Subtraction-free complexity and cluster algebras
(joint work with S. Fomin, G. Koshevoy)

Dima Grigoriev (Lille)

CNRS

18/05/2016, Saint-Petersbourg

Complexity of rational functions with respect to a subset of arithmetic operations

Let $M \subset \{+, -, \times, /\}$. Complexity $C_M(f)$ for a rational function $f \in \mathbb{Q}(X_1, \ldots, X_n)$ is defined as the minimal number of operations from M necessary to compute f, provided it is finite.

Problem. For given $M \subset M_1 \subset \{+, -, \times, /\}$ how big can be $C_M(f)$ in comparison with $C_{M_1}(f)$?

This problem is non-trivial just for three pairs of $M \subset M_1$.

Theorem

$C_{+,-,\times}(f) \leq O(C_{+,-,\times,/}(f) \cdot \deg(f))$
*for a polynomial f (***V. Strassen, 1973***)*

("ring complexity" $\leq O$ ("field complexity" $\cdot \deg$))
Question. Is the bound sharp for big $\deg(f)$?

Corollary

$C_{+,-,\times}(\det) \leq O(n^4)$ (**D. K. Faddeev, V. N. Faddeeva, 1960**)

Complexity of rational functions with respect to a subset of arithmetic operations

Let $M \subset \{+, -, \times, /\}$. Complexity $C_M(f)$ for a rational function $f \in \mathbb{Q}(X_1, \dots, X_n)$ is defined as the minimal number of operations from M necessary to compute f, provided it is finite.

Problem. For given $M \subset M_1 \subset \{+, -, \times, /\}$ how big can be $C_M(f)$ in comparison with $C_{M_1}(f)$?

This problem is non-trivial just for three pairs of $M \subset M_1$.

Theorem

$C_{+,-,\times}(f) \leq O(C_{+,-,\times,/}(f) \cdot \deg(f))$
*for a polynomial f (**V. Strassen, 1973**)*

("ring complexity" $\leq O$ ("field complexity" $\cdot \deg$))
Question. Is the bound sharp for big $\deg(f)$?

Corollary

$C_{+,-,\times}(\det) \leq O(n^4)$ (**D. K. Faddeev, V. N. Faddeeva, 1960**)

Complexity of rational functions with respect to a subset of arithmetic operations

Let $M \subset \{+, -, \times, /\}$. Complexity $C_M(f)$ for a rational function $f \in \mathbb{Q}(X_1, \dots, X_n)$ is defined as the minimal number of operations from M necessary to compute f, provided it is finite.

Problem. For given $M \subset M_1 \subset \{+, -, \times, /\}$ how big can be $C_M(f)$ in comparison with $C_{M_1}(f)$?

This problem is non-trivial just for three pairs of $M \subset M_1$.

Theorem

$C_{+,-,\times}(f) \leq O(C_{+,-,\times,/}(f) \cdot \deg(f))$
*for a polynomial f (***V. Strassen, 1973***)*

("ring complexity" $\leq O$ ("field complexity" $\cdot \deg$))

Question. Is the bound sharp for big $\deg(f)$?

Corollary

$C_{+,-,\times}(\det) \leq O(n^4)$ *(***D. K. Faddeev, V. N. Faddeeva, 1960***)*

Complexity of rational functions with respect to a subset of arithmetic operations

Let $M \subset \{+, -, \times, /\}$. Complexity $C_M(f)$ for a rational function $f \in \mathbb{Q}(X_1, \dots, X_n)$ is defined as the minimal number of operations from M necessary to compute f, provided it is finite.

Problem. For given $M \subset M_1 \subset \{+, -, \times, /\}$ how big can be $C_M(f)$ in comparison with $C_{M_1}(f)$?

This problem is non-trivial just for three pairs of $M \subset M_1$.

Theorem

$C_{+,-,\times}(f) \leq O(C_{+,-,\times,/}(f) \cdot \deg(f))$
*for a polynomial f (**V. Strassen, 1973**)*

("ring complexity" $\leq O$ ("field complexity" $\cdot \deg$))
Question. Is the bound sharp for big $\deg(f)$?

Corollary

$C_{+,-,\times}(\det) \leq O(n^4)$ (**D. K. Faddeev, V. N. Faddeeva, 1960**)

Complexity of rational functions with respect to a subset of arithmetic operations

Let $M \subset \{+, -, \times, /\}$. Complexity $C_M(f)$ for a rational function $f \in \mathbb{Q}(X_1, \dots, X_n)$ is defined as the minimal number of operations from M necessary to compute f, provided it is finite.

Problem. For given $M \subset M_1 \subset \{+, -, \times, /\}$ how big can be $C_M(f)$ in comparison with $C_{M_1}(f)$?

This problem is non-trivial just for three pairs of $M \subset M_1$.

Theorem

$C_{+,-,\times}(f) \leq O(C_{+,-,\times,/}(f) \cdot \deg(f))$
*for a polynomial f (**V. Strassen, 1973**)*

("ring complexity" $\leq O$ ("field complexity" $\cdot \deg$))

Question. Is the bound sharp for big $\deg(f)$?

Corollary

$C_{+,-,\times}(\det) \leq O(n^4)$ (**D. K. Faddeev, V. N. Faddeeva, 1960**)

Complexity of rational functions with respect to a subset of arithmetic operations

Let $M \subset \{+, -, \times, /\}$. Complexity $C_M(f)$ for a rational function $f \in \mathbb{Q}(X_1, \dots, X_n)$ is defined as the minimal number of operations from M necessary to compute f, provided it is finite.

Problem. For given $M \subset M_1 \subset \{+, -, \times, /\}$ how big can be $C_M(f)$ in comparison with $C_{M_1}(f)$?

This problem is non-trivial just for three pairs of $M \subset M_1$.

Theorem

$C_{+,-,\times}(f) \leq O(C_{+,-,\times,/}(f) \cdot \deg(f))$
*for a polynomial f (**V. Strassen, 1973**)*

("ring complexity" $\leq O$ ("field complexity" $\cdot \deg$))

Question. Is the bound sharp for big $\deg(f)$?

Corollary

$C_{+,-,\times}(\det) \leq O(n^4)$ (**D. K. Faddeev, V. N. Faddeeva, 1960**)

Complexity of rational functions with respect to a subset of arithmetic operations

Let $M \subset \{+, -, \times, /\}$. Complexity $C_M(f)$ for a rational function $f \in \mathbb{Q}(X_1, \dots, X_n)$ is defined as the minimal number of operations from M necessary to compute f, provided it is finite.

Problem. For given $M \subset M_1 \subset \{+, -, \times, /\}$ how big can be $C_M(f)$ in comparison with $C_{M_1}(f)$?

This problem is non-trivial just for three pairs of $M \subset M_1$.

Theorem

$C_{+,-,\times}(f) \leq O(C_{+,-,\times,/}(f) \cdot \deg(f))$
*for a polynomial f (**V. Strassen, 1973**)*

("ring complexity" $\leq O$ ("field complexity" $\cdot \deg$))

Question. Is the bound sharp for big $\deg(f)$?

Corollary

$C_{+,-,\times}(\det) \leq O(n^4)$ (**D. K. Faddeev, V. N. Faddeeva, 1960**)

Complexity of rational functions with respect to a subset of arithmetic operations

Let $M \subset \{+, -, \times, /\}$. Complexity $C_M(f)$ for a rational function $f \in \mathbb{Q}(X_1, \dots, X_n)$ is defined as the minimal number of operations from M necessary to compute f, provided it is finite.

Problem. For given $M \subset M_1 \subset \{+, -, \times, /\}$ how big can be $C_M(f)$ in comparison with $C_{M_1}(f)$?

This problem is non-trivial just for three pairs of $M \subset M_1$.

Theorem

$C_{+,-,\times}(f) \leq O(C_{+,-,\times,/}(f) \cdot \deg(f))$
*for a polynomial f (**V. Strassen, 1973**)*

("ring complexity" $\leq O$ ("field complexity" $\cdot \deg$))

Question. Is the bound sharp for big $\deg(f)$?

Corollary

$C_{+,-,\times}(\det) \leq O(n^4)$ (**D. K. Faddeev, V. N. Faddeeva, 1960**)

Subtraction-free computations

- $C_{+,\times}(f)$ can be exponentially bigger than $C_{+,\times,/}(f)$
- $C_{+,\times,/}(g)$ can be exponentially bigger than $C_{+,-,\times,/}(g)$.

L. Valiant, 1980 has constructed a polynomial h (with positive coefficients) such that $C_{+,\times}(h)$ is exponentially bigger than $C_{+,-,\times,/}(h)$. Funny consequence: at least one of two above • results holds. But which one?: A growth of $C_{+,\times,/}(h)$ is not known.

Subtraction-free computations have a meaning in numerical analysis: if a computation deals with positive numbers c given with a relative error, i. e. $(1-\epsilon)\cdot b < c < (1+\epsilon)\cdot b$ where ϵ is a relative error then one can estimate easily relative error after operations $+,\ \times,\ /$.

Polynomial in n variables is called *numerically positive* if it is positive everywhere on the positive orthant $(0,\infty)^n$.

Theorem. Polynomial f has a finite subtraction-free complexity $C_{+,\times,/}(f) < \infty$ iff for each face P of its Newton polytope the restriction $f|_P$ is numerically positive (**Handelman, 1985**).

Corollary. Subtraction-free complexity is computable.

Subtraction-free computations

- $C_{+,\times}(f)$ can be exponentially bigger than $C_{+,\times,/}(f)$
- $C_{+,\times,/}(g)$ can be exponentially bigger than $C_{+,-,\times,/}(g)$.

L. Valiant, 1980 has constructed a polynomial h (with positive coefficients) such that $C_{+,\times}(h)$ is exponentially bigger than $C_{+,-,\times,/}(h)$. Funny consequence: at least one of two above • results holds. But which one?: A growth of $C_{+,\times,/}(h)$ is not known.

Subtraction-free computations have a meaning in numerical analysis: if a computation deals with positive numbers c given with a relative error, i. e. $(1-\epsilon)\cdot b < c < (1+\epsilon)\cdot b$ where ϵ is a relative error then one can estimate easily relative error after operations $+, \times, /$.

Polynomial in n variables is called *numerically positive* if it is positive everywhere on the positive orthant $(0,\infty)^n$.

Theorem. Polynomial f has a finite subtraction-free complexity $C_{+,\times,/}(f) < \infty$ iff for each face P of its Newton polytope the restriction $f|_P$ is numerically positive (**Handelman, 1985**).

Corollary. Subtraction-free complexity is computable.

Subtraction-free computations

- $C_{+,\times}(f)$ can be exponentially bigger than $C_{+,\times,/}(f)$
- $C_{+,\times,/}(g)$ can be exponentially bigger than $C_{+,-,\times,/}(g)$.

L. Valiant, 1980 has constructed a polynomial h (with positive coefficients) such that $C_{+,\times}(h)$ is exponentially bigger than $C_{+,-,\times,/}(h)$. Funny consequence: at least one of two above • results holds. But which one?: A growth of $C_{+,\times,/}(h)$ is not known.

Subtraction-free computations have a meaning in numerical analysis: if a computation deals with positive numbers c given with a relative error, i. e. $(1-\epsilon)\cdot b < c < (1+\epsilon)\cdot b$ where ϵ is a relative error then one can estimate easily relative error after operations $+, \times, /$.

Polynomial in n variables is called *numerically positive* if it is positive everywhere on the positive orthant $(0,\infty)^n$.

Theorem. Polynomial f has a finite subtraction-free complexity $C_{+,\times,/}(f) < \infty$ iff for each face P of its Newton polytope the restriction $f|_P$ is numerically positive (**Handelman, 1985**).

Corollary. Subtraction-free complexity is computable.

Subtraction-free computations

- $C_{+,\times}(f)$ can be exponentially bigger than $C_{+,\times,/}(f)$
- $C_{+,\times,/}(g)$ can be exponentially bigger than $C_{+,-,\times,/}(g)$.

L. Valiant, 1980 has constructed a polynomial h (with positive coefficients) such that $C_{+,\times}(h)$ is exponentially bigger than $C_{+,-,\times,/}(h)$. Funny consequence: at least one of two above • results holds. But which one?: A growth of $C_{+,\times,/}(h)$ is not known.

Subtraction-free computations have a meaning in numerical analysis: if a computation deals with positive numbers c given with a relative error, i. e. $(1-\epsilon)\cdot b < c < (1+\epsilon)\cdot b$ where ϵ is a relative error then one can estimate easily relative error after operations $+, \times, /$.

Polynomial in n variables is called *numerically positive* if it is positive everywhere on the positive orthant $(0,\infty)^n$.

Theorem. Polynomial f has a finite subtraction-free complexity $C_{+,\times,/}(f) < \infty$ iff for each face P of its Newton polytope the restriction $f|_P$ is numerically positive (**Handelman, 1985**).

Corollary. Subtraction-free complexity is computable.

Subtraction-free computations

- $C_{+,\times}(f)$ can be exponentially bigger than $C_{+,\times,/}(f)$
- $C_{+,\times,/}(g)$ can be exponentially bigger than $C_{+,-,\times,/}(g)$.

L. Valiant, 1980 has constructed a polynomial h (with positive coefficients) such that $C_{+,\times}(h)$ is exponentially bigger than $C_{+,-,\times,/}(h)$. Funny consequence: at least one of two above • results holds. But which one?: A growth of $C_{+,\times,/}(h)$ is not known.

