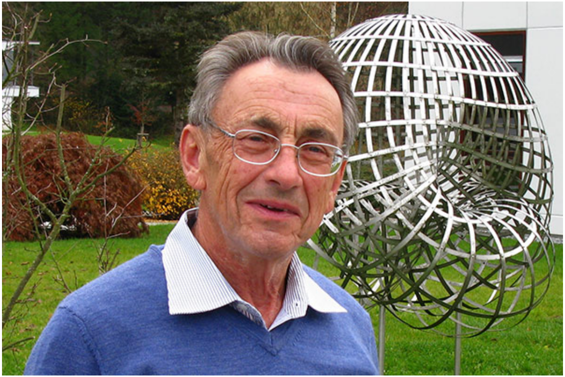
Regular properties and the existence of proof systems

Proof Complexity Workshop
St Petersburg, 17–20 May, 2016

Rosalie Iemhoff
Utrecht University

In memory of Grigori Mints (1939–2014)



An old question: When does a logic have a (decent) proof system?

Applications:
Syntactic description, consistency, decidability, explicit witnesses, ...

Georg Kreisel: proof theory begins where recursion theory ends.

Focus on particular proof systems, such as sequent calculi or natural deduction.

A good proof system: You know it when you see it.

Proof-Theoretic Semantics is an area concerned with the development of good proof systems, building on the philosophy of Michael Dummett.

Another old question:
When does a logic have a (cut-free) sequent calculus?

Observation: *Many positive instances. Less negative ones.*

Negative answers based on a mismatch between the complexity of the logic and the sequent calculus, are not the aim. Rather, to obtain negative answers for those logics to which this does not apply.

(Negri) Fix a labelled sequent calculus and determine which axioms, when added, preserve cut-elimination.

(Ciabattoni, Galatos, Terui) Fix a sequent calculus and determine which structural rules, when added, preserve cut-elimination.

Aim 1: Formulate regular properties *that, when violated by a logic, imply that the logic does not have a sequent calculus of a certain form.*

In other words, prove theorems of the form: if a logic has such and such a calculus, then it has such and such a property.

Logics: Although wider applicable, only modal and intermediate propositional logics will be considered.

Aim 2: Prove that certain logics satisfy certain regular properties.

The logics considered are classical and intermediate (modal) logics.

Dfn *The language of the logics consists of atoms* $p, q, r, \ldots$, *constants* $\top$ *and* $\bot$, *connectives* $\wedge, \vee, \to$, *and possibly a modal operator* $\Box$. $\neg\varphi$ *is defined as* $\varphi \to \bot$.

All logics are extensions of intuitionistic logic IPC. *In the case of modal logics we do not require them to be normal.*

Dfn Sequents *are expressions* $\Gamma \Rightarrow \Delta$*, where* Γ *and* Δ *are finite multisets, interpreted as* $I(\Gamma \Rightarrow \Delta) = (\bigwedge \Gamma \to \bigvee \Delta)$*.*

A sequent calculus *is a collection of axioms and rules that consist of sequents, such as*

$$\frac{}{\varphi \Rightarrow \varphi} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi} \qquad \frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \to \psi, \Delta}$$

GC is a standard cut-free calculus for classical propositional logic CPC. It has no structural rules, which are admissible.

GI is a standard cut-free calculus for intuitionistic propositional logic IPC. It has no structural rules, which are admissible, and every sequent has at most one formula at the right, such as in the rules

$$\frac{\Gamma \Rightarrow \varphi \quad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \wedge \psi} \qquad \frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \to \psi}$$

A regular property *of a logic is not a precisely defined notion.*

I use it in this talk for properties that can be proved to hold in all logics that have a certain sequent calculus.

Examples are Skolemization, Herbrand's Theorem, interpolation, uniform interpolation, . . .

Aim 1: Formulate regular properties *that, when violated by a logic, imply that the logic does not have a sequent calculus of a certain form.*

In other words, prove theorems of the form: if a logic has such and such a calculus, then it has such and such a property.

Aim 2: Prove that certain logics satisfy certain regular properties.

Logics: Although wider applicable, only modal and intermediate propositional logics will be considered.

Regular property: Uniform interpolation.

Note The broader the class of calculi, the stronger the result.

Dfn *A logic has* interpolation *if, whenever* $\vdash \varphi \to \psi$ *there is a* χ *in the common language of* φ *and* ψ *such that* $\vdash \varphi \to \chi$ *and* $\vdash \chi \to \psi$.

What the common language is depends on the logic and the approach. In case of propositional logic: the variables that occur in both φ *and* ψ *and all constants and connectives.*

Ex *The interpolants for* $(p \wedge q \to p \vee r)$ *and* $(p \wedge \neg p \to q)$ *are* p *and* $\bot$.

William Craig proved in 1957 that classical predicate logic has interpolation.

Later it was shown by many others (including Gabbay, Maxsimova, Schütte, Smoryński) that the following modal and intermediate logics have interpolation:

$$CPC, IPC, KC, LC, K, K4, GL, S4.$$

Thm (Maxsimova '79)
There are exactly 7 propositional intermediate logics with interpolation.

Thm (Maxsimova '79)
There are at most 37 extensions of $S4$ with interpolation.

Thm (Mints, Olkhovikov, Urquhart '13)
The intermediate logic of constant domains does not have interpolation.

