Subtraction-free complexity and cluster algebras (joint work with S. Fomin, G. Koshevoy)

Dima Grigoriev (Lille)

CNRS

18/05/2016, Saint-Petersbourg

Let *M* ⊂ {+, −, ×, /}. Complexity $C_M(f)$ for a rational function $f \in \mathbb{Q}(X_1, \ldots, X_n)$ is defined as the minimal number of operations from *M* necessary to compute *f*, provided it is finite.

Problem. For given $M \subset M_1 \subset \{+, -, \times, /\}$ how big can be $C_M(f)$ in

Let $M \subset \{+, -, \times, / \}$. Complexity $C_M(f)$ for a rational function $f \in \mathbb{Q}(X_1, \ldots, X_n)$ is defined as the minimal number of operations from *M* necessary to compute *f*, provided it is finite.

Problem. For given $M \subset M_1 \subset \{+, -, \times, \Lambda\}$ how big can be $C_M(f)$ in

This problem is non-trivial just for three pairs of $M \subset M_1$.

Let $M \subset \{+, -, \times, / \}$. Complexity $C_M(f)$ for a rational function $f \in \mathbb{Q}(X_1, \ldots, X_n)$ is defined as the minimal number of operations from *M* necessary to compute *f*, provided it is finite.

Problem. For given $M \subset M_1 \subset \{+, -, \times, /\}$ how big can be $C_M(f)$ in comparison with $C_{M_1}(f)$?

This problem is non-trivial just for three pairs of $M \subset M_1$.

*for a polynomial f (***V. Strassen, 1973***)*

Let $M \subset \{+, -, \times, / \}$. Complexity $C_M(f)$ for a rational function $f \in \mathbb{Q}(X_1, \ldots, X_n)$ is defined as the minimal number of operations from *M* necessary to compute *f*, provided it is finite.

Problem. For given $M \subset M_1 \subset \{+, -, \times, /\}$ how big can be $C_M(f)$ in comparison with $C_{M_1}(f)$?

This problem is non-trivial just for three pairs of $M \subset M_1$.

*for a polynomial f (***V. Strassen, 1973***)*

Let $M \subset \{+, -, \times, / \}$. Complexity $C_M(f)$ for a rational function $f \in \mathbb{Q}(X_1, \ldots, X_n)$ is defined as the minimal number of operations from *M* necessary to compute *f*, provided it is finite.

Problem. For given $M \subset M_1 \subset \{+, -, \times, /\}$ how big can be $C_M(f)$ in comparison with $C_{M_1}(f)$?

This problem is non-trivial just for three pairs of $M \subset M_1$.

Theorem

 $C_{+,-, \times}(f)$ ≤ $O(C_{+,-, \times}/(f) \cdot deg(f))$ *for a polynomial f (***V. Strassen, 1973***)*

Question. Is the bound sharp for big deg(*f*)?

Let $M \subset \{+, -, \times, / \}$. Complexity $C_M(f)$ for a rational function $f \in \mathbb{Q}(X_1, \ldots, X_n)$ is defined as the minimal number of operations from *M* necessary to compute *f*, provided it is finite.

Problem. For given $M \subset M_1 \subset \{+, -, \times, /\}$ how big can be $C_M(f)$ in comparison with $C_{M_1}(f)$?

This problem is non-trivial just for three pairs of $M \subset M_1$.

Theorem

 $C_{+,-, \times}(f)$ ≤ $O(C_{+,-, \times}/(f) \cdot deg(f))$ *for a polynomial f (***V. Strassen, 1973***)*

("ring complexity" \leq O ("field complexity" \cdot deg))

Question. Is the bound sharp for big deg(*f*)?

C+, [−], [×](det) ≤ *O*(*n* 4) *(***D. K. Faddeev, V. N. F[add](#page-5-0)[eeva, 1960](#page-0-0)***[\)](#page-0-0)*

Let $M \subset \{+, -, \times, / \}$. Complexity $C_M(f)$ for a rational function $f \in \mathbb{Q}(X_1, \ldots, X_n)$ is defined as the minimal number of operations from *M* necessary to compute *f*, provided it is finite.

Problem. For given $M \subset M_1 \subset \{+, -, \times, /\}$ how big can be $C_M(f)$ in comparison with $C_{M_1}(f)$?

This problem is non-trivial just for three pairs of $M \subset M_1$.

Theorem

 $C_{+,-, \times}(f) \leq O(C_{+,-, \times/}(f) \cdot \deg(f))$ *for a polynomial f (***V. Strassen, 1973***)*

("ring complexity" $\leq O$ ("field complexity" \cdot deg)) **Question**. Is the bound sharp for big deg(*f*)?

C+, [−], [×](det) ≤ *O*(*n* 4) *(***D. K. Faddeev, V. N. F[add](#page-6-0)[eeva, 1960](#page-0-0)***[\)](#page-0-0)*

Let $M \subset \{+, -, \times, / \}$. Complexity $C_M(f)$ for a rational function $f \in \mathbb{Q}(X_1, \ldots, X_n)$ is defined as the minimal number of operations from *M* necessary to compute *f*, provided it is finite.

Problem. For given $M \subset M_1 \subset \{+, -, \times, /\}$ how big can be $C_M(f)$ in comparison with $C_{M_1}(f)$?

This problem is non-trivial just for three pairs of $M \subset M_1$.

Theorem

 $C_{+,-, \times}(f) \leq O(C_{+,-, \times/}(f) \cdot \deg(f))$ *for a polynomial f (***V. Strassen, 1973***)*

("ring complexity" \leq O ("field complexity" \cdot deg))

Question. Is the bound sharp for big deg(*f*)?

Corollary

C+, [−], [×](det) ≤ *O*(*n* 4) *(***D. K. Faddeev, V. N. F[add](#page-7-0)[eeva, 1960](#page-0-0)***[\)](#page-0-0)*

• $C_{+,\times}(f)$ can be exponentially bigger than $C_{+,\times}$, (f)

Corollary. Subtraction-free complexity is com[pu](#page-8-0)t[able.](#page-0-0) (E) (E)

• $C_{+,\times}(f)$ can be exponentially bigger than $C_{+,\times}$, (f)

• $C_{+,\times}$, (g) can be exponentially bigger than $C_{+,-,\times}$, (g) .

Corollary. Subtraction-free complexity is com[pu](#page-9-0)t[able.](#page-0-0) (E) (E)

• $C_{+,\times}(f)$ can be exponentially bigger than $C_{+,\times}(f)$

• $C_{+,\times}$, (g) can be exponentially bigger than $C_{+,-,\times}$, (g) .

L. Valiant, 1980 has constructed a polynomial *h* (with positive

Corollary. Subtraction-free complexity is com[pu](#page-10-0)t[able.](#page-0-0) (E) (E)

- • $C_{+,\times}(f)$ can be exponentially bigger than $C_{+,\times}$, (f)
- $C_{+,\times}$, (g) can be exponentially bigger than $C_{+,-,\times}$, (g) .

L. Valiant, 1980 has constructed a polynomial *h* (with positive *C*_{+, −, ×, /}(*h*). Funny consequence: at least one of two above • results

Corollary. Subtraction-free complexity is com[pu](#page-11-0)t[able.](#page-0-0) \cdot = \cdot + = \cdot

- • $C_{+,\times}(f)$ can be exponentially bigger than $C_{+,\times}$, (f)
- $C_{+,\times}$, (g) can be exponentially bigger than $C_{+,-,\times}$, (g) .
- **L. Valiant, 1980** has constructed a polynomial *h* (with positive coefficients) such that $C_{+,\times}(h)$ is exponentially bigger than *C*_{+, −, x}, (h). Funny consequence: at least one of two above • results
-
-
-

Corollary. Subtraction-free complexity is com[pu](#page-12-0)t[able.](#page-0-0) \cdot = \cdot + = \cdot

- • $C_{+,\times}(f)$ can be exponentially bigger than $C_{+,\times}(f)$
- $C_{+,\times}$, (g) can be exponentially bigger than $C_{+,-,\times}$, (g) .

L. Valiant, 1980 has constructed a polynomial *h* (with positive coefficients) such that $C_{+,\times}(h)$ is exponentially bigger than *C*+, [−], [×], /(*h*). Funny consequence: at least one of two above • results **holds.** But which one?: A growth of $C_{+\times}$, (h) is not known.

Corollary. Subtraction-free complexity is com[pu](#page-13-0)t[able.](#page-0-0) \cdot = \cdot + = \cdot

- • $C_{+,\times}(f)$ can be exponentially bigger than $C_{+,\times}(f)$
- $C_{+,\times}$, (g) can be exponentially bigger than $C_{+,-,\times}$, (g) .

L. Valiant, 1980 has constructed a polynomial *h* (with positive coefficients) such that $C_{+,\times}(h)$ is exponentially bigger than *C*+, [−], [×], /(*h*). Funny consequence: at least one of two above • results **holds. But which one?:** A growth of $C_{+\times}$, (h) is not known.

Corollary. Subtraction-free complexity is com[pu](#page-14-0)t[able.](#page-0-0) \cdot = \cdot + = \cdot

- • $C_{+,\times}(f)$ can be exponentially bigger than $C_{+,\times}(f)$
- $C_{+,\times}$, (g) can be exponentially bigger than $C_{+,-,\times}$, (g) .

L. Valiant, 1980 has constructed a polynomial *h* (with positive coefficients) such that $C_{+,\times}(h)$ is exponentially bigger than *C*+, [−], [×], /(*h*). Funny consequence: at least one of two above • results holds. But which one?: A growth of $C_{+,\times}$, (h) is not known.

error, i. e. $(1 - \epsilon) \cdot b < c < (1 + \epsilon) \cdot b$ where ϵ is a relative error then

Corollary. Subtraction-free complexity is com[pu](#page-15-0)t[able.](#page-0-0) \cdot = \cdot + = \cdot

- • $C_{+,\times}(f)$ can be exponentially bigger than $C_{+,\times}(f)$
- $C_{+,\times}$, (g) can be exponentially bigger than $C_{+,-,\times}$, (g) .