Subtraction-free computations have a meaning in numerical analysis: if a computation deals with positive numbers c given with a relative error, i. e. $(1-\epsilon)\cdot b < c < (1+\epsilon)\cdot b$ where ϵ is a relative error then one can estimate easily relative error after operations $+,\ \times,\ /$.

Polynomial in n variables is called *numerically positive* if it is positive everywhere on the positive orthant $(0,\infty)^n$.

Theorem. Polynomial f has a finite subtraction-free complexity $C_{+,\times,/}(f) < \infty$ iff for each face P of its Newton polytope the restriction $f|_P$ is numerically positive (**Handelman, 1985**).

Corollary. Subtraction-free complexity is computable.

Subtraction-free computations

- $C_{+,\times}(f)$ can be exponentially bigger than $C_{+,\times,/}(f)$
- $C_{+,\times,/}(g)$ can be exponentially bigger than $C_{+,-,\times,/}(g)$.

L. Valiant, 1980 has constructed a polynomial h (with positive coefficients) such that $C_{+,\times}(h)$ is exponentially bigger than $C_{+,-,\times,/}(h)$. Funny consequence: at least one of two above • results holds. But which one?: A growth of $C_{+,\times,/}(h)$ is not known.

Subtraction-free computations have a meaning in numerical analysis: if a computation deals with positive numbers c given with a relative error, i. e. $(1-\epsilon)\cdot b < c < (1+\epsilon)\cdot b$ where ϵ is a relative error then one can estimate easily relative error after operations $+, \times, /$.

Polynomial in n variables is called *numerically positive* if it is positive everywhere on the positive orthant $(0,\infty)^n$.

Theorem. Polynomial f has a finite subtraction-free complexity $C_{+,\times,/}(f) < \infty$ iff for each face P of its Newton polytope the restriction $f|_P$ is numerically positive (**Handelman, 1985**).

Corollary. Subtraction-free complexity is computable.

Subtraction-free computations

- $C_{+,\times}(f)$ can be exponentially bigger than $C_{+,\times,/}(f)$
- $C_{+,\times,/}(g)$ can be exponentially bigger than $C_{+,-,\times,/}(g)$.

L. Valiant, 1980 has constructed a polynomial h (with positive coefficients) such that $C_{+,\times}(h)$ is exponentially bigger than $C_{+,-,\times,/}(h)$. Funny consequence: at least one of two above • results holds. But which one?: A growth of $C_{+,\times,/}(h)$ is not known.

Subtraction-free computations have a meaning in numerical analysis: if a computation deals with positive numbers c given with a relative error, i. e. $(1-\epsilon)\cdot b < c < (1+\epsilon)\cdot b$ where ϵ is a relative error then one can estimate easily relative error after operations $+, \times, /$.

Polynomial in n variables is called *numerically positive* if it is positive everywhere on the positive orthant $(0,\infty)^n$.

Theorem. Polynomial f has a finite subtraction-free complexity $C_{+,\times,/}(f) < \infty$ iff for each face P of its Newton polytope the restriction $f|_P$ is numerically positive (**Handelman, 1985**).

Corollary. Subtraction-free complexity is computable.

Subtraction-free computations

- $C_{+,\times}(f)$ can be exponentially bigger than $C_{+,\times,/}(f)$
- $C_{+,\times,/}(g)$ can be exponentially bigger than $C_{+,-,\times,/}(g)$.

L. Valiant, 1980 has constructed a polynomial h (with positive coefficients) such that $C_{+,\times}(h)$ is exponentially bigger than $C_{+,-,\times,/}(h)$. Funny consequence: at least one of two above • results holds. But which one?: A growth of $C_{+,\times,/}(h)$ is not known.

Subtraction-free computations have a meaning in numerical analysis: if a computation deals with positive numbers c given with a relative error, i. e. $(1-\epsilon)\cdot b < c < (1+\epsilon)\cdot b$ where ϵ is a relative error then one can estimate easily relative error after operations $+, \times, /$.

Polynomial in n variables is called *numerically positive* if it is positive everywhere on the positive orthant $(0,\infty)^n$.

Theorem. Polynomial f has a finite subtraction-free complexity $C_{+,\times,/}(f) < \infty$ iff for each face P of its Newton polytope the restriction $f|_P$ is numerically positive (**Handelman, 1985**).

Corollary. Subtraction-free complexity is computable.

Subtraction-free computations

- $C_{+,\times}(f)$ can be exponentially bigger than $C_{+,\times,/}(f)$
- $C_{+,\times,/}(g)$ can be exponentially bigger than $C_{+,-,\times,/}(g)$.

L. Valiant, 1980 has constructed a polynomial h (with positive coefficients) such that $C_{+,\times}(h)$ is exponentially bigger than $C_{+,-,\times,/}(h)$. Funny consequence: at least one of two above • results holds. But which one?: A growth of $C_{+,\times,/}(h)$ is not known.

Subtraction-free computations have a meaning in numerical analysis: if a computation deals with positive numbers c given with a relative error, i. e. $(1-\epsilon)\cdot b < c < (1+\epsilon)\cdot b$ where ϵ is a relative error then one can estimate easily relative error after operations $+, \times, /$.

Polynomial in n variables is called *numerically positive* if it is positive everywhere on the positive orthant $(0,\infty)^n$.

Theorem. Polynomial f has a finite subtraction-free complexity $C_{+,\times,/}(f) < \infty$ iff for each face P of its Newton polytope the restriction $f|_P$ is numerically positive (**Handelman, 1985**).

Corollary. Subtraction-free complexity is computable.

Subtraction-free computations

- $C_{+,\times}(f)$ can be exponentially bigger than $C_{+,\times,/}(f)$
- $C_{+,\times,/}(g)$ can be exponentially bigger than $C_{+,-,\times,/}(g)$.

L. Valiant, 1980 has constructed a polynomial h (with positive coefficients) such that $C_{+,\times}(h)$ is exponentially bigger than $C_{+,-,\times,/}(h)$. Funny consequence: at least one of two above • results holds. But which one?: A growth of $C_{+,\times,/}(h)$ is not known.

Subtraction-free computations have a meaning in numerical analysis: if a computation deals with positive numbers c given with a relative error, i. e. $(1-\epsilon)\cdot b < c < (1+\epsilon)\cdot b$ where ϵ is a relative error then one can estimate easily relative error after operations $+, \times, /$.

Polynomial in n variables is called *numerically positive* if it is positive everywhere on the positive orthant $(0,\infty)^n$.

Theorem. Polynomial f has a finite subtraction-free complexity $C_{+,\times,/}(f) < \infty$ iff for each face P of its Newton polytope the restriction $f|_P$ is numerically positive (**Handelman, 1985**).

Corollary. Subtraction-free complexity is computable.

Subtraction-free computations

- $C_{+,\times}(f)$ can be exponentially bigger than $C_{+,\times,/}(f)$
- $C_{+,\times,/}(g)$ can be exponentially bigger than $C_{+,-,\times,/}(g)$.

L. Valiant, 1980 has constructed a polynomial h (with positive coefficients) such that $C_{+,\times}(h)$ is exponentially bigger than $C_{+,-,\times,/}(h)$. Funny consequence: at least one of two above • results holds. But which one?: A growth of $C_{+,\times,/}(h)$ is not known.

Subtraction-free computations have a meaning in numerical analysis: if a computation deals with positive numbers c given with a relative error, i. e. $(1-\epsilon)\cdot b < c < (1+\epsilon)\cdot b$ where ϵ is a relative error then one can estimate easily relative error after operations $+,\ \times,\ /$.

Polynomial in n variables is called *numerically positive* if it is positive everywhere on the positive orthant $(0,\infty)^n$.

Theorem. Polynomial f has a finite subtraction-free complexity $C_{+,\times,/}(f) < \infty$ iff for each face P of its Newton polytope the restriction $f|_P$ is numerically positive (**Handelman, 1985**).

Corollary. Subtraction-free complexity is computable.

Subtraction-free computations

- $C_{+,\times}(f)$ can be exponentially bigger than $C_{+,\times,/}(f)$
- $C_{+,\times,/}(g)$ can be exponentially bigger than $C_{+,-,\times,/}(g)$.

L. Valiant, 1980 has constructed a polynomial h (with positive coefficients) such that $C_{+,\times}(h)$ is exponentially bigger than $C_{+,-,\times,/}(h)$. Funny consequence: at least one of two above • results holds. But which one?: A growth of $C_{+,\times,/}(h)$ is not known.

Subtraction-free computations have a meaning in numerical analysis: if a computation deals with positive numbers c given with a relative error, i. e. $(1-\epsilon)\cdot b < c < (1+\epsilon)\cdot b$ where ϵ is a relative error then one can estimate easily relative error after operations $+,\ \times,\ /$.

Polynomial in n variables is called *numerically positive* if it is positive everywhere on the positive orthant $(0,\infty)^n$.

Theorem. Polynomial f has a finite subtraction-free complexity $C_{+,\times,/}(f) < \infty$ iff for each face P of its Newton polytope the restriction $f|_P$ is numerically positive (**Handelman, 1985**).

Corollary. Subtraction-free complexity is computable.

Subtraction-free computations

- $C_{+,\times}(f)$ can be exponentially bigger than $C_{+,\times,/}(f)$
- $C_{+,\times,/}(g)$ can be exponentially bigger than $C_{+,-,\times,/}(g)$.

L. Valiant, 1980 has constructed a polynomial h (with positive coefficients) such that $C_{+,\times}(h)$ is exponentially bigger than $C_{+,-,\times,/}(h)$. Funny consequence: at least one of two above • results holds. But which one?: A growth of $C_{+,\times,/}(h)$ is not known.

Subtraction-free computations have a meaning in numerical analysis: if a computation deals with positive numbers c given with a relative error, i. e. $(1-\epsilon)\cdot b < c < (1+\epsilon)\cdot b$ where ϵ is a relative error then one can estimate easily relative error after operations $+,\ \times,\ /$.

Polynomial in n variables is called *numerically positive* if it is positive everywhere on the positive orthant $(0,\infty)^n$.

Theorem. Polynomial f has a finite subtraction-free complexity $C_{+,\times,/}(f) < \infty$ iff for each face P of its Newton polytope the restriction $f|_P$ is numerically positive (**Handelman, 1985**).

Corollary. Subtraction-free complexity is computable.

Cluster algebras

Consider $n \times n$ matrix $X = (X_{ij})$ with variable entries. For a set $I \subset \{1, \dots, n\}$ of size $|I| = k$ denote by Δ_I the determinant of $k \times k$ submatrix of X formed by first k rows and by columns $i \in I$. Then the **flag algebra** $F[\{\Delta_I\}_I]$ is the ring of regular functions on the flag variety.

Clusters are special families of flag minors Δ_I (being bases of the flag algebra). To describe clusters define relation $I \prec J$ for $I, J \subset \{1, \dots, n\}$ if for any pair $i \in I \setminus J$, $j \in J \setminus I$ we have $i < j$. We say that I, J are **strongly separated** if either $I \prec J$ or $J \prec I$. Cluster is a maximal (wrt inclusion) family of pairwise strongly separated flag minors (excluding $I = \{1, \dots, n\}$). Each cluster contains $n(n+1)/2 - 1$ flag minors (**Fomin-Zelevinsky**).

Shorthand: for $j \notin I$ denote $Ij =: I \cup \{j\}$.

Cluster algebras

Consider $n \times n$ matrix $X = (X_{ij})$ with variable entries. For a set $I \subset \{1, \ldots, n\}$ of size $|I| = k$ denote by Δ_I the determinant of $k \times k$ submatrix of X formed by first k rows and by columns $i \in I$. Then the **flag algebra** $F[\{\Delta_I\}_I]$ is the ring of regular functions on the flag variety.

Clusters are special families of flag minors Δ_I (being bases of the flag algebra). To describe clusters define relation $I \prec J$ for $I, J \subset \{1, \ldots, n\}$ if for any pair $i \in I \setminus J$, $j \in J \setminus I$ we have $i < j$. We say that I, J are **strongly separated** if either $I \prec J$ or $J \prec I$. Cluster is a maximal (wrt inclusion) family of pairwise strongly separated flag minors (excluding $I = \{1, \ldots, n\}$). Each cluster contains $n(n+1)/2 - 1$ flag minors (**Fomin-Zelevinsky**).

Shorthand: for $j \notin I$ denote $Ij =: I \cup \{j\}$.

Cluster algebras

Consider $n \times n$ matrix $X = (X_{ij})$ with variable entries. For a set $I \subset \{1, \ldots, n\}$ of size $|I| = k$ denote by Δ_I the determinant of $k \times k$ submatrix of X formed by first k rows and by columns $i \in I$. Then the **flag algebra** $F[\{\Delta_I\}_I]$ is the ring of regular functions on the flag variety.

Clusters are special families of flag minors Δ_I (being bases of the flag algebra). To describe clusters define relation $I \prec J$ for $I, J \subset \{1, \ldots, n\}$ if for any pair $i \in I \setminus J$, $j \in J \setminus I$ we have $i < j$. We say that I, J are **strongly separated** if either $I \prec J$ or $J \prec I$. Cluster is a maximal (wrt inclusion) family of pairwise strongly separated flag minors (excluding $I = \{1, \ldots, n\}$). Each cluster contains $n(n+1)/2 - 1$ flag minors (**Fomin-Zelevinsky**).

Shorthand: for $j \notin I$ denote $Ij =: I \cup \{j\}$.

