qquad \{x \mid \exists \pi : (x, \pi) \in L_k\} = L$$

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Def: Let Z be a set of functions $\mathbb{N} \to \mathbb{N}$. (eg: $Z = \Omega(2^n)$)
A pf system ensemble $(L_k)_{k\geqslant 1}$ requires proofs of size Z on instances $\Phi_1, \Phi_2, \ldots$ if $\forall k \geqslant 1, \exists z \in Z$ where

$$(\forall n \geqslant 1, \forall \pi) \qquad (\Phi_n, \pi) \in L_k \Rightarrow |\pi| \geqslant z(n)$$

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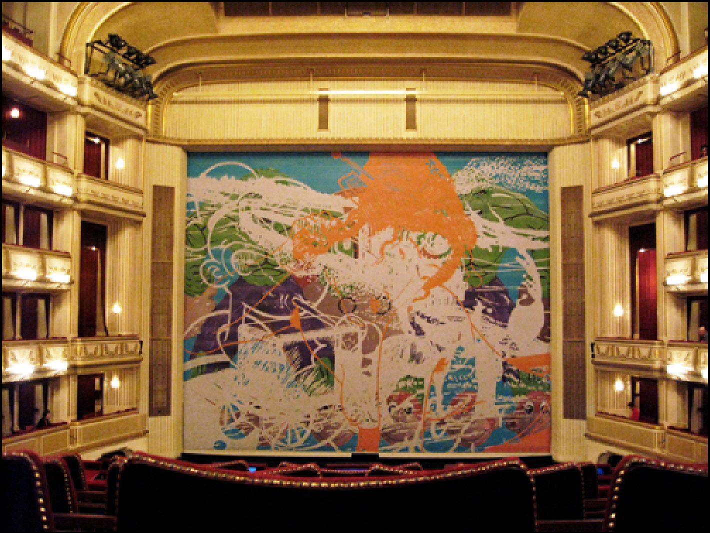
Relationship between this framework & PH vs. PSPACE qtn

is analogous to

the relationship between SAT proof complexity &
NP vs. coNP qtn



Act: Relaxing QU-resolution

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To use prop: need to detect when $\Phi[a]$ is false
...but this is hard in general!

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Example relaxations: $\forall y \forall y' \exists x_1 \exists x_2 \exists x_3 \psi$, $\exists x_1 \forall y' \exists x_2 \forall y \exists x_3 \psi$

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Remarks:

- This makes sense even if Φ is not clausal, i.e., even if Φ has the form $Q_1 v_1 \ldots Q_n v_n$(circuit)

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Def: For $k \geqslant 2$, define $H(\Phi, \Pi_k)$ as the set

$$\{\text{clause}(a) \mid \Phi[a] \text{ has a false } \Pi_k \text{ relaxation}\}$$

We have $H(\Phi, \Pi_2) \subseteq H(\Phi, \Pi_3) \subseteq \cdots$

Def: Relaxing QU-resolution is the proof system ensemble $(L_k)_{k \geqslant 2}$ where L_k is defined as

$$\{(\Phi, \pi) \mid \pi \text{ is a QU-res proof of } \Phi \text{ from } H(\Phi, \Pi_k)\}$$

Remarks:

- This makes sense even if Φ is not clausal, i.e., even if Φ has the form $Q_1 v_1 \ldots Q_n v_n$(circuit)
- This way of “lifting” to an enhanced set of clauses can be used to define relaxed versions of any clause-based QBF proof system

Contributions

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1. Framework – proof system ensembles
2. Definition of *relaxing QU-resolution*,
   a particular ensemble obtained by “lifting” QU-resolution
3. Two technical results on relaxing QU-resolution:
   exponential lower bound for general version,
   exponential separation of general/tree-like versions

Questions

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Are there natural ways to define other pf system ensembles?

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What constitutes a good/reasonable/natural/etc. definition of a proof system ensemble?

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end [*fin*]