L. Valiant, 1980 has constructed a polynomial *h* (with positive coefficients) such that $C_{+,\times}(h)$ is exponentially bigger than *C*+, [−], [×], /(*h*). Funny consequence: at least one of two above • results holds. But which one?: A growth of $C_{+,\times}$, (h) is not known.

Subtraction-free computations have a meaning in numerical analysis: error, i. e. $(1 - \epsilon) \cdot b < c < (1 + \epsilon) \cdot b$ where ϵ is a relative error then

- • $C_{+,\times}(f)$ can be exponentially bigger than $C_{+,\times}(f)$
- $C_{+,\times}$, (g) can be exponentially bigger than $C_{+,-,\times}$, (g) .

L. Valiant, 1980 has constructed a polynomial *h* (with positive coefficients) such that $C_{+,\times}(h)$ is exponentially bigger than *C*+, [−], [×], /(*h*). Funny consequence: at least one of two above • results holds. But which one?: A growth of $C_{+,\times}$, (h) is not known.

Subtraction-free computations have a meaning in numerical analysis: if a computation deals with positive numbers *c* given with a relative error, i. e. $(1 - \epsilon) \cdot b < c < (1 + \epsilon) \cdot b$ where ϵ is a relative error then one can estimate easily relative error after operations $+$, \times , $/$.

Theorem. Polynomial *f* has a finite subtraction-free complexity $C_{+,\times}$, $(f) < \infty$ iff for each face *P* of its Newton polytope the restriction *f*|*^P* is numerically positive (**Handelman, 1985**).

Conduction-free complexity is com[pu](#page-17-0)t[able.](#page-0-0)

 QQ

- • $C_{+,\times}(f)$ can be exponentially bigger than $C_{+,\times}(f)$
- $C_{+,\times}$, (g) can be exponentially bigger than $C_{+,-,\times}$, (g) .

L. Valiant, 1980 has constructed a polynomial *h* (with positive coefficients) such that $C_{+,\times}(h)$ is exponentially bigger than *C*+, [−], [×], /(*h*). Funny consequence: at least one of two above • results holds. But which one?: A growth of $C_{+,\times}$, (h) is not known.

Subtraction-free computations have a meaning in numerical analysis: if a computation deals with positive numbers *c* given with a relative error, i. e. $(1 - \epsilon) \cdot b < c < (1 + \epsilon) \cdot b$ where ϵ is a relative error then one can estimate easily relative error after operations $+$, \times , $/$.

Polynomial in *n* variables is called *numerically positive* if it is positive everywhere on the positive orthant $(0,\infty)^n$.

Theorem. Polynomial *f* has a finite subtraction-free complexity $C_{+,\times}$, (f) < ∞ iff for each face *P* of its Newton polytope the restriction *f*|*^P* is numerically positive (**Handelman, 1985**).

Corollary. Subtraction-free complexity is com[pu](#page-18-0)t[able.](#page-0-0) $\rightarrow \rightarrow \rightarrow \rightarrow \rightarrow$ **Dima Grigoriev (CNRS) [Subtraction-free complexity](#page-0-0) 18.05.16 3 / 1**

 QQ

- • $C_{+,\times}(f)$ can be exponentially bigger than $C_{+,\times}(f)$
- $C_{+,\times}$, (g) can be exponentially bigger than $C_{+,-,\times}$, (g) .

L. Valiant, 1980 has constructed a polynomial *h* (with positive coefficients) such that $C_{+,\times}(h)$ is exponentially bigger than *C*+, [−], [×], /(*h*). Funny consequence: at least one of two above • results holds. But which one?: A growth of $C_{+,\times}$, (h) is not known.

Subtraction-free computations have a meaning in numerical analysis: if a computation deals with positive numbers *c* given with a relative error, i. e. $(1 - \epsilon) \cdot b < c < (1 + \epsilon) \cdot b$ where ϵ is a relative error then one can estimate easily relative error after operations $+$, \times , $/$.

Polynomial in *n* variables is called *numerically positive* if it is positive everywhere on the positive orthant $(0,\infty)^n$.

Theorem. Polynomial *f* has a finite subtraction-free complexity $C_{+,\times}$, $(f) < \infty$ iff for each face *P* of its Newton polytope the restriction *f*|*^P* is numerically positive (**Handelman, 1985**).

Corollary. Subtraction-free complexity is com[pu](#page-19-0)t[able.](#page-0-0) \cdot + ≥ + + ≥ + = ≥

 QQ

- $C_{+,\times}(f)$ can be exponentially bigger than $C_{+,\times}(f)$
- $C_{+,\times}$, (g) can be exponentially bigger than $C_{+,-,\times}$, (g) .

L. Valiant, 1980 has constructed a polynomial *h* (with positive coefficients) such that $C_{+,\times}(h)$ is exponentially bigger than *C*+, [−], [×], /(*h*). Funny consequence: at least one of two above • results holds. But which one?: A growth of $C_{+,\times}$, (h) is not known.

Subtraction-free computations have a meaning in numerical analysis: if a computation deals with positive numbers *c* given with a relative error, i. e. $(1 - \epsilon) \cdot b < c < (1 + \epsilon) \cdot b$ where ϵ is a relative error then one can estimate easily relative error after operations $+$, \times , $/$.

Polynomial in *n* variables is called *numerically positive* if it is positive everywhere on the positive orthant $(0,\infty)^n$.

Theorem. Polynomial *f* has a finite subtraction-free complexity $C_{+,\times}$, $(f) < \infty$ iff for each face P of its Newton polytope the restriction *f*|*^P* is numerically positive (**Handelman, 1985**).

Corollary. Subtraction-free complexity is com[pu](#page-20-0)t[able.](#page-0-0) QQ

Consider $n \times n$ matrix $X = (X_i)$ with variable entries. For a set *I* ⊂ $\{1,\ldots,n\}$ of size $|I| = k$ denote by Δ_I the determinant of $k \times k$

 $($ ロ } $($ $($ $)$ } $($ $)$

Consider $n \times n$ **matrix** $X = (X_{ij})$ **with variable entries.** For a set

I ⊂ $\{1,\ldots,n\}$ of size $|I| = k$ denote by Δ_I the determinant of $k \times k$ **flag algebra** *F*[{∆*I*}*^I*] is the ring of regular functions on the flag variety.

イロト イ押ト イラト イラト

Consider $n \times n$ matrix $X = (X_i)$ with variable entries. For a set *I* ⊂ $\{1,\ldots,n\}$ of size $|I| = k$ denote by Δ_I the determinant of $k \times k$ submatrix of *X* formed by first *k* rows and by columns $i \in I$. Then the **flag algebra** *F*[{∆*I*}*^I*] is the ring of regular functions on the flag variety.

Clusters are special families of flag minors ∆*^I* (being bases of the flag

イロト イ押ト イラト イラト

Consider $n \times n$ matrix $X = (X_i)$ with variable entries. For a set *I* ⊂ $\{1,\ldots,n\}$ of size $|I| = k$ denote by Δ_I the determinant of $k \times k$ submatrix of *X* formed by first *k* rows and by columns $i \in I$. Then the **flag algebra** *F*[{∆*I*}*^I*] is the ring of regular functions on the flag variety.

Clusters are special families of flag minors ∆*^I* (being bases of the flag algebra). To describe clusters define relation $I \prec J$ for $I, J \subset \{1, \ldots, n\}$

റെ റ

Consider $n \times n$ matrix $X = (X_i)$ with variable entries. For a set *I* ⊂ $\{1,\ldots,n\}$ of size $|I| = k$ denote by Δ_I the determinant of $k \times k$ submatrix of *X* formed by first *k* rows and by columns $i \in I$. Then the **flag algebra** *F*[{∆*I*}*^I*] is the ring of regular functions on the flag variety.

Clusters are special families of flag minors ∆*^I* (being bases of the flag **algebra).** To describe clusters define relation $I \prec J$ for $I, J \subset \{1, \ldots, n\}$ **strongly separated** if either $I \prec J$ or $J \prec I$. Cluster is a maximal (wrt

 Ω

イロト イ押 トイラト イラトー

Consider $n \times n$ matrix $X = (X_i)$ with variable entries. For a set *I* ⊂ $\{1,\ldots,n\}$ of size $|I| = k$ denote by Δ_I the determinant of $k \times k$ submatrix of *X* formed by first *k* rows and by columns $i \in I$. Then the **flag algebra** *F*[{∆*I*}*^I*] is the ring of regular functions on the flag variety.

Clusters are special families of flag minors ∆*^I* (being bases of the flag algebra). To describe clusters define relation $I \prec J$ for *I*, $J \subset \{1, \ldots, n\}$ **if for any pair** $i \in I \setminus J$ **,** $j \in J \setminus I$ **we have** $i < j$ **.** We say that *I*, *J* are **strongly separated** if either $I \prec J$ or $J \prec I$. Cluster is a maximal (wrt

 Ω

Consider $n \times n$ matrix $X = (X_i)$ with variable entries. For a set *I* ⊂ $\{1,\ldots,n\}$ of size $|I| = k$ denote by Δ_I the determinant of $k \times k$ submatrix of *X* formed by first *k* rows and by columns $i \in I$. Then the **flag algebra** *F*[{∆*I*}*^I*] is the ring of regular functions on the flag variety.

Clusters are special families of flag minors ∆*^I* (being bases of the flag algebra). To describe clusters define relation $I \prec J$ for *I*, $J \subset \{1, \ldots, n\}$ if for any pair $i \in I \setminus J$, $j \in J \setminus I$ we have $i < j$. We say that *I*, *J* are **strongly separated** if either *I* ≺ *J* or *J* ≺ *I*. Cluster is a maximal (wrt *I* = {1, ..., *n*}). Each cluster contains *n*(*n* + 1)/2 − 1 flag minors

 Ω

Consider $n \times n$ matrix $X = (X_i)$ with variable entries. For a set *I* ⊂ $\{1,\ldots,n\}$ of size $|I| = k$ denote by Δ_I the determinant of $k \times k$ submatrix of *X* formed by first *k* rows and by columns $i \in I$. Then the **flag algebra** *F*[{∆*I*}*^I*] is the ring of regular functions on the flag variety.