Cluster algebras

Consider $n \times n$ matrix $X = (X_{ij})$ with variable entries. For a set $I \subset \{1, \ldots, n\}$ of size $|I| = k$ denote by Δ_I the determinant of $k \times k$ submatrix of X formed by first k rows and by columns $i \in I$. Then the **flag algebra** $F[\{\Delta_I\}_I]$ is the ring of regular functions on the flag variety.

Clusters are special families of flag minors Δ_I (being bases of the flag algebra). To describe clusters define relation $I \prec J$ for $I, J \subset \{1, \ldots, n\}$ if for any pair $i \in I \setminus J$, $j \in J \setminus I$ we have $i < j$. We say that I, J are **strongly separated** if either $I \prec J$ or $J \prec I$. Cluster is a maximal (wrt inclusion) family of pairwise strongly separated flag minors (excluding $I = \{1, \ldots, n\}$). Each cluster contains $n(n+1)/2 - 1$ flag minors (**Fomin-Zelevinsky**).

Shorthand: for $j \notin I$ denote $Ij =: I \cup \{j\}$.

Cluster algebras

Consider $n \times n$ matrix $X = (X_{ij})$ with variable entries. For a set $I \subset \{1, \dots, n\}$ of size $|I| = k$ denote by Δ_I the determinant of $k \times k$ submatrix of X formed by first k rows and by columns $i \in I$. Then the **flag algebra** $F[\{\Delta_I\}_I]$ is the ring of regular functions on the flag variety.

Clusters are special families of flag minors Δ_I (being bases of the flag algebra). To describe clusters define relation $I \prec J$ for $I, J \subset \{1, \dots, n\}$ if for any pair $i \in I \setminus J$, $j \in J \setminus I$ we have $i < j$. We say that I, J are **strongly separated** if either $I \prec J$ or $J \prec I$. Cluster is a maximal (wrt inclusion) family of pairwise strongly separated flag minors (excluding $I = \{1, \dots, n\}$). Each cluster contains $n(n+1)/2 - 1$ flag minors (**Fomin-Zelevinsky**).

Shorthand: for $j \notin I$ denote $Ij =: I \cup \{j\}$.

Cluster algebras

Consider $n \times n$ matrix $X = (X_{ij})$ with variable entries. For a set $I \subset \{1, \dots, n\}$ of size $|I| = k$ denote by Δ_I the determinant of $k \times k$ submatrix of X formed by first k rows and by columns $i \in I$. Then the **flag algebra** $F[\{\Delta_I\}_I]$ is the ring of regular functions on the flag variety.

Clusters are special families of flag minors Δ_I (being bases of the flag algebra). To describe clusters define relation $I \prec J$ for $I, J \subset \{1, \dots, n\}$ if for any pair $i \in I \setminus J$, $j \in J \setminus I$ we have $i < j$. We say that I, J are **strongly separated** if either $I \prec J$ or $J \prec I$. Cluster is a maximal (wrt inclusion) family of pairwise strongly separated flag minors (excluding $I = \{1, \dots, n\}$). Each cluster contains $n(n+1)/2 - 1$ flag minors (**Fomin-Zelevinsky**).

Shorthand: for $j \notin I$ denote $Ij =: I \cup \{j\}$.

Cluster algebras

Consider $n \times n$ matrix $X = (X_{ij})$ with variable entries. For a set $I \subset \{1, \dots, n\}$ of size $|I| = k$ denote by Δ_I the determinant of $k \times k$ submatrix of X formed by first k rows and by columns $i \in I$. Then the **flag algebra** $F[\{\Delta_I\}_I]$ is the ring of regular functions on the flag variety.

Clusters are special families of flag minors Δ_I (being bases of the flag algebra). To describe clusters define relation $I \prec J$ for $I, J \subset \{1, \dots, n\}$ if for any pair $i \in I \setminus J$, $j \in J \setminus I$ we have $i < j$. We say that I, J are **strongly separated** if either $I \prec J$ or $J \prec I$. Cluster is a maximal (wrt inclusion) family of pairwise strongly separated flag minors (excluding $I = \{1, \dots, n\}$). Each cluster contains $n(n+1)/2 - 1$ flag minors (**Fomin-Zelevinsky**).

Shorthand: for $j \notin I$ denote $Ij =: I \cup \{j\}$.

Cluster algebras

Consider $n \times n$ matrix $X = (X_{ij})$ with variable entries. For a set $I \subset \{1, \ldots, n\}$ of size $|I| = k$ denote by Δ_I the determinant of $k \times k$ submatrix of X formed by first k rows and by columns $i \in I$. Then the **flag algebra** $F[\{\Delta_I\}_I]$ is the ring of regular functions on the flag variety.

Clusters are special families of flag minors Δ_I (being bases of the flag algebra). To describe clusters define relation $I \prec J$ for $I, J \subset \{1, \ldots, n\}$ if for any pair $i \in I \setminus J$, $j \in J \setminus I$ we have $i < j$. We say that I, J are **strongly separated** if either $I \prec J$ or $J \prec I$. Cluster is a maximal (wrt inclusion) family of pairwise strongly separated flag minors (excluding $I = \{1, \ldots, n\}$). Each cluster contains $n(n+1)/2 - 1$ flag minors (**Fomin-Zelevinsky**).

Shorthand: for $j \notin I$ denote $Ij =: I \cup \{j\}$.

Cluster algebras

Consider $n \times n$ matrix $X = (X_{ij})$ with variable entries. For a set $I \subset \{1, \dots, n\}$ of size $|I| = k$ denote by Δ_I the determinant of $k \times k$ submatrix of X formed by first k rows and by columns $i \in I$. Then the **flag algebra** $F[\{\Delta_I\}_I]$ is the ring of regular functions on the flag variety.

Clusters are special families of flag minors Δ_I (being bases of the flag algebra). To describe clusters define relation $I \prec J$ for $I, J \subset \{1, \dots, n\}$ if for any pair $i \in I \setminus J$, $j \in J \setminus I$ we have $i < j$. We say that I, J are **strongly separated** if either $I \prec J$ or $J \prec I$. Cluster is a maximal (wrt inclusion) family of pairwise strongly separated flag minors (excluding $I = \{1, \dots, n\}$). Each cluster contains $n(n+1)/2 - 1$ flag minors (**Fomin-Zelevinsky**).

Shorthand: for $j \notin I$ denote $Ij =: I \cup \{j\}$.

Cluster algebras

Consider $n \times n$ matrix $X = (X_{ij})$ with variable entries. For a set $I \subset \{1, \dots, n\}$ of size $|I| = k$ denote by Δ_I the determinant of $k \times k$ submatrix of X formed by first k rows and by columns $i \in I$. Then the **flag algebra** $F[\{\Delta_I\}_I]$ is the ring of regular functions on the flag variety.

Clusters are special families of flag minors Δ_I (being bases of the flag algebra). To describe clusters define relation $I \prec J$ for $I, J \subset \{1, \dots, n\}$ if for any pair $i \in I \setminus J$, $j \in J \setminus I$ we have $i < j$. We say that I, J are **strongly separated** if either $I \prec J$ or $J \prec I$. Cluster is a maximal (wrt inclusion) family of pairwise strongly separated flag minors (excluding $I = \{1, \dots, n\}$). Each cluster contains $n(n+1)/2 - 1$ flag minors (**Fomin-Zelevinsky**).

Shorthand: for $j \notin I$ denote $Ij =: I \cup \{j\}$.

Cluster graph

Plücker relations: $\Delta_{Iik} \cdot \Delta_{Ij} = \Delta_{Iij} \cdot \Delta_{Ik} + \Delta_{Ijk} \cdot \Delta_{Ii}$, $i, j, k \notin I$; $i < j < k$

Clusters form a directed acyclic **cluster graph**: there is an edge from one cluster to another if they are, respectively, of the types

$\{\Delta_{Iik}, \Delta_{Iij}, \Delta_{Ik}, \Delta_{Ijk}, \Delta_{Ii}, \dots\}$;

$\{\Delta_{Ij}, \Delta_{Iij}, \Delta_{Ik}, \Delta_{Ijk}, \Delta_{Ii}, \dots\}$

Cluster graph is graded according to the sum of sizes $|I|$. A birational (with integer positive coefficients) transformation from one cluster to another (along an edge or back) is called a **flip**. Cluster graph has the unique minimal element corresponding to I being all the proper intervals of $\{1, \dots, n\}$ and the unique maximal element corresponding to all the complements of the proper intervals.

Fomin-Zelevinsky: any non-minimal cluster contains a 5-tuple $\Delta_{Iik}, \Delta_{Iij}, \Delta_{Ik}, \Delta_{Ijk}, \Delta_{Ii}$, therefore, a flip can be applied to the cluster. Any flag minor belongs to some cluster. Thus, any flag minor can be computed with $O(n^3)$ flips starting with the interval flag minors, and a flag minor is a Laurent polynomial in the interval flag minors with integer positive coefficients.

Cluster graph

Plücker relations: $\Delta_{Iik} \cdot \Delta_{Ij} = \Delta_{Iij} \cdot \Delta_{Ik} + \Delta_{Ijk} \cdot \Delta_{Ii}$, $i, j, k \notin I$; $i < j < k$

Clusters form a directed acyclic **cluster graph**: there is an edge from one cluster to another if they are, respectively, of the types
$\{\Delta_{Iik}, \Delta_{Iij}, \Delta_{Ik}, \Delta_{Ijk}, \Delta_{Ii}, \dots\}$;
$\{\Delta_{Ij}, \Delta_{Iij}, \Delta_{Ik}, \Delta_{Ijk}, \Delta_{Ii}, \dots\}$

Cluster graph is graded according to the sum of sizes $|I|$. A birational (with integer positive coefficients) transformation from one cluster to another (along an edge or back) is called a **flip**. Cluster graph has the unique minimal element corresponding to I being all the proper intervals of $\{1, \dots, n\}$ and the unique maximal element corresponding to all the complements of the proper intervals.

Fomin-Zelevinsky: any non-minimal cluster contains a 5-tuple $\Delta_{Iik}, \Delta_{Iij}, \Delta_{Ik}, \Delta_{Ijk}, \Delta_{Ii}$, therefore, a flip can be applied to the cluster. Any flag minor belongs to some cluster. Thus, any flag minor can be computed with $O(n^3)$ flips starting with the interval flag minors, and a flag minor is a Laurent polynomial in the interval flag minors with integer positive coefficients.

Cluster graph

Plücker relations: $\Delta_{Iik} \cdot \Delta_{Ij} = \Delta_{Iij} \cdot \Delta_{Ik} + \Delta_{Ijk} \cdot \Delta_{Ii}$, $i, j, k \notin I$; $i < j < k$

Clusters form a directed acyclic **cluster graph**: there is an edge from one cluster to another if they are, respectively, of the types

$\{\Delta_{Iik}, \Delta_{Iij}, \Delta_{Ik}, \Delta_{Ijk}, \Delta_{Ii}, \dots\}$;

$\{\Delta_{Ij}, \Delta_{Iij}, \Delta_{Ik}, \Delta_{Ijk}, \Delta_{Ii}, \dots\}$

Cluster graph is graded according to the sum of sizes $|I|$. A birational (with integer positive coefficients) transformation from one cluster to another (along an edge or back) is called a **flip**. Cluster graph has the unique minimal element corresponding to I being all the proper intervals of $\{1, \dots, n\}$ and the unique maximal element corresponding to all the complements of the proper intervals.

Fomin-Zelevinsky: any non-minimal cluster contains a 5-tuple $\Delta_{Iik}, \Delta_{Iij}, \Delta_{Ik}, \Delta_{Ijk}, \Delta_{Ii}$, therefore, a flip can be applied to the cluster. Any flag minor belongs to some cluster. Thus, any flag minor can be computed with $O(n^3)$ flips starting with the interval flag minors, and a flag minor is a Laurent polynomial in the interval flag minors with integer positive coefficients.

Cluster graph

Plücker relations: $\Delta_{Iik} \cdot \Delta_{Ij} = \Delta_{Iij} \cdot \Delta_{Ik} + \Delta_{Ijk} \cdot \Delta_{Ii},\ i, j, k \notin I;\ i < j < k$

Clusters form a directed acyclic **cluster graph**: there is an edge from one cluster to another if they are, respectively, of the types

$\{\Delta_{Iik}, \Delta_{Iij}, \Delta_{Ik}, \Delta_{Ijk}, \Delta_{Ii}, \dots\}$;

$\{\Delta_{Ij}, \Delta_{Iij}, \Delta_{Ik}, \Delta_{Ijk}, \Delta_{Ii}, \dots\}$

Cluster graph is graded according to the sum of sizes $|I|$. A birational (with integer positive coefficients) transformation from one cluster to another (along an edge or back) is called a **flip**. Cluster graph has the unique minimal element corresponding to I being all the proper intervals of $\{1, \dots, n\}$ and the unique maximal element corresponding to all the complements of the proper intervals.

Fomin-Zelevinsky: any non-minimal cluster contains a 5-tuple $\Delta_{Iik}, \Delta_{Iij}, \Delta_{Ik}, \Delta_{Ijk}, \Delta_{Ii}$, therefore, a flip can be applied to the cluster. Any flag minor belongs to some cluster. Thus, any flag minor can be computed with $O(n^3)$ flips starting with the interval flag minors, and a flag minor is a Laurent polynomial in the interval flag minors with integer positive coefficients.