Dfn *A logic has* uniform interpolation *if the interpolant depends only on the premiss or the conclusion: For all* φ *there are formulas* $\exists p\varphi$ *and* $\forall p\varphi$ *in* $\mathcal{L}(\varphi)$ *not containing* p *such that for all* χ *not containing* p*:*

$$\vdash \varphi \to \chi \;\Leftrightarrow \vdash \exists p\varphi \to \chi \qquad \vdash \chi \to \varphi \;\Leftrightarrow \vdash \chi \to \forall p\varphi.$$

$\exists p\varphi$ *is the* post *or* right *(uniform) interpolant,* $\forall p\varphi$ *the* pre *or* left *interpolant.*

$\forall p$ *satisfies the standard rules for universal quantification:*

$$\forall p\varphi \to \varphi \qquad \varphi \to \exists p\varphi \qquad \frac{\psi \to \varphi}{\psi \to \forall p\varphi}\ (p \text{ not in } \psi) \qquad \frac{\varphi \to \psi}{\exists p\varphi \to \psi}\ (p \text{ not in } \psi)$$

Ex *In CPC:* $\forall p p = \bot$ *and* $\exists p p = \top$. $\forall p(p \vee q) = q$ *and* $\exists p(p \vee q) = \top$.

Prop *Uniform interpolation implies interpolation: Two interpolants of* $\varphi(\bar{p}, \bar{q}) \to \psi(\bar{p}, \bar{r})$*, where* $\bar{p}$*,* $\bar{q}$ *and* $\bar{r}$ *are pairwise disjunct, are*

$$\exists \bar{q}\varphi(\bar{p}, \bar{q}) \qquad \forall \bar{r}\psi(\bar{p}, \bar{r}).$$

Develop a method to obtain uniform interpolants from certain sequent calculi.

Show that logics without uniform interpolation cannot have calculi of that form.

Side benefit: Modular proofs of uniform interpolation.

Prop CPC has uniform interpolation.

Thm (Pitts '92) IPC has uniform interpolation.

Thm (Shavrukov '94) GL has uniform interpolation.

Thm (Ghilardi & Zawadowski '95)
K has uniform interpolation. S4 does not. There are exactly six extensions of S4 with uniform interpolation.

Thm (Bílková '06) KT has uniform interpolation. K4 does not.

Thm (Maxsimova '77, Ghilardi & Zawadowski '02)
There are exactly seven intermediate logics with (uniform) interpolation:

$$\mathsf{IPC}, \mathsf{Sm}, \mathsf{GSc}, \mathsf{LC}, \mathsf{KC}, \mathsf{Bd}_2, \mathsf{CPC}.$$

Pitts uses Dyckhoff's '92 sequent calculus for IPC.

Given a collection of sequent calculi, we sketch (seriously simplifying) how to prove that a logic with a sequent calculus in that class has uniform interpolation.

Given a sequent calculus G and a logic L, define for every instance

$$\frac{S_1 \quad \dots \quad S_n}{S_0}\ R$$

of an axiom or rule in G, the formula $\forall\overset{R}{p}S_0$ *in terms of* $\forall p S_i$ $(i > 0)$.

$$\forall p S \quad \equiv_{df} \quad \bigvee\{\forall\overset{R}{p}S \mid R \text{ is an instance of a rule with conclusion } S\} \vee \bigvee \forall p^{not} S \vee \forall p^{at} S.$$

Likewise for $\exists$.

Dfn *A calculus is* terminating *if there exists a well-founded order* $\prec$ *on sequents such that in every rule the premisses come before the conclusion, sequents come after proper subsequents, and any sequent is the conclusion of at most finitely many instances of rules in G.*

Dfn A rule is focussed *if it is of the form*

$$\frac{S \cdot S_1 \quad \dots \quad S \cdot S_n}{S \cdot S_0}$$

where S, S_i *are sequents and* S_0 *contains exactly one formula, which is not an atom.* $(\Gamma \Rightarrow \Delta) \cdot (\Pi \Rightarrow \Sigma) = (\Gamma, \Pi \Rightarrow \Delta, \Sigma)$

Ex Focussed:

$$\frac{\Gamma \Rightarrow \varphi, \Delta \quad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \wedge \psi, \Delta} \qquad \frac{\Gamma, \varphi \to \chi, \psi \to \chi \Rightarrow \eta}{\Gamma, \varphi \vee \psi \to \chi \Rightarrow \eta}$$

Every rule in Dyckhoff's '92 calculus except one is focussed.

Similar notion of modal focussed rules*.*

Thm *A modal logic with a balanced terminating calculus all of which axioms and rules are (modal) focussed has uniform interpolation.*

Cor *CPC, K, KD have uniform interpolation* (K first proved by Visser).

Cor *The logic K4 does not have a balanced terminating sequent calculus that consists of (modal) focussed axioms and rules. The same holds for all extensions of S4, except for the six that have uniform interpolation.*

Dfn *Over K, the logics K4 and KD are axiomatized by*

$$\Box\varphi \to \Box\Box\varphi \qquad \Box\varphi \to \neg\Box\neg\varphi,$$

respectively. Logic S4 is K4 plus $\Box\varphi \to \varphi$*. Possible sequent rule for K4:*

$$\frac{\Box\Gamma, \Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi}$$

Thm Any (modal) intermediate logic with a terminating normal calculus that consists of Dyckhoff's calculus, focussed rules, and nonisolated modal focussed rules, has uniform interpolation.

Cor IPC, iK and iKD have uniform interpolation (IPC proved by Pitts).

Cor Except for the seven intermediate logics that have interpolation, no intermediate logic has a sequent calculus as in the theorem.

- *Extend the method to other modal logics, such as GL, KT, iGL.*
- *Find alternative regular properties to apply method to.*
- *Extend the method to predicate theories.*
- *Use other proof systems than sequent calculi.*

Finis