Clusters are special families of flag minors ∆*^I* (being bases of the flag algebra). To describe clusters define relation $I \prec J$ for $I, J \subset \{1, \ldots, n\}$ if for any pair $i \in I \setminus J$, $j \in J \setminus I$ we have $i < j$. We say that *I*, *J* are **strongly separated** if either $I \prec J$ or $J \prec I$. Cluster is a maximal (wrt inclusion) family of pairwise strongly separated flag minors (excluding *I* = {1,...,*n*}). Each cluster contains *n*(*n* + 1)/2 − 1 flag minors

 QQ

 $\mathbf{A} \cap \mathbf{B} \rightarrow \mathbf{A} \oplus \mathbf{B} \rightarrow \mathbf{A} \oplus \mathbf{B} \rightarrow \mathbf{B} \oplus \mathbf{B}$

Consider $n \times n$ matrix $X = (X_i)$ with variable entries. For a set *I* ⊂ $\{1,\ldots,n\}$ of size $|I| = k$ denote by Δ_I the determinant of $k \times k$ submatrix of *X* formed by first *k* rows and by columns $i \in I$. Then the **flag algebra** *F*[{∆*I*}*^I*] is the ring of regular functions on the flag variety.

Clusters are special families of flag minors ∆*^I* (being bases of the flag algebra). To describe clusters define relation $I \prec J$ for *I*, $J \subset \{1, \ldots, n\}$ if for any pair $i \in I \setminus J$, $j \in J \setminus I$ we have $i < j$. We say that *I*, *J* are **strongly separated** if either $I \prec J$ or $J \prec I$. Cluster is a maximal (wrt inclusion) family of pairwise strongly separated flag minors (excluding *I* = {1, ..., *n*}). Each cluster contains *n*(*n* + 1)/2 − 1 flag minors (**Fomin-Zelevinsky**).

 QQQ

イロメイ 御 メイ君 メイ君 メー 君

Consider $n \times n$ matrix $X = (X_i)$ with variable entries. For a set *I* ⊂ $\{1,\ldots,n\}$ of size $|I| = k$ denote by Δ_I the determinant of $k \times k$ submatrix of *X* formed by first *k* rows and by columns $i \in I$. Then the **flag algebra** *F*[{∆*I*}*^I*] is the ring of regular functions on the flag variety.

Clusters are special families of flag minors ∆*^I* (being bases of the flag algebra). To describe clusters define relation $I \prec J$ for *I*, $J \subset \{1, \ldots, n\}$ if for any pair $i \in I \setminus J$, $j \in J \setminus I$ we have $i < j$. We say that *I*, *J* are **strongly separated** if either $I \prec J$ or $J \prec I$. Cluster is a maximal (wrt inclusion) family of pairwise strongly separated flag minors (excluding *I* = {1, ..., *n*}). Each cluster contains *n*(*n* + 1)/2 − 1 flag minors (**Fomin-Zelevinsky**).

Shorthand: for $j \notin I$ denote $Ij =: I \cup \{j\}$.

 QQQ

 $\mathbf{A} \oplus \mathbf{A} \rightarrow \mathbf{A} \oplus \mathbf{A} \rightarrow \mathbf{A} \oplus \mathbf{A} \rightarrow \mathbf{A} \oplus \mathbf{A}$

Plücker relations: $\Delta_{lik} \cdot \Delta_{li} = \Delta_{lii} \cdot \Delta_{lk} + \Delta_{lik} \cdot \Delta_{li}$, $i, j, k \notin l; i < j < k$ Clusters form a directed acyclic **cluster graph**: there is an edge from

Plücker relations: $\Delta_{lik} \cdot \Delta_{lj} = \Delta_{lij} \cdot \Delta_{lk} + \Delta_{ljk} \cdot \Delta_{li}$, $i, j, k \notin l; i < j < k$

Clusters form a directed acyclic **cluster graph**: there is an edge from

Plücker relations: $\Delta_{lik} \cdot \Delta_{lj} = \Delta_{lij} \cdot \Delta_{lk} + \Delta_{ljk} \cdot \Delta_{li}$, $i, j, k \notin l; i < j < k$ Clusters form a directed acyclic **cluster graph**: there is an edge from one cluster to another if they are, respectively, of the types

Plücker relations: $\Delta_{lik} \cdot \Delta_{lj} = \Delta_{lij} \cdot \Delta_{lk} + \Delta_{ljk} \cdot \Delta_{li}$, $i, j, k \notin l; i < j < k$ Clusters form a directed acyclic **cluster graph**: there is an edge from one cluster to another if they are, respectively, of the types {∆*Iik* , ∆*Iij*, ∆*Ik* , ∆*Ijk* , ∆*Ii*, . . . }; {∆*Ij*, ∆*Iij*, ∆*Ik* , ∆*Ijk* , ∆*Ii*, . . . }

Cluster graph is graded according to the sum of sizes |*I*|. A birational another (along an edge or back) is called a **flip**. Cluster graph has the
Plücker relations: $\Delta_{lik} \cdot \Delta_{lj} = \Delta_{lij} \cdot \Delta_{lk} + \Delta_{ljk} \cdot \Delta_{li}$, $i, j, k \notin l; i < j < k$ Clusters form a directed acyclic **cluster graph**: there is an edge from one cluster to another if they are, respectively, of the types {∆*Iik* , ∆*Iij*, ∆*Ik* , ∆*Ijk* , ∆*Ii*, . . . }; {∆*Ij*, ∆*Iij*, ∆*Ik* , ∆*Ijk* , ∆*Ii*, . . . }

Cluster graph is graded according to the sum of sizes |*I*|. A birational another (along an edge or back) is called a **flip**. Cluster graph has the

K ロ ▶ K 個 ▶ K 君 ▶ K 君 ▶ … Ω

Plücker relations: $\Delta_{lik} \cdot \Delta_{lj} = \Delta_{lij} \cdot \Delta_{lk} + \Delta_{ljk} \cdot \Delta_{lj}$, *i*, *j*, *k* ∉ *l*; *i* < *j* < *k* Clusters form a directed acyclic **cluster graph**: there is an edge from one cluster to another if they are, respectively, of the types {∆*Iik* , ∆*Iij*, ∆*Ik* , ∆*Ijk* , ∆*Ii*, . . . }; {∆*Ij*, ∆*Iij*, ∆*Ik* , ∆*Ijk* , ∆*Ii*, . . . }

Cluster graph is graded according to the sum of sizes |*I*|. A birational (with integer positive coefficients) transformation from one cluster to another (along an edge or back) is called a **flip**. Cluster graph has the

K ロ ▶ K 御 ▶ K 唐 ▶ K 唐 ▶ ... Ω

Plücker relations: $\Delta_{lik} \cdot \Delta_{lj} = \Delta_{lij} \cdot \Delta_{lk} + \Delta_{ljk} \cdot \Delta_{lj}$, *i*, *j*, *k* ∉ *l*; *i* < *j* < *k* Clusters form a directed acyclic **cluster graph**: there is an edge from one cluster to another if they are, respectively, of the types {∆*Iik* , ∆*Iij*, ∆*Ik* , ∆*Ijk* , ∆*Ii*, . . . }; {∆*Ij*, ∆*Iij*, ∆*Ik* , ∆*Ijk* , ∆*Ii*, . . . }

Cluster graph is graded according to the sum of sizes |*I*|. A birational (with integer positive coefficients) transformation from one cluster to another (along an edge or back) is called a **flip**. Cluster graph has the unique minimal element corresponding to *I* being all the proper intervals of $\{1,\ldots,n\}$ and the unique maximal element corresponding

Fomin-Zelevinsky: any non-minimal cluster contains a 5-tuple K ロ ▶ K 御 ▶ K 唐 ▶ K 唐 ▶ │ 唐 Ω

Plücker relations: $\Delta_{lik} \cdot \Delta_{lj} = \Delta_{lij} \cdot \Delta_{lk} + \Delta_{ljk} \cdot \Delta_{lj}$, *i*, *j*, *k* ∉ *l*; *i* < *j* < *k* Clusters form a directed acyclic **cluster graph**: there is an edge from one cluster to another if they are, respectively, of the types {∆*Iik* , ∆*Iij*, ∆*Ik* , ∆*Ijk* , ∆*Ii*, . . . }; {∆*Ij*, ∆*Iij*, ∆*Ik* , ∆*Ijk* , ∆*Ii*, . . . }

Cluster graph is graded according to the sum of sizes |*I*|. A birational (with integer positive coefficients) transformation from one cluster to another (along an edge or back) is called a **flip**. Cluster graph has the unique minimal element corresponding to *I* being all the proper intervals of $\{1, \ldots, n\}$ and the unique maximal element corresponding to all the complements of the proper intervals.

Fomin-Zelevinsky: any non-minimal cluster contains a 5-tuple K ロ ▶ K 御 ▶ K 唐 ▶ K 唐 ▶ │ 唐 QQ

Plücker relations: $\Delta_{lik} \cdot \Delta_{lj} = \Delta_{lij} \cdot \Delta_{lk} + \Delta_{ljk} \cdot \Delta_{lj}$, *i*, *j*, *k* ∉ *l*; *i* < *j* < *k* Clusters form a directed acyclic **cluster graph**: there is an edge from one cluster to another if they are, respectively, of the types {∆*Iik* , ∆*Iij*, ∆*Ik* , ∆*Ijk* , ∆*Ii*, . . . }; {∆*Ij*, ∆*Iij*, ∆*Ik* , ∆*Ijk* , ∆*Ii*, . . . }

Cluster graph is graded according to the sum of sizes |*I*|. A birational (with integer positive coefficients) transformation from one cluster to another (along an edge or back) is called a **flip**. Cluster graph has the unique minimal element corresponding to *I* being all the proper intervals of $\{1, \ldots, n\}$ and the unique maximal element corresponding to all the complements of the proper intervals.