Cluster graph

Plücker relations: $\Delta_{Iik} \cdot \Delta_{Ij} = \Delta_{Iij} \cdot \Delta_{Ik} + \Delta_{Ijk} \cdot \Delta_{Ii}, \; i, j, k \notin I; \; i < j < k$

Clusters form a directed acyclic **cluster graph**: there is an edge from one cluster to another if they are, respectively, of the types
$\{\Delta_{Iik}, \Delta_{Iij}, \Delta_{Ik}, \Delta_{Ijk}, \Delta_{Ii}, \dots\}$;
$\{\Delta_{Ij}, \Delta_{Iij}, \Delta_{Ik}, \Delta_{Ijk}, \Delta_{Ii}, \dots\}$

Cluster graph is graded according to the sum of sizes $|I|$. A birational (with integer positive coefficients) transformation from one cluster to another (along an edge or back) is called a **flip**. Cluster graph has the unique minimal element corresponding to I being all the proper intervals of $\{1, \dots, n\}$ and the unique maximal element corresponding to all the complements of the proper intervals.

Fomin-Zelevinsky: any non-minimal cluster contains a 5-tuple $\Delta_{Iik}, \Delta_{Iij}, \Delta_{Ik}, \Delta_{Ijk}, \Delta_{Ii}$, therefore, a flip can be applied to the cluster. Any flag minor belongs to some cluster. Thus, any flag minor can be computed with $O(n^3)$ flips starting with the interval flag minors, and a flag minor is a Laurent polynomial in the interval flag minors with integer positive coefficients.

Cluster graph

Plücker relations: $\Delta_{Iik} \cdot \Delta_{Ij} = \Delta_{Iij} \cdot \Delta_{Ik} + \Delta_{Ijk} \cdot \Delta_{Ii}$, $i, j, k \notin I$; $i < j < k$

Clusters form a directed acyclic **cluster graph**: there is an edge from one cluster to another if they are, respectively, of the types

$\{\Delta_{Iik}, \Delta_{Iij}, \Delta_{Ik}, \Delta_{Ijk}, \Delta_{Ii}, \dots\}$;

$\{\Delta_{Ij}, \Delta_{Iij}, \Delta_{Ik}, \Delta_{Ijk}, \Delta_{Ii}, \dots\}$

Cluster graph is graded according to the sum of sizes $|I|$. A birational (with integer positive coefficients) transformation from one cluster to another (along an edge or back) is called a **flip**. Cluster graph has the unique minimal element corresponding to I being all the proper intervals of $\{1, \dots, n\}$ and the unique maximal element corresponding to all the complements of the proper intervals.

Fomin-Zelevinsky: any non-minimal cluster contains a 5-tuple $\Delta_{Iik}, \Delta_{Iij}, \Delta_{Ik}, \Delta_{Ijk}, \Delta_{Ii}$, therefore, a flip can be applied to the cluster. Any flag minor belongs to some cluster. Thus, any flag minor can be computed with $O(n^3)$ flips starting with the interval flag minors, and a flag minor is a Laurent polynomial in the interval flag minors with integer positive coefficients.

Cluster graph

Plücker relations: $\Delta_{Iik} \cdot \Delta_{Ij} = \Delta_{Iij} \cdot \Delta_{Ik} + \Delta_{Ijk} \cdot \Delta_{Ii}$, $i, j, k \notin I$; $i < j < k$

Clusters form a directed acyclic **cluster graph**: there is an edge from one cluster to another if they are, respectively, of the types
$\{\Delta_{Iik}, \Delta_{Iij}, \Delta_{Ik}, \Delta_{Ijk}, \Delta_{Ii}, \dots\}$;
$\{\Delta_{Ij}, \Delta_{Iij}, \Delta_{Ik}, \Delta_{Ijk}, \Delta_{Ii}, \dots\}$

Cluster graph is graded according to the sum of sizes $|I|$. A birational (with integer positive coefficients) transformation from one cluster to another (along an edge or back) is called a **flip**. Cluster graph has the unique minimal element corresponding to I being all the proper intervals of $\{1, \dots, n\}$ and the unique maximal element corresponding to all the complements of the proper intervals.

Fomin-Zelevinsky: any non-minimal cluster contains a 5-tuple $\Delta_{Iik}, \Delta_{Iij}, \Delta_{Ik}, \Delta_{Ijk}, \Delta_{Ii}$, therefore, a flip can be applied to the cluster. Any flag minor belongs to some cluster. Thus, any flag minor can be computed with $O(n^3)$ flips starting with the interval flag minors, and a flag minor is a Laurent polynomial in the interval flag minors with integer positive coefficients.

Cluster graph

Plücker relations: $\Delta_{Iik} \cdot \Delta_{Ij} = \Delta_{Iij} \cdot \Delta_{Ik} + \Delta_{Ijk} \cdot \Delta_{Ii},\ i, j, k \notin I;\ i < j < k$

Clusters form a directed acyclic **cluster graph**: there is an edge from one cluster to another if they are, respectively, of the types
$\{\Delta_{Iik}, \Delta_{Iij}, \Delta_{Ik}, \Delta_{Ijk}, \Delta_{Ii}, \dots\}$;
$\{\Delta_{Ij}, \Delta_{Iij}, \Delta_{Ik}, \Delta_{Ijk}, \Delta_{Ii}, \dots\}$

Cluster graph is graded according to the sum of sizes $|I|$. A birational (with integer positive coefficients) transformation from one cluster to another (along an edge or back) is called a **flip**. Cluster graph has the unique minimal element corresponding to I being all the proper intervals of $\{1, \dots, n\}$ and the unique maximal element corresponding to all the complements of the proper intervals.

Fomin-Zelevinsky: any non-minimal cluster contains a 5-tuple $\Delta_{Iik}, \Delta_{Iij}, \Delta_{Ik}, \Delta_{Ijk}, \Delta_{Ii}$, therefore, a flip can be applied to the cluster. Any flag minor belongs to some cluster. Thus, any flag minor can be computed with $O(n^3)$ flips starting with the interval flag minors, and a flag minor is a Laurent polynomial in the interval flag minors with integer positive coefficients.

Cluster graph

Plücker relations: $\Delta_{Iik} \cdot \Delta_{Ij} = \Delta_{Iij} \cdot \Delta_{Ik} + \Delta_{Ijk} \cdot \Delta_{Ii},\ i, j, k \notin I;\ i < j < k$

Clusters form a directed acyclic **cluster graph**: there is an edge from one cluster to another if they are, respectively, of the types
$\{\Delta_{Iik},\ \Delta_{Iij},\ \Delta_{Ik},\ \Delta_{Ijk},\ \Delta_{Ii}, \dots\}$;
$\{\Delta_{Ij},\ \Delta_{Iij},\ \Delta_{Ik},\ \Delta_{Ijk},\ \Delta_{Ii}, \dots\}$

Cluster graph is graded according to the sum of sizes $|I|$. A birational (with integer positive coefficients) transformation from one cluster to another (along an edge or back) is called a **flip**. Cluster graph has the unique minimal element corresponding to I being all the proper intervals of $\{1, \dots, n\}$ and the unique maximal element corresponding to all the complements of the proper intervals.

Fomin-Zelevinsky: any non-minimal cluster contains a 5-tuple $\Delta_{Iik},\ \Delta_{Iij},\ \Delta_{Ik},\ \Delta_{Ijk},\ \Delta_{Ii}$, therefore, a flip can be applied to the cluster. Any flag minor belongs to some cluster. Thus, any flag minor can be computed with $O(n^3)$ flips starting with the interval flag minors, and a flag minor is a Laurent polynomial in the interval flag minors with integer positive coefficients.

Cluster graph

Plücker relations: $\Delta_{Iik} \cdot \Delta_{Ij} = \Delta_{Iij} \cdot \Delta_{Ik} + \Delta_{Ijk} \cdot \Delta_{Ii}$, $i, j, k \notin I$; $i < j < k$

Clusters form a directed acyclic **cluster graph**: there is an edge from one cluster to another if they are, respectively, of the types
$\{\Delta_{Iik}, \Delta_{Iij}, \Delta_{Ik}, \Delta_{Ijk}, \Delta_{Ii}, \dots\}$;
$\{\Delta_{Ij}, \Delta_{Iij}, \Delta_{Ik}, \Delta_{Ijk}, \Delta_{Ii}, \dots\}$

Cluster graph is graded according to the sum of sizes $|I|$. A birational (with integer positive coefficients) transformation from one cluster to another (along an edge or back) is called a **flip**. Cluster graph has the unique minimal element corresponding to I being all the proper intervals of $\{1, \dots, n\}$ and the unique maximal element corresponding to all the complements of the proper intervals.

Fomin-Zelevinsky: any non-minimal cluster contains a 5-tuple $\Delta_{Iik}, \Delta_{Iij}, \Delta_{Ik}, \Delta_{Ijk}, \Delta_{Ii}$, therefore, a flip can be applied to the cluster.

Any flag minor belongs to some cluster. Thus, any flag minor can be computed with $O(n^3)$ flips starting with the interval flag minors, and a flag minor is a Laurent polynomial in the interval flag minors with integer positive coefficients.

Cluster graph

Plücker relations: $\Delta_{Iik} \cdot \Delta_{Ij} = \Delta_{Iij} \cdot \Delta_{Ik} + \Delta_{Ijk} \cdot \Delta_{Ii}$, $i, j, k \notin I$; $i < j < k$

Clusters form a directed acyclic **cluster graph**: there is an edge from one cluster to another if they are, respectively, of the types
$\{\Delta_{Iik}, \Delta_{Iij}, \Delta_{Ik}, \Delta_{Ijk}, \Delta_{Ii}, \dots\}$;
$\{\Delta_{Ij}, \Delta_{Iij}, \Delta_{Ik}, \Delta_{Ijk}, \Delta_{Ii}, \dots\}$

Cluster graph is graded according to the sum of sizes $|I|$. A birational (with integer positive coefficients) transformation from one cluster to another (along an edge or back) is called a **flip**. Cluster graph has the unique minimal element corresponding to I being all the proper intervals of $\{1, \dots, n\}$ and the unique maximal element corresponding to all the complements of the proper intervals.

Fomin-Zelevinsky: any non-minimal cluster contains a 5-tuple $\Delta_{Iik}, \Delta_{Iij}, \Delta_{Ik}, \Delta_{Ijk}, \Delta_{Ii}$, therefore, a flip can be applied to the cluster. Any flag minor belongs to some cluster. Thus, any flag minor can be computed with $O(n^3)$ flips starting with the interval flag minors, and a flag minor is a Laurent polynomial in the interval flag minors with integer positive coefficients.

Cluster graph

Plücker relations: $\Delta_{Iik} \cdot \Delta_{Ij} = \Delta_{Iij} \cdot \Delta_{Ik} + \Delta_{Ijk} \cdot \Delta_{Ii}$, $i, j, k \notin I$; $i < j < k$

Clusters form a directed acyclic **cluster graph**: there is an edge from one cluster to another if they are, respectively, of the types
$\{\Delta_{Iik}, \Delta_{Iij}, \Delta_{Ik}, \Delta_{Ijk}, \Delta_{Ii}, \dots\}$;
$\{\Delta_{Ij}, \Delta_{Iij}, \Delta_{Ik}, \Delta_{Ijk}, \Delta_{Ii}, \dots\}$
Cluster graph is graded according to the sum of sizes $|I|$. A birational (with integer positive coefficients) transformation from one cluster to another (along an edge or back) is called a **flip**. Cluster graph has the unique minimal element corresponding to I being all the proper intervals of $\{1, \dots, n\}$ and the unique maximal element corresponding to all the complements of the proper intervals.

Fomin-Zelevinsky: any non-minimal cluster contains a 5-tuple $\Delta_{Iik}, \Delta_{Iij}, \Delta_{Ik}, \Delta_{Ijk}, \Delta_{Ii}$, therefore, a flip can be applied to the cluster. Any flag minor belongs to some cluster. Thus, any flag minor can be computed with $O(n^3)$ flips starting with the interval flag minors, and a flag minor is a Laurent polynomial in the interval flag minors with integer positive coefficients.

Cluster graph

Plücker relations: $\Delta_{Iik} \cdot \Delta_{Ij} = \Delta_{Iij} \cdot \Delta_{Ik} + \Delta_{Ijk} \cdot \Delta_{Ii},\ i, j, k \notin I;\ i < j < k$

Clusters form a directed acyclic **cluster graph**: there is an edge from one cluster to another if they are, respectively, of the types
$\{\Delta_{Iik}, \Delta_{Iij}, \Delta_{Ik}, \Delta_{Ijk}, \Delta_{Ii}, \dots\}$;
$\{\Delta_{Ij}, \Delta_{Iij}, \Delta_{Ik}, \Delta_{Ijk}, \Delta_{Ii}, \dots\}$

Cluster graph is graded according to the sum of sizes $|I|$. A birational (with integer positive coefficients) transformation from one cluster to another (along an edge or back) is called a **flip**. Cluster graph has the unique minimal element corresponding to I being all the proper intervals of $\{1, \dots, n\}$ and the unique maximal element corresponding to all the complements of the proper intervals.

Fomin-Zelevinsky: any non-minimal cluster contains a 5-tuple $\Delta_{Iik}, \Delta_{Iij}, \Delta_{Ik}, \Delta_{Ijk}, \Delta_{Ii}$, therefore, a flip can be applied to the cluster. Any flag minor belongs to some cluster. Thus, any flag minor can be computed with $O(n^3)$ flips starting with the interval flag minors, and a flag minor is a Laurent polynomial in the interval flag minors with integer positive coefficients.

Totally positive matrices

As a consequence from cluster transformations we conclude that for a real matrix if all the interval flag minors are positive then all flag minors are positive as well.