Fomin-Zelevinsky: any non-minimal cluster contains a 5-tuple

∆*Iik* , ∆*Iij*, ∆*Ik* , ∆*Ijk* , ∆*Ii*, therefore, a flip can be applied to the cluster. K ロ ▶ K 御 ▶ K 唐 ▶ K 唐 ▶ │ 唐 Ω

Plücker relations: $\Delta_{lik} \cdot \Delta_{lj} = \Delta_{lij} \cdot \Delta_{lk} + \Delta_{ljk} \cdot \Delta_{lj}$, *i*, *j*, *k* ∉ *l*; *i* < *j* < *k* Clusters form a directed acyclic **cluster graph**: there is an edge from one cluster to another if they are, respectively, of the types {∆*Iik* , ∆*Iij*, ∆*Ik* , ∆*Ijk* , ∆*Ii*, . . . }; {∆*Ij*, ∆*Iij*, ∆*Ik* , ∆*Ijk* , ∆*Ii*, . . . }

Cluster graph is graded according to the sum of sizes |*I*|. A birational (with integer positive coefficients) transformation from one cluster to another (along an edge or back) is called a **flip**. Cluster graph has the unique minimal element corresponding to *I* being all the proper intervals of $\{1, \ldots, n\}$ and the unique maximal element corresponding to all the complements of the proper intervals.

Fomin-Zelevinsky: any non-minimal cluster contains a 5-tuple ∆*Iik* , ∆*Iij*, ∆*Ik* , ∆*Ijk* , ∆*Ii*, therefore, a flip can be applied to the cluster. integer positive coefficients. K ロ ▶ K 御 ▶ K 唐 ▶ K 唐 ▶ │ 唐 Ω

Plücker relations: $\Delta_{lik} \cdot \Delta_{li} = \Delta_{lii} \cdot \Delta_{lk} + \Delta_{lik} \cdot \Delta_{li}$, $i, j, k \notin l$; $i < j < k$ Clusters form a directed acyclic **cluster graph**: there is an edge from one cluster to another if they are, respectively, of the types {∆*Iik* , ∆*Iij*, ∆*Ik* , ∆*Ijk* , ∆*Ii*, . . . }; {∆*Ij*, ∆*Iij*, ∆*Ik* , ∆*Ijk* , ∆*Ii*, . . . }

Cluster graph is graded according to the sum of sizes |*I*|. A birational (with integer positive coefficients) transformation from one cluster to another (along an edge or back) is called a **flip**. Cluster graph has the unique minimal element corresponding to *I* being all the proper intervals of $\{1, \ldots, n\}$ and the unique maximal element corresponding to all the complements of the proper intervals.

Fomin-Zelevinsky: any non-minimal cluster contains a 5-tuple ∆*Iik* , ∆*Iij*, ∆*Ik* , ∆*Ijk* , ∆*Ii*, therefore, a flip can be applied to the cluster. Any flag minor belongs to some cluster. Thus, any flag minor can be K ロ X K @ X K 등 X K 등 X → 등 QQ

Plücker relations: $\Delta_{lik} \cdot \Delta_{li} = \Delta_{lii} \cdot \Delta_{lk} + \Delta_{lik} \cdot \Delta_{li}$, $i, j, k \notin l$; $i < j < k$ Clusters form a directed acyclic **cluster graph**: there is an edge from one cluster to another if they are, respectively, of the types {∆*Iik* , ∆*Iij*, ∆*Ik* , ∆*Ijk* , ∆*Ii*, . . . }; {∆*Ij*, ∆*Iij*, ∆*Ik* , ∆*Ijk* , ∆*Ii*, . . . }

Cluster graph is graded according to the sum of sizes |*I*|. A birational (with integer positive coefficients) transformation from one cluster to another (along an edge or back) is called a **flip**. Cluster graph has the unique minimal element corresponding to *I* being all the proper intervals of $\{1, \ldots, n\}$ and the unique maximal element corresponding to all the complements of the proper intervals.

Fomin-Zelevinsky: any non-minimal cluster contains a 5-tuple ∆*Iik* , ∆*Iij*, ∆*Ik* , ∆*Ijk* , ∆*Ii*, therefore, a flip can be applied to the cluster. Any flag minor belongs to some cluster. Thus, any flag minor can be computed with $O(n^3)$ flips starting with the interval flag minors, and a

◆ロメ ◆個メ ◆唐メ ◆唐メン語

 QQ

Plücker relations: $\Delta_{lik} \cdot \Delta_{lj} = \Delta_{lij} \cdot \Delta_{lk} + \Delta_{ljk} \cdot \Delta_{lj}$, *i*, *j*, *k* ∉ *l*; *i* < *j* < *k* Clusters form a directed acyclic **cluster graph**: there is an edge from one cluster to another if they are, respectively, of the types {∆*Iik* , ∆*Iij*, ∆*Ik* , ∆*Ijk* , ∆*Ii*, . . . }; {∆*Ij*, ∆*Iij*, ∆*Ik* , ∆*Ijk* , ∆*Ii*, . . . }

Cluster graph is graded according to the sum of sizes |*I*|. A birational (with integer positive coefficients) transformation from one cluster to another (along an edge or back) is called a **flip**. Cluster graph has the unique minimal element corresponding to *I* being all the proper intervals of $\{1, \ldots, n\}$ and the unique maximal element corresponding to all the complements of the proper intervals.

Fomin-Zelevinsky: any non-minimal cluster contains a 5-tuple ∆*Iik* , ∆*Iij*, ∆*Ik* , ∆*Ijk* , ∆*Ii*, therefore, a flip can be applied to the cluster. Any flag minor belongs to some cluster. Thus, any flag minor can be computed with $O(n^3)$ flips starting with the interval flag minors, and a flag minor is a Laurent polynomial in the interval flag minors with integer positive coefficients. K ロ X K 御 X K 唐 X K 唐 X (唐 QQ

Totally positive matrices

As a consequence from cluster transformations we conclude that for a real matrix if all the interval flag minors are positive then all flag minors are positive as well.

 4 d \rightarrow 4 \equiv \rightarrow 4 \equiv

Totally positive matrices

As a consequence from cluster transformations we conclude that for a real matrix if all the interval flag minors are positive then all flag minors are positive as well.

A real matrix is called *totally positive* if all its minors are positive.

 Ω

イロト イ母 トイラト イラト

Totally positive matrices

As a consequence from cluster transformations we conclude that for a real matrix if all the interval flag minors are positive then all flag minors are positive as well.

A real matrix is called *totally positive* if all its minors are positive. Cluster transformation entail that for total positivity it suffices to verify positivity of minors formed by sets of rows *I* and of columns *J* for all pairs of intervals *I*, $J \subset \{1, \ldots, n\}$.

Substitute for the entries of the matrix $X_{ij} = x_j^j$

 $\Delta := \Delta_{\{1,...,k\}} = \prod_{1 \leq i < j \leq k} (x_j - x_i)$ is Vandermond determinant. The

 $\left\{ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right.$

Substitute for the entries of the matrix $X_{ij} = x_{ij}^{j}$ *i* . Then

 $\Delta := \Delta_{\{1,...,k\}} = \prod_{1 \leq i < j \leq k} (x_j - x_i)$ is Vandermond determinant. The quotient $S_I = \Delta_I/\Delta$ is a **Schur polynomial** having integer positive

 $\left\{ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right.$

Substitute for the entries of the matrix $X_{ij} = x_i^j.$ Then $\Delta := \Delta_{\{1,\ldots,k\}} = \prod_{1 \leq i < j \leq k} (x_j - x_i)$ is Vandermond determinant. The quotient $S_I = \Delta_I/\Delta$ is a **Schur polynomial** having integer positive

Substitute for the entries of the matrix $X_{ij} = x_i^j.$ Then $\Delta := \Delta_{\{1,\ldots,k\}} = \prod_{1 \leq i < j \leq k} (x_j - x_i)$ is Vandermond determinant. The quotient $S_I = \Delta_I/\Delta$ is a **Schur polynomial** having integer positive coefficients (called **Kostka numbers** being #*P*-hard to compute).

Substitute for the entries of the matrix $X_{ij} = x_{ij}^{j}$ *i* . Then $\Delta := \Delta_{\{1,...,k\}} = \prod_{1 \leq i < j \leq k} (x_j - x_i)$ is Vandermond determinant. The quotient $S_I = \Delta_I/\Delta$ is a **Schur polynomial** having integer positive coefficients (called **Kostka numbers** being #*P*-hard to compute).

Since for an interval $I = [i, i + 1, \ldots, j]$ the interval Schur polynomial is

 $(1 + 4\sqrt{10})$ $(1 + 4\sqrt{10})$

Substitute for the entries of the matrix $X_{ij} = x_{ij}^{j}$ *i* . Then $\Delta := \Delta_{\{1,...,k\}} = \prod_{1 \leq i < j \leq k} (x_j - x_i)$ is Vandermond determinant. The quotient $S_I = \Delta_I/\Delta$ is a **Schur polynomial** having integer positive coefficients (called **Kostka numbers** being #*P*-hard to compute).

Since for an interval $I = [i, i + 1, \ldots, j]$ the interval Schur polynomial is

 $(1 + 4\sqrt{10})$ $(1 + 4\sqrt{10})$ $(1 + 4\sqrt{10})$

Substitute for the entries of the matrix $X_{ij} = x_{ij}^{j}$ *i* . Then

 $\Delta := \Delta_{\{1,...,k\}} = \prod_{1 \leq i < j \leq k} (x_j - x_i)$ is Vandermond determinant. The quotient $S_I = \Delta_I/\Delta$ is a **Schur polynomial** having integer positive coefficients (called **Kostka numbers** being #*P*-hard to compute).