A real matrix is called *totally positive* if all its minors are positive. Cluster transformation entail that for total positivity it suffices to verify positivity of minors formed by sets of rows I and of columns J for all pairs of intervals $I, J \subset \{1, \dots, n\}$.

Totally positive matrices

As a consequence from cluster transformations we conclude that for a real matrix if all the interval flag minors are positive then all flag minors are positive as well.

A real matrix is called *totally positive* if all its minors are positive. Cluster transformation entail that for total positivity it suffices to verify positivity of minors formed by sets of rows I and of columns J for all pairs of intervals $I, J \subset \{1, \ldots, n\}$.

Totally positive matrices

As a consequence from cluster transformations we conclude that for a real matrix if all the interval flag minors are positive then all flag minors are positive as well.

A real matrix is called *totally positive* if all its minors are positive. Cluster transformation entail that for total positivity it suffices to verify positivity of minors formed by sets of rows I and of columns J for all pairs of intervals $I, J \subset \{1, \dots, n\}$.

Cubic subtraction-free complexity of Schur polynomials

Substitute for the entries of the matrix $X_{ij} = x_i^j$. Then $\Delta := \Delta_{\{1,\ldots,k\}} = \prod_{1 \le i < j \le k} (x_j - x_i)$ is Vandermond determinant. The quotient $S_I = \Delta_I / \Delta$ is a **Schur polynomial** having integer positive coefficients (called **Kostka numbers** being $\#P$-hard to compute).

Since for an interval $I = [i, i+1, \ldots, j]$ the interval Schur polynomial is the monomial $S_I = (x_1 \cdots x_{j-i+1})^i$ being easy to compute, we get

Corollary

Subtraction-free complexity of a Schur polynomial $C_{+,\times,/}(S_I) \le O(n^3 \cdot \log n)$.

This does not yet imply an exponential gap between $C_{+,\times,/}$ and $C_{+,\times}$ because we don't know a lower bound on the complexity $C_{+,\times}(S_I)$. To establish this gap we proceed to another class of cluster transformations.

Cubic subtraction-free complexity of Schur polynomials

Substitute for the entries of the matrix $X_{ij} = x_i^j$. Then $\Delta := \Delta_{\{1,\ldots,k\}} = \prod_{1 \le i < j \le k} (x_j - x_i)$ is Vandermond determinant. The quotient $S_I = \Delta_I / \Delta$ is a **Schur polynomial** having integer positive coefficients (called **Kostka numbers** being $\#P$-hard to compute).

Since for an interval $I = [i, i+1, \ldots, j]$ the interval Schur polynomial is the monomial $S_I = (x_1 \cdots x_{j-i+1})^i$ being easy to compute, we get

Corollary

Subtraction-free complexity of a Schur polynomial $C_{+,\times,/}(S_I) \le O(n^3 \cdot \log n)$.

This does not yet imply an exponential gap between $C_{+,\times,/}$ and $C_{+,\times}$ because we don't know a lower bound on the complexity $C_{+,\times}(S_I)$. To establish this gap we proceed to another class of cluster transformations.

Cubic subtraction-free complexity of Schur polynomials

Substitute for the entries of the matrix $X_{ij} = x_i^j$. Then $\Delta := \Delta_{\{1,\dots,k\}} = \prod_{1 \le i < j \le k}(x_j - x_i)$ is Vandermond determinant. The quotient $S_I = \Delta_I / \Delta$ is a **Schur polynomial** having integer positive coefficients (called **Kostka numbers** being $\#P$-hard to compute).

Since for an interval $I = [i, i+1, \dots, j]$ the interval Schur polynomial is the monomial $S_I = (x_1 \cdots x_{j-i+1})^i$ being easy to compute, we get

Corollary

Subtraction-free complexity of a Schur polynomial $C_{+,\times,/}(S_I) \le O(n^3 \cdot \log n)$.

This does not yet imply an exponential gap between $C_{+,\times,/}$ and $C_{+,\times}$ because we don't know a lower bound on the complexity $C_{+,\times}(S_I)$. To establish this gap we proceed to another class of cluster transformations.

Cubic subtraction-free complexity of Schur polynomials

Substitute for the entries of the matrix $X_{ij} = x_i^j$. Then $\Delta := \Delta_{\{1,\dots,k\}} = \prod_{1 \le i < j \le k}(x_j - x_i)$ is Vandermond determinant. The quotient $S_I = \Delta_I / \Delta$ is a **Schur polynomial** having integer positive coefficients (called **Kostka numbers** being $\#P$-hard to compute).

Since for an interval $I = [i, i+1, \dots, j]$ the interval Schur polynomial is the monomial $S_I = (x_1 \cdots x_{j-i+1})^i$ being easy to compute, we get

Corollary

Subtraction-free complexity of a Schur polynomial $C_{+,\times,/}(S_I) \le O(n^3 \cdot \log n)$.

This does not yet imply an exponential gap between $C_{+,\times,/}$ and $C_{+,\times}$ because we don't know a lower bound on the complexity $C_{+,\times}(S_I)$. To establish this gap we proceed to another class of cluster transformations.

Cubic subtraction-free complexity of Schur polynomials

Substitute for the entries of the matrix $X_{ij} = x_i^j$. Then $\Delta := \Delta_{\{1,\dots,k\}} = \prod_{1\le i<j\le k}(x_j - x_i)$ is Vandermond determinant. The quotient $S_I = \Delta_I/\Delta$ is a **Schur polynomial** having integer positive coefficients (called **Kostka numbers** being $\#P$-hard to compute).

Since for an interval $I = [i, i+1, \dots, j]$ the interval Schur polynomial is the monomial $S_I = (x_1 \cdots x_{j-i+1})^i$ being easy to compute, we get

Corollary

Subtraction-free complexity of a Schur polynomial $C_{+,\times,/}(S_I) \le O(n^3 \cdot \log n)$.

This does not yet imply an exponential gap between $C_{+,\times,/}$ and $C_{+,\times}$ because we don't know a lower bound on the complexity $C_{+,\times}(S_I)$. To establish this gap we proceed to another class of cluster transformations.

Cubic subtraction-free complexity of Schur polynomials

Substitute for the entries of the matrix $X_{ij} = x_i^j$. Then $\Delta := \Delta_{\{1,\dots,k\}} = \prod_{1 \le i < j \le k} (x_j - x_i)$ is Vandermond determinant. The quotient $S_I = \Delta_I / \Delta$ is a **Schur polynomial** having integer positive coefficients (called **Kostka numbers** being $\#P$-hard to compute).

Since for an interval $I = [i, i+1, \dots, j]$ the interval Schur polynomial is the monomial $S_I = (x_1 \cdots x_{j-i+1})^i$ being easy to compute, we get

Corollary

Subtraction-free complexity of a Schur polynomial $C_{+,\times,/}(S_I) \le O(n^3 \cdot \log n)$.

This does not yet imply an exponential gap between $C_{+,\times,/}$ and $C_{+,\times}$ because we don't know a lower bound on the complexity $C_{+,\times}(S_I)$. To establish this gap we proceed to another class of cluster transformations.

Cubic subtraction-free complexity of Schur polynomials

Substitute for the entries of the matrix $X_{ij} = x_i^j$. Then $\Delta := \Delta_{\{1,\dots,k\}} = \prod_{1 \le i < j \le k}(x_j - x_i)$ is Vandermond determinant. The quotient $S_I = \Delta_I / \Delta$ is a **Schur polynomial** having integer positive coefficients (called **Kostka numbers** being $\#P$-hard to compute).

Since for an interval $I = [i, i+1, \dots, j]$ the interval Schur polynomial is the monomial $S_I = (x_1 \cdots x_{j-i+1})^i$ being easy to compute, we get

Corollary

Subtraction-free complexity of a Schur polynomial $C_{+,\times,/}(S_I) \le O(n^3 \cdot \log n)$.

This does not yet imply an exponential gap between $C_{+,\times,/}$ and $C_{+,\times}$ because we don't know a lower bound on the complexity $C_{+,\times}(S_I)$. To establish this gap we proceed to another class of cluster transformations.

Cubic subtraction-free complexity of Schur polynomials

Substitute for the entries of the matrix $X_{ij} = x_i^j$. Then $\Delta := \Delta_{\{1,\dots,k\}} = \prod_{1 \le i < j \le k}(x_j - x_i)$ is Vandermond determinant. The quotient $S_I = \Delta_I / \Delta$ is a **Schur polynomial** having integer positive coefficients (called **Kostka numbers** being $\#P$-hard to compute).

Since for an interval $I = [i,\, i+1,\, \dots,\, j]$ the interval Schur polynomial is the monomial $S_I = (x_1 \cdots x_{j-i+1})^i$ being easy to compute, we get

Corollary

Subtraction-free complexity of a Schur polynomial $C_{+,\times,/}(S_I) \le O(n^3 \cdot \log n)$.

This does not yet imply an exponential gap between $C_{+,\times,/}$ and $C_{+,\times}$ because we don't know a lower bound on the complexity $C_{+,\times}(S_I)$. To establish this gap we proceed to another class of cluster transformations.

Cubic subtraction-free complexity of Schur polynomials

Substitute for the entries of the matrix $X_{ij} = x_i^j$. Then $\Delta := \Delta_{\{1,\dots,k\}} = \prod_{1\le i<j\le k}(x_j - x_i)$ is Vandermond determinant. The quotient $S_I = \Delta_I/\Delta$ is a **Schur polynomial** having integer positive coefficients (called **Kostka numbers** being $\#P$-hard to compute).

Since for an interval $I = [i,\ i+1,\ \dots,\ j]$ the interval Schur polynomial is the monomial $S_I = (x_1 \cdots x_{j-i+1})^i$ being easy to compute, we get

Corollary

Subtraction-free complexity of a Schur polynomial $C_{+,\times,/}(S_I) \le O(n^3 \cdot \log n)$.

This does not yet imply an exponential gap between $C_{+,\times,/}$ and $C_{+,\times}$ because we don't know a lower bound on the complexity $C_{+,\times}(S_I)$. To establish this gap we proceed to another class of cluster transformations.

Cubic subtraction-free complexity of Schur polynomials

Substitute for the entries of the matrix $X_{ij} = x_i^j$. Then $\Delta := \Delta_{\{1,\dots,k\}} = \prod_{1 \le i < j \le k}(x_j - x_i)$ is Vandermond determinant. The quotient $S_I = \Delta_I / \Delta$ is a **Schur polynomial** having integer positive coefficients (called **Kostka numbers** being $\#P$-hard to compute).

Since for an interval $I = [i,\, i+1,\, \dots,\, j]$ the interval Schur polynomial is the monomial $S_I = (x_1 \cdots x_{j-i+1})^i$ being easy to compute, we get

Corollary

Subtraction-free complexity of a Schur polynomial $C_{+,\times,/}(S_I) \le O(n^3 \cdot \log n)$.

This does not yet imply an exponential gap between $C_{+,\times,/}$ and $C_{+,\times}$ because we don't know a lower bound on the complexity $C_{+,\times}(S_I)$. To establish this gap we proceed to another class of cluster transformations.

Star-mesh transformations

Let edges of a complete graph G with n vertices (viewed as an electrical circuit) be endowed with **conductances** $x_{i,j}$. Denote by $cond_{i,j}(G)$ the conductance of the circuit between vertices i, j.

For a spanning tree T of G denote by X^T the monomial being the product of $x_{i,j}$ for all edges i, j of T. The generating polynomial $f(G)$ is the sum of the monomials X^T over all spanning trees T.

Example: for G with 3 vertices $f(G) = x_{1,2} \cdot x_{2,3} + x_{1,2} \cdot x_{1,3} + x_{2,3} \cdot x_{1,3}$.

For vertices v, w of G denote by $G_{v,w}$ the graph obtained from G by gluing v, w into a vertex u with new conductances $x_{i,u} := x_{i,v} + x_{i,w}$.

Kirchhoff (1847): $cond_{i,j}(G) = f(G)/f(G_{i,j})$.

Star-mesh transformation: let a vertex $v \neq i, j$ of G. Denote by G_v a graph obtained from G by removing v and replacing each conductance $x_{p,q}$ by $x_{p,q} + x_{p,v} \cdot x_{q,v} \cdot \sum_k (x_{k,v})^{-1}$. Then $cond_{i,j}(G) = cond_{i,j}(G_v)$.

Corollary

Subtraction-free complexity $C_{+,\times,/}(cond_{i,j}(G), f(G)) \leq O(n^3)$.

Star-mesh transformations

Let edges of a complete graph G with n vertices (viewed as an electrical circuit) be endowed with **conductances** $x_{i,j}$. Denote by $cond_{i,j}(G)$ the conductance of the circuit between vertices i, j.

For a spanning tree T of G denote by X^T the monomial being the product of $x_{i,j}$ for all edges i, j of T. The generating polynomial $f(G)$ is the sum of the monomials X^T over all spanning trees T.

Example: for G with 3 vertices $f(G) = x_{1,2} \cdot x_{2,3} + x_{1,2} \cdot x_{1,3} + x_{2,3} \cdot x_{1,3}$.

For vertices v, w of G denote by $G_{v,w}$ the graph obtained from G by gluing v, w into a vertex u with new conductances $x_{i,u} := x_{i,v} + x_{i,w}$.

Kirchhoff (1847): $cond_{i,j}(G) = f(G)/f(G_{i,j})$.

Star-mesh transformation: let a vertex $v \neq i, j$ of G. Denote by G_v a graph obtained from G by removing v and replacing each conductance $x_{p,q}$ by $x_{p,q} + x_{p,v} \cdot x_{q,v} \cdot \sum_k (x_{k,v})^{-1}$. Then $cond_{i,j}(G) = cond_{i,j}(G_v)$.

Corollary

Subtraction-free complexity $C_{+,\times,/}(cond_{i,j}(G), f(G)) \leq O(n^3)$.

Star-mesh transformations

Let edges of a complete graph G with n vertices (viewed as an electrical circuit) be endowed with **conductances** $x_{i,j}$. Denote by $cond_{i,j}(G)$ the conductance of the circuit between vertices i, j.

For a spanning tree T of G denote by X^T the monomial being the product of $x_{i,j}$ for all edges i, j of T. The generating polynomial $f(G)$ is the sum of the monomials X^T over all spanning trees T.

Example: for G with 3 vertices $f(G) = x_{1,2} \cdot x_{2,3} + x_{1,2} \cdot x_{1,3} + x_{2,3} \cdot x_{1,3}$.

For vertices v, w of G denote by $G_{v,w}$ the graph obtained from G by gluing v, w into a vertex u with new conductances $x_{i,u} := x_{i,v} + x_{i,w}$.

Kirchhoff (1847): $cond_{i,j}(G) = f(G)/f(G_{i,j})$.

Star-mesh transformation: let a vertex $v \neq i, j$ of G. Denote by G_v a graph obtained from G by removing v and replacing each conductance $x_{p,q}$ by $x_{p,q} + x_{p,v} \cdot x_{q,v} \cdot \sum_k (x_{k,v})^{-1}$. Then $cond_{i,j}(G) = cond_{i,j}(G_v)$.

Corollary

Subtraction-free complexity $C_{+,\times,/}(cond_{i,j}(G), f(G)) \leq O(n^3)$.

Star-mesh transformations

Let edges of a complete graph G with n vertices (viewed as an electrical circuit) be endowed with **conductances** $x_{i,j}$. Denote by $cond_{i,j}(G)$ the conductance of the circuit between vertices i, j.

For a spanning tree T of G denote by X^T the monomial being the product of $x_{i,j}$ for all edges i, j of T. The generating polynomial $f(G)$ is the sum of the monomials X^T over all spanning trees T.

Example: for G with 3 vertices $f(G) = x_{1,2} \cdot x_{2,3} + x_{1,2} \cdot x_{1,3} + x_{2,3} \cdot x_{1,3}$.

For vertices v, w of G denote by $G_{v,w}$ the graph obtained from G by gluing v, w into a vertex u with new conductances $x_{i,u} := x_{i,v} + x_{i,w}$.

Kirchhoff (1847): $cond_{i,j}(G) = f(G)/f(G_{i,j})$.

Star-mesh transformation: let a vertex $v \neq i, j$ of G. Denote by G_v a graph obtained from G by removing v and replacing each conductance $x_{p,q}$ by $x_{p,q} + x_{p,v} \cdot x_{q,v} \cdot \sum_k (x_{k,v})^{-1}$. Then $cond_{i,j}(G) = cond_{i,j}(G_v)$.

Corollary

Subtraction-free complexity $C_{+,\times,/}(cond_{i,j}(G), f(G)) \leq O(n^3)$.

Star-mesh transformations

Let edges of a complete graph G with n vertices (viewed as an electrical circuit) be endowed with **conductances** $x_{i,j}$. Denote by $cond_{i,j}(G)$ the conductance of the circuit between vertices i, j.

For a spanning tree T of G denote by X^T the monomial being the product of $x_{i,j}$ for all edges i, j of T. The generating polynomial $f(G)$ is the sum of the monomials X^T over all spanning trees T.

Example: for G with 3 vertices $f(G) = x_{1,2} \cdot x_{2,3} + x_{1,2} \cdot x_{1,3} + x_{2,3} \cdot x_{1,3}$.

For vertices v, w of G denote by $G_{v,w}$ the graph obtained from G by gluing v, w into a vertex u with new conductances $x_{i,u} := x_{i,v} + x_{i,w}$.

Kirchhoff (1847): $cond_{i,j}(G) = f(G)/f(G_{i,j})$.

Star-mesh transformation: let a vertex $v \neq i, j$ of G. Denote by G_v a graph obtained from G by removing v and replacing each conductance $x_{p,q}$ by $x_{p,q} + x_{p,v} \cdot x_{q,v} \cdot \sum_k (x_{k,v})^{-1}$. Then $cond_{i,j}(G) = cond_{i,j}(G_v)$.

Corollary

Subtraction-free complexity $C_{+,\times,/}(cond_{i,j}(G), f(G)) \leq O(n^3)$.

Star-mesh transformations

Let edges of a complete graph G with n vertices (viewed as an electrical circuit) be endowed with **conductances** $x_{i,j}$. Denote by $cond_{i,j}(G)$ the conductance of the circuit between vertices i, j.

For a spanning tree T of G denote by X^T the monomial being the product of $x_{i,j}$ for all edges i, j of T. The generating polynomial $f(G)$ is the sum of the monomials X^T over all spanning trees T.

Example: for G with 3 vertices $f(G) = x_{1,2} \cdot x_{2,3} + x_{1,2} \cdot x_{1,3} + x_{2,3} \cdot x_{1,3}$.

For vertices v, w of G denote by $G_{v,w}$ the graph obtained from G by gluing v, w into a vertex u with new conductances $x_{i,u} := x_{i,v} + x_{i,w}$.

Kirchhoff (1847): $cond_{i,j}(G) = f(G)/f(G_{i,j})$.

Star-mesh transformation: let a vertex $v \neq i, j$ of G. Denote by G_v a graph obtained from G by removing v and replacing each conductance $x_{p,q}$ by $x_{p,q} + x_{p,v} \cdot x_{q,v} \cdot \sum_k (x_{k,v})^{-1}$. Then $cond_{i,j}(G) = cond_{i,j}(G_v)$.

Corollary

Subtraction-free complexity $C_{+,\times,/}(cond_{i,j}(G), f(G)) \leq O(n^3)$.

Star-mesh transformations

Let edges of a complete graph G with n vertices (viewed as an electrical circuit) be endowed with **conductances** $x_{i,j}$. Denote by $cond_{i,j}(G)$ the conductance of the circuit between vertices i, j.

For a spanning tree T of G denote by X^T the monomial being the product of $x_{i,j}$ for all edges i, j of T. The generating polynomial $f(G)$ is the sum of the monomials X^T over all spanning trees T.

Example: for G with 3 vertices $f(G) = x_{1,2} \cdot x_{2,3} + x_{1,2} \cdot x_{1,3} + x_{2,3} \cdot x_{1,3}$.

For vertices v, w of G denote by $G_{v,w}$ the graph obtained from G by gluing v, w into a vertex u with new conductances $x_{i,u} := x_{i,v} + x_{i,w}$.

Kirchhoff (1847): $cond_{i,j}(G) = f(G)/f(G_{i,j})$.

Star-mesh transformation: let a vertex $v \neq i, j$ of G. Denote by G_v a graph obtained from G by removing v and replacing each conductance $x_{p,q}$ by $x_{p,q} + x_{p,v} \cdot x_{q,v} \cdot \sum_k (x_{k,v})^{-1}$. Then $cond_{i,j}(G) = cond_{i,j}(G_v)$.

Corollary

Subtraction-free complexity $C_{+,\times,/}(cond_{i,j}(G), f(G)) \leq O(n^3)$.

Star-mesh transformations

Let edges of a complete graph G with n vertices (viewed as an electrical circuit) be endowed with **conductances** $x_{i,j}$. Denote by $cond_{i,j}(G)$ the conductance of the circuit between vertices i, j.

For a spanning tree T of G denote by X^T the monomial being the product of $x_{i,j}$ for all edges i, j of T. The generating polynomial $f(G)$ is the sum of the monomials X^T over all spanning trees T.

Example: for G with 3 vertices $f(G) = x_{1,2} \cdot x_{2,3} + x_{1,2} \cdot x_{1,3} + x_{2,3} \cdot x_{1,3}$.

For vertices v, w of G denote by $G_{v,w}$ the graph obtained from G by gluing v, w into a vertex u with new conductances $x_{i,u} := x_{i,v} + x_{i,w}$.

Kirchhoff (1847): $cond_{i,j}(G) = f(G)/f(G_{i,j})$.

Star-mesh transformation: let a vertex $v \neq i, j$ of G. Denote by G_v a graph obtained from G by removing v and replacing each conductance $x_{p,q}$ by $x_{p,q} + x_{p,v} \cdot x_{q,v} \cdot \sum_k (x_{k,v})^{-1}$. Then $cond_{i,j}(G) = cond_{i,j}(G_v)$.

Corollary

Subtraction-free complexity $C_{+,\times,/}(cond_{i,j}(G), f(G)) \le O(n^3)$.

Star-mesh transformations

Let edges of a complete graph G with n vertices (viewed as an electrical circuit) be endowed with **conductances** $x_{i,j}$. Denote by $cond_{i,j}(G)$ the conductance of the circuit between vertices i, j.

For a spanning tree T of G denote by X^T the monomial being the product of $x_{i,j}$ for all edges i, j of T. The generating polynomial $f(G)$ is the sum of the monomials X^T over all spanning trees T.

Example: for G with 3 vertices $f(G) = x_{1,2} \cdot x_{2,3} + x_{1,2} \cdot x_{1,3} + x_{2,3} \cdot x_{1,3}$.

For vertices v, w of G denote by $G_{v,w}$ the graph obtained from G by gluing v, w into a vertex u with new conductances $x_{i,u} := x_{i,v} + x_{i,w}$.

Kirchhoff (1847): $cond_{i,j}(G) = f(G)/f(G_{i,j})$.

Star-mesh transformation: let a vertex $v \neq i, j$ of G. Denote by G_v a graph obtained from G by removing v and replacing each conductance $x_{p,q}$ by $x_{p,q} + x_{p,v} \cdot x_{q,v} \cdot \sum_k (x_{k,v})^{-1}$. Then $cond_{i,j}(G) = cond_{i,j}(G_v)$.

Corollary

Subtraction-free complexity $C_{+,\times,/}(cond_{i,j}(G), f(G)) \leq O(n^3)$.

Star-mesh transformations

Let edges of a complete graph G with n vertices (viewed as an electrical circuit) be endowed with **conductances** $x_{i,j}$. Denote by $cond_{i,j}(G)$ the conductance of the circuit between vertices i, j.

For a spanning tree T of G denote by X^T the monomial being the product of $x_{i,j}$ for all edges i, j of T. The generating polynomial $f(G)$ is the sum of the monomials X^T over all spanning trees T.

Example: for G with 3 vertices $f(G) = x_{1,2} \cdot x_{2,3} + x_{1,2} \cdot x_{1,3} + x_{2,3} \cdot x_{1,3}$.

For vertices v, w of G denote by $G_{v,w}$ the graph obtained from G by gluing v, w into a vertex u with new conductances $x_{i,u} := x_{i,v} + x_{i,w}$.

Kirchhoff (1847): $cond_{i,j}(G) = f(G)/f(G_{i,j})$.

Star-mesh transformation: let a vertex $v \neq i, j$ of G. Denote by G_v a graph obtained from G by removing v and replacing each conductance $x_{p,q}$ by $x_{p,q} + x_{p,v} \cdot x_{q,v} \cdot \sum_k (x_{k,v})^{-1}$. Then $cond_{i,j}(G) = cond_{i,j}(G_v)$.

Corollary

Subtraction-free complexity $C_{+,\times,/}(cond_{i,j}(G), f(G)) \leq O(n^3)$.

Star-mesh transformations

Let edges of a complete graph G with n vertices (viewed as an electrical circuit) be endowed with **conductances** $x_{i,j}$. Denote by $cond_{i,j}(G)$ the conductance of the circuit between vertices i, j.

For a spanning tree T of G denote by X^T the monomial being the product of $x_{i,j}$ for all edges i, j of T. The generating polynomial $f(G)$ is the sum of the monomials X^T over all spanning trees T.

Example: for G with 3 vertices $f(G) = x_{1,2} \cdot x_{2,3} + x_{1,2} \cdot x_{1,3} + x_{2,3} \cdot x_{1,3}$.

For vertices v, w of G denote by $G_{v,w}$ the graph obtained from G by gluing v, w into a vertex u with new conductances $x_{i,u} := x_{i,v} + x_{i,w}$.

Kirchhoff (1847): $cond_{i,j}(G) = f(G)/f(G_{i,j})$.

Star-mesh transformation: let a vertex $v \neq i, j$ of G. Denote by G_v a graph obtained from G by removing v and replacing each conductance $x_{p,q}$ by $x_{p,q} + x_{p,v} \cdot x_{q,v} \cdot \sum_k (x_{k,v})^{-1}$. Then $cond_{i,j}(G) = cond_{i,j}(G_v)$.

Corollary

Subtraction-free complexity $C_{+,\times,/}(cond_{i,j}(G), f(G)) \leq O(n^3)$.

Star-mesh transformations

Let edges of a complete graph G with n vertices (viewed as an electrical circuit) be endowed with **conductances** $x_{i,j}$. Denote by $cond_{i,j}(G)$ the conductance of the circuit between vertices i, j.

For a spanning tree T of G denote by X^T the monomial being the product of $x_{i,j}$ for all edges i, j of T. The generating polynomial $f(G)$ is the sum of the monomials X^T over all spanning trees T.

Example: for G with 3 vertices $f(G) = x_{1,2} \cdot x_{2,3} + x_{1,2} \cdot x_{1,3} + x_{2,3} \cdot x_{1,3}$.