Since for an interval $I = [i, i+1, \ldots, j]$ the interval Schur polynomial is the monomial $S_l = (x_1 \cdots x_{j - i + 1})^l$ being easy to compute, we get

Substitute for the entries of the matrix $X_{ij} = x_{ij}^{j}$ *i* . Then

 $\Delta := \Delta_{\{1,...,k\}} = \prod_{1 \leq i < j \leq k} (x_j - x_i)$ is Vandermond determinant. The quotient $S_I = \Delta_I/\Delta$ is a **Schur polynomial** having integer positive coefficients (called **Kostka numbers** being #*P*-hard to compute).

Since for an interval $I = [i, i+1, \ldots, j]$ the interval Schur polynomial is the monomial $S_l = (x_1 \cdots x_{j - i + 1})^l$ being easy to compute, we get

Corollary

Subtraction-free complexity of a Schur polynomial $C_{+,\times, /}(S_l) \leq O(n^3 \cdot \log n).$

 QQ

Substitute for the entries of the matrix $X_{ij} = x_{ij}^{j}$ *i* . Then

 $\Delta := \Delta_{\{1,...,k\}} = \prod_{1 \leq i < j \leq k} (x_j - x_i)$ is Vandermond determinant. The quotient $S_I = \Delta_I/\Delta$ is a **Schur polynomial** having integer positive coefficients (called **Kostka numbers** being #*P*-hard to compute).

Since for an interval $I = [i, i+1, \ldots, j]$ the interval Schur polynomial is the monomial $S_l = (x_1 \cdots x_{j - i + 1})^l$ being easy to compute, we get

Corollary

Subtraction-free complexity of a Schur polynomial $C_{+,\times, /}(S_l) \leq O(n^3 \cdot \log n).$

This does not yet imply an exponential gap between $C_{+,\times}$, and $C_{+,\times}$ because we don't know a lower bound on the complexity $C_{+,\times}(S_l)$. To $\mathbf{A} \cap \mathbf{B} \rightarrow \mathbf{A} \oplus \mathbf{B} \rightarrow \mathbf{A} \oplus \mathbf{B} \rightarrow \mathbf{B} \oplus \mathbf{B}$ QQQ

Substitute for the entries of the matrix $X_{ij} = x_{ij}^{j}$ *i* . Then

 $\Delta := \Delta_{\{1,...,k\}} = \prod_{1 \leq i < j \leq k} (x_j - x_i)$ is Vandermond determinant. The quotient $S_I = \Delta_I/\Delta$ is a **Schur polynomial** having integer positive coefficients (called **Kostka numbers** being #*P*-hard to compute).

Since for an interval $I = [i, i+1, \ldots, j]$ the interval Schur polynomial is the monomial $S_l = (x_1 \cdots x_{j - i + 1})^l$ being easy to compute, we get

Corollary

Subtraction-free complexity of a Schur polynomial $C_{+,\times, /}(S_l) \leq O(n^3 \cdot \log n).$

This does not yet imply an exponential gap between $C_{+,\times}$, and $C_{+,\times}$ because we don't know a lower bound on the complexity $C_{+,\times}(S_l)$. To establish this gap we proceed to another class of cluster transformations. イロト イ母 トイラ トイラ トーラ QQQ

electrical circuit) be endowed with **conductances** *xi*,*^j* . Denote by

Let edges of a complete graph *G* with *n* vertices (viewed as an electrical circuit) be endowed with **conductances** *xi*,*^j* . Denote by

For a spanning tree $\mathcal T$ of G denote by $X^{\mathcal T}$ the monomial being the product of *xi*,*^j* for all edges *i*, *j* of *T*. The generating polynomial *f*(*G*) is

Let edges of a complete graph *G* with *n* vertices (viewed as an electrical circuit) be endowed with **conductances** *xi*,*^j* . Denote by *cond*_{*i*},*j*(*G*) the conductance of the circuit between vertices *i*, *j*.

For a spanning tree $\mathcal T$ of G denote by $X^{\mathcal T}$ the monomial being the product of *xi*,*^j* for all edges *i*, *j* of *T*. The generating polynomial *f*(*G*) is the sum of the monomials $X^{\mathcal{T}}$ over all spanning trees $\mathcal{T}.$

Let edges of a complete graph *G* with *n* vertices (viewed as an electrical circuit) be endowed with **conductances** *xi*,*^j* . Denote by *cond*_{*i*},*j*(*G*) the conductance of the circuit between vertices *i*, *j*.

For a spanning tree T of G denote by X^T the monomial being the product of *xi*,*^j* for all edges *i*, *j* of *T*. The generating polynomial *f*(*G*) is the sum of the monomials $X^{\mathcal{T}}$ over all spanning trees $\mathcal{T}.$ **Example**: for *G* with 3 vertices $f(G) = x_1 \cdot x_2 \cdot x_3 + x_1 \cdot x_2 \cdot x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_5$.

Let edges of a complete graph *G* with *n* vertices (viewed as an electrical circuit) be endowed with **conductances** *xi*,*^j* . Denote by *cond*_{*i*},*j*(*G*) the conductance of the circuit between vertices *i*, *j*.

For a spanning tree T of G denote by X^T the monomial being the product of *xi*,*^j* for all edges *i*, *j* of *T*. The generating polynomial *f*(*G*) is the sum of the monomials $X^{\mathcal{T}}$ over all spanning trees $\mathcal{T}.$

Example: for *G* with 3 vertices $f(G) = x_1 \cdot x_2 \cdot x_3 + x_1 \cdot x_3 + x_2 \cdot x_1 \cdot x_2$ gluing *v*, *w* into a vertex *u* with new conductances $x_{i,u} := x_{i,v} + x_{i,w}$.

Let edges of a complete graph *G* with *n* vertices (viewed as an electrical circuit) be endowed with **conductances** *xi*,*^j* . Denote by *cond*_{*i*},*j*(*G*) the conductance of the circuit between vertices *i*, *j*.

For a spanning tree T of G denote by X^T the monomial being the product of *xi*,*^j* for all edges *i*, *j* of *T*. The generating polynomial *f*(*G*) is the sum of the monomials $X^{\mathcal{T}}$ over all spanning trees $\mathcal{T}.$

Example: for *G* with 3 vertices $f(G) = x_{1,2} \cdot x_{2,3} + x_{1,2} \cdot x_{1,3} + x_{2,3} \cdot x_{1,3}$.

gluing *v*, *w* into a vertex *u* with new conductances $x_{i,y} := x_{i,y} + x_{i,w}$.

Kirchhoff (1847): *cond*_{*i*,*j*}(*G*) = $f(G)/f(G_{i,j})$.

Let edges of a complete graph *G* with *n* vertices (viewed as an electrical circuit) be endowed with **conductances** *xi*,*^j* . Denote by *cond*_{*i*},*j*(*G*) the conductance of the circuit between vertices *i*, *j*.

For a spanning tree T of G denote by X^T the monomial being the product of *xi*,*^j* for all edges *i*, *j* of *T*. The generating polynomial *f*(*G*) is the sum of the monomials $X^{\mathcal{T}}$ over all spanning trees $\mathcal{T}.$

Example: for *G* with 3 vertices $f(G) = x_{1,2} \cdot x_{2,3} + x_{1,2} \cdot x_{1,3} + x_{2,3} \cdot x_{1,3}$. For vertices *v*, *w* of *G* denote by *Gv*,*^w* the graph obtained from *G* by gluing *v*, *w* into a vertex *u* with new conductances $x_{i,u} = x_{i,v} + x_{i,w}$.

Kirchhoff (1847): *cond*_{*i*,*j*}(*G*) = $f(G)/f(G_{i,j})$.

Star-mesh transformation: let a vertex $v \neq i$, *j* of *G*. Denote by G_v a

Let edges of a complete graph *G* with *n* vertices (viewed as an electrical circuit) be endowed with **conductances** *xi*,*^j* . Denote by *cond*_{*i*},*j*(*G*) the conductance of the circuit between vertices *i*, *j*.

For a spanning tree T of G denote by X^T the monomial being the product of *xi*,*^j* for all edges *i*, *j* of *T*. The generating polynomial *f*(*G*) is the sum of the monomials $X^{\mathcal{T}}$ over all spanning trees $\mathcal{T}.$

Example: for *G* with 3 vertices $f(G) = x_{1,2} \cdot x_{2,3} + x_{1,2} \cdot x_{1,3} + x_{2,3} \cdot x_{1,3}$.

For vertices *v*, *w* of *G* denote by $G_{V,W}$ the graph obtained from *G* by gluing *v*, *w* into a vertex *u* with new conductances $x_{i,u} := x_{i,v} + x_{i,w}$.

Kirchhoff (1847): $cond_{i,j}(G) = f(G)/f(G_{i,j}).$

Star-mesh transformation: let a vertex $v \neq i$, *j* of *G*. Denote by G_v a graph obtained from *G* by removing *v* and replacing each conductance $x_{p,q}$ by $x_{p,q}+x_{p,\nu}\cdot x_{q,\nu}\cdot \sum_k (x_{k,\nu})^{-1}.$ Then $cond_{i,j}(G)=cond_{i,j}(G_\nu).$

Let edges of a complete graph *G* with *n* vertices (viewed as an electrical circuit) be endowed with **conductances** *xi*,*^j* . Denote by *cond*_{*i*},*j*(*G*) the conductance of the circuit between vertices *i*, *j*.

For a spanning tree T of G denote by X^T the monomial being the product of *xi*,*^j* for all edges *i*, *j* of *T*. The generating polynomial *f*(*G*) is the sum of the monomials $X^{\mathcal{T}}$ over all spanning trees $\mathcal{T}.$

Example: for *G* with 3 vertices $f(G) = x_{1,2} \cdot x_{2,3} + x_{1,2} \cdot x_{1,3} + x_{2,3} \cdot x_{1,3}$.

For vertices *v*, *w* of *G* denote by *Gv*,*^w* the graph obtained from *G* by gluing *v*, *w* into a vertex *u* with new conductances $x_{i,u} := x_{i,v} + x_{i,w}$.

Kirchhoff (1847): $cond_{i,j}(G) = f(G)/f(G_{i,j}).$

Star-mesh transformation: let a vertex $v \neq i$ **,** *j* **of** *G***. Denote by** G_v **a** graph obtained from *G* by removing *v* and replacing each conductance $x_{p,q}$ by $x_{p,q}+x_{p,\nu}\cdot x_{q,\nu}\cdot \sum_k (x_{k,\nu})^{-1}.$ Then $cond_{i,j}(G)=cond_{i,j}(G_\nu).$

Let edges of a complete graph *G* with *n* vertices (viewed as an electrical circuit) be endowed with **conductances** *xi*,*^j* . Denote by *cond*_{*i*},*j*(*G*) the conductance of the circuit between vertices *i*, *j*.

For a spanning tree T of G denote by X^T the monomial being the product of *xi*,*^j* for all edges *i*, *j* of *T*. The generating polynomial *f*(*G*) is the sum of the monomials $X^{\mathcal{T}}$ over all spanning trees $\mathcal{T}.$

Example: for *G* with 3 vertices $f(G) = x_{1,2} \cdot x_{2,3} + x_{1,2} \cdot x_{1,3} + x_{2,3} \cdot x_{1,3}$.

For vertices *v*, *w* of *G* denote by *Gv*,*^w* the graph obtained from *G* by gluing *v*, *w* into a vertex *u* with new conductances $x_{i,u} := x_{i,v} + x_{i,w}$.

Kirchhoff (1847): $cond_{i,j}(G) = f(G)/f(G_{i,j}).$

Star-mesh transformation: let a vertex $v \neq i$, *j* of *G*. Denote by G_v a graph obtained from *G* by removing *v* and replacing each conductance $x_{p,q}$ by $x_{p,q} + x_{p,\textbf{v}} \cdot x_{q,\textbf{v}} \cdot \sum_k (x_{k,\textbf{v}})^{-1}.$ Then $cond_{i,j}(G) = cond_{i,j}(G_{\textbf{v}}).$

Subtraction-[f](#page-66-0)ree complexity $C_{+,\times}$ *, [\(](#page-0-0)co[n](#page-0-0)d_{<i>i,j*}(*[G](#page-0-0)*[\)](#page-0-0), $f(G)$) $\leq O(n^3)$ $\leq O(n^3)$ $\leq O(n^3)$ $\leq O(n^3)$ $\leq O(n^3)$ *[.](#page-0-0)*

Let edges of a complete graph *G* with *n* vertices (viewed as an electrical circuit) be endowed with **conductances** *xi*,*^j* . Denote by *cond*_{*i*},*j*(*G*) the conductance of the circuit between vertices *i*, *j*.

For a spanning tree T of G denote by X^T the monomial being the product of *xi*,*^j* for all edges *i*, *j* of *T*. The generating polynomial *f*(*G*) is the sum of the monomials $X^{\mathcal{T}}$ over all spanning trees $\mathcal{T}.$

Example: for *G* with 3 vertices $f(G) = x_{1,2} \cdot x_{2,3} + x_{1,2} \cdot x_{1,3} + x_{2,3} \cdot x_{1,3}$.

For vertices *v*, *w* of *G* denote by *Gv*,*^w* the graph obtained from *G* by gluing *v*, *w* into a vertex *u* with new conductances $x_{i,u} := x_{i,v} + x_{i,w}$.

Kirchhoff (1847): $cond_{i,j}(G) = f(G)/f(G_{i,j}).$

Star-mesh transformation: let a vertex $v \neq i$, *j* of *G*. Denote by G_v a graph obtained from *G* by removing *v* and replacing each conductance $x_{\rho,q}$ by $x_{\rho,q}+x_{\rho,\mathrm{v}}\cdot x_{q,\mathrm{v}}\cdot \sum_k (x_{k,\mathrm{v}})^{-1}.$ Then $cond_{i,j}(G)=cond_{i,j}(G_\mathrm{v}).$

Subtraction-[f](#page-67-0)ree complexity $C_{+,\times}$ *, [\(](#page-0-0)co[n](#page-0-0)d_{<i>i,j*}(*[G](#page-0-0)*[\)](#page-0-0), $f(G)$) $\leq O(n^3)$ $\leq O(n^3)$ $\leq O(n^3)$ $\leq O(n^3)$ $\leq O(n^3)$ *[.](#page-0-0)*

Let edges of a complete graph *G* with *n* vertices (viewed as an electrical circuit) be endowed with **conductances** *xi*,*^j* . Denote by *cond*_{*i*},*j*(*G*) the conductance of the circuit between vertices *i*, *j*.

For a spanning tree T of G denote by X^T the monomial being the product of *xi*,*^j* for all edges *i*, *j* of *T*. The generating polynomial *f*(*G*) is the sum of the monomials $X^{\mathcal{T}}$ over all spanning trees $\mathcal{T}.$

Example: for *G* with 3 vertices $f(G) = x_{1,2} \cdot x_{2,3} + x_{1,2} \cdot x_{1,3} + x_{2,3} \cdot x_{1,3}$.

For vertices *v*, *w* of *G* denote by *Gv*,*^w* the graph obtained from *G* by gluing *v*, *w* into a vertex *u* with new conductances $x_{i,u} := x_{i,v} + x_{i,w}$.

Kirchhoff (1847): $cond_{i,j}(G) = f(G)/f(G_{i,j}).$

Star-mesh transformation: let a vertex $v \neq i$, *j* of *G*. Denote by G_v a graph obtained from *G* by removing *v* and replacing each conductance $x_{\rho,q}$ by $x_{\rho,q}+x_{\rho,\mathrm{v}}\cdot x_{q,\mathrm{v}}\cdot \sum_k (x_{k,\mathrm{v}})^{-1}.$ Then $cond_{i,j}(G)=cond_{i,j}(G_\mathrm{v}).$

Corollary

Subtraction-[f](#page-68-0)ree complexity $C_{+,\times,}/(cond_{i,j}(G), f(G)) \leq O(n^3)$ $C_{+,\times,}/(cond_{i,j}(G), f(G)) \leq O(n^3)$ *[.](#page-0-0)*

Dima Grigoriev (CNRS) [Subtraction-free complexity](#page-0-0) 18.05.16 8 / 1

Arborescences

Now *G* is a complete graph with *n* vertices and edges *i*, *j* endowed with two variables $x_{i,j}\neq x_{j,i}.$ Fix a vertex r of G and consider any spanning tree *T* of *G* as having its root at *r* and all its edges directed towards *r*

Arborescences

Now *G* is a complete graph with *n* vertices and edges *i*, *j* endowed with $\tt two\ variables$ $x_{i,j} \neq x_{j,i}.$ Fix a vertex r of G and consider any spanning tree *T* of *G* as having its root at *r* and all its edges directed towards *r X ^T* corresponding to this tree is the product of all *xi*,*^j* over all edges *i*, *j* of *T* according to the chosen direction of *T*. The generating polynomial
Now *G* is a complete graph with *n* vertices and edges *i*, *j* endowed with two variables $x_{i,j} \neq x_{j,i}.$ Fix a vertex r of G and consider any spanning tree *T* of *G* as having its root at *r* and all its edges directed towards *r* (such directed trees are called *arborescences*). Then the monomial

X ^T corresponding to this tree is the product of all *xi*,*^j* over all edges *i*, *j* of *T* according to the chosen direction of *T*. The generating polynomial $\phi(\bm{G})$ of arborescences is the sum of all the monomials $X^{\mathsf{T}}.$

Now *G* is a complete graph with *n* vertices and edges *i*, *j* endowed with two variables $x_{i,j} \neq x_{j,i}.$ Fix a vertex r of G and consider any spanning tree *T* of *G* as having its root at *r* and all its edges directed towards *r* (such directed trees are called *arborescences*). Then the monomial $X^{\mathcal{T}}$ corresponding to this tree is the product of all $x_{i,j}$ over all edges *i*, *j* of *T* according to the chosen direction of *T*. The generating polynomial $\phi(\bm{G})$ of arborescences is the sum of all the monomials $X^{\mathsf{T}}.$

Example: if $n = 3$, $r = 2$ then $\phi(G) = x_{1,2} \cdot x_{3,2} + x_{1,2} \cdot x_{3,1} + x_{1,3} \cdot x_{3,2}$.

Now *G* is a complete graph with *n* vertices and edges *i*, *j* endowed with two variables $x_{i,j} \neq x_{j,i}.$ Fix a vertex r of G and consider any spanning tree *T* of *G* as having its root at *r* and all its edges directed towards *r* (such directed trees are called *arborescences*). Then the monomial $X^{\mathcal{T}}$ corresponding to this tree is the product of all $x_{i,j}$ over all edges *i*, *j* of *T* according to the chosen direction of *T*. The generating polynomial $\phi(G)$ of arborescences is the sum of all the monomials $X^{\mathcal{T}}.$

Example: if $n = 3$, $r = 2$ then $\phi(G) = x_{1,2} \cdot x_{3,2} + x_{1,2} \cdot x_{3,1} + x_{1,3} \cdot x_{3,2}$.

Star-mesh transformation: let a vertex $v \neq r$ of *G*. Denote by G_v a

Now *G* is a complete graph with *n* vertices and edges *i*, *j* endowed with two variables $x_{i,j} \neq x_{j,i}.$ Fix a vertex r of G and consider any spanning tree *T* of *G* as having its root at *r* and all its edges directed towards *r* (such directed trees are called *arborescences*). Then the monomial $X^{\mathcal{T}}$ corresponding to this tree is the product of all $x_{i,j}$ over all edges *i*, *j* of *T* according to the chosen direction of *T*. The generating polynomial $\phi(G)$ of arborescences is the sum of all the monomials $X^{\mathcal{T}}.$

Example: if $n = 3$, $r = 2$ then $\phi(G) = x_{1,2} \cdot x_{3,2} + x_{1,2} \cdot x_{3,1} + x_{1,3} \cdot x_{3,2}$.

Star-mesh transformation: let a vertex $v \neq r$ of *G*. Denote by G_v a graph obtained from *G* by removing *v* and replacing each *xp*,*^q* by $x_{p,q} + x_{p,v} \cdot x_{v,q} \cdot (\sum_k x_{v,k})^{-1}.$ Then $\phi(G) = \sum_k x_{v,k} \cdot \phi(G_v).$

Now *G* is a complete graph with *n* vertices and edges *i*, *j* endowed with two variables $x_{i,j} \neq x_{j,i}.$ Fix a vertex r of G and consider any spanning tree *T* of *G* as having its root at *r* and all its edges directed towards *r* (such directed trees are called *arborescences*). Then the monomial $X^{\mathcal{T}}$ corresponding to this tree is the product of all $x_{i,j}$ over all edges *i*, *j* of *T* according to the chosen direction of *T*. The generating polynomial $\phi(G)$ of arborescences is the sum of all the monomials $X^{\mathcal{T}}.$

Example: if $n = 3$, $r = 2$ then $\phi(G) = x_{1,2} \cdot x_{3,2} + x_{1,2} \cdot x_{3,1} + x_{1,3} \cdot x_{3,2}$.

Star-mesh transformation: let a vertex $v \neq r$ **of** *G***. Denote by** G_v **a**

graph obtained from *G* by removing *v* and replacing each *xp*,*^q* by $x_{p,q} + x_{p,v} \cdot x_{v,q} \cdot (\sum_k x_{v,k})^{-1}.$ Then $\phi(G) = \sum_k x_{v,k} \cdot \phi(G_v).$

Now *G* is a complete graph with *n* vertices and edges *i*, *j* endowed with two variables $x_{i,j} \neq x_{j,i}.$ Fix a vertex r of G and consider any spanning tree *T* of *G* as having its root at *r* and all its edges directed towards *r* (such directed trees are called *arborescences*). Then the monomial $X^{\mathcal{T}}$ corresponding to this tree is the product of all $x_{i,j}$ over all edges *i*, *j* of *T* according to the chosen direction of *T*. The generating polynomial $\phi(G)$ of arborescences is the sum of all the monomials $X^{\mathcal{T}}.$

Example: if $n = 3$, $r = 2$ then $\phi(G) = x_{1,2} \cdot x_{3,2} + x_{1,2} \cdot x_{3,1} + x_{1,3} \cdot x_{3,2}$.

Star-mesh transformation: let a vertex $v \neq r$ of *G*. Denote by G_v a graph obtained from *G* by removing *v* and replacing each $x_{p,q}$ by $\bm{x_{p,q}} + \bm{x_{p,v}} \cdot \bm{x_{v,q}} \cdot (\sum_{k} x_{v,k})^{-1}.$ Then $\phi(G) = \sum_{k} \chi_{v,k} \cdot \phi(G_v).$

Now *G* is a complete graph with *n* vertices and edges *i*, *j* endowed with two variables $x_{i,j} \neq x_{j,i}.$ Fix a vertex r of G and consider any spanning tree *T* of *G* as having its root at *r* and all its edges directed towards *r* (such directed trees are called *arborescences*). Then the monomial $X^{\mathcal{T}}$ corresponding to this tree is the product of all $x_{i,j}$ over all edges *i*, *j* of *T* according to the chosen direction of *T*. The generating polynomial $\phi(G)$ of arborescences is the sum of all the monomials $X^{\mathcal{T}}.$

Example: if $n = 3$, $r = 2$ then $\phi(G) = x_{1,2} \cdot x_{3,2} + x_{1,2} \cdot x_{3,1} + x_{1,3} \cdot x_{3,2}$.

Star-mesh transformation: let a vertex $v \neq r$ of *G*. Denote by G_v a graph obtained from *G* by removing *v* and replacing each $x_{p,q}$ by $x_{p,q} + x_{p,\nu} \cdot x_{\nu,q} \cdot (\sum_k x_{\nu,k})^{-1}$. Then $\phi(G) = \sum_k x_{\nu,k} \cdot \phi(G_\nu)$.

S[c](#page-0-0)hnorr-Valiant-Jerrum-Snir: $C_{+,\times}(\phi(G)) \geq c^n$ [for some](#page-0-0) $c > 1$ $c > 1$ $c > 1$.

 QQ

Now *G* is a complete graph with *n* vertices and edges *i*, *j* endowed with two variables $x_{i,j} \neq x_{j,i}.$ Fix a vertex r of G and consider any spanning tree *T* of *G* as having its root at *r* and all its edges directed towards *r* (such directed trees are called *arborescences*). Then the monomial $X^{\mathcal{T}}$ corresponding to this tree is the product of all $x_{i,j}$ over all edges *i*, *j* of *T* according to the chosen direction of *T*. The generating polynomial $\phi(G)$ of arborescences is the sum of all the monomials $X^{\mathcal{T}}.$

Example: if $n = 3$, $r = 2$ then $\phi(G) = x_{1,2} \cdot x_{3,2} + x_{1,2} \cdot x_{3,1} + x_{1,3} \cdot x_{3,2}$.

Star-mesh transformation: let a vertex $v \neq r$ of *G*. Denote by G_v a graph obtained from *G* by removing *v* and replacing each $x_{p,q}$ by $x_{p,q} + x_{p,\nu} \cdot x_{\nu,q} \cdot (\sum_k x_{\nu,k})^{-1}$. Then $\phi(G) = \sum_k x_{\nu,k} \cdot \phi(G_\nu)$.

Corollary

$$
C_{+,\times,/}(\phi(G))\leq O(n^3).
$$

S[c](#page-0-0)hnorr-Valiant-Jerrum-Snir: $C_{+,\times}(\phi(G)) \geq c^n$ [for some](#page-0-0) $c > 1$ $c > 1$ $c > 1$.

 QQ

Now *G* is a complete graph with *n* vertices and edges *i*, *j* endowed with two variables $x_{i,j} \neq x_{j,i}.$ Fix a vertex r of G and consider any spanning tree *T* of *G* as having its root at *r* and all its edges directed towards *r* (such directed trees are called *arborescences*). Then the monomial $X^{\mathcal{T}}$ corresponding to this tree is the product of all $x_{i,j}$ over all edges *i*, *j* of *T* according to the chosen direction of *T*. The generating polynomial $\phi(G)$ of arborescences is the sum of all the monomials $X^{\mathcal{T}}.$

Example: if $n = 3$, $r = 2$ then $\phi(G) = x_{1,2} \cdot x_{3,2} + x_{1,2} \cdot x_{3,1} + x_{1,3} \cdot x_{3,2}$.

Star-mesh transformation: let a vertex $v \neq r$ of *G*. Denote by G_v a graph obtained from *G* by removing *v* and replacing each *xp*,*^q* by $x_{p,q} + x_{p,\nu} \cdot x_{\nu,q} \cdot (\sum_k x_{\nu,k})^{-1}$. Then $\phi(G) = \sum_k x_{\nu,k} \cdot \phi(G_\nu)$.

Corollary

$$
C_{+,\times,/}(\phi(G))\leq O(n^3).
$$

 $\operatorname{\mathsf{Schnorr-Valiant-Jerrum-Snir: } } C_{+,\,\times}(\phi(G)) \geq c^n$ $\operatorname{\mathsf{Schnorr-Valiant-Jerrum-Snir: } } C_{+,\,\times}(\phi(G)) \geq c^n$ $\operatorname{\mathsf{Schnorr-Valiant-Jerrum-Snir: } } C_{+,\,\times}(\phi(G)) \geq c^n$ [for some](#page-0-0) $c>1.$ $c>1.$ $c>1.$

 QQ

Polya (1923): any homogeneous polynomial $h \in \mathbb{R}[x_1, \ldots, x_n]$ positive as a fraction $P/(x_1+\cdots+x_n)^N$ for suitable integer N and a polynomial

 $\deg(Q)>2^{c2^n}$ for some constant $c>0.1$

(Polya's theorem for h_n) $\infty > C_{+,\times,7}(h_n) > c2^n$ *(lemma)*

most doubles). On the other hand, evidently, *C*[+](#page-80-0), [−](#page-0-0), [×](#page-0-0)[, /](#page-0-0)[\(](#page-0-0)*[h](#page-0-0)[n](#page-0-0)*[\)](#page-0-0) [<](#page-0-0) *[O](#page-0-0)*[\(](#page-0-0)*[n](#page-0-0)*)[.](#page-0-0) 000

Polya (1923): any homogeneous polynomial $h \in \mathbb{R}[x_1, \ldots, x_n]$ positive **everywhere on the non-negative orthant** $\mathbb{R}^n_{\geq 0} \setminus \{0\}$ **can be represented**

as a fraction $P/(x_1+\cdots+x_n)^N$ for suitable integer N and a polynomial

Example: $x^2 - xy + y^2 = (x^3 + y^3)/(x + y)$.

 $\deg(Q)>2^{c2^n}$ for some constant $c>0.1$

(Polya's theorem for h_n) $\infty > C_{+,\times,7}(h_n) > c2^n$ *(lemma)*

most doubles). On the other hand, evidently, *C*[+](#page-81-0), [−](#page-0-0), [×](#page-0-0)[, /](#page-0-0)[\(](#page-0-0)*[h](#page-0-0)[n](#page-0-0)*[\)](#page-0-0) [<](#page-0-0) *[O](#page-0-0)*[\(](#page-0-0)*[n](#page-0-0)*)[.](#page-0-0) 000

Polya (1923): any homogeneous polynomial $h \in \mathbb{R}[x_1, \ldots, x_n]$ positive everywhere on the non-negative orthant $\mathbb{R}^n_{\geq 0}\setminus\{0\}$ can be represented as a fraction $P/(x_1+\cdots+x_n)^N$ for suitable integer N and a polynomial *P* with positive coefficients.

Example: $x^2 - xy + y^2 = (x^3 + y^3)/(x + y)$.

 $\deg(Q)>2^{c2^n}$ for some constant $c>0.$

(Polya's theorem for h_n) $\infty > C_{+,\times,7}(h_n) > c2^n$ *(lemma)*

most doubles). On the other hand, evidently, *C*[+](#page-82-0), [−](#page-0-0), [×](#page-0-0)[, /](#page-0-0)[\(](#page-0-0)*[h](#page-0-0)[n](#page-0-0)*[\)](#page-0-0) [<](#page-0-0) *[O](#page-0-0)*[\(](#page-0-0)*[n](#page-0-0)*)[.](#page-0-0) 000

Polya (1923): any homogeneous polynomial $h \in \mathbb{R}[x_1, \ldots, x_n]$ positive everywhere on the non-negative orthant $\mathbb{R}^n_{\geq 0}\setminus\{0\}$ can be represented as a fraction $P/(x_1+\cdots+x_n)^N$ for suitable integer N and a polynomial *P* with positive coefficients.

Example: $x^2 - xy + y^2 = (x^3 + y^3)/(x + y)$.

 $\deg(Q) > 2^{c2^n}$ for some constant $c > 0$.

(Polya's theorem for h_n) $\infty > C_{+,\times,7}(h_n) > c2^n$ *(lemma)*

most doubles). On the other hand, evidently, *C*[+](#page-83-0), [−](#page-0-0), [×](#page-0-0)[, /](#page-0-0)[\(](#page-0-0)*[h](#page-0-0)[n](#page-0-0)*[\)](#page-0-0) [<](#page-0-0) *[O](#page-0-0)*[\(](#page-0-0)*[n](#page-0-0)*)[.](#page-0-0) 000

Polya (1923): any homogeneous polynomial $h \in \mathbb{R}[x_1, \ldots, x_n]$ positive everywhere on the non-negative orthant $\mathbb{R}^n_{\geq 0}\setminus\{0\}$ can be represented as a fraction $P/(x_1+\cdots+x_n)^N$ for suitable integer N and a polynomial *P* with positive coefficients.

Example: $x^2 - xy + y^2 = (x^3 + y^3)/(x + y)$.

Lemma

For any representation $h_n := (t - x_1)^4 + (x_1 - 2x_2)^4 + (x_2^2 - tx_3)^2 +$ $(x_3^2 - tx_4)^2 + \cdots + (x_{n-1}^2 - tx_n)^2 + 2x_n^4 + 4x_n^2(t - x_1)^2 = P/Q$ where *P*, *Q being polynomials with positive coefficients we have*

(Polya's theorem for h_n) $\infty > C_{+,\times,7}(h_n) > c2^n$ *(lemma)*

most doubles). On the other hand, evidently, *C*[+](#page-84-0), [−](#page-0-0), [×](#page-0-0)[, /](#page-0-0)[\(](#page-0-0)*[h](#page-0-0)[n](#page-0-0)*[\)](#page-0-0) [<](#page-0-0) *[O](#page-0-0)*[\(](#page-0-0)*[n](#page-0-0)*)[.](#page-0-0) 000

Polya (1923): any homogeneous polynomial $h \in \mathbb{R}[x_1, \ldots, x_n]$ positive everywhere on the non-negative orthant $\mathbb{R}^n_{\geq 0}\setminus\{0\}$ can be represented as a fraction $P/(x_1+\cdots+x_n)^N$ for suitable integer N and a polynomial *P* with positive coefficients.

Example: $x^2 - xy + y^2 = (x^3 + y^3)/(x + y)$.

Lemma

For any representation $h_n := (t - x_1)^4 + (x_1 - 2x_2)^4 + (x_2^2 - tx_3)^2 +$ $(x_3^2 - tx_4)^2 + \cdots + (x_{n-1}^2 - tx_n)^2 + 2x_n^4 + 4x_n^2(t - x_1)^2 = P/Q$ where *P*, *Q being polynomials with positive coefficients we have* $\deg(Q) > 2^{c2^n}$ for some constant $c > 0$.

(Polya's theorem for h_n) $\infty > C_{+,\times,7}(h_n) > c2^n$ *(lemma)*

most doubles). On the other hand, evidently, *C*[+](#page-85-0), [−](#page-0-0), [×](#page-0-0)[, /](#page-0-0)[\(](#page-0-0)*[h](#page-0-0)[n](#page-0-0)*[\)](#page-0-0) [<](#page-0-0) *[O](#page-0-0)*[\(](#page-0-0)*[n](#page-0-0)*)[.](#page-0-0) 000

Polya (1923): any homogeneous polynomial $h \in \mathbb{R}[x_1, \ldots, x_n]$ positive everywhere on the non-negative orthant $\mathbb{R}^n_{\geq 0}\setminus\{0\}$ can be represented as a fraction $P/(x_1+\cdots+x_n)^N$ for suitable integer N and a polynomial *P* with positive coefficients.

Example: $x^2 - xy + y^2 = (x^3 + y^3)/(x + y)$.

Lemma

For any representation $h_n := (t - x_1)^4 + (x_1 - 2x_2)^4 + (x_2^2 - tx_3)^2 +$ $(x_3^2 - tx_4)^2 + \cdots + (x_{n-1}^2 - tx_n)^2 + 2x_n^4 + 4x_n^2(t - x_1)^2 = P/Q$ where *P*, *Q being polynomials with positive coefficients we have* $\deg(Q) > 2^{c2^n}$ for some constant $c > 0$.

Corollary

(Polya's theorem for h_n) $\infty > C_{+,\times,/}(h_n) > c2^n$ (lemma)

most doubles). On the other hand, evidently, *C*[+](#page-86-0), [−](#page-0-0), [×](#page-0-0)[, /](#page-0-0)[\(](#page-0-0)*[h](#page-0-0)[n](#page-0-0)*[\)](#page-0-0) [<](#page-0-0) *[O](#page-0-0)*[\(](#page-0-0)*[n](#page-0-0)*)[.](#page-0-0) QQ

Polya (1923): any homogeneous polynomial $h \in \mathbb{R}[x_1, \ldots, x_n]$ positive everywhere on the non-negative orthant $\mathbb{R}^n_{\geq 0}\setminus\{0\}$ can be represented as a fraction $P/(x_1+\cdots+x_n)^N$ for suitable integer N and a polynomial *P* with positive coefficients.

Example: $x^2 - xy + y^2 = (x^3 + y^3)/(x + y)$.

Lemma

For any representation $h_n := (t - x_1)^4 + (x_1 - 2x_2)^4 + (x_2^2 - tx_3)^2 +$ $(x_3^2 - tx_4)^2 + \cdots + (x_{n-1}^2 - tx_n)^2 + 2x_n^4 + 4x_n^2(t - x_1)^2 = P/Q$ where *P*, *Q being polynomials with positive coefficients we have* $\deg(Q) > 2^{c2^n}$ for some constant $c > 0$.

Corollary

(Polya's theorem for h_n) $\infty > C_{+,\times,/}(h_n) > c2^n$ *(lemma)*

most doubles). On t[h](#page-0-0)e other ha[n](#page-0-0)d, evidently, C_{\pm} , $\sum_{i=1}^{\infty}$ /[\(](#page-0-0)*h[<](#page-0-0)sub>n</sub>*[\)](#page-0-0) $\leq Q(n)$ [.](#page-0-0) 000

Polya (1923): any homogeneous polynomial $h \in \mathbb{R}[x_1, \ldots, x_n]$ positive everywhere on the non-negative orthant $\mathbb{R}^n_{\geq 0}\setminus\{0\}$ can be represented as a fraction $P/(x_1+\cdots+x_n)^N$ for suitable integer N and a polynomial *P* with positive coefficients.

Example: $x^2 - xy + y^2 = (x^3 + y^3)/(x + y)$.

Lemma

For any representation $h_n := (t - x_1)^4 + (x_1 - 2x_2)^4 + (x_2^2 - tx_3)^2 +$ $(x_3^2 - tx_4)^2 + \cdots + (x_{n-1}^2 - tx_n)^2 + 2x_n^4 + 4x_n^2(t - x_1)^2 = P/Q$ where *P*, *Q being polynomials with positive coefficients we have* $\deg(Q) > 2^{c2^n}$ for some constant $c > 0$.

Corollary

(Polya's theorem for
$$
h_n
$$
) $\infty > C_{+, \times, /}(h_n) > c2^n$ (lemma)

(since after each operation the degree of a rational representation at most doubles). On t[h](#page-0-0)e other ha[n](#page-0-0)d, evidently, C_{\pm} , $\chi_{\rm eff}(h_n) \leq Q(n)$ $\chi_{\rm eff}(h_n) \leq Q(n)$ $\chi_{\rm eff}(h_n) \leq Q(n)$ $\chi_{\rm eff}(h_n) \leq Q(n)$ $\chi_{\rm eff}(h_n) \leq Q(n)$ [.](#page-0-0)

Polya (1923): any homogeneous polynomial $h \in \mathbb{R}[x_1, \ldots, x_n]$ positive everywhere on the non-negative orthant $\mathbb{R}^n_{\geq 0}\setminus\{0\}$ can be represented as a fraction $P/(x_1+\cdots+x_n)^N$ for suitable integer N and a polynomial *P* with positive coefficients.

Example: $x^2 - xy + y^2 = (x^3 + y^3)/(x + y)$.

Lemma

For any representation $h_n := (t - x_1)^4 + (x_1 - 2x_2)^4 + (x_2^2 - tx_3)^2 +$ $(x_3^2 - tx_4)^2 + \cdots + (x_{n-1}^2 - tx_n)^2 + 2x_n^4 + 4x_n^2(t - x_1)^2 = P/Q$ where *P*, *Q being polynomials with positive coefficients we have* $\deg(Q) > 2^{c2^n}$ for some constant $c > 0$.

Corollary

(Polya's theorem for
$$
h_n
$$
) $\infty > C_{+, \times, /}(h_n) > c2^n$ (lemma)

(since after each operation the degree of a rational representation at most doubles). On t[h](#page-0-0)e other ha[n](#page-0-0)d, evidently, $C_{+,-, \times}$ $C_{+,-, \times}$ $C_{+,-, \times}$, $(h_n) < O(n)$ $(h_n) < O(n)$ [.](#page-0-0) Ω