For vertices v, w of G denote by $G_{v,w}$ the graph obtained from G by gluing v, w into a vertex u with new conductances $x_{i,u} := x_{i,v} + x_{i,w}$.

Kirchhoff (1847): $cond_{i,j}(G) = f(G)/f(G_{i,j})$.

Star-mesh transformation: let a vertex $v \neq i, j$ of G. Denote by G_v a graph obtained from G by removing v and replacing each conductance $x_{p,q}$ by $x_{p,q} + x_{p,v} \cdot x_{q,v} \cdot \sum_k (x_{k,v})^{-1}$. Then $cond_{i,j}(G) = cond_{i,j}(G_v)$.

Corollary

Subtraction-free complexity $C_{+,\times,/}(cond_{i,j}(G),\ f(G)) \leq O(n^3)$.

Arborescences

Now G is a complete graph with n vertices and edges i, j endowed with two variables $x_{i,j} \neq x_{j,i}$. Fix a vertex r of G and consider any spanning tree T of G as having its root at r and all its edges directed towards r (such directed trees are called *arborescences*). Then the monomial X^T corresponding to this tree is the product of all $x_{i,j}$ over all edges i, j of T according to the chosen direction of T. The generating polynomial $\phi(G)$ of arborescences is the sum of all the monomials X^T.

Example: if $n = 3$, $r = 2$ then $\phi(G) = x_{1,2} \cdot x_{3,2} + x_{1,2} \cdot x_{3,1} + x_{1,3} \cdot x_{3,2}$.

Star-mesh transformation: let a vertex $v \neq r$ of G. Denote by G_v a graph obtained from G by removing v and replacing each $x_{p,q}$ by $x_{p,q} + x_{p,v} \cdot x_{v,q} \cdot (\sum_k x_{v,k})^{-1}$. Then $\phi(G) = \sum_k x_{v,k} \cdot \phi(G_v)$.

Corollary

$C_{+,\times,/}(\phi(G)) \leq O(n^3)$.

Schnorr-Valiant-Jerrum-Snir: $C_{+,\times}(\phi(G)) \geq c^n$ for some $c > 1$.

Arborescences

Now G is a complete graph with n vertices and edges i, j endowed with two variables $x_{i,j} \neq x_{j,i}$. Fix a vertex r of G and consider any spanning tree T of G as having its root at r and all its edges directed towards r (such directed trees are called *arborescences*). Then the monomial X^T corresponding to this tree is the product of all $x_{i,j}$ over all edges i, j of T according to the chosen direction of T. The generating polynomial $\phi(G)$ of arborescences is the sum of all the monomials X^T.

Example: if $n = 3,\ r = 2$ then $\phi(G) = x_{1,2} \cdot x_{3,2} + x_{1,2} \cdot x_{3,1} + x_{1,3} \cdot x_{3,2}$.

Star-mesh transformation: let a vertex $v \neq r$ of G. Denote by G_v a graph obtained from G by removing v and replacing each $x_{p,q}$ by $x_{p,q} + x_{p,v} \cdot x_{v,q} \cdot (\sum_k x_{v,k})^{-1}$. Then $\phi(G) = \sum_k x_{v,k} \cdot \phi(G_v)$.

Corollary

$C_{+,\times,/}(\phi(G)) \leq O(n^3)$.

Schnorr-Valiant-Jerrum-Snir: $C_{+,\times}(\phi(G)) \geq c^n$ for some $c > 1$.

Arborescences

Now G is a complete graph with n vertices and edges i, j endowed with two variables $x_{i,j} \neq x_{j,i}$. Fix a vertex r of G and consider any spanning tree T of G as having its root at r and all its edges directed towards r (such directed trees are called *arborescences*). Then the monomial X^T corresponding to this tree is the product of all $x_{i,j}$ over all edges i, j of T according to the chosen direction of T. The generating polynomial $\phi(G)$ of arborescences is the sum of all the monomials X^T.

Example: if $n = 3$, $r = 2$ then $\phi(G) = x_{1,2} \cdot x_{3,2} + x_{1,2} \cdot x_{3,1} + x_{1,3} \cdot x_{3,2}$.

Star-mesh transformation: let a vertex $v \neq r$ of G. Denote by G_v a graph obtained from G by removing v and replacing each $x_{p,q}$ by $x_{p,q} + x_{p,v} \cdot x_{v,q} \cdot (\sum_k x_{v,k})^{-1}$. Then $\phi(G) = \sum_k x_{v,k} \cdot \phi(G_v)$.

Corollary

$C_{+,\times,/}(\phi(G)) \leq O(n^3)$.

Schnorr-Valiant-Jerrum-Snir: $C_{+,\times}(\phi(G)) \geq c^n$ for some $c > 1$.

Arborescences

Now G is a complete graph with n vertices and edges i, j endowed with two variables $x_{i,j} \neq x_{j,i}$. Fix a vertex r of G and consider any spanning tree T of G as having its root at r and all its edges directed towards r (such directed trees are called *arborescences*). Then the monomial X^T corresponding to this tree is the product of all $x_{i,j}$ over all edges i, j of T according to the chosen direction of T. The generating polynomial $\phi(G)$ of arborescences is the sum of all the monomials X^T.

Example: if $n = 3$, $r = 2$ then $\phi(G) = x_{1,2} \cdot x_{3,2} + x_{1,2} \cdot x_{3,1} + x_{1,3} \cdot x_{3,2}$.

Star-mesh transformation: let a vertex $v \neq r$ of G. Denote by G_v a graph obtained from G by removing v and replacing each $x_{p,q}$ by $x_{p,q} + x_{p,v} \cdot x_{v,q} \cdot (\sum_k x_{v,k})^{-1}$. Then $\phi(G) = \sum_k x_{v,k} \cdot \phi(G_v)$.

Corollary

$C_{+,\times,/}(\phi(G)) \leq O(n^3)$.

Schnorr-Valiant-Jerrum-Snir: $C_{+,\times}(\phi(G)) \geq c^n$ for some $c > 1$.

Arborescences

Now G is a complete graph with n vertices and edges i, j endowed with two variables $x_{i,j} \neq x_{j,i}$. Fix a vertex r of G and consider any spanning tree T of G as having its root at r and all its edges directed towards r (such directed trees are called *arborescences*). Then the monomial X^T corresponding to this tree is the product of all $x_{i,j}$ over all edges i, j of T according to the chosen direction of T. The generating polynomial $\phi(G)$ of arborescences is the sum of all the monomials X^T.

Example: if $n = 3$, $r = 2$ then $\phi(G) = x_{1,2} \cdot x_{3,2} + x_{1,2} \cdot x_{3,1} + x_{1,3} \cdot x_{3,2}$.

Star-mesh transformation: let a vertex $v \neq r$ of G. Denote by G_v a graph obtained from G by removing v and replacing each $x_{p,q}$ by $x_{p,q} + x_{p,v} \cdot x_{v,q} \cdot (\sum_k x_{v,k})^{-1}$. Then $\phi(G) = \sum_k x_{v,k} \cdot \phi(G_v)$.

Corollary

$C_{+,\times,/}(\phi(G)) \leq O(n^3)$.

Schnorr-Valiant-Jerrum-Snir: $C_{+,\times}(\phi(G)) \geq c^n$ for some $c > 1$.

Arborescences

Now G is a complete graph with n vertices and edges i, j endowed with two variables $x_{i,j} \neq x_{j,i}$. Fix a vertex r of G and consider any spanning tree T of G as having its root at r and all its edges directed towards r (such directed trees are called *arborescences*). Then the monomial X^T corresponding to this tree is the product of all $x_{i,j}$ over all edges i, j of T according to the chosen direction of T. The generating polynomial $\phi(G)$ of arborescences is the sum of all the monomials X^T.

Example: if $n = 3,\ r = 2$ then $\phi(G) = x_{1,2} \cdot x_{3,2} + x_{1,2} \cdot x_{3,1} + x_{1,3} \cdot x_{3,2}$.

Star-mesh transformation: let a vertex $v \neq r$ of G. Denote by G_v a graph obtained from G by removing v and replacing each $x_{p,q}$ by $x_{p,q} + x_{p,v} \cdot x_{v,q} \cdot (\sum_k x_{v,k})^{-1}$. Then $\phi(G) = \sum_k x_{v,k} \cdot \phi(G_v)$.

Corollary

$C_{+,\times,/}(\phi(G)) \leq O(n^3)$.

Schnorr-Valiant-Jerrum-Snir: $C_{+,\times}(\phi(G)) \geq c^n$ for some $c > 1$.

Arborescences

Now G is a complete graph with n vertices and edges $i,\ j$ endowed with two variables $x_{i,j} \neq x_{j,i}$. Fix a vertex r of G and consider any spanning tree T of G as having its root at r and all its edges directed towards r (such directed trees are called *arborescences*). Then the monomial X^T corresponding to this tree is the product of all $x_{i,j}$ over all edges $i,\ j$ of T according to the chosen direction of T. The generating polynomial $\phi(G)$ of arborescences is the sum of all the monomials X^T.

Example: if $n = 3,\ r = 2$ then $\phi(G) = x_{1,2} \cdot x_{3,2} + x_{1,2} \cdot x_{3,1} + x_{1,3} \cdot x_{3,2}$.

Star-mesh transformation: let a vertex $v \neq r$ of G. Denote by G_v a graph obtained from G by removing v and replacing each $x_{p,q}$ by $x_{p,q} + x_{p,v} \cdot x_{v,q} \cdot (\sum_k x_{v,k})^{-1}$. Then $\phi(G) = \sum_k x_{v,k} \cdot \phi(G_v)$.

Corollary

$C_{+,\times,/}(\phi(G)) \leq O(n^3)$.

Schnorr-Valiant-Jerrum-Snir: $C_{+,\times}(\phi(G)) \geq c^n$ for some $c > 1$.

Arborescences

Now G is a complete graph with n vertices and edges i, j endowed with two variables $x_{i,j} \neq x_{j,i}$. Fix a vertex r of G and consider any spanning tree T of G as having its root at r and all its edges directed towards r (such directed trees are called *arborescences*). Then the monomial X^T corresponding to this tree is the product of all $x_{i,j}$ over all edges i, j of T according to the chosen direction of T. The generating polynomial $\phi(G)$ of arborescences is the sum of all the monomials X^T.

Example: if $n = 3,\ r = 2$ then $\phi(G) = x_{1,2} \cdot x_{3,2} + x_{1,2} \cdot x_{3,1} + x_{1,3} \cdot x_{3,2}$.

Star-mesh transformation: let a vertex $v \neq r$ of G. Denote by G_v a graph obtained from G by removing v and replacing each $x_{p,q}$ by $x_{p,q} + x_{p,v} \cdot x_{v,q} \cdot (\sum_k x_{v,k})^{-1}$. Then $\phi(G) = \sum_k x_{v,k} \cdot \phi(G_v)$.

Corollary

$C_{+,\times,/}(\phi(G)) \leq O(n^3)$.

Schnorr-Valiant-Jerrum-Snir: $C_{+,\times}(\phi(G)) \geq c^n$ for some $c > 1$.

Arborescences

Now G is a complete graph with n vertices and edges i, j endowed with two variables $x_{i,j} \neq x_{j,i}$. Fix a vertex r of G and consider any spanning tree T of G as having its root at r and all its edges directed towards r (such directed trees are called *arborescences*). Then the monomial X^T corresponding to this tree is the product of all $x_{i,j}$ over all edges i, j of T according to the chosen direction of T. The generating polynomial $\phi(G)$ of arborescences is the sum of all the monomials X^T.

Example: if $n = 3,\ r = 2$ then $\phi(G) = x_{1,2} \cdot x_{3,2} + x_{1,2} \cdot x_{3,1} + x_{1,3} \cdot x_{3,2}$.

Star-mesh transformation: let a vertex $v \neq r$ of G. Denote by G_v a graph obtained from G by removing v and replacing each $x_{p,q}$ by $x_{p,q} + x_{p,v} \cdot x_{v,q} \cdot (\sum_k x_{v,k})^{-1}$. Then $\phi(G) = \sum_k x_{v,k} \cdot \phi(G_v)$.

Corollary

$C_{+,\times,/}(\phi(G)) \leq O(n^3)$.

Schnorr-Valiant-Jerrum-Snir: $C_{+,\times}(\phi(G)) \geq c^n$ for some $c > 1$.

Arborescences

Now G is a complete graph with n vertices and edges i, j endowed with two variables $x_{i,j} \neq x_{j,i}$. Fix a vertex r of G and consider any spanning tree T of G as having its root at r and all its edges directed towards r (such directed trees are called *arborescences*). Then the monomial X^T corresponding to this tree is the product of all $x_{i,j}$ over all edges i, j of T according to the chosen direction of T. The generating polynomial $\phi(G)$ of arborescences is the sum of all the monomials X^T.

Example: if $n = 3,\ r = 2$ then $\phi(G) = x_{1,2} \cdot x_{3,2} + x_{1,2} \cdot x_{3,1} + x_{1,3} \cdot x_{3,2}$.

Star-mesh transformation: let a vertex $v \neq r$ of G. Denote by G_v a graph obtained from G by removing v and replacing each $x_{p,q}$ by $x_{p,q} + x_{p,v} \cdot x_{v,q} \cdot (\sum_k x_{v,k})^{-1}$. Then $\phi(G) = \sum_k x_{v,k} \cdot \phi(G_v)$.

Corollary

$C_{+,\times,/}(\phi(G)) \leq O(n^3)$.

Schnorr-Valiant-Jerrum-Snir: $C_{+,\times}(\phi(G)) \geq c^n$ for some $c > 1$.

Arborescences

Now G is a complete graph with n vertices and edges i, j endowed with two variables $x_{i,j} \neq x_{j,i}$. Fix a vertex r of G and consider any spanning tree T of G as having its root at r and all its edges directed towards r (such directed trees are called *arborescences*). Then the monomial X^T corresponding to this tree is the product of all $x_{i,j}$ over all edges i, j of T according to the chosen direction of T. The generating polynomial $\phi(G)$ of arborescences is the sum of all the monomials X^T.

Example: if $n = 3,\ r = 2$ then $\phi(G) = x_{1,2} \cdot x_{3,2} + x_{1,2} \cdot x_{3,1} + x_{1,3} \cdot x_{3,2}$.

Star-mesh transformation: let a vertex $v \neq r$ of G. Denote by G_v a graph obtained from G by removing v and replacing each $x_{p,q}$ by $x_{p,q} + x_{p,v} \cdot x_{v,q} \cdot (\sum_k x_{v,k})^{-1}$. Then $\phi(G) = \sum_k x_{v,k} \cdot \phi(G_v)$.

Corollary

$C_{+,\times,/}(\phi(G)) \leq O(n^3)$.

Schnorr-Valiant-Jerrum-Snir: $C_{+,\times}(\phi(G)) \geq c^n$ for some $c > 1$.

Subtraction can be exponentially powerful

Polya (1923): any homogeneous polynomial $h \in \mathbb{R}[x_1, \dots, x_n]$ positive everywhere on the non-negative orthant $\mathbb{R}^n_{\geq 0} \setminus \{0\}$ can be represented as a fraction $P/(x_1 + \dots + x_n)^N$ for suitable integer N and a polynomial P with positive coefficients.

Example: $x^2 - xy + y^2 = (x^3 + y^3)/(x + y)$.

Lemma

For any representation $h_n := (t - x_1)^4 + (x_1 - 2x_2)^4 + (x_2^2 - tx_3)^2 + (x_3^2 - tx_4)^2 + \dots + (x_{n-1}^2 - tx_n)^2 + 2x_n^4 + 4x_n^2(t - x_1)^2 = P/Q$ *where* P, Q *being polynomials with positive coefficients we have* $\deg(Q) > 2^{c2^n}$ *for some constant* $c > 0$.

Corollary

(Polya's theorem for h_n*)* $\infty > C_{+,\times,/}(h_n) > c2^n$ *(lemma)*

(since after each operation the degree of a rational representation at most doubles). On the other hand, evidently, $C_{+,-,\times,/}(h_n) \leq O(n)$.

Subtraction can be exponentially powerful

Polya (1923): any homogeneous polynomial $h \in \mathbb{R}[x_1, \ldots, x_n]$ positive everywhere on the non-negative orthant $\mathbb{R}^n_{\geq 0} \setminus \{0\}$ can be represented as a fraction $P/(x_1 + \cdots + x_n)^N$ for suitable integer N and a polynomial P with positive coefficients.

Example: $x^2 - xy + y^2 = (x^3 + y^3)/(x+y)$.

Lemma

For any representation $h_n := (t - x_1)^4 + (x_1 - 2x_2)^4 + (x_2^2 - tx_3)^2 + (x_3^2 - tx_4)^2 + \cdots + (x_{n-1}^2 - tx_n)^2 + 2x_n^4 + 4x_n^2(t - x_1)^2 = P/Q$ *where* P, Q *being polynomials with positive coefficients we have* $\deg(Q) > 2^{c2^n}$ *for some constant* $c > 0$.

Corollary

(Polya's theorem for h_n*)* $\infty > C_{+,\times,/}(h_n) > c2^n$ *(lemma)*

(since after each operation the degree of a rational representation at most doubles). On the other hand, evidently, $C_{+,-,\times,/}(h_n) \le O(n)$.

Subtraction can be exponentially powerful

Polya (1923): any homogeneous polynomial $h \in \mathbb{R}[x_1, \dots, x_n]$ positive everywhere on the non-negative orthant $\mathbb{R}^n_{\geq 0} \setminus \{0\}$ can be represented as a fraction $P/(x_1 + \cdots + x_n)^N$ for suitable integer N and a polynomial P with positive coefficients.

Example: $x^2 - xy + y^2 = (x^3 + y^3)/(x + y)$.

Lemma

For any representation $h_n := (t - x_1)^4 + (x_1 - 2x_2)^4 + (x_2^2 - tx_3)^2 + (x_3^2 - tx_4)^2 + \cdots + (x_{n-1}^2 - tx_n)^2 + 2x_n^4 + 4x_n^2(t - x_1)^2 = P/Q$ *where* P, Q *being polynomials with positive coefficients we have* $\deg(Q) > 2^{c2^n}$ *for some constant* $c > 0$.

Corollary

(Polya's theorem for h_n) $\infty > C_{+,\times,/}(h_n) > c2^n$ *(lemma)*

(since after each operation the degree of a rational representation at most doubles). On the other hand, evidently, $C_{+,-,\times,/}(h_n) \leq O(n)$.

Subtraction can be exponentially powerful

Polya (1923): any homogeneous polynomial $h \in \mathbb{R}[x_1, \dots, x_n]$ positive everywhere on the non-negative orthant $\mathbb{R}^n_{\geq 0} \setminus \{0\}$ can be represented as a fraction $P/(x_1 + \cdots + x_n)^N$ for suitable integer N and a polynomial P with positive coefficients.

Example: $x^2 - xy + y^2 = (x^3 + y^3)/(x + y)$.

Lemma

For any representation $h_n := (t - x_1)^4 + (x_1 - 2x_2)^4 + (x_2^2 - tx_3)^2 + (x_3^2 - tx_4)^2 + \cdots + (x_{n-1}^2 - tx_n)^2 + 2x_n^4 + 4x_n^2(t - x_1)^2 = P/Q$ *where* P, Q *being polynomials with positive coefficients we have* $\deg(Q) > 2^{c2^n}$ *for some constant* $c > 0$.

Corollary

(Polya's theorem for h_n*)* $\infty > C_{+,\times,/}(h_n) > c2^n$ *(lemma)*

(since after each operation the degree of a rational representation at most doubles). On the other hand, evidently, $C_{+,-,\times,/}(h_n) \leq O(n)$.

Subtraction can be exponentially powerful

Polya (1923): any homogeneous polynomial $h \in \mathbb{R}[x_1, \dots, x_n]$ positive everywhere on the non-negative orthant $\mathbb{R}^n_{\geq 0} \setminus \{0\}$ can be represented as a fraction $P/(x_1 + \dots + x_n)^N$ for suitable integer N and a polynomial P with positive coefficients.

Example: $x^2 - xy + y^2 = (x^3 + y^3)/(x + y)$.

Lemma

For any representation $h_n := (t - x_1)^4 + (x_1 - 2x_2)^4 + (x_2^2 - tx_3)^2 + (x_3^2 - tx_4)^2 + \dots + (x_{n-1}^2 - tx_n)^2 + 2x_n^4 + 4x_n^2(t - x_1)^2 = P/Q$ *where* P, Q *being polynomials with positive coefficients we have* $\deg(Q) > 2^{c2^n}$ *for some constant* $c > 0$.

Corollary

(Polya's theorem for h_n) $\infty > C_{+,\times,/}(h_n) > c2^n$ *(lemma)*

(since after each operation the degree of a rational representation at most doubles). On the other hand, evidently, $C_{+,-,\times,/}(h_n) \leq O(n)$.

Subtraction can be exponentially powerful

Polya (1923): any homogeneous polynomial $h \in \mathbb{R}[x_1, \dots, x_n]$ positive everywhere on the non-negative orthant $\mathbb{R}^n_{\geq 0} \setminus \{0\}$ can be represented as a fraction $P/(x_1 + \cdots + x_n)^N$ for suitable integer N and a polynomial P with positive coefficients.

Example: $x^2 - xy + y^2 = (x^3 + y^3)/(x + y)$.

Lemma

For any representation $h_n := (t - x_1)^4 + (x_1 - 2x_2)^4 + (x_2^2 - tx_3)^2 + (x_3^2 - tx_4)^2 + \cdots + (x_{n-1}^2 - tx_n)^2 + 2x_n^4 + 4x_n^2(t - x_1)^2 = P/Q$ *where* P, Q *being polynomials with positive coefficients we have* $\deg(Q) > 2^{c2^n}$ *for some constant* $c > 0$.

Corollary

(Polya's theorem for h_n*)* $\infty > C_{+,\times,/}(h_n) > c2^n$ *(lemma)*

(since after each operation the degree of a rational representation at most doubles). On the other hand, evidently, $C_{+,-,\times,/}(h_n) \leq O(n)$.

Subtraction can be exponentially powerful

Polya (1923): any homogeneous polynomial $h \in \mathbb{R}[x_1, \dots, x_n]$ positive everywhere on the non-negative orthant $\mathbb{R}^n_{\geq 0} \setminus \{0\}$ can be represented as a fraction $P/(x_1 + \cdots + x_n)^N$ for suitable integer N and a polynomial P with positive coefficients.

Example: $x^2 - xy + y^2 = (x^3 + y^3)/(x + y)$.

Lemma

For any representation $h_n := (t - x_1)^4 + (x_1 - 2x_2)^4 + (x_2^2 - tx_3)^2 + (x_3^2 - tx_4)^2 + \cdots + (x_{n-1}^2 - tx_n)^2 + 2x_n^4 + 4x_n^2(t - x_1)^2 = P/Q$ *where* P, Q *being polynomials with positive coefficients we have* $\deg(Q) > 2^{c2^n}$ *for some constant* $c > 0$.

Corollary

(Polya's theorem for h_n*)* $\infty > C_{+,\times,/}(h_n) > c2^n$ *(lemma)*

(since after each operation the degree of a rational representation at most doubles). On the other hand, evidently, $C_{+,-,\times,/}(h_n) \le O(n)$.

Subtraction can be exponentially powerful

Polya (1923): any homogeneous polynomial $h \in \mathbb{R}[x_1, \ldots, x_n]$ positive everywhere on the non-negative orthant $\mathbb{R}^n_{\geq 0} \setminus \{0\}$ can be represented as a fraction $P/(x_1 + \cdots + x_n)^N$ for suitable integer N and a polynomial P with positive coefficients.

Example: $x^2 - xy + y^2 = (x^3 + y^3)/(x + y)$.

Lemma

For any representation $h_n := (t - x_1)^4 + (x_1 - 2x_2)^4 + (x_2^2 - tx_3)^2 + (x_3^2 - tx_4)^2 + \cdots + (x_{n-1}^2 - tx_n)^2 + 2x_n^4 + 4x_n^2(t - x_1)^2 = P/Q$ *where* P, Q *being polynomials with positive coefficients we have* $\deg(Q) > 2^{c2^n}$ *for some constant* $c > 0$.

Corollary

(Polya's theorem for h_n*)* $\infty > C_{+,\times,/}(h_n) > c2^n$ *(lemma)*

(since after each operation the degree of a rational representation at most doubles). On the other hand, evidently, $C_{+,-,\times,/}(h_n) \leq O(n)$.

Subtraction can be exponentially powerful

Polya (1923): any homogeneous polynomial $h \in \mathbb{R}[x_1, \dots, x_n]$ positive everywhere on the non-negative orthant $\mathbb{R}^n_{\geq 0} \setminus \{0\}$ can be represented as a fraction $P/(x_1 + \cdots + x_n)^N$ for suitable integer N and a polynomial P with positive coefficients.

Example: $x^2 - xy + y^2 = (x^3 + y^3)/(x + y)$.

Lemma

For any representation $h_n := (t - x_1)^4 + (x_1 - 2x_2)^4 + (x_2^2 - tx_3)^2 + (x_3^2 - tx_4)^2 + \cdots + (x_{n-1}^2 - tx_n)^2 + 2x_n^4 + 4x_n^2(t - x_1)^2 = P/Q$ *where* P, Q *being polynomials with positive coefficients we have* $\deg(Q) > 2^{c2^n}$ *for some constant* $c > 0$.

Corollary

(Polya's theorem for h_n*)* $\infty > C_{+,\times,/}(h_n) > c2^n$ *(lemma)*

(since after each operation the degree of a rational representation at most doubles). On the other hand, evidently, $C_{+,-,\times,/}(h_n) \leq O(n)$.

Subtraction can be exponentially powerful

Polya (1923): any homogeneous polynomial $h \in \mathbb{R}[x_1, \dots, x_n]$ positive everywhere on the non-negative orthant $\mathbb{R}^n_{\geq 0} \setminus \{0\}$ can be represented as a fraction $P/(x_1 + \cdots + x_n)^N$ for suitable integer N and a polynomial P with positive coefficients.

Example: $x^2 - xy + y^2 = (x^3 + y^3)/(x + y)$.

Lemma

For any representation $h_n := (t - x_1)^4 + (x_1 - 2x_2)^4 + (x_2^2 - tx_3)^2 + (x_3^2 - tx_4)^2 + \cdots + (x_{n-1}^2 - tx_n)^2 + 2x_n^4 + 4x_n^2(t - x_1)^2 = P/Q$ *where* P, Q *being polynomials with positive coefficients we have* $\deg(Q) > 2^{c2^n}$ *for some constant* $c > 0$.

Corollary

(Polya's theorem for h_n*)* $\infty > C_{+,\times,/}(h_n) > c2^n$ *(lemma)*

(since after each operation the degree of a rational representation at most doubles). On the other hand, evidently, $C_{+,-,\times,/}(h_n) < O(n